A uniqueness result for maximal surfaces in Minkowski 3-space

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Abstract

In this paper, we study the Dirichlet problem associated to the maximal surface equation. We prove the uniqueness of bounded solutions to this problem in unbounded domain in $\mathbb{R}^2$.

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Introduction

We consider the Minkowski space-time $\mathbb{L}^3$ *i.e.* $\mathbb{R}^3$ with the following Lorentzian metric $\langle x, y\rangle_{\mathbb{L}} = x_1y_1 + x_2y_2 - x_3y_3$. We define $|x|^2_{\mathbb{L}} = \langle x, x\rangle$. A vector is said to be *spacelike* if $|x|^2_{\mathbb{L}} > 0$ and a surface S of class C^1 is said to be spacelike if $|\cdot|^2_{\mathbb{L}}$ is positive definite on the tangent space to S. Such a surface is locally the graph of a function over a domain in $\mathbb{R}^2$.

If v is a function in a domain Ω in $\mathbb{R}^2$, the graph of v is spacelike if and only if $|\nabla v| < 1$. The function v is then Lipschitz continuous and it extends to the closure $\overline{\Omega}$. In the paper, we assume that $\partial\Omega$ is sufficiently regular for such an extension to exist: for example, $\Omega = \widetilde{\Omega} - \{\text{points}\}$ with $\partial\widetilde{\Omega}$ smooth. We denote by φ the trace $v|_{\partial\Omega}$ of v on the boundary. The maximal area problem in the class of spacelike surfaces consists in solving the following variational problem:

$$\max_v \int_\Omega \sqrt{1 - |\nabla v|^2}\,\mathrm{d}x, \quad v|_{\partial\Omega} = \varphi$$

The critical points of this functional are the solutions of the maximal surface equation :

$$\operatorname{div} \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} = 0 \tag{$*$}$$

The maximal area problem is then linked to the Dirichlet problem associated to $(*)$: to find a solution v of $(*)$ in Ω such that $v|_{\partial\Omega} = \varphi$. This Dirichlet problem has been already studied by several authors, for example see [BS] and [KM].

In this paper, we prove the uniqueness of bounded solutions to the Dirichlet problem. More precisely, if Ω is an unbounded domain and φ is a bounded continuous function on $\partial\Omega$, we prove that, if it exists, a bounded solution v of $(*)$ in Ω with $v|_{\partial\Omega} = \varphi$ is unique (Theorem 2). The study of the uniqueness is important in the construction of certain moduli spaces of maximal surfaces (see [FLS1]and [FLS2]).

In fact our result has already been stated by A. A. Klyachin in [Kl]. In his note, Klyachin states several results about the Dirichlet problem for $(*)$ in unbounded domains and any dimension. But he does not give any proof. He uses the notion of capacity : if Ω is a bounded domain and $P, Q \subset \Omega$ satisfy $P \cap Q = \emptyset$, the capacity of $(P, Q; \Omega)$ is :

$$\mathrm{cap}(P, Q; \Omega) = \inf \int_\Omega |\nabla u|^2$$

where the inf is taken on all Lipschitz continuous functions u on Ω with $u = 0$ on P and $u = 1$ on Q. If Ω is unbounded Ω is said to be parabolic if for any compact subset $P \in \Omega$: $\lim_{\mathbb{R}\to+\infty} \mathrm{cap}(P, \partial B_R \cap \Omega; B_R \cap \Omega) = 0$ where B_R denotes the centered ball of radius R. One of the results stated by Klyachin is the following one (Theorem 3 in [Kl]).

Theorem. Let $\Omega \subset \mathbb{R}^n$ be an unbounded parabolic domain. Let v_1 and v_2 be two bounded solutions of $(*)$ in Ω. We define $\varphi_1 = v_1|_{\partial\Omega}$ and $\varphi_2 = v_2|_{\partial\Omega}$. Then : $v_1(x) \le v_2(x) + \sup_{\partial\Omega}(\varphi_1 - \varphi_2)$.

Since every unbounded domain in $\mathbb{R}^2$ is parabolic, this theorem implies our uniqueness result.

