The half space property for cmc 1/2 graphs in $\mathbb{E}(-1,\tau)$

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Abstract

In this paper, we prove a half-space theorem with respect to constant mean curvature 1/2 entire graphs in $\mathbb{E}(-1,\tau)$. If Σ is such an entire graph and Σ' is a properly immersed constant mean curvature 1/2 surface included in the mean convex side of Σ then Σ' is a vertical translate of Σ. We also have an equivalent statement for the non mean convex side of Σ.

1 Introduction

In the theory of constant mean curvature surfaces, the half-space property is a problem that have received many contributions in recent years. If Σ is a properly embedded constant mean curvature H_0 surface in a Riemannian 3-manifold M, we may wonder about the existence of an other properly embedded constant mean curvature H_0 surface Σ' which has no intersection with Σ. If such a surface exists, can we say anything about its geometry?

One of the first results about this question is the half-space theorem of Hoffman and Meeks [8]. It says that a minimal surface of $\mathbb{R}^3$ on one side of a plane P is a plane parallel to P. This result can also be interpreted as a maximum principle at infinity: if the minimal surface has a boundary, the distance from the surface to P is reached along its boundary (for a general result, see [12]). The same type of half-space results have been proved by several authors in other homogeneous spaces (see [16, 17, 7, 3, 4, 14, 11, 18]). In most of these results, the half-space property is studied with respect to a surface Σ which is parabolic. The parabolicity is really an important hypothesis as it has been proved in [11] and [18].

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The first result involving non parabolic surfaces is due to Daniel, Meeks and Rosenberg [4] and concerns minimal surfaces in the Heisenberg space Nil_3. As a Riemannian homogeneous space, Nil_3 is a Killing Riemannian submersion over $\mathbb{R}^2$. So, in Nil_3, we can consider surfaces called graphs that are images of sections of the submersion. Among these surfaces, the minimal entire graphs (graph over the whole $\mathbb{R}^2$) have been classified by Fernandez and Mira [6]; certain ones are parabolic and others are not (see Examples 1 and 2 below). Actually, Daniel, Meeks and Rosenberg proved that if Σ' is a properly immersed minimal surface in Nil_3 on one side of an entire minimal graph Σ then Σ' is a vertical translate of Σ.

An other situation where interesting non parabolic surfaces appears is in considering entire constant mean curvature $1/2$ graphs in $\mathbb{H}^2 \times \mathbb{R}$. In fact, Daniel and Hauswirth [3] proved that there is an isometric correspondence between such surfaces and entire minimal graphs in Nil_3. This suggests that a half-space property similar to the one of Nil_3 should be true for these surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Actually, we have partial results in this direction, the half-space property had been established by Nelli and Sa Earp [13] for a particular entire graph which is rotationally symmetric (Example 1). More recently Cartier and Hauswirth [1] proved also the half-space property for a family of entire graphs that can be obtained by deforming the rotationally symmetric one.

In this paper, we prove the half-space property for all entire cmc $1/2$ graphs in $\mathbb{H}^2\times\mathbb{R}$. In fact, we prove this for entire cmc $1/2$ graphs of $\mathbb{E}(-1,\tau)$. $\mathbb{E}(-1,\tau)$ denotes the family of simply connected homogeneous spaces which are Killing Riemannian submersions over $\mathbb{H}^2$; we have $\mathbb{H}^2 \times \mathbb{R} = \mathbb{E}(-1,0)$. Our main theorem (Theorem 3) is then

> **Theorem.** Let Σ be an entire constant mean curvature $1/2$ graph in $\mathbb{E}(-1,\tau)$ and Σ' be a properly immersed constant mean curvature $1/2$ surface in $\mathbb{E}(-1,\tau)$. If Σ' is included in the mean convex side of Σ then Σ' is a vertical translate of Σ.

We also have a statement when Σ' is included in the non-mean convex side of Σ.

Our strategy of proof is similar to the one of [4] and [18] which consists in constructing a family of barriers that converges to a vertical translate of Σ. The main difficulty is to prove that the barriers we construct actually converge to a vertical translate of Σ. So the main part of the proof (Proposition 5) is devoted to a uniqueness result for the exterior Dirichlet problem associate to the constant mean curvature $1/2$ equation. This uniqueness result is in fact a maximum principle at infinity for a certain partial differential equation. If $u_0 \leq u_1$ are two solutions of our exterior Dirichlet problem, we construct $(u_t)_{0\leq t\leq 1}$ a family of solutions of the exterior problem that goes continuously from u_0 to u_1. Then $\partial_t u_t$ defines a Jacobi field on the graph of u_t. We use the associated family introduced by Daniel in [2] to study this Jacobi field on a minimal surface in Nil_3. In fact, we prove uniqueness of this Jacobi field. In some sense, it can be interpreted as an infinitesimal version of the half-space theorem of Daniel, Meeks and Rosenberg. Using this uniqueness, we can reintegrate $\partial_t u_t$ in order to prove that u_0 and u_1 should differ by a constant which leads to the conclusion.

