Minimal surfaces in finite volume non compact hyperbolic 3-manifolds

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Abstract

We prove there exists a compact embedded minimal surface in a complete finite volume hyperbolic 3-manifold $\mathcal{N}$. We also obtain a least area, incompressible, properly embedded, finite topology, 2-sided surface. We prove a properly embedded minimal surface of bounded curvature has finite topology. This determines its asymptotic behavior. Some rigidity theorems are obtained.

1 Introduction

There has been considerable progress on the study of properly embedded minimal surfaces in euclidean 3-space. We now know all such orientable surfaces that are planar domains; they are planes, helicoids, catenoids and Riemanns'minimal surfaces. Also we understand the geometry of properly embedded periodic minimal surfaces, that are finite topology in the quotient.

In hyperbolic 3-space, there is no classification of this nature. A continuous rectifiable curve in (the boundary at infinity of $\mathbb{H}^3$) is the asymptotic boundary of a least area embedded simply connected surface.

In this paper we study the existence of periodic minimal surfaces in $\mathbb{H}^3$. More precisely, we consider surfaces in complete non compact hyperbolic 3-manifolds $\mathcal{N}$ of finite volume. In the following of the paper, we will refer to such manifolds $\mathcal{N}$ as hyperbolic cusp manifolds. In a closed hyperbolic manifold (or any closed Riemannian 3-manifold), there is always a compact embedded minimal surface [10]. They cannot be of genus zero or one, but there are many higher genus such surfaces. The existence and deformation theory of such surfaces was initiated by K. Uhlenbeck [15].

Hyperbolic cusp manifolds play an important role in the theory of closed hyperbolic 3-manifolds. Many link complements in the unit 3-sphere have such a finite volume hyperbolic structure. Given any $V > 0$, Jorgensen proved there are a finite number of such $\mathcal{N}$ of volume V. Then Thurston proved that a closed hyperbolic 3-manifold of volume less than V can be obtained from this finite number of manifolds $\mathcal{N}$ given by Jorgensen, by hyperbolic Dehn surgery on at least one of the cusp ends (see [2] for details).

We will prove there is a compact embedded minimal surface in any complete hyperbolic 3-manifold of finite volume. Since such a non compact manifold $\mathcal{N}$ is "not convex at infinity", minimization techniques do not produce such a minimal surface. To understand this the reader can verify that on a complete hyperbolic 3-punctured 2-sphere, there is no simple closed geodesic. In dimension 3, a min-max technique, together with several maximum principles in the cusp ends of $\mathcal{N}$, will produce compact embedded minimal surfaces.

We will give two existence results of embedded compact minimal surfaces.

Theorem A. *There is a compact embedded minimal surface Σ in $\mathcal{N}$.*

Theorem B. *Let S be a closed orientable embedded surface in $\mathcal{N}$ which is not a 2-sphere or a torus. If S is incompressible and non-separating, then S is isotopic to a least area embedded minimal surface.*

Concerning properly embedded non compact minimal surfaces, there are already existence results due to Hass, Rubinstein and Wang [7] and Ruberman [12]. Using different arguments, we give an other proof of Ruberman's minimization result.

Theorem. *Let S be a properly embedded, non compact, finite topology, incompressible, non separating surface in $\mathcal{N}$. Then S is isotopic to a least area embedded minimal surface.*

The surfaces produced by the above theorem have bounded curvature. Actually the techniques we develop enable us to prove:

Theorem C. *Let Σ be a properly embedded minimal surface in $\mathcal{N}$ of bounded curvature. Then Σ has finite topology.*

Since stable minimal surfaces have bounded curvature we conclude:

Corollary 1. *A properly embedded stable minimal surface in $\mathcal{N}$ has finite topology.*

Finite topology is particularly interesting here due to the Finite Total curvature theorem below that describes the geometry of the ends of a properly immersed minimal surface in $\mathcal{N}$ of finite topology

Theorem 2 (Collin, Hauswirth, Rosenberg [4]). *A properly immersed minimal surface Σ in $\mathcal{N}$ of finite topology has finite total curvature*

$$\int_\Sigma K_\Sigma = 2\pi\chi(\Sigma)$$

Moreover, each end A of Σ is asymptotic to a totally geodesic 2-cusp end in an end C of $\mathcal{N}$.

We will make precise these notions.

The simplest example of a surface Σ with finite topology appearing in the above Theorem is a 3-punctured sphere. Actually, minimal 3-punctured spheres are totally geodesic.

Theorem D. *A proper minimal immersion of a 3-punctured sphere in $\mathcal{N}$ is totally geodesic.*

The paper is organized as follows. In Section 2, we make some general remarks on the geometry of cusp manifolds stating some results of Jorgensen, Thurston and Adams. In Section 3, we consider 3-punctured spheres in hyperbolic cusp manifolds and prove Theorem D. In Section 4, we study minimal surfaces entering the ends of hyperbolic cusp manifolds $\mathcal{N}$. We prove two maximum principles which govern the geometry of minimal surfaces in the ends of $\mathcal{N}$. We also establish a transversality result which is used to study annular ends of minimal surfaces. Section 5 proves Theorems A and B, the existence of compact embedded minimal surfaces in hyperbolic cusp manifolds. Section 6 proves the minimization result in the non compact case. Then in Section 7, we present several examples to illustrate these theorems.

2   Some discussion of the manifolds $\mathcal{N}$

In this section we recall some facts about the geometry of a non compact hyperbolic 3-manifold $\mathcal{N}$ of finite volume.

Such $\mathcal{N}$ are the union of a compact submanifold bounded by mean concave mean curvature one tori, and a finite number of ends, each end isometric to a quotient of a horoball of $\mathbb{H}^3$ by a $\mathbb{Z}^2$ group of parabolic isometries leaving

the horoball invariant. The horospheres in this horoball quotient to mean curvature one tori in $\mathcal{N}$.

An end of $\mathcal{N}$ can be parametrized by $M = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 1/2\}$ modulo a group $G = G(v_1, v_2)$, generated by two translations by linearly independent horizontal vectors $v_1, v_2 \in \mathbb{R}^2 \times \{0\}$.

The end $C = M/G$ is endowed with the quotient of the hyperbolic metric of M :

$$g_\mathbb{H} = \frac{1}{z^2}(dx^2 + dy^2 + dz^2) = \frac{1}{z^2}dX^2.$$

The horospheres $\{z = c\}$ quotient to tori $T(c)$, of mean curvature one with respect to the unit normal vector $z\partial z$. The vertical curves $\{(x, y) = \text{constant}\}$ are geodesics orthogonal to the tori $T(c)$, with arc length given by $s = \ln z$. The induced metric on $T(c)$ is flat and lengths on $T(c)$ decrease exponentially as $s \to \infty$.

We will denote by $T(a, b)$ the subset $\{a \leq z \leq b\}$ of C.

The Euclidean planes $\{ax + by = c\}$ are totally geodesic surfaces in C. When they are properly embedded in C, they are the totally geodesic 2-cusp ends in C that appears in the Finite Total Curvature Theorem above.

Define $\Lambda(C) = \max\{|v_1|, |v_2|\}$ with $|v|$ the Euclidean norm. We notice that we have made a choice of generators v_1, v_2 of the group G, so the value $\Lambda(C)$ depends on this choice (we can minimize the value of Λ among all choices but it is not important in the following).

Remark 1. The above notations are well adapted to study the geometry close to $z = 1$. For z_0 larger than 1, let H be the map $(x, y, z) \mapsto (z_0 x, z_0 y, z_0 z)$ which sends M to $\mathbb{R}^2 \times [z_0/2, +\infty)$. This map gives us then a chart of $C' = \{z \geq z_0/2\} \subset C$ parametrized by $\{z' \geq 1/2\}$ with $\Lambda(C') = \Lambda(C)/z_0$. So, considering a part of the end that is sufficiently far away, we can always assume that $\Lambda(C)$ is small.