This uniqueness result for the maximal surface equation is also important for the study of the Dirichlet problem associated to the minimal surface equation. The graph of a function u over a domain $\Omega \subset \mathbb{R}^2$ is a surface in $\mathbb{R}^3$ with its standard euclidean metric. It has vanishing mean curvature if u satisfies the following partial differential equation:

$$\mathrm{div}\, \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \qquad (**)$$

This equation implies that there exists locally a function v such that:

$$\mathrm{d}v = \mathrm{d}\Psi_u = \frac{u_x}{\sqrt{1 + |\nabla u|^2}}\mathrm{d}y - \frac{u_y}{\sqrt{1 + |\nabla u|^2}}\mathrm{d}x$$

(here u_x and u_y are the first derivatives of u). $v = \Psi_u$ is called the conjugate function to u and a simple computation shows that v is a solution of $(*)$.

Then the uniqueness for solutions of $(*)$ implies uniqueness for solutions of $(**)$.

The proof of our uniqueness result uses the same idea as P. Collin and R. Krust in [CK]. But to apply this idea, we need to prove an estimate for the first derivatives of v in a subdomain of Ω; this is done in Lemma 3.

1 The uniqueness result

Let $\Omega \subset \mathbb{R}^2$ be a domain and v a solution of the maximal surface equation :

$$\operatorname{div} \frac{\nabla v}{\sqrt{1-|\nabla v|^2}} = 0 \tag{$*$}$$

In the following, the quantity $\sqrt{1-|\nabla v|^2}$ will be denoted by w_v. We define the 1-form α_v as follows:

$$\alpha_v = \frac{v_x}{w_v}\mathrm{d}y - \frac{v_y}{w_v}\mathrm{d}x$$

where v_x and v_y are the first derivatives of v. The maximal surface equation is then equivalent to $\mathrm{d}\alpha_v = 0$.

First, we need a technical lemma.

Lemma 1. *Let v and v' be two functions. Let P be a point in Ω and $\varepsilon > 0$ such that $|\nabla v|(P) \leq 1-\varepsilon$ and $|\nabla v'|(P) \leq 1-\varepsilon$. Then there exists a constant $C(\varepsilon)$ that depends only on ε such that, at the point P, we have:*

$$\left((\nabla v - \nabla v') \cdot \left(\frac{\nabla v}{w_v} - \frac{\nabla v'}{w_{v'}}\right)\right) \geq C(\varepsilon) \left|\frac{\nabla v}{w_v} - \frac{\nabla v'}{w_{v'}}\right|^2 \tag{1}$$

Let us notice that in Collin-Krust's paper [CK], there is the same lemma, but they do not need any bound on the gradient to get the estimate (1).

Proof. Let w denote w_v and w' denote $w_{v'}$. We define $n = (-v_x, -v_y, -1)/w$ and $n' = (-v'_x, -v'_y, -1)/w'$. We have $|n|^2_{\mathbb{L}} = -1$ and $|n'|^2_{\mathbb{L}} = -1$, so:

$$\begin{aligned}\left((\nabla v - \nabla v') \cdot \left(\frac{\nabla v}{w} - \frac{\nabla v'}{w'}\right)\right) &= \langle (w'n' - wn), (n' - n)\rangle \\ &= (w + w')(-1 - \langle n, n'\rangle) \\ &= \frac{w + w'}{2}|(n' - n)|^2_{\mathbb{L}}\end{aligned}$$

Since $|\nabla v| \leq 1-\varepsilon$ and $|\nabla v'| \leq 1-\varepsilon$ there exists $C_1(\varepsilon) > 0$ such that

$$(w + w')/2 \geq C_1(\varepsilon) \tag{2}$$

Moreover:

$$|(n'-n)|_{\mathbb{L}}^2 = \left|\frac{\nabla v}{w} - \frac{\nabla v'}{w'}\right|^2 - \left(\frac{1}{w} - \frac{1}{w'}\right)^2$$

Let $x \in \mathbb{R}^2$ be $\nabla v/w$ and x' be $\nabla v'/w'$. Thus $1/w = \sqrt{1+|x|^2}$ and $1/w' = \sqrt{1+|x'|^2}$. Since ∇v and $\nabla v'$ are bounded by $1-\varepsilon$, there exists $R(\varepsilon)$ such that $|x|$ and $|x'|$ are bounded by $R(\varepsilon)$. Hence:

$$\begin{aligned}
\frac{|(n'-n)|_{\mathbb{L}}^2}{\left|\frac{\nabla v}{w} - \frac{\nabla v'}{w'}\right|^2} &= 1 - \frac{\left(\frac{1}{w} - \frac{1}{w'}\right)^2}{|x-x'|^2} \\
&= 1 - \frac{\left(\sqrt{1+|x|^2} - \sqrt{1+|x'|^2}\right)^2}{|x-x'|^2} \\
&= 1 - \frac{\left(|x|^2 - |x'|^2\right)^2}{|x-x'|^2\left(\sqrt{1+|x|^2} + \sqrt{1+|x'|^2}\right)^2} \\
&= 1 - \left(\frac{|x| - |x'|}{|x-x'|}\right)^2 \left(\frac{|x| + |x'|}{\left(\sqrt{1+|x|^2} + \sqrt{1+|x'|^2}\right)}\right)^2 \\
&\geq 1 - \left(\frac{|x| + |x'|}{\left(\sqrt{1+|x|^2} + \sqrt{1+|x'|^2}\right)}\right)^2 > 0
\end{aligned}$$