In Section 2, we recall some definitions about the $\mathbb{E}(\kappa,\tau)$ spaces and explain what are the entire constant mean curvature graphs in them. Section 3 is devoted to two gradient estimates for solutions of the constant mean curvature equation, these estimates are used in the construction of the barriers and in the proof of the uniqueness result. In Section 4, we prove our main theorem assuming the uniqueness result (Proposition 5). This uniqueness result is proved in Section 5. Finally, in Appendix A, we give some computations concerning Killing Riemannian submersions and, in Appendix B, we construct two barriers that are used in Section 4.

2 Entire cmc $1/2$ graphs in $\mathbb{E}(-1,\tau)$

2.1 The ambient spaces $\mathbb{E}(\kappa,\tau)$

In this section, we give a quick introduction to the ambient space $\mathbb{E}(\kappa,\tau)$ when $\kappa \leq 0$; for a complete description we refer to [2].

The space $\mathbb{E}(\kappa,\tau)$ is a simply connected homogeneous space with an isometry group of dimension at least 4. For $\kappa \leq 0$, we define $D_\kappa = \{(x,y)\in \mathbb{R}^2 \,|\, 1+\kappa(x^2+y^2)\geq 0\}$ ($D_0=\mathbb{R}^2$). A model for the ambient space $\mathbb{E}(\kappa,\tau)$ is then $D_\kappa\times\mathbb{R}$ with the following complete Riemannian metric

$$\mathrm{d}s^2_{\kappa,\tau} = \lambda_\kappa^2(\mathrm{d}x^2+\mathrm{d}y^2) + (2\tau\lambda_\kappa(y\mathrm{d}x - x\mathrm{d}y)+\mathrm{d}z)^2$$

where

$$\lambda_\kappa = \frac{2}{1+\kappa(x^2+y^2)}$$

The map $\pi : \mathbb{E}(\kappa,\tau)\to (D_\kappa, \lambda_\kappa^2(\mathrm{d}x^2+\mathrm{d}y^2)), (x,y,z)\mapsto (x,y)$ is then a Killing Riemannian submersion (see definitions in Appendix A) over either the hyperbolic space $\mathbb{H}^2(\kappa)$ of curvature κ or the Euclidean space $\mathbb{R}^2$ ($\kappa=0$). Moreover the unit Killing vector field is $\xi=\partial_z$.

If $\mu > 0$, the map $h_\mu : (x, y, z) \mapsto \mu(x, y, z)$ is a diffeomorphism from $\mathbb{E}(\kappa, \tau)$ to $\mathbb{E}(\frac{\kappa}{\mu^2}, \frac{\tau}{\mu})$ such that

$$h_\mu^*(\mathrm{d}s^2_{\frac{\kappa}{\mu^2},\frac{\tau}{\mu}}) = \mu^2 \mathrm{d}s^2_{\kappa,\tau}$$

So the study of the geometry of the spaces $\mathbb{E}(\kappa, \tau)$ when $\kappa < 0$ reduces to the one of the spaces $\mathbb{E}(-1, \tau)$; so in the following we only focus to these spaces. We will denote $\mathbb{H}^2 = \mathbb{H}^2(-1)$ and $\lambda = \lambda_{-1}$, we also use $\overline{\nabla}$ to denote the Levi-Civita connection on $\mathbb{E}(-1, \tau)$.

The vector field ξ generates a flow $(\varphi_t)_t$, in the following we will denote by $p + t$ the point $\varphi_t(p)$ where $p \in \mathbb{E}(-1, \tau)$. This notation is coherent with the identification of sections with functions in the model.

The coordinate z defines a function on $\mathbb{E}(-1, \tau)$. In the following, we will consider the gradient of this function. So we introduce the vector field ζ with the following expression

$$\zeta = \overline{\nabla} z = -2\tau y F_1 + 2\tau x F_2 + \xi \tag{1}$$

where $F_1 = \frac{1}{\lambda}\partial_x - 2\tau y \partial_z$ and $F_2 = \frac{1}{\lambda}\partial_y + 2\tau x \partial_z$ are the horizontal lift of an orthonormal frame of $\mathbb{H}^2$ (let us notice that (F_1, F_2, ξ) is an orthonormal frame of $\mathbb{E}(-1, \tau)$).

2.2 The mean curvature of graphs in $\mathbb{E}(-1, \tau)$

In $\mathbb{E}(-1, \tau)$, a surface is called a graph if it is the image of a smooth section $\sigma : \Omega \subset \mathbb{H}^2 \to \mathbb{E}(-1, \tau)$, this image is called the graph of σ (see Appendix A). For such a section, we can define a vector field $G\sigma$ on Ω by the following property:

$$(G\sigma(p), X)_{\mathbb{H}^2} = (\xi, \mathrm{d}_p\sigma(X))_{\mathbb{E}(-1,\tau)} \text{ for any } X \in T_p\mathbb{H}^2$$

When $\tau = 0$ and σ is a function, $G\sigma$ is the gradient of σ; in general, $G\sigma$ will play the role of the gradient.