We mention two theorems concerning the manifolds $\mathcal{N}$.

Theorem 3 (Jorgensen). *Given $V > 0$, there exist a finite number of such manifolds $\mathcal{N}$ whose volume is equal to V.*

Theorem 4 (Thurston). *Any compact hyperbolic 3-manifold M^3, $\partial M^3 = \emptyset$, with $Vol(M) < V$ is obtained from the finite number of $\mathcal{N}$ given by Jorgensens' theorem, by hyperbolic Dehn surgery on at least one of the cusp ends.*

Concerning surface theory in $\mathcal{N}$, we mention one theorem that inspired Theorem D.

Theorem 5 (Adams [1])**.** *Let Σ be a properly embedded 3-punctured sphere in $\mathcal{N}$, Σ incompressible. Then Σ is isotopic to a totally geodesic 3-punctured sphere in $\mathcal{N}$.*

3   Minimal 3-punctured spheres are totally geodesic

In this section we prove that, under some hypotheses, a minimal surface is totally geodesic. We first have the following result.

Theorem D. *A proper minimal immersion of a 3-punctured sphere in $\mathcal{N}$ is totally geodesic. Moreover, it is π_1 injective.*

Proof. Let $\Sigma \subset \mathcal{N}$ be a properly immersed minimal 3-punctured sphere in $\mathcal{N}$. Let $x_0 \in \Sigma$ and α, β, γ be three loops at x_0 that are freely homotopic to embedded loops in the different ends of Σ.

Let $\pi : \mathbb{H}^3 \to \mathcal{N}$ be a universal covering map and $\tilde{x}_0$ be in $\pi^{-1}(x_0)$. Let $\widetilde{\Sigma}$ be the lift of Σ passing through $\tilde{x}_0$. The choice of $\tilde{x}_0$ induces a monomorphism $\varphi : \pi_1(\Sigma, x_0) \to \mathrm{Isom}^+(\mathbb{H}^3)$. $\widetilde{\Sigma}$ is then a proper immersion of the quotient of the universal cover of Σ by $\ker \varphi$. Let Γ be the image of the monomorphism φ. As a consequence $\widetilde{\Sigma}$ is properly immersed in $\mathbb{H}^3$. Let us denote by T_α , T_β and T_γ the maps in Γ associated by φ to $[\alpha], [\beta], [\gamma]$.

By the Finite Total Curvature Theorem 2, we know each end is asymptotic to $\mu \times \mathbb{R}_+$ where μ is a geodesic in some $T(c)$ in a cusp end of $\mathcal{N}$; $\mu \times \mathbb{R}_+$ is a totally geodesic annulus in this cusp end. The inclusion of $T(c)$ into $\mathcal{N}$ induces an injection of the fundamental group of $T(c)$ into that of $\mathcal{N}$. Hence α, β and γ are sent to non zero parabolic elements of $\mathrm{Isom}^+(\mathbb{H}^3)$ by φ.

Next we will prove the limit set of $\widetilde{\Sigma}$ is a circle C in $\partial_\infty \mathbb{H}^3 \simeq \mathbb{S}^2$. Then the maximum principle yields that $\widetilde{\Sigma}$ is the totally geodesic plane P bounded by C, thus proving Theorem C. More precisely, foliate $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ minus two points by totally geodesic planes and their asymptotic boundaries so that P is one leaf of the foliation. This foliation at $\partial_\infty \mathbb{H}^3$ is a foliation by circles with two "poles" p and q. The circles close to p bound hyperbolic planes Q in $\mathbb{H}^3$ that are disjoint from $\widetilde{\Sigma}$. As the circles in the foliations of $\partial_\infty \mathbb{H}^3$ go from p to C, there can be no first point of contact of the planes with $\widetilde{\Sigma}$ ($\widetilde{\Sigma}$ is proper and the limit set of $\widetilde{\Sigma}$ is C). Hence $\widetilde{\Sigma}$ is in the half space of $\mathbb{H}^3 \setminus P$ containing q. The same argument with planes coming from q to C shows that $\widetilde{\Sigma} = P$. So $\widetilde{\Sigma}$ is simply connected which implies $\ker \varphi = \{1\}$ and Σ is π_1 injective.

Let us now prove the existence of C. We have the following claim whose proof is based on Adams work [1].

Claim 1. *There is a circle C in $\partial_\infty \mathbb{H}^3$ which is invariant by Γ. The limit set of $\widetilde{\Sigma}$ is C.*

Proof. Using the half space model for $\mathbb{H}^3$ with ∞ the fixed point of T_α and using the $SL_2(\mathbb{C})$ representation of $\text{Isom}^+(\mathbb{H}^3)$ we can write,

$$T_\alpha = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \qquad T_\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $w \in \mathbb{C}^*$, $a, b, c, d \in \mathbb{C}$ such that $ad - bc = 1$ and $a + d = 2$. Then $T_\gamma = T_\alpha \cdot T_\beta = \begin{pmatrix} a + cw & b + dw \\ c & d \end{pmatrix}$ is parabolic so it must satisfy to $\lambda = a + cw + d = \pm 2$.

Since $a + d = 2$ we have $c = 0$ if $\lambda = 2$ or $c = -4/w$ if $\lambda = -2$. If $c = 0$, T_α, T_β and T_γ would fix the point ∞ and all elements in Γ would have ∞ as fixed point and $\{\infty\}$ is the limit set of Γ. We will rule out this possibility below.

If $c = -4/w$, the fixed point of T_β is $x_\beta = \frac{w(d-a)}{8}$ and the fixed point of T_γ is $x_\gamma = \frac{w(d-a)}{8} + \frac{w}{2}$. T_α leaves invariant the circle $C = \{\frac{w(d-a)}{8} + tw, t \in \mathbb{R}\} \cup \{\infty\}$. Also we have

$$T_\beta(\infty) = -\frac{wa}{4} = \frac{w(d-a)}{8} = \frac{w}{4} \in C$$

$$T_\beta(x_\gamma) = T_\alpha^{-1}(x_\gamma) = \frac{w(d-a)}{8} - \frac{w}{2} \in C$$

Hence T_β leaves C invariant. Thus Γ leaves C invariant. Actually, C is the limit set of Γ.

Now let us see that the limit set of $\widetilde{\Sigma}$ is the limit set $\partial \Gamma$ of Γ. We split Σ in the union of a compact part K containing x_0 and three cusp ends C_i. So $\widetilde{\Sigma}$ split in the union of the lift $\widetilde{K}$ of K and the union of pieces contained in disjoint horoballs H_α with boundary along the horoball. Because of the asymptotic behavior of Σ, the lift g_i of ∂C_i in ∂H_α is not homeomorphic to a circle. Thus there is a non trivial $\gamma \in \Gamma$ that leaves g_i and then H_α invariant. So the center of H_α belongs to $\partial \Gamma$.

If $\partial \Gamma$ is only one point (case $c = 0$) it means that there is only one horoball H_α: it is $\{z \geq c\}$. Since any point in $\widetilde{K}$ is at a finite distance from H_α and $\widetilde{K}$ is periodic, the z function reaches its minimum somewhere. The maximum principle then get a contradiction. So $c \neq 0$ and $\partial \Gamma = C$.

Let (p_i) be a proper sequence of points in $\widetilde{\Sigma}$ and assume it converges to some point $p_\infty \in \partial_\infty \mathbb{H}^3$. If all the p_i belong to $\widetilde{K}$, there is a sequence of elements $\gamma_i \in \Gamma$ such that the distance between p_i and $\gamma_i \cdot \tilde{x}_0$ stays bounded.