By continuity and since $|x|$ and $|x'|$ are bounded by $R(\varepsilon)$, there exists a constant $C_2(\varepsilon) > 0$ such that:

$$1 - \left(\frac{|x| + |x'|}{\left(\sqrt{1+|x|^2} + \sqrt{1+|x'|^2}\right)}\right)^2 > C_2(\varepsilon) \tag{3}$$

Then in combining (2) and (3), we get (1) with $C(\varepsilon) = C_1(\varepsilon)C_2(\varepsilon)$. □

We denote by d the usual distance in $\mathbb{R}^2$ and by d_Ω the intrinsic metric in Ω *i.e.* $d_\Omega(p,q)$ is the infimum of the length of all paths in Ω going from p to q. Let $\delta > 0$, we denote by Ω_δ the set $\{p \in \Omega \,|\, d_\Omega(p, \partial\Omega) > \delta\}$. We then can write our uniqueness result.

Theorem 2. *Let Ω be an unbounded domain in $\mathbb{R}^2$ and φ a bounded continuous function on $\partial\Omega$. Let v and v' be two bounded solutions of $(*)$ in Ω with $v|_{\partial\Omega} = \varphi = v'|_{\partial\Omega}$. Then $v = v'$.*

Proof. Let v and v' be two such solutions. We assume that $\sup v - v' > 0$ and we denote this supremum by 4δ. Let $a \in [2\delta, 3\delta]$ be chosen such that $\widetilde{\Omega} = \{v > v' + a\}$ has smooth boundary. Since $2\delta \le a \le 3\delta$ and v and v' are 1-Lipschitz continuous $\widetilde{\Omega} \subset \Omega_\delta$. We then have the following lemma.

Lemma 3. *There exists $\varepsilon > 0$ such that, in $\widetilde{\Omega}$, $|\nabla v| \le 1-\varepsilon$ and $|\nabla v'| \le 1-\varepsilon$.*

Before proving this lemma, we finish Theorem 2 proof. Let $\tilde{v}$ denote $v - v' - a$ and $\tilde{\alpha}$ denote $\alpha_v - \alpha_{v'}$.

For $r > 0$, we define $\widetilde{\Omega}_r = \{p \in \widetilde{\Omega} \,|\, |p| < r\}$ and $C_r = \{p \in \widetilde{\Omega} \,|\, |p| = r\}$. Since $\tilde{v} = 0$ on $\partial\widetilde{\Omega}_r \backslash C_r$ and $\tilde{\alpha}$ is closed, we have :

$$\int_{C_r} \tilde{v}\tilde{\alpha} = \int_{\partial\widetilde{\Omega}_r} \tilde{v}\tilde{\alpha} = \iint_{\widetilde{\Omega}_r} \mathrm{d}\tilde{v} \wedge \tilde{\alpha}$$

Since $\mathrm{d}\tilde{v} \wedge \tilde{\alpha} = \Big((\nabla v - \nabla v') \cdot \Big(\frac{\nabla v}{w_v} - \frac{\nabla v'}{w_{v'}}\Big)\Big)\,\mathrm{d}x \wedge \mathrm{d}y$, Lemma 1 and Lemma 3 imply that:

$$C(\varepsilon) \iint_{\widetilde{\Omega}_r} |\tilde{\alpha}|^2 \le \int_{C_r} \tilde{v}\tilde{\alpha}$$

Let r_0 be such that $\mu = C(\varepsilon) \iint_{\widetilde{\Omega}_{r_0}} |\tilde{\alpha}|^2 > 0$. In $\widetilde{\Omega}$, $\tilde{v}$ is bounded by 2δ so :

$$\mu + C(\varepsilon) \iint_{\widetilde{\Omega}_r \backslash \widetilde{\Omega}_{r_0}} |\tilde{\alpha}|^2 \le 2\delta \int_{C_r} |\tilde{\alpha}|$$