In fact, the graph of σ has constant mean curvature H_0 if

$$\mathrm{div}_{\mathbb{H}^2}\left(\frac{G\sigma}{\sqrt{1 + \|G\sigma\|^2}}\right) = 2H_0 \tag{2}$$

where the mean curvature is computed with respect to the upward pointing normal (see (14) in Appendix A for a proof of (2)).

If we use the coordinates given by the model, a section σ can be identified to a function (*i.e.* $z = \sigma(x, y)$) and $G\sigma$ has the following expression:

$$G\sigma = \frac{1}{\lambda^2}(\sigma_x + 2\tau\lambda y)\partial_x + \frac{1}{\lambda^2}(\sigma_y - 2\tau\lambda x)\partial_y$$

2.3 Entire cmc $1/2$ graphs in $\mathbb{E}(-1,\tau)$

If σ is a section defined on the whole $\mathbb{H}^2$ and its mean curvature is constant H_0, it is known that this mean curvature is less than $1/2$ (see [15]). In fact, we are interested in entire graphs with mean curvature $1/2$; so we consider sections σ defined on the whole $\mathbb{H}^2$ which are solution of

$$\mathrm{div}_{\mathbb{H}^2}\left(\frac{G\sigma}{\sqrt{1+\|G\sigma\|^2}}\right)=1 \tag{3}$$

Such an entire graph Σ bounds two connected components of $\mathbb{E}(-1,\tau)$, one is above the graph, it is the mean convex side of Σ, and one is below.

In fact, the space of all constant mean curvature $1/2$ entire graphs in $\mathbb{E}(-1,\tau)$ is classified (see [3], [7] and [14]). Let us give some explanations on the classification.

Let $X:\Sigma\to\mathbb{E}(-1,\tau)$ be a simply connected constant mean curvature $1/2$ surface in $\mathbb{E}(-1,\tau)$. On this surface, we can consider four geometric data: they are its metric g, its shape operator S, a function ν and a vector field T such that along Σ the vector field ξ can be written $\xi=T+\nu N$ where N is the unit normal vector to Σ. In [2], Daniel prove that these data satisfies to equations which are necessary and sufficient for the existence of the immersion X.

Now if we consider θ such that $\tau+\frac{i}{2}=e^{i\theta}\sqrt{\tau^2+\frac{1}{4}}$, we can consider the following data

$$\begin{aligned}
g'&=g\\
S'&=e^{\theta J}(S-\frac{1}{2}I)\\
\nu'&=\nu\\
T'&=e^{\theta J}T
\end{aligned}$$

These new data solve the conditions introduced by Daniel [2] for the existence of a minimal immersion X' of Σ in $\mathbb{E}(0,\tau')$ with $\tau'=\sqrt{\tau^2+\frac{1}{4}}$. Actually this gives an isometric correspondence between minimal immersions in $\mathbb{E}(0,\tau')$ and cmc $1/2$ immersions in $\mathbb{E}(-1,\tau)$. By the work of Daniel, Hauswirth, Rosenberg, Spruck and Penafiel [3, 7, 14], we know that being an entire graph is preserved by this correspondence. So every entire cmc $1/2$ graph in $\mathbb{E}(-1,\tau)$ corresponds to an entire minimal graph in $\mathbb{E}(0,\tau')$. By the work of Fernandez and Mira [6], we know that the space of entire minimal graphs in $\mathbb{E}(0,\tau')$ is in correspondence with the space of quadratic holomorphic differential over $\mathbb{C}$ or the unit disk. This gives the classification of entire cmc 1/2 graphs in $\mathbb{E}(-1,\tau)$

Example 1 (The rotational example). Using the Poincaré disk model D_{-1}, the function

$$\sigma(x,y) = \frac{2}{\sqrt{1-x^2-y^2}}$$

is a solution of (3) for $\tau = 0$. So its graph is an entire 1/2 graph in $\mathbb{H}^2 \times \mathbb{R}$. This graph is invariant by rotations around a vertical axis. This graph is conformal to the unit disk *i.e.* it is a non-parabolic surface.

Example 2 (The translational example). In the half-space model for $\mathbb{H}^2$, the function

$$u(x,y) = \frac{\sqrt{x^2+y^2}}{y}$$

is a solution of (3) for $\tau = 0$. So its graph is an entire 1/2 graph in $\mathbb{H}^2 \times \mathbb{R}$. This graph is invariant by translations along a certain horizontal geodesic line. This graph is conformal to $\mathbb{C}$ *i.e.* it is a parabolic surface.