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So (p_i) and $(\gamma_i \cdot \tilde{x}_0)$ have the same limit so p_∞ is in the limit set of Γ so in C.

So we can assume that $p_i \in H_{\alpha_i}$ for all i. If the sequence (α_i) is finite; p_∞ is a center of one of the H_α so it is in C. If the sequence (α_i) is not finite the the distance from $\tilde{x}_0$ to H_{α_i} goes to ∞. There is a neighborhood N_m of C in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ that contains all horoballs centered on C whose distance to $\tilde{x}_\infty$ is larger than m and such that $\cap_{m>0} N_m = C$. So $p_\infty \in C$. $\qquad \square$

$\square$

The proof of the preceding result is based on the study of the group representation φ. It can be controlled under some other hypotheses.

Proposition 6. *Let S_0 and S_1 be two properly immersed minimal surfaces in $\mathcal{N}$ such that the immersions are homotopic. If S_0 is totally geodesic, then $S_0 = S_1$.*

Proof. Let $f_t : S \times [0,1] \to \mathcal{N}$ be the homotopy between the two minimal immersions. Let $x_0 \in S$ and $\tilde{x}_0$ be a point in $\pi^{-1}(f_0(x_0))$ where $\pi : \mathbb{H}^3 \to \mathcal{N}$ is a covering map. Let $g_t : \widetilde{S} \times [0,1] \to \mathcal{N}$ be the lift of f_t such that $g_0(x_0) = \tilde{x}_0$. This defines a group representation $\varphi : \pi_1(S \times [0,1], (x_0,0)) = \pi_1(S, x_0) \to \text{Isom}^+(\mathbb{H}^3)$ such that g is φ-equivariant. Since $f_0(S)$ is totally geodesic, $g_0(\widetilde{S})$ is a totally geodesic disk. This implies that the image of φ has a circle C as limit set. As in the proof of Claim 1, it implies that $g_1(\widetilde{S})$ has C as asymptotic boundary. Then, as in the proof of Theorem C, $g_1(\widetilde{S})$ is totally geodesic and equals $g_0(\widetilde{S})$. So $f_0(S) = f_1(S)$. $\qquad \square$

4   Minimal surfaces in the cusp ends of $\mathcal{N}$

In this section, we will analyse the behaviour of embedded minimal surfaces that enter cusp ends of $\mathcal{N}$. In dimension 2, the situation is simple. If N^2 is a 2-cusp (*i.e.* a quotient of a horodisk of $\mathbb{H}^2$ by a parabolic isometry leaving the horodisk invariant) then a geodesic that enters N^2 either goes straight to the cusp (*i.e.* it is an orthogonal trajectory of the horocycles of the cusp) or it leaves N^2 in a finite time. In dimension 3, for the moment we know that a properly immersed minimal annulus that enters a cusp end of $\mathcal{N}^3$ is asymptotic to a 2-cusp of the end $(\gamma \times [c,\infty)$, γ a compact geodesic of $T(c))$, or the intersection of the minimal annulus with the end of $\mathcal{N}$ is compact.

We will establish two maximum principles in the ends of $\mathcal{N}$ which will control the geometry of embedded minimal surfaces in the ends.

Let $C = M/G(v_1, v_2)$ be an end of $\mathcal{N}$, parametrized by the quotient of $M = \{(x, y, z) \in \mathbb{R}^3 | z \geq 1/2\}$ as in Section 2, with $\Lambda(C) = \Lambda(v_1, v_2)$ the diameter of $T(1)$. Recall that we can make $\Lambda(C)$ as small as we wish by passing to a subend C' of C defined by $z \geq z_0$, z_0 large.

We modify the metric on C introducing smooth functions $\Psi : [1/2, \infty) \to \mathbb{R}$ satisfying $\Psi(z) = z$ for $1/2 \leq z \leq 1$, and Ψ non decreasing. There will be other conditions on Ψ as we proceed.

Let $g_\Psi = \frac{1}{\Psi^2(X)} dX^2$ be a new metric on C; g_Ψ is the hyperbolic metric of $\mathcal{N}$ for $1/2 \geq z \geq 1$.

The mean curvature of the torus $T(z)$ in the metric g_Ψ equals $\Psi'(z)$, with respect to the unit normal $\Psi(z)\partial_z$ (so points towards the cusp: perhaps it is zero). The sectional curvatures for g_Ψ are:

$$K_{g_\Psi} = \begin{cases} -\Psi'(z)^2 \text{ for the } (\partial_x, \partial_y) \text{ plane} \\ \Psi(z)\Psi''(z) - \Psi'(z)^2 \text{ for the } (\partial_x, \partial_z) \text{ and } (\partial_y, \partial_z) \text{ planes} \end{cases}$$

We will always introduce Ψ's such that $|\Psi'|$ and $|\Psi\Psi''|$ are bounded by some fixed constant. Hence the sectional curvatures of the new metrics will be uniformly bounded as well. Then given $\varepsilon_0 > 0$, there is a $k_0 > 0$ such that a stable minimal surface in (C, g_Ψ) has curvature bounded by k_0 at all points at least at a distance ε_0 from the boundary. The bound k_0 depends only on the bound of the sectional curvatures and ε_0 not on the injectivity radius [11]

Remark 2. The pull back of the g_Ψ by the map H defined in Remark 1 is $H^* g_\Psi = g_{\Psi_{z_0}}$ where $\Psi_{z_0}(z) = \frac{1}{z_0}\Psi(z_0 z)$. This modification does not change the estimates on Ψ' and $\Psi\Psi''$.

4.1 Maximum principles

In the section we prove maximum principles for a cusp end C endowed with a metric g_Ψ. The following estimates will depend on an upper-bound on $|\Psi'|$ and $|\Psi\Psi''|$.

We have a first result.

Proposition 7 (Maximum principle I). *Let $k_0, \varepsilon_0 > 0$. There is a $\Lambda_0 = \Lambda(k_0, \varepsilon_0)$ such that if Σ is an embedded minimal surface in (C, g_Ψ) with $|A_\Sigma| \leq k_0$ and $\Lambda(C) \leq \Lambda_0$. Then if $p \in \Sigma$ is at least an intrinsic distance ε_0 from $\partial\Sigma$ and if $z(q) \leq z(p)$ for all q in the intrinsic ε_0-disc centered at p, then $\Sigma = \{z = z(p)\}$ and $\Psi'(z(p)) = 0$.*

Proof. Let $\pi : M \to C$ be the covering projection and $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$. We may suppose $p = \pi(0, 0, z(p))$.

Since the curvature of $\widetilde{\Sigma}$ is bounded by k_0, $\widetilde{\Sigma}$ is a graph of bounded geometry in a neighborhood of p. Hence the exists $\mu = \mu(k_0, \varepsilon_0)$ and a smooth function $u : D_\mu(0,0) \to \mathbb{R}$, $D_\mu(0,0) = \{x^2 + y^2 \leq \mu\}$, $u(0,0) = z(p)$ and the graph of u in M is a subset of $\widetilde{\Sigma}$. μ can be chosen such that, if $q \in \mathrm{graph}(u)$, $d_\Sigma(\pi(q), p) < \varepsilon_0$. Hence $z(p)$ is a maximum value of u in $D_\mu(0,0)$.

Now $\widetilde{\Sigma}$ is invariant by $G(v_1, v_2)$ so if $\Lambda_0 < \mu/2$, we have $v_1, v_2 \in D_{\mu/2}(0,0)$. So $D_\mu(0,0) \cap D_\mu(v_1) = D \neq \emptyset$.