Let us denote $\int_{C_r} |\tilde{\alpha}|$ by $\eta(r)$. By Schwartz's Lemma :

$$\eta^2(r) \le \ell(C_r) \int_{C_r} |\tilde{\alpha}|^2 \le 2\pi r \int_{C_r} |\tilde{\alpha}|^2$$

Hence $\displaystyle\int_{C_r} |\tilde{\alpha}|^2 \ge \frac{\eta^2(r)}{2\pi r}$ and

$$\int_{r_0}^{r} \frac{\eta^2(t)}{2\pi t} \le \iint_{\widetilde{\Omega}_r \backslash \widetilde{\Omega}_{r_0}} |\tilde{\alpha}|^2$$

Finally :

$$\mu + C(\varepsilon) \int_{r_0}^{r} \frac{\eta^2(t)}{2\pi t} \le 2\delta\eta(r) \tag{4}$$

Let y be the solution of the following Cauchy problem :

$$y'(t) = C(\varepsilon)\frac{y^2(t)}{4\pi\delta t}, \quad y(r_0) = \frac{\mu}{4\delta}$$

y is defined on $[r_0, r_1)$ with $r_1 = r_0 \exp(\frac{16\pi\delta^2}{\mu C(\varepsilon)})$ and satisfies :

$$\frac{4\delta}{\mu} - \frac{1}{y(t)} = \frac{C(\varepsilon)}{4\pi\delta} \ln \frac{t}{r_0}$$

By (4), $\eta(t) \geq y(t)$ on $[r_0, r_1)$ and, since $\lim_{t\to r_1} y(t) = +\infty$, we get a contradiction, indeed η is bounde. Then $v = v'$. □

As said in the introduction Theorem 2 has a consequence for solution of the minimal surface equation.

Corollary 4. *Let Ω be an unbounded simply-connected domain in $\mathbb{R}^2$. Let u and u' be two solutions of $(**)$ in Ω such that Ψ_u and $\Psi_{u'}$ are bounded in Ω and $\Psi_u = \Psi_{u'}$ on $\partial\Omega$. Then $u - u'$ is constant.*

We need the simple-connectedness hypothesis to ensure that Ψ_u and $\Psi_{u'}$ are well defined.

Proof. Ψ_u and $\Psi_{u'}$ are two solutions of $(*)$ in Ω, then, by theorem 2, $\Psi_u = \Psi_{u'}$. Then $\nabla u = \nabla u'$ and $u - u'$ is constant. □

To end Theorem 2 proof, we have to prove Lemma 3.

2 The gradient estimate

This section is devoted to the proof of the gradient estimate in Lemma 3; This is the last step in Theorem 2 proof.

Proof of Lemma 3. If Lemma 3 is not true, we can assume that $\sup_{\widetilde{\Omega}} |\nabla v| = 1$. Thus there exists (p_n) a sequence in $\widetilde{\Omega}$ such that $|\nabla v|(p_n) \to 1$. Let O be the point $(0,0)$. Let r_n be the affine rotation in $\mathbb{R}^2$ such that $r_n(O) = p_n$ and $R_n^{-1}\big(\nabla v(p_n)\big) = (|\nabla v|(p_n), 0)$ (R_n is the linear rotation associated to r_n). We define $v_n = v \circ r_n$ which is a solution of $(*)$ in $\Omega_n = r_n^{-1}\Omega$. We have $\nabla v_n = R_n^{-1}\nabla v$ so $\nabla v_n(O) \to (1,0)$. In the same way we define $v'_n = v' \circ r_n$.

Let $I(a,b) \subset \mathbb{R}^2$ be the segment $[a,b] \times \{0\}$ $(a < b)$. Let ε be positive, ε will be fixed later but let us notice that ε/δ will be small. Let $D(a,b)$

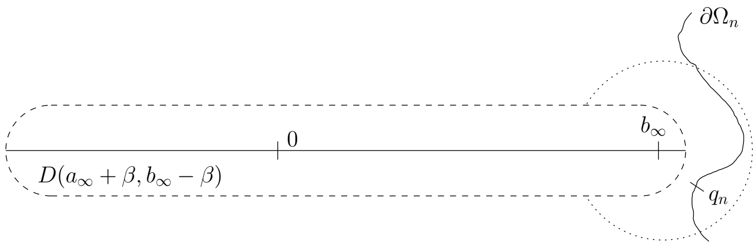

Figure 1:

denote the set $\{p \in \mathbb{R}^2 \,|\, d(p, I(a,b)) < \varepsilon\}$, $D(a,b)$ is the union of a rectangle of width 2ε and length $b-a$ and two half-disks of radius ε (see Figure 1).