3 Gradient estimates

In the sequel, we need several gradient estimates for constant mean curvature 1/2 graphs in $\mathbb{E}(-1,\tau)$. Actually, we consider these surfaces as graphs of functions in the model; in consequence the estimates we get depend on our choice of the model.

3.1 Some preliminary computations

Let σ be a section of $\mathbb{E}(-1,\tau)$ whose graph has constant mean curvature H_0, σ is a solution of (2). Let N denote the upward pointing unit normal to the graph Σ of σ. On Σ, we consider the function $\nu = (N, \xi)$. Since ξ is a unit killing vector field, ν is a Jacobi function on Σ so:

$$\Delta_\Sigma \nu = -(Ric(N,N) + \|S\|^2)\nu.$$

where Ric is the Ricci tensor of $\mathbb{E}(-1,\tau)$. Actually, we have $Ric(N,N) = -(1+2\tau^2) + \nu^2(1+4\tau^2)$ (see [2]), thus we get:

$$\Delta_\Sigma \nu = -(-(1+2\tau^2) + \nu^2(1+4\tau^2) + \|S\|^2)\nu.$$

We also have

$$\Delta_\Sigma \frac{1}{\nu} = -\frac{\Delta_\Sigma \nu}{\nu^2} + 2\frac{\|\nabla_\Sigma \nu^2\|^2}{\nu^3}.$$

So if we introduce the operator $Lu = \Delta_\Sigma u - 2\nu(\nabla_\Sigma \frac{1}{\nu}, \nabla_\Sigma u)$, we have:

$$L\frac{1}{\nu} = (-(1+2\tau^2) + \nu^2(1+4\tau^2) + \|S\|^2)\frac{1}{\nu} \geq -\frac{1+2\tau^2}{\nu}.$$

Using the model, we denote by h the restriction of the z coordinate to Σ. We then have

$$\nabla_\Sigma h = \zeta^\top$$

where $X^\top$ denotes the orthogonal projection of X on $T\Sigma$. Using the expression (1), we obtain the following estimate:

$$\|\nabla_\Sigma h\| = \|\zeta\|^2 - (\zeta, N)^2 \geq 1 - \nu^2 - c_1\nu \tag{4}$$

where c_1 is a positive constant. For the Laplacian of h we have:

$$|\Delta_\Sigma h| \leq c_2(x,y) \tag{5}$$

with $c_2(x,y)$ a smooth function that depends on τ and H_0.

Let p be a point in $\mathbb{H}^2$ and d denote the hyperbolic distance from p. The function d can be extended to the whole $\mathbb{E}(-1,\tau)$ by considering $d \circ \pi$. We then have $\overline{\nabla} d = \widetilde{\nabla d}$ where $\widetilde{X}$ denote the horizontal lift of X (see Appendix A) and ∇ is the gradient operator on $\mathbb{H}^2$. We then have

$$\|\nabla_\Sigma d^2\| = \|\overline{\nabla} d^{2^\top}\| \leq d \tag{6}$$

and because of formulas (11), (12) and (13)

$$|\Delta_\Sigma d^2| \leq c_3(x,y,d) \tag{7}$$

with c_3 a smooth function depending on τ and H_0

3.2 A gradient estimate close to the boundary

In this section, we give a first gradient estimate which is similar to Lemma 3.1 proved by Rosenberg, Schulze and Spruck in [18]. This result will be used to control the gradient of a section close to the boundary of the domain.

Proposition 1. *Let σ be a section of $\mathbb{E}(-1,\tau)$ satisfying to (2) which is defined on a bounded domain $\Omega \subset \mathbb{H}^2$ and is C^1 up to the boundary. Then there are two positive constants M and α that depends only on Ω and τ such that*

$$\sup_\Omega W \leq \max(M, \sup_{\partial\Omega} W) \sup_\Omega e^{-\alpha\sigma} \sup_\Omega e^{\alpha\sigma}.$$

where $W = \sqrt{1+\|G\sigma\|^2}$. The quantity $e^{\pm\alpha\sigma}$ are computed by identifying σ with a function in the model.

Proof. Let $\eta = e^{\alpha h}$ where α will be chosen later. From estimates (4) and (5), there is a positive constant k that depends only on Ω and τ such that

$$\Delta_\Sigma \eta = (\alpha^2 \|\nabla_\Sigma h\|^2 + \alpha \Delta_\Sigma h)\eta \geq (\alpha^2(1-\nu^2-k\nu) - \alpha k)\eta$$

Thus

$$L\frac{\eta}{\nu} = (L\frac{1}{\nu})\eta + \frac{1}{\nu}\Delta_\Sigma \eta \geq (-(1+2\tau^2) + \alpha^2(1-\nu^2-k\nu) - \alpha k)\frac{\eta}{\nu}$$

There is ν_0 that depends only on k such that, for $\nu \leq \nu_0$:

$$-(1+2\tau^2) + \alpha^2(1-\nu^2-k\nu) - \alpha k \geq -(1+2\tau^2) + \alpha^2/2 - \alpha k$$