Let $u_1 : D \to \mathbb{R}$, be $u_1(q) = u(q - v_1)$; the graph of u_1 is contained in $\widetilde{\Sigma}$. Then $u_1(v_1) = u(O) \geq u(v_1)$ since u reaches its maximum at $O = (0,0)$. Also $O \in D$, $u_1(O) = u(-v_1) \leq u(O)$. Thus the graphs of u and u_1 over D must intersect and since $\widetilde{\Sigma}$ is embedded, $u = u_1$ on D. Hence u_1 is a smooth continuation of u to $D_\mu(O) \cup D_\mu(v_1)$. Repeating this with $G = \mathbb{Z}v_1 + \mathbb{Z}v_2$, we see that u extends smoothly to an entire minimal graph contained in $\widetilde{\Sigma}$. This graph is periodic with respect to G hence bounded below. The maximum principle at a minimum point of u implies that u is constant. Hence $u = u(0,0) = z(p)$ and $T(z(p))$ is minimal so $\Psi'(z(p)) = 0$. $\qquad\square$

Next we use the maximum principle I to prove a compact embedded minimal surface can not go far into a cusp end; no *a priori* curvature bound assumed. More precisely we have the following statement.

Proposition 8 (Maximum principle II)**.** *Let* $0 < t_0 < 1/2$*. There is a* $\Lambda_0 = \Lambda_0(t_0)$ *such that if* $\Lambda(C) \leq \Lambda_0$ *and* Σ *is a compact embedded minimal surface in* (C, g_Ψ) *with* $\partial\Sigma \subset T(1 - t_0)$ *(*Σ *being transverse to* $T(1 - t_0)$*) then* $\Sigma \subset \{z \leq 1\}$*.*

Proof. First suppose Σ is a stable minimal surface. Then by curvature bounds for stable surfaces [11], we know there is a k_0 such that $|A_\Sigma| \leq k_0$ on $\Sigma \cap \{z \geq 1 - t_0/2\}$; k_0 depends on our assumed bounds on $\Psi', \Psi\Psi''$. By the maximum principle I, there is a Λ_0, only depending on t_0, such that if $\Lambda(C) \leq \Lambda_0$ then z has no maximum larger than 1. Hence $\Sigma \subset \{z \leq 1\}$.

Now suppose that Σ is not stable. Choose c_0 and c so that $z < c_0 < c$ on Σ and consider $\Sigma \subset X = \{1/2 \leq z \leq c\}$. We remark that Σ separates $\overset{\circ}{X}$, the interior of X. Indeed any loop in $\overset{\circ}{X}$ is homologous to a loop in $T(c_0)$ which does not intersect Σ. So the intersection number mod 2 of a loop with Σ is always 0. Then denote by A the connected component of $X \setminus \Sigma$ which contains $\{z = c\}$. *A priori* the boundary of A is mean convex except

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for $\{z = c\}$. But we can modify the function Ψ for $c_0 \leq z \leq c$ such that $\Psi'(c) = 0$ and keeping Ψ non-decreasing and the bounds on Ψ' and $\Psi\Psi''$ (c should be assumed very large). Then $T(c)$ is now minimal and A has mean convex boundary. In A, there exist a least area surface $\widetilde{\Sigma}$ with $\partial\Sigma = \partial\widetilde{\Sigma}$. Now the maximum of the z function on $\widetilde{\Sigma}$ is larger than the one on Σ. Since $\widetilde{\Sigma}$ is stable we already know that $\widetilde{\Sigma} \subset \{z \leq 1\}$, so Σ as well. $\qquad\square$

4.2 Transversality

Now we will see that embedded minimal surfaces of bounded curvature are "strongly transversal" to $T(c)$ in C endowed with the hyperbolic metric.

Proposition 9. *Let $k_0, \varepsilon_0 > 0$ be given. There exist constants Λ_0 and θ_0 such that if Σ is an embedded minimal surface in $(C, g_{\mathbb{H}})$, $\Lambda(C) \leq \Lambda_0$, $|A_\Sigma| \leq k_0$, with $\partial\Sigma$ at an intrinsic distance distance greater than ε_0 of the points of Σ in $T(1)$, then the angle between Σ and $T(1)$ is at least θ_0. The constant Λ_0 and θ_0 only depend on k_0 and ε_0.*

Proof. If this proposition fails, there exists $\Sigma_n, p_n \in \Sigma_n \cap T(1)$ in a hyperbolic cusp C_n satisfying the hypotheses, such that $\Lambda(C_n)$ and the angle between Σ_n and $T(1)$ at p_n goes to zero. Lift Σ_n to M so that $p_n = (0,0,1)$. The curvature bound gives the existence of a disk $D = D_\mu(0,0) \subset \mathbb{R}^2$ and smooth functions u_n on D whose graphs are contained in Σ_n (for large n). These functions have bounded $C^{2,\alpha}$ norm by the curvature bound and the fact that their gradient at $(0,0)$ converges to zero. Hence a subsequence of the u_n converges to a minimal graph u over D and the graph of u is tangent to $T(1)$ at $(0,0,1)$.

Let v_1^n, v_2^n be the generator of the group leaving C_n invariant. Let v_0 be in D. Since $\Lambda(C_n) \to 0$, there is a sequence $(a_1^n, a_2^n)_{n \in \mathbb{N}}$ in $\mathbb{Z}^2$ such that $a_1^n v_1^n + a_2^n v_2^n \to v_0$. The graph of $u_n(\cdot - (a_1^n v_1^n + a_2^n v_2^n))$ over $D + a_1^n v_1^n + a_2^n v_2^n$ is also a part of a lift of Σ_n. Since Σ_n is embedded, its lift is also embedded. So, for any n, we have either $u_n(\cdot) \leq u_n(\cdot - (a_1^n v_1^n + a_2^n v_2^n))$ or $u_n(\cdot) \geq u_n(\cdot - (a_1^n v_1^n + a_2^n v_2^n))$. Thus at the limit, $u(\cdot) \leq u(\cdot - v_0)$ or $u(\cdot) \geq u(\cdot - v_0)$ on $D \cap (D + v_0)$.

Let S be the totally geodesic surface in M tangent to $\{z = 1\}$ at $(0,0,1)$. Over D, S can be described as the graph of a radial function h. We have $h(0,0) = 1$ and there is $\alpha > 0$ such that, over D, $h((x,y) \leq 1 - \alpha(x^2 + y^2)$. The functions u and h are two solutions of the minimal surface equation with the same value and the same gradient at the origin. So the function $u - h$ looks like a harmonic polynomial of degree at least 2.

If the degree is 2, one can find $v_0 \in D \setminus \{(0,0)\}$ such that $(u-h)(v_0) < 0$ and $(u-h)(-v_0) < 0$. Then we have

$$u(v_0) < h(v_0) < h(0,0) = u(v_0 - v_0)$$
$$u((0,0) - v_0) < h(-v_0) < h(0,0) = u(0,0)$$

This contradicts $u(\cdot) \le u(\cdot - v_0)$ or $u(\cdot) \ge u(\cdot - v_0)$ on the whole $D \cap (D + v_0)$.

If the degree is larger than 3, the growth at the origin of h implies that there is a disk D' centered at the origin included in D such that $u < 1$ in $D' \setminus \{(0,0)\}$. So if $v_0 \in D' \setminus \{(0,0)\}$ we have

$$u(v_0) < u(0,0) = u(v_0 - v_0) \text{ and } u((0,0) - v_0) = u(-v_0) < u(0,0)$$

This gives also a contradiction $u(\cdot) \le u(\cdot - v_0)$ or $u(\cdot) \ge u(\cdot - v_0)$ on the whole $D \cap (D + v_0)$. $\qquad\square$

We notice that Remark 1 can be used to get strong transversality with $T(c)$ for $c \ge 1$.