For every n, we define a_n and b_n by: $a_n = \inf\{a \leq 0 \,|\, D(a,0) \subset \Omega_n\}$ and $b_n = \sup\{b \geq 0 \,|\, D(0,b) \subset \Omega_n\}$. Since $\varepsilon < \delta$ and $O \in \Omega_{n\delta}$ (because $p_n \in \Omega_\delta$), $b_n > 0$ and $a_n < 0$; moreover $D(a_n, b_n) \subset \Omega_n$. We define $b_\infty = \liminf b_n$, $b_\infty > 0$, b_∞ may take the value $+\infty$; by taking a subsequence, we assume that $b_\infty = \lim b_n$. Then we define $a_\infty = \limsup a_n$, $a_\infty < 0$, a_∞ may take the value $-\infty$; as above we can assume that $a_\infty = \lim a_n$. Let $\beta \leq \min(\varepsilon/2, |a_\infty|, b_\infty)$, let A denote $a_\infty + \beta$ if $a_\infty > -\infty$ and any negative number if not and B denote $b_\infty - \beta$ if $b_\infty < +\infty$ and any positive number if not. For n large enough, $D(A,B) \subset \Omega_n$ (see Figure 1).

Since $D(A,B)$ is simply connected, for each large n in $\mathbb{N}$, there exists u_n a function on $D(A,B)$ such that $\mathrm{d}u_n = \alpha_{v_n}$. Besides the function u_n satisfies the minimal surface equation:

$$\operatorname{div} \frac{\nabla u_n}{\sqrt{1+|\nabla u_n|^2}} = 0 \tag{$**$}$$

The graph of u_n is a minimal surface in $\mathbb{R}^3$ with the euclidean metric. We have

$$\mathrm{d}v_n = \frac{u_{ny}}{\sqrt{1+|\nabla u_n|^2}}\mathrm{d}x - \frac{u_{nx}}{\sqrt{1+|\nabla u_n|^2}}\mathrm{d}y$$

Thus v_n is the opposite of the conjugate function to u_n. Since $\nabla v_n(O) \to (1,0)$, $|\nabla u_n|(O) \to +\infty$ and $\frac{\nabla u_n}{|\nabla u_n|}(O) \to (0,1)$. Then $\{y=0\} \cap D(A,B)$ is a line of divergence for the sequence (u_n) (see [Ma1, Ma2]). This implies that if $A - \varepsilon < s < t < B + \varepsilon$:

$$\lim v_n(t,0) - v_n(s,0) = t - s \tag{5}$$

By hypothesis, v is bounded by some $M > 0$ so v_n is bounded by M. This implies that A and B are bounded thus a_∞ and b_∞ can not take infinite value; indeed (5) implies $B - A \leq 2M$. Hence $A = a_\infty + \beta$ and $B = b_\infty - \beta$. By the definition of b_∞, the point $(b_\infty, 0)$ which is in $D(a_\infty + \beta, b_\infty - \beta)$ is at a distance less than 2ε from $\partial\Omega_n$ for big n (see Figure 1). So there exists, for each large n, a point q_n in $\partial\Omega_n$ such that $d_{\Omega_n}(q_n, (b_\infty, 0)) \leq 2\varepsilon$. By (5), we can assume that for n large enough:

$$v_n(b_\infty, 0) - v_n(O) \geq b_\infty - \varepsilon$$

thus :

$$\begin{aligned} v_n(O) &= v_n(O) - v_n(b_\infty, 0) + v_n(b_\infty, 0) \\ &\leq \varepsilon - b_\infty + v_n(b_\infty, 0) \\ &\leq \varepsilon - b_\infty + 2\varepsilon + \varphi(q_n) = 3\varepsilon - b_\infty + \varphi(q_n) \end{aligned}$$

we recall that φ is the boundary value of v and v'. Moreover

$$v'_n(O) \geq \varphi(q_n) - d_{\Omega_n}(O, q_n) \geq \varphi(q_n) - 2\varepsilon - b_\infty$$

So $v_n(O) - v'_n(O) \leq 5\varepsilon$. Since the sequence (p_n) is in $\widetilde{\Omega}$, $v(p_n) - v'(p_n) > a$ and $v_n(O) - v'_n(O) > a$. Hence if ε is chosen such that $\varepsilon < a/5$, we get a contradiction and Lemma 3 is proved. □

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