Thus, if α is chosen sufficiently large (depending only on k and τ) we get $L\frac{\eta}{\nu} \geq 0$ where $\nu \leq \nu_0$. By the maximum principle, this implies that the maximum of $\frac{\eta}{\nu}$ is reached on the boundary of Σ or in $\{\nu \geq \nu_0\}$. This implies that

$$\frac{\eta}{\nu} \leq \max(\sup_{\partial\Sigma} \frac{\eta}{\nu}, \frac{1}{\nu_0}\sup_{\Sigma} \eta)$$

This is the expected estimate since $\nu = W^{-1}$. □

3.3 A gradient estimate inside the domain

We give now an other gradient estimate which is used to control a solution far from the boundary. It is similar but less precise than the one given by Korevaar in [9] or Spruck in [20]

Proposition 2. *Let σ be a section of $\mathbb{E}(-1,\tau)$ satisfying to* (2) *which is defined on a geodesic disk of $\mathbb{H}^2$ centered at $p = (x_p, y_p) \in D_{-1}$ and of hyperbolic radius R. We assume that σ viewed as a function in the model is positive. Then there exists a positive constant M that depends on $x_p^2 + y_p^2$, R, $\sigma(p)$ and H_0 such that*

$$W(p) \leq M$$

Proof. Let d be the hyperbolic distance from p in $\mathbb{H}^2$ and we extend it to the whole $\mathbb{E}(-1,\tau)$.

Let us define, on Σ, $\varphi = (-\frac{h}{2h_0} + 3/4 - (\frac{d}{R})^2)^+$ which is less than $3/4$ and where $h_0 = \sigma(p)$. If $P = (p, \sigma(p))$, we have $\varphi(P) = 1/4$ and $\varphi = 0$ close to $\partial\Sigma$. Let us consider $\eta = e^{K\varphi} - 1$ for a constant K that will be chosen below. Let us define $u = \frac{\eta}{\nu}$, we see that $\max u$ is positive and is reached inside the support of φ.

We have

$$\begin{aligned} Lu &= (L\frac{1}{\nu})\eta + (\frac{1}{\nu})\Delta_\Sigma \eta \\ &\geq -(1+2\tau^2)\frac{\eta}{\nu} + \frac{1}{\nu}(K^2\|\nabla_\Sigma \varphi\|^2 + K\Delta_\Sigma\varphi)e^{K\varphi} \\ &\geq (K^2\|\nabla_\Sigma \varphi\|^2 + K\Delta_\Sigma\varphi - (1+2\tau^2))\frac{e^{K\varphi}}{\nu} + \frac{1+2\tau^2}{\nu} \\ &\geq (K^2\|\nabla_\Sigma \varphi\|^2 + K\Delta_\Sigma\varphi - (1+2\tau^2))\frac{e^{K\varphi}}{\nu} \end{aligned}$$

Let us see that K can be chosen such that the first factor in the above expression is positive.

Let us estimate the different terms. We have

$$\nabla_\Sigma \varphi = -\frac{\zeta^\top}{2h_0} - 2\frac{\nabla_\Sigma d^2}{R^2}$$

Thus, by (1) and (6), there is a constant k_1 that depends on p and R such that

$$(\nabla_\Sigma\varphi, \xi) \leq -\frac{1}{2h_0}(1-k_1\nu) + k_1\nu$$

So there exists $\nu_1 > 0$ that depends on k_1 and h_0 such that if $\nu \leq \nu_1$

$$(\nabla_\Sigma\varphi, \xi) \leq -\frac{1}{2\sqrt{2}h_0} \quad \text{and} \quad \|\nabla_\Sigma\varphi\|^2 \geq \frac{1}{8h_0^2}$$

Besides, by (5) and (7), there is a constant k_2 that depends only on the domain p, R, h_0 such that $|\Delta_\Sigma\varphi| \leq k_2$. This implies that for $\nu \leq \nu_1$ we have

$$K^2\|\nabla_\Sigma \varphi\|^2 + K\Delta_\Sigma\varphi - (1+2\tau^2) \geq K^2\frac{1}{8h_0^2} - Kk_2 - (1+2\tau^2)$$

So there exists $K > 0$ that depends only on h_0, k_2 and τ such that, for $\nu \leq \nu_1$,

$$K^2\|\nabla_\Sigma \varphi\|^2 + K\Delta_\Sigma\varphi - (1+2\tau^2) > 0$$

Thus applying the maximum principle for the operator L, we get that the maximum of u is reached at a point Q where $\nu \geq \nu_1$. Thus

$$\frac{e^{K/4}-1}{\nu(P)} \leq u(Q) \leq \frac{e^{3K/4}-1}{\nu_1}$$

So we get the expected estimate:

$$\frac{1}{\nu(P)} \leq \frac{1}{\nu_1}\frac{e^{3K/4}-1}{e^{K/4}-1}$$

□

4 The half-space theorem

In this section, using Proposition 5, we prove the half space theorem with respect to constant mean curvature $1/2$ entire graph in $\mathbb{E}(-1,\tau)$. The theorem is the following.