A consequence of Proposition 9 is then the following result.

Theorem C. *Let Σ be a properly embedded minimal surface in $\mathcal{N}$ of bounded curvature. Then Σ has finite topology.*

Proof. If k_0 is an upper bound of the norm of the second fundamental form of Σ, Proposition 9 gives a constant Λ_0. Now $\mathcal{N}$ can be decomposed as the union of a compact part K and a finite number of cusp-ends C_i with $\Lambda(C_i) \le \Lambda_0$. Since Σ is transversal to the tori $T_i(c)$, Σ has the same topology as $\Sigma \cap \overset{\circ}{K}$; so it has finite topology. $\qquad\square$

5 Existence of compact embedded minimal surfaces in $\mathcal{N}$

Producing minimal surfaces is often done by minimizing the area in a certain class of surfaces. In order to ensure the compactness of our surface in $\mathcal{N}$, a min-max argument is more suitable in our proof of the following existence result.

Theorem A. *There exists a compact embedded minimal surface in any $\mathcal{N}$.*

Proof. Let $C_1, \cdots, C_k$ be the cusp ends of $\mathcal{N}$. Let z_i be the z-coordinates in C_i and assume that $\Lambda(C_i) \le \Lambda_0$, for $1 \le i \le k$; Λ_0 the constant given by the maximum principle II. This can always be realized by Remarks 1 and 2.

Now we change the hyperbolic metric in each end C_i as follows. Let $\Psi : [1/2, \infty) \to \mathbb{R}$ satisfy $\Psi(z) = z$ for $1/2 \leq z \leq 1$, $\Psi'(z) > 0$ and $\lim_{z \to \infty} \Psi(z) = 3/2$.

Let L be large (to be specified later) and modify the metric g_Ψ in $[L, L+1]$ so the new metric gives a compactification of C_i by removing $\{z_i \geq L\}$ and attaching a solid torus to $T(L)$. The precise way to do this will be explained below. With this new metric, for $L \leq z \leq L + 1$ the mean curvature of the tori $T_i(z)$ is increasing; going from $\Psi'(L)$ at $z = L$ to ∞ as $z \to L+1$. $z = L+1$ corresponds to the core of the solid torus. We do this in each cusp and get a compact manifold $\tilde{\mathcal{N}}$ without boundary endowed with a certain metric. We notice that the manifold does not depend on L but the metric does.

Now we can choose a Morse function f on $\tilde{\mathcal{N}}$ such that all the tori $T_i(z)$, $1/2 \leq z \leq L$, $1 \leq i \leq k$, are level surfaces of f.

This Morse function f defines a sweep-out of the manifold $\tilde{\mathcal{N}}$ and

$$M_0 = \max_{t \in \mathbb{R}} \mathcal{H}^2(f^{-1}(t))$$

essentially does not depend on L (in fact it can decrease when L increases) ($\mathcal{H}^2$ is the 2-dimensional Hausdorff measure).

Almgren-Pitts min-max theory applies to this sweep-out and gives a compact embedded minimal surface Σ in $\tilde{\mathcal{N}}$ whose area is at most M_0 (see theorem 1.6 in [3]). Let us see now that Σ actually lies in the hyperbolic part of $\tilde{\mathcal{N}}$ so in $\mathcal{N}$.

Since $\Psi \to 3/2$, the metric on $T_i(k, k + 1)$ is uniformly controlled and close to being flat. As a consequence of the monotonicity formula for minimal surfaces (see Theorem 17.6 in [13]), if $\Sigma \cap T_i(k + 1/2) \neq \emptyset$ ($1 \leq k \leq L - 1$), the area of $\Sigma \cap T_i(k, k + 1)$ is at least $c_0 > 0$. The constant c_0 only depends on the ambient sectional curvature bound and the v_j^i's (the vectors in the end C_i).

This monotonicity formula gives at least linear growth for Σ. More precisely, if a connected component of Σ intersect $T_i(1)$ and $T_i(L)$ is has area at least $c_0(L - 1)$. So by choosing, L larger than M_0/c_0 there is no component of Σ meeting both $T_i(1)$ and $T_i(L)$.

Also, no connected component lies entirely in $\{z_i \geq 1\}$. Indeed, the z_i would have a minimum on the component which is impossible by the classical maximum principle and the sign of the mean curvature on $T_i(z)$. Thus Σ stays out of $\{z_i \geq L\}$. Hence by the maximum principle II, Σ does not enter in any $\{z_i \geq 1\}$ which completes the proof.

Let us now give the definition of the new metric on $[L, L+1]$. The tori $T_i(c)$ are the quotient of $\mathbb{R}^2$ by v_1^i, v_2^i so they can be parametrized by $u\frac{v_1^i}{2\pi} + v\frac{v_2^i}{2\pi}$ where $(u, v) \in \mathbb{S}^1 \times \mathbb{S}^1$. With this parametrization, the metric g_Ψ on C_i is then

$$\frac{1}{4\pi^2 \Psi(z_i)^2}(|v_1|^2 du^2 + 2(v_1, v_2)dudv + |v_2|^2 dv^2 + dz_i^2) =$$

$$\frac{1}{\Psi(z_i)^2}(a^2 du^2 + 2b dudv + c^2 dv^2 + dz_i^2)$$

Let φ be a smooth non increasing function on $[L, L+1]$ such that $\varphi(z) = 1$ near L and $\varphi(z) = ((L+1) - z)/a$ near $L+1$. We then change the metric on $\{L \leq z_i \leq L+1\}$ by

$$\frac{1}{\Psi^2(z_i)}(dz_i^2 + a^2 \varphi(z_i)^2 du^2 + 2b\varphi(z_i)dudv + c^2 dv^2) \tag{1}$$

Actually, this change consists in cutting $\{z_i \geq L\}$ from the cusp end C_i and gluing a solid torus along $T(L)$. To see this, let D be the unit disk with its polar coordinates $(r, \theta) \in [0, 1] \times \mathbb{S}^1$ and let us define the map $h : D \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 \times [L, L+1]$ by $(r, \theta, v) \mapsto (\theta, v, L+1-r)$. The induced metric by h from the one in (1) for r near 0 is

$$\frac{1}{\Psi^2(L+1-r)}(dr^2 + a^2 \frac{r^2}{a^2}d\theta^2 + 2b\frac{r}{a}d\theta dv + c^2 dv^2)$$

$$= \frac{1}{\Psi^2(L+1-r)}(dr^2 + r^2 d\theta^2 + 2\frac{b}{a}rd\theta dv + c^2 dv^2)$$

This is a well defined metric on the solid torus $D \times \mathbb{S}^1$.

With this new metric, the tori $T_i(c) = \{z_i = c\}$ ($c \in [L, L+1)$) have constant mean curvature $\Psi'(c) - \frac{\varphi'(c)}{2\varphi(c)}\Psi(c) > 0$ with respect to $\Psi(z)\partial_z$. $\quad\square$

A minimization argument can be done under some hypotheses to produce compact minimal surfaces.

Theorem B. *Let S be a closed orientable embedded surface in $\mathcal{N}$ which is not a 2-sphere or a torus. If S is incompressible and non-separating, then S is isotopic to a least area embedded minimal surface.*

Proof. Let $C_1, \cdots, C_k$ be the cusp ends of $\mathcal{N}$. Let z_i be the z-coordinates in C_i such that the surface S does not enter in $\{z_i \geq 1\}$. We assume that

$\Lambda(C_i) \leq \Lambda_0$, for $1 \leq i \leq k$; Λ_0 the constant given by the maximum principle II for the function Ψ below.