Theorem 3. *Let* Σ *be a constant mean curvature* $1/2$ *entire graph in* $\mathbb{E}(-1,\tau)$. *Let* Σ' *be a properly immersed constant mean curvature* $1/2$ *surface in* $\mathbb{E}(-1,\tau)$ *such that* $\Sigma \cap \Sigma' = \emptyset$.

- *if* Σ' *is above* Σ *then* $\Sigma' = \Sigma + t$ *for some* $t > 0$.
- *if* Σ' *is below* Σ *and is well oriented with respect to* Σ *then* $\Sigma' = \Sigma - t$ *for some* $t > 0$.

Being well oriented means that the mean curvature vector of Σ' points in the connected component of $\mathbb{E}(-1,\tau)$ bounded by Σ and Σ'. Since Σ' is only immersed, this condition has a meaning only for points of Σ' lying on the boundary of the connected component.

In the following, we write the proof only for the first case. We will just make remarks to explain where the orientation hypothesis is used in the second case. The approach is very similar to the one used by Rosenberg, Schulze and Spruck in [18] (see also Daniel, Meeks and Rosenberg [4]).

4.1 Construction of the barriers

We are working in the model. Let us consider an increasing sequence $0 < r_0 < r_1 < \cdots < r_n < \cdots$ with $r_n < 1$ and $\lim r_n = 1$ and let A_n be the annulus $\{r_0 \le r \le r_n\} \subset D_{-1}$ and $A = \{r_0 \le r < 1\}$ ($r^2 = x^2 + y^2$). We denote by σ the function that defines the graph Σ.

Using the implicit function theorem, there is $\delta > 0$ such that there is a smooth family $(u_{t,1})_{0 \le t \le \delta}$ of smooth solutions $u_{t,1}$ of (3) on A_1 with $u_{t,1} = \sigma + t$ on $\{r = r_0\}$ and $u_{t,1} = \sigma$ on $\{r = r_1\}$.

In fact, the same construction can be made for every n with the same δ (see Figure 1).

Lemma 4. *There is* $\delta > 0$ *such that, for every* $n \ge 1$ *and* $0 \le t \le \delta$, *there exists a smooth solution* $u_{t,n}$ *of* (3) *on* A_n *such that* $u_{t,n} = \sigma + t$ *on* $\{r = r_0\}$ *and* $u_{t,n} = \sigma$ *on* $\{r = r_n\}$.

Moreover, for any $k > 0$ *the family of functions* $(u_{t,n})_{0 \le t \le \delta, n \ge 1}$ *is uniformly bounded in the* $C^{2,\alpha}$ *norm on* A_k.

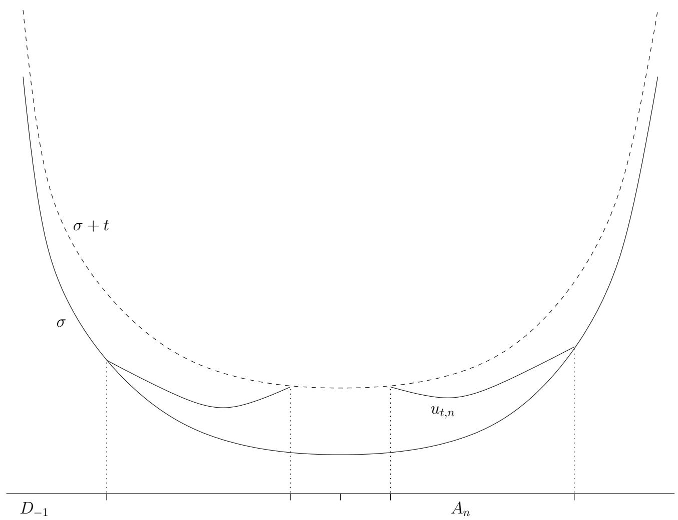

Figure 1: The barriers $u_{t,n}$

Because of the maximum principle, we have uniqueness of solutions to a Dirichlet problem on compact domains; so the above solutions $u_{t,n}$ are unique. Thus $u_{0,n} = \sigma$.

Proof. Let δ be given by the construction of $u_{t,1}$. Let us prove that this constant works also for other n. Let us define $u_{0,n} = \sigma$ on A_n, the existence of $u_{t,n}$ will come from the method of continuity. In order to apply this method, we need some *a priori* estimates for the solutions.

So let us consider a solution $u_{t,n}$ ($t \leq \delta$). By the maximum principle, we have $\sigma \leq u_{t,n} \leq \sigma + t \leq \sigma + \delta$, so there is a constant $c_1(n) > 0$ such that $|u_{t,n}| \leq c_1(n)$ on A_n.