Let $\Psi : \mathbb{R}_+^* \to \mathbb{R}$ be a smooth increasing function such that $\Psi(z) = z$ on $(0, 1]$ and $\Psi'(2) = 0$.

For each $a \geq 1$, let $\mathcal{N}(a)$ be $\mathcal{N}$ with each cusp end truncated at $z_i = a$; *i.e.* $\mathcal{N}(a) = \mathcal{N} \setminus \cup_{1 \leq i \leq k}\{z_i > a\}$. We remark that the $\mathcal{N}(a)$ are all diffeomorphic to each other.

Let n be an integer. In each cusp end C_i, we change the metric on $\mathcal{N}(2n)$ by using a function $\Psi_n : [1/2, 2n] \to \mathbb{R}; z \mapsto n\Psi(\frac{z}{n})$. So $\Psi_n(z) = z$ on $[1/2, n]$ and $\Psi_n'(2n) = 0$; the torus $T_j(2n)$ minimal. We notice that the metric on $\mathcal{N}(n)$ is not modified.

Let us minimize the area in the isotopy class of S in the manifold with minimal boundary $\mathcal{N}(2n)$. By Theorem 5.1 and remarks before Theorem 6.12 in [8], there is a least area surface Σ_n in $\mathcal{N}(2n)$ which is isotopic to S. Theorem 5.1 in [8] can be applied because $\mathcal{N}(2n)$ is P^2-irreducible ($\mathcal{N}(2n)$ is orientable and its universal cover is diffeomorphic to $\mathbb{R}^3$). Moreover the minimization process does not produce a non-orientable surface since, in that case, S would be isotopic to the boundary of the tubular neighborhood of it, hence S would separate $\mathcal{N}$. Finally Σ_n is not one connected component of $\partial\mathcal{N}(2n)$ since S is not a torus.

In $\mathcal{N}(2n)$, $(\{z = c\})_{c \in [1,2n]}$ is a mean convex foliation so $\Sigma_n \cap \mathcal{N}(1) \neq \emptyset$. By the maximum principle II, it implies that $\Sigma_n \subset \mathcal{N}(1)$ so in a piece of $\mathcal{N}$ where the metric never changes. *A priori*, the surfaces Σ_n could be different. But, since they all lie in $\mathcal{N}(1)$, they all appear in the minimization process in $\mathcal{N}(2)$ so they all have the same area. So Σ_1 is a least area surface in the isotopy class of S in $\mathcal{N}$ with the hyperbolic metric. $\square$

Remark 3. We can notice that there is a uniform lower bound for the area of minimal surfaces in manifolds $\mathcal{N}$. The point is that the thick part of such a manifold $\mathcal{N}$ is not empty. So at each point in the thick part there is an embedded geodesic ball of radius $\varepsilon_3/2$ centered at that point where ε_3 is the Margulis constant of hyperbolic 3-manifolds.

Each connected component of the thin part is either a hyperbolic cusp or the tubular neighborhood of a closed geodesic. So it is foliated by mean convex surfaces and a minimal surface Σ can not be included in such a component. So there is $x \in \Sigma$ in the thick part. Thus we can apply a monotonicity formula (see [13]) to conclude that the area of the part of Σ inside the geodesic ball of radius $\varepsilon_3/2$ and center x is larger than a constant $c_3 > 0$ that depends only on the geometry of the hyperbolic $\varepsilon_3/2$ ball.

When the compact minimal surface Σ is stable, we can be more precise

(see [6] where Hass attributes this estimates to Uhlenbeck). Applying the stability inequality to the constant function 1. We get that

$$\int_\Sigma -(\mathrm{Ric}(N,N) + |A|^2) \geq 0.$$

Since $|A|^2 = -2(K_\Sigma + 1)$, the Gauss-Bonnet formula gives

$$Area(\Sigma) \geq -\frac{1}{2}\int_\Sigma K_\Sigma = -\frac{1}{2}\chi(\Sigma) = 2\pi(g-1)$$

with g the genus of Σ. Morover, by Gauss formula $K_\Sigma \leq -1$, so the Gauss-Bonnet formula gives $Area(\Sigma) \leq 4\pi(g-1)$.

6   Existence of non compact embedded minimal surfaces in $\mathcal{N}$

In [7], Hass, Rubinstein and Wang construct proper minimal surfaces in manifolds $\mathcal{N}$ by a minimization argument in homotopy classes. In [12], Ruberman constructs least area surfaces in the isotopy class. Here we make use of results in Section 4 to give a different approach on the proof of this second result.

First we remark that, in manifolds $\mathcal{N}$, there is always a "Seifert" surface. $\mathcal{N}$ is topologically the interior of a compact manifold $\overline{\mathcal{N}}$ with tori boundary components and each boundary torus is incompressible. By Lemma 6.8 in [9], there is a compact embedded surface $\overline{S}$ in $\overline{\mathcal{N}}$ with non empty boundary which is incompressible and 2-sided; moreover it is non-separating. Then $S = \overline{S} \cap \mathcal{N}$ is a properly embedded smooth surface in $\mathcal{N}$, S incompressible, of finite topology, non compact, non-separating and 2-sided.

The result is the following statement.

Theorem 10. *Let S be a properly embedded, non compact, finite topology, incompressible, non separating surface in $\mathcal{N}$. Then S is isotopic to a least area embedded minimal surface.*

Proof. S has a finite number of annular ends $A_1, \cdots, A_p$, each one being included in one cusp end C_i of $\mathcal{N}$. Since A_j is incompressible in C_i, we can isotope S so that each annular end A_j is totally geodesic in the end C_i it enters. We still call S this new surface and we notice that its area is finite for the hyperbolic metric.

Let $\Psi : \mathbb{R}_+^* \to \mathbb{R}$ be a smooth increasing function such that $\Psi(z) = z$ on $(0,1]$ and $\Psi'(4/3) = 0$. Let Λ_0 be the constant given by the Maximum

principle II and the transversality lemma (Propositions 8 and 9). Assume the ends of $\mathcal{N}$ are chosen so that $\Lambda(C_i) \leq \Lambda_0$ for each end C_i.

As in the proof of Theorem B, we denote $\mathcal{N}(a) = \mathcal{N} \setminus \cup_{1 \leq i \leq k}\{z_i > a\}$. We remark that $\mathcal{N}(a)$ is diffeomorphic to $\overline{\mathcal{N}}$.

Let n be a large integer. In each cusp end C_i, we change the metric on $\mathcal{N}(4n)$ by using a function $\Psi_n : [1/2, 4n] \to \mathbb{R}; z \mapsto 3n\Psi(\frac{z}{3n})$. So $\Psi_n(z) = z$ on $[1/2, 3n]$ and $\Psi'_n(4n) = 0$; the torus $T_j(4n)$ minimal. We notice that the metric on $\mathcal{N}(3n)$ is not modified.

Let $S(4n) = S \cap \mathcal{N}(4n)$; the area of $S(4n)$ is bounded by some constant A independent of n. By Theorem 6.12 in [8], there is a least area surface $\Sigma(4n)$ in $\mathcal{N}(4n)$, isotopic to $S(4n)$ and $\partial\Sigma(4n) = \partial S(4n)$. We remark that $\Sigma(4n)$ is stable so has bounded curvature away from its boundary (independent of n) (see [11]).

In each cusp end C_i, Proposition 9 implies $\Sigma(4n)$ is transverse to the tori $T_1(a)$, $1 \leq a \leq 2n$ (see Remark 1, in order to apply Proposition 9). So each intersection $\Sigma(4n) \cap T_1(a)$ is composed of the same number of Jordan curves for $1 \leq a \leq 2n$. The next claims prove that this number is equal to the number of boundary components of $\Sigma(4n)$ on $T_1(4n)$.