By the maximum principle, on A_1, we have $u_{\delta,1} + t - \delta \leq u_{t,n} \leq \sigma + t$. But these three functions coincide on $\{r = r_0\}$ so the gradient of $u_{t,n}$ on $\{r = r_0\}$ is bounded by a constant c_2 that does not depend on n and t.

From Appendix B, we know that there exists a smooth function h on A_n such that $h = \sigma$ on $\{r = r_n\}$, $h \geq c_1(n)$ on $\{r = r_0\}$ and

$$\mathrm{div}_{\mathbb{H}^2}\left(\frac{Gh}{\sqrt{1 + \|Gh\|^2}}\right) \leq 1$$

so by the maximum principle, $\sigma \leq u_{t,n} \leq h$ on A_n. These functions coincide on $\{r = r_n\}$ so the gradient of $u_{t,n}$ on $\{r = r_n\}$ is bounded by a constant

$c_3(n)$. Thus by Proposition 1, there is a constant $c_4(n)$ such that $\|\nabla u_{t,n}\| \leq c_4(n)$ on A_n for any $t \in [0, \delta]$. This implies that the equation solved by $u_{t,n}$ is uniformly elliptic.

Then the DeGiorgi-Nash-Moser and Schauder estimates implies *a priori* bounds for higher derivatives of $u_{t,n}$ on A_n. The method of continuity can then be applied to prove the existence of the solutions. We notice that the estimates we just get depend on n but in the lemma we want estimates that are also independent of n.

For the C^1 estimate, let us consider $k \geq 1$, as above $\sigma \leq u_{t,n} \leq \sigma + \delta$ so the family $(u_{t,n})_n$ is uniformly bounded on A_{k+1}. Thus by Proposition 2, the gradient of $u_{t,n}$ is uniformly bounded on $\{r = r_k\}$. Since $\|\nabla u_{t,n}\| \leq c_2$ on $\{r = r_0\}$; Proposition 1 tells that $\nabla u_{t,n}$ is uniformly bounded on A_k. As above this gives uniform estimates for higher derivatives on A_k. □

Since $(u_{\delta,n})$ and their derivatives are uniformly bounded, a diagonal process gives a smooth solution u_δ of (3) on A and a subsequence $(u_{\delta,n'})$ such that $u_{\delta,n'} \to u_\delta$ where the convergence is smooth in all compact subsets of A.

By construction, we have $u_\delta = \sigma + \delta$ on ∂A and $\sigma \leq u_\delta \leq \sigma + \delta$. We remark that $\sigma + \delta$ is an other solution of (3) with the same property; so by Proposition 5 (see Section 5), $u_\delta = \sigma + \delta$.

Remark 1. For the case Σ' below Σ, we need to construct solutions $(u_{-\delta,n})$ which are below σ. The proof is similar, we just use the k barriers of Appendix B instead of the h barriers.

4.2 Proof of the half space theorem

Let Σ' be a properly immersed constant mean curvature $1/2$ surface in $\mathbb{E}(-1, \tau)$ which is above Σ. By replacing Σ by $\Sigma + t$ if necessary, we can assume that Σ' is not above $\Sigma + \varepsilon$ for any $\varepsilon > 0$.

If Σ and Σ' touches, the maximum principle implies that $\Sigma' = \Sigma$; so we can assume that the two surfaces do not meet. Let $\delta' > 0$ be such that Σ' does not meet $\Sigma + t$ for $t \in [0, \delta']$ over $\{r \leq r_0\}$. Let $\delta \leq \delta'$ such that Lemma 4 is true and let us consider a subsequence of $(u_{\delta,n})$ such that $u_{\delta,n'} \to \sigma + \delta$.

Let us denote by $\Sigma_{\delta,n}$ the graph of $u_{\delta,n}$. The surface $\Sigma_{\delta,n} - \delta$ is below Σ so below Σ'. Besides the boundary of $\Sigma_{\delta,n} - t$ for $t \in [0, \delta]$ never meet Σ', thus, by the maximum principle, $\Sigma_{\delta,n}$ is below Σ'. Letting n tends to $+\infty$ along the chosen subsequence implies that $\Sigma + \delta = \lim \Sigma_{\delta,n'}$ is below Σ'. This gives a contradiction and Theorem 3 is proved.

Remark 2. When Σ' is below Σ, the orientation hypothesis is used in order to apply the maximum principle between Σ' and Σ and between Σ' and $\Sigma_{-\delta,n}+t$.

5 A uniqueness exterior result

In this section we prove a uniqueness result for (3) in an exterior domain on $\mathbb{H}^2$.

We still consider the model for $\mathbb{E}(-1,\tau)$. Let us consider $r_0\in(0,1)$ and $A=\{r\ge r_0\}\in D_{-1}=\mathbb{H}^2$. We have the following uniqueness result.