Claim 2. *Let Ω be a domain in $\Sigma(4n)$ with boundary in $T_1(a)$ $(1 \leq a \leq n)$. Then Ω does not enter in any $\{z_i \geq a\}$.*

Proof. If Σ enters in one $\{z_i \geq a\}$, by transversality, it enters in $\{z_i \geq 2n\}$. So the function z_i will have a maximum larger than $2n$ which is impossible by Proposition 8 (see also Remark 1). $\square$

Claim 3. *Let γ be a connected component of $\Sigma(4n) \cap T_1(a)$ $(1 \leq a \leq n)$. Then γ is not trivial in $\pi_1(T_1(a))$.*

Proof. Assume that γ is trivial in $\pi_1(T_1(a))$. Since $\Sigma(4n)$ is incompressible, γ bounds a disk Δ in $\Sigma(4n)$. By Claim 2, Δ stays in $\mathcal{N}(a)$ where the metric is still hyperbolic. So we can lift Δ to a minimal disk Δ' in $\mathbb{R}^2 \times \mathbb{R}_+$ (with the hyperbolic metric) with boundary in $z_1 = a$ and entirely included in $\{z_1 \leq a\}$. This is impossible by the maximum principle since $\{z_1 = s\}$ has constant mean curvature one. $\square$

Claim 4. *Let Σ be a connected component of $\Sigma(4n) \cap \{n \leq z_1 \leq 4n\}$. Then Σ is an annulus with one boundary component in $T_1(n)$ and one in $T_1(4n)$.*

Proof. Let us first prove that the inclusion map of Σ in $\{n \leq z_1 \leq 4n\}$ is π_1-injective. So let γ be a loop in Σ which bounds a disk in $\{n \leq z_1 \leq 4n\}$. Since $\Sigma(4n)$ is incompressible, there is a disk Δ in $\Sigma(4n)$ bounded by γ. If

Δ is in Σ, we are done. If not, there is a subdisk Δ' of Δ with boundary in $T_1(n)$; but this is impossible by Claim 3. So the inclusion map is π_1-injective. We notice that $\pi_1(\{n \leq z_1 \leq 4n\})$ is Abelian, so $\pi_1(\Sigma)$ is Abelian. This implies that Σ is topologically a sphere, a disk, an annulus or a torus. The sphere and the torus are not possible since Σ has a non-empty boundary. Claim 2 implies that Σ must have a boundary component on $T_1(4n)$. If the whole boundary of Σ is in $T_1(4n)$, the z_1 function admits a minimum on Σ that is impossible by the maximum principle since the $T_1(c)$ have positive mean curvature. So Σ is an annulus with one boundary component in $T_1(n)$ and one in $T_1(4n)$. $\qquad\square$

With these claims, we have thus proved that $\Sigma(4n) \cap \mathcal{N}(n)$ is isotopic to $S \cap \mathcal{N}(n)$ (here, we allow the boundary to move). We also notice that because of the curvature estimate on $\Sigma(4n)$ and the transversality estimate given by Proposition 9, the intersection curves $\Sigma(4n) \cap T_1(a)$ $(1 \leq a \leq n)$ have bounded curvature. So they have a well controlled geometry far in the cusp. More precisely, there is a_0 such that $\Sigma(4n) \cap \{a_0 \leq z_1 \leq n\}$ is a graph over $S \cap \{a_0 \leq z_1 \leq n\}$. So the sequence $\Sigma(4n) \cap \mathcal{N}(n)$ is a sequence of surfaces with uniformly bounded area and curvature whose behavior in the cusps is well controlled. Thus a subsequence converges to a minimal surface Σ. This convergence says that $\Sigma(4n) \cap \mathcal{N}(k)$ can be written as a graph or a double graph over $\Sigma \cap \mathcal{N}(k)$. In the first case, the surface Σ is then isotopic to S. In the second case, $\Sigma(4n) \cap \mathcal{N}(k)$ is isotopic to the boundary of a tubular neighborhood of $\Sigma \cap \mathcal{N}(k)$ in $\mathcal{N}(k)$; this implies that $\Sigma(4n) \cap \mathcal{N}(k)$ is a separating surface which is impossible by the properties of S. $\qquad\square$

We notice that the area estimate given in Remark 3 are also true for non compact minimal surface. Indeed, because of the asymptotic behaviour of a stable minimal surface, the constant function 1 can be used as a test function even in the non compact case.

7 Some examples

In this section, we give some "explicit" examples that illustrate the above theorems.

H. Schwarz and A. Novius constructed periodic minimal surfaces in $\mathbb{R}^3$ by constructing minimal surfaces in a cube possessing the symmetries of the cube. These surfaces then extend to $\mathbb{R}^3$ by symmetry in the faces.

K. Polthier constructed periodic embedded minimal surfaces in $\mathbb{H}^3$ in an analogous manner. Let P be a finite side polyhedron of $\mathbb{H}^3$ such that sym-

metry in the faces of P tessellate $\mathbb{H}^3$. If Σ_0 is an embedded minimal surface in P, meeting the faces of P orthogonally and with the same symmetry as P. Then Σ extends to an embedded minimal surface in $\mathbb{H}^3$ by symmetry in the faces. Polthier makes this work for many polyhedron P; *e.g.* for all the regular ideal Platonic solids whose vertices are on the spheres at infinity. Among these examples, one can obtain examples in complete hyperbolic 3-manifolds of finite volume.

We first describe how this technique yields an embedded genus 3 compact minimal surface in the figure eight knot complement $\mathcal{N}$.

Let T be an ideal regular tetrahedron of $\mathbb{H}^3$; all the dihedral angles are $2\pi/3$. In the Klein model of $\mathbb{H}^3$ (the unit ball of $\mathbb{R}^3$), T is a regular Euclidean tetrahedron with its four vertices on the unit sphere. Label the faces of T and two vertices of T, as in Figure 1a. Then identify face A with face B by a rotation by $2\pi/3$ about v , and identify D with C by a rotation by $2\pi/3$ about w.

The quotient of T by these face matchings, produces a non orientable hyperbolic 3-manifold of finite volume. There is one vertex and its link is a Klein bottle. This manifold $\mathcal{N}$ was discovered by Giesekind in 1912.

The orientable 2-sheeted cover $\mathcal{N}'$ of the Giesekind manifold is diffeomorphic to the complement of the figure eight knot in $\mathbb{S}^3$; hence is a complete hyperbolic manifold of finite volume. In [14], Thurston explains how $\mathcal{N}'$ is homeomoprhic to the complement of the figure eight knot (see also [5]).

We construct an embedded compact minimal surface in $\mathcal{N}$ that lifts to a surface of genus 3 in $\mathcal{N}'$.

The geodesics from each vertex of T to its opposite face, all meet at one point p in T. Join p to each edge of T by the minimizing geodesic. Also join p to each vertex of T by a geodesic. This produces the edges of a tessellation of T by 24 congruent tetrahedra.

Consider the tetrahedron T_1 of this tessellation as in Figure 1b. By a conjugate surface technique, Polthier proved there exists an embedded minimal disk D_1 in T_1 meeting the boundary of T_1 orthogonally as in Figure 1c. Symmetry by the faces of T_1 (and the faces of the symmetric tetrahedron of the tessellation of T) extend D_1 to an embedded minimal surface S meeting each face of T in one embedded Jordan curve in the interior of the face. S is topologically a sphere minus 4 points.

The face identification on T send $S \cap A$ to $S \cap B$ and $S \cap D$ to $S \cap C$. Hence S passes to the quotient in $\mathcal{N}$ to a compact embedded minimal surface whose topology is the connected sum of two Klein bottles. The lift of this to $\mathcal{N}'$ is a genus 3 compact embedded minimal surface.

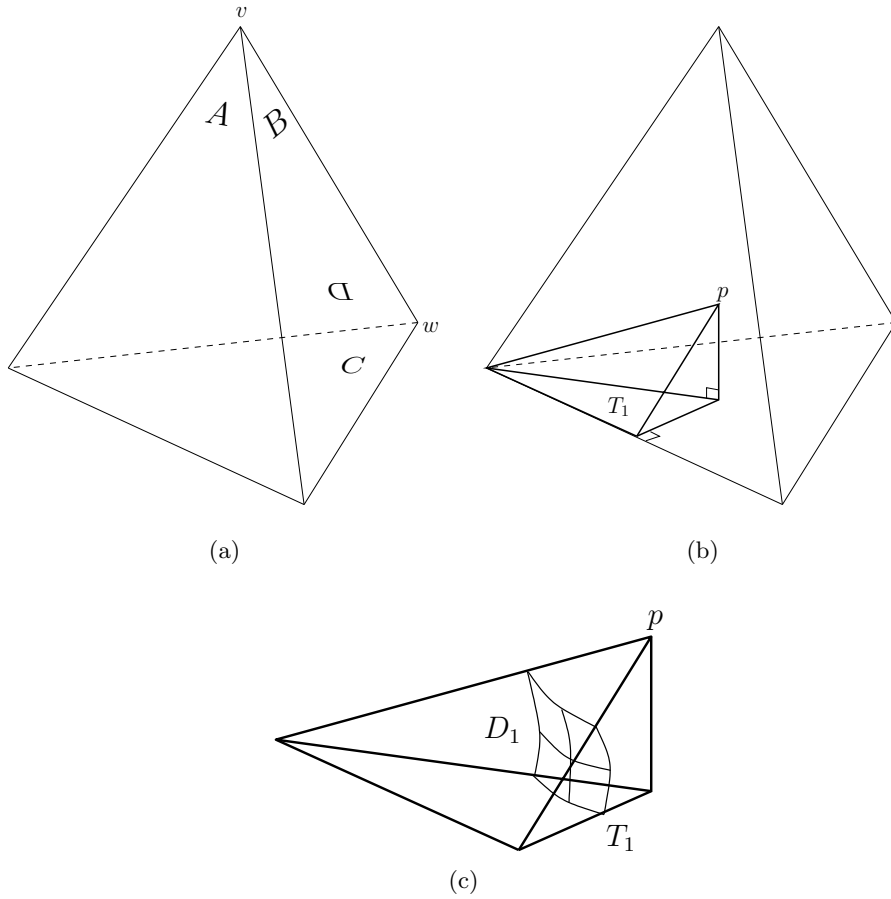A Seifert surface for the figure eight knot is an incompressible surface

Figure 1: a minimal surface in the Giesekind manifold

homeomorphic to a once punctured torus. Applying Theorem 10 gives a properly embedded minimal one punctured torus in the complement of the figure eight knot.

Theorem 4 of C. Adams [1] yields many totally geodesic properly embedded 3-punctured spheres in complete hyperbolic 3-manifolds $\mathcal{N}$ of finite volume. Suppose $\mathcal{N}$ arises as a link or knot complement that contains an embedded incompressible 3-punctured sphere (so by Adams, it is isotopic to a totally geodesic one). For example if the link or the knot contains a part as in Figure 2a such that the disk D with 2 punctures is a 3-punctured incompressible sphere in $\mathcal{N}$. An example is the Whitehead link (Figure 2b).
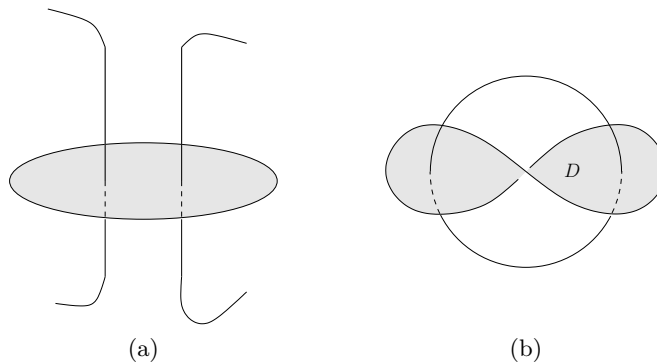
Figure 2: Incompressible 3-punctured sphere in general position and in the complement of Whitehead link

The Borromean rings is also a hyperbolic link. Its complement contains an embedded incompressible thrice punctured sphere (Figure 3a) and an embedded once punctured torus (Figure 3b) which is isotopic to a properly embedded minimal once punctured torus by Theorem 10.

It will be interesting to estimate the areas of the minimal surfaces obtained by Theorems A, B and 10 as in Remark 3. For examples, consider the figure eight knot complement $\mathcal{N}$. We know there is a properly embedded minimal once punctured torus Σ in $\mathcal{N}$ by Theorem 10 (Figure 3c). The Finite Total Curvature Theorem 2 and the Gauss equation tells us the area of Σ is strictly less than 2π (there are no embedded totally geodesic surfaces in $\mathcal{N}$).

What is the area of Σ ? What is the properly embedded, non compact, minimal surface of smallest area (it exists) in $\mathcal{N}$? And in all such manifolds $\mathcal{N}$?

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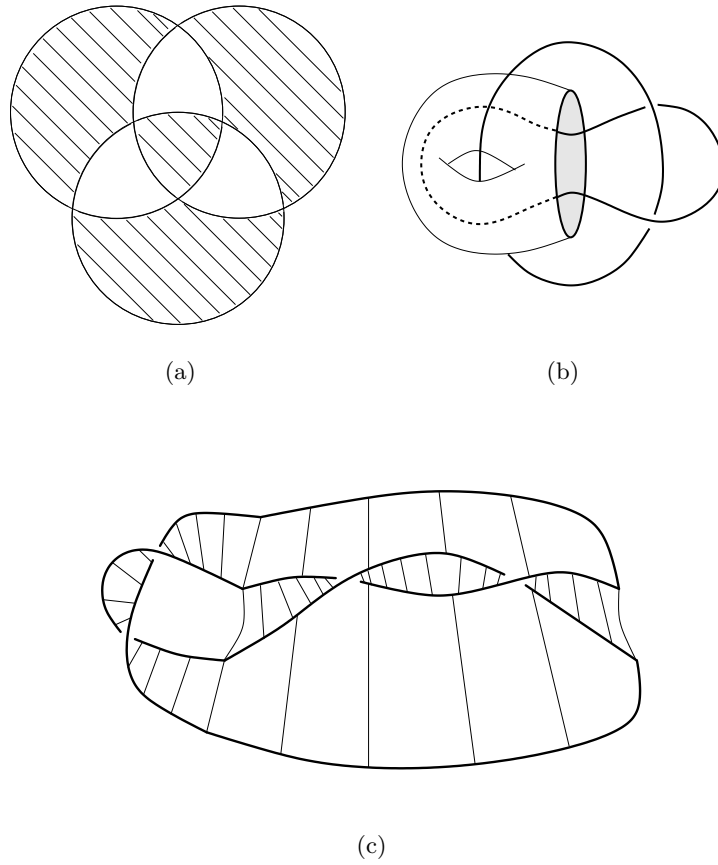(a)   (b)

(c)

Figure 3: Incompressible 3-punctured sphere and 1-punctured torus in the complement of Borromean rings and an incompressible 1-punctured torus in the figure eight knot complement