Proposition 5. *Let u,v be two smooth solutions of* (3) *on A such that $u=v$ on ∂A and $|u-v|$ is bounded on A then $u=v$.*

The rest of this section is devoted to the proof of this result.

Let u and v be as in the proposition. If $u\ne v$ and exchanging u and v if necessary, there is a $t_0>0$ such that $v+t_0\ge u$ and $\inf_A(v-u+t_0)=0$. Let us denote $\tilde u=v+t_0$. We have $\tilde u=u+t_0$ on ∂A and $\inf_A(\tilde u-u)=0$.

5.1 Construction of a foliation

Let $r_0<r_1<\cdots$ be an increasing sequence with $\lim r_n=1$. As above, we denote $A_n=\{r_0\le r\le r_n\}$

Let $\delta<t_0$ be given by Lemma 4 with $\sigma=u$. Then we have the associated family of solutions $u_{t,n}$ ($t\in[0,\delta]$) of (3).

We remark that if $t\le t'$, the maximum principle tells that $u_{t,n}\le u_{t',n}\le u_{t,n}+t'-t\le\tilde u$. By Lemma 4, for any k, the family $(u_{t,n})_{0\le t\le\delta,n\ge1}$ is uniformly bounded in the $C^{2,\alpha}$ norm over A_k.

Thus by a diagonal process, there is a family of smooth solutions $(u_{q\delta})_{q\in\mathbb{Q}\cap[0,1]}$ of (3) in A and a subsequence such that $u_{q\delta,n'}\to u_{q\delta}$ for any $q\in\mathbb{Q}\cap[0,1]$ (the convergence is smooth in any compact subset of A). If $q\le q'\in\mathbb{Q}\cap[0,1]$, we have $u_{q\delta}\le u_{q'\delta}\le u_{q\delta}+(q'-q)\delta$. Besides the family $(u_{q\delta})_{q\in\mathbb{Q}\cap[0,1]}$ is uniformly bounded in the $C^{2,\alpha}$ norm over A_k. These two properties imply that, for $t\in[0,\delta]$, the function

$$u_t=\sup_{\substack{q\in\mathbb{Q}\cap[0,1]\\ q\delta\le t}}u_{q\delta}=\inf_{\substack{q\in\mathbb{Q}\cap[0,1]\\ q\delta\ge t}}u_{q\delta}$$

is well defined and is a smooth solution of (3) on A (we notice that this definition coincide with the original one when $t/\delta\in\mathbb{Q}$). By construction, if $t\le t'$, $u_t\le u_{t'}\le u_t+(t'-t)$ and $u_t=u+t=u_0+t$ on ∂A. Moreover, for any k, the family $(u_t)_{t\in[0,\delta]}$ is uniformly bounded in the $C^{2,\alpha}$ norm over A_k. Thus $\lim_{t\to t'} u_t = u_{t'}$ with smooth convergence on any compact subset of A.

We also have $u = u_0 \le u_t \le u_\delta \le \tilde{u}$; this implies that $\inf_A(u_t - u) = 0$.

5.2 Derivation of the foliation

In the preceding subsection, we have constructed the family (u_t) and proved that it depends continuously in the parameter t. In this subsection, we study the differentiability with respect to t.

Since all the functions u_t satisfies to (3), a derivative $v = \frac{du_t}{dt}|_{t=\bar{t}}$ is *a priori* a solution of

$$\mathrm{div}_{\mathbb{H}^2}\left(\frac{\nabla v - \chi_{u_{\bar{t}}}(\nabla v, \chi_{u_{\bar{t}}})}{\sqrt{1+\|Gu_{\bar{t}}\|^2}}\right) = 0 \tag{8}$$

where $\chi_u = \dfrac{Gu}{\sqrt{1+\|Gu\|^2}}$ (see Appendix A). We will prove that this is in fact the case.

Lemma 6. *Let $\bar{t} \in [0,\delta]$ and (t_k) a sequence converging to $\bar{t}$ then there is a solution v of (8) and a subsequence such that*

$$\frac{u_{t_{k'}} - u_{\bar{t}}}{t_{k'} - \bar{t}} \longrightarrow v$$

with a smooth convergence in any compact subset of A.

The idea of the proof comes from the work of Solomon about the regularity of minimal foliations [19].

Proof. Let v_k be equal to $u_{t_k} - u_{\bar{t}}$. We then have $Gu_{t_k} = Gu_{\bar{t}} + \nabla v_k$. Thus v_k is a solution of the following equation

$$\begin{aligned} 0 &= \mathrm{div}_{\mathbb{H}^2}\left(\frac{\nabla v_k + Gu_{\bar{t}}}{\sqrt{1+\|\nabla v_k + Gu_{\bar{t}}\|^2}} - \frac{Gu_{\bar{t}}}{\sqrt{1+\|Gu_{\bar{t}}\|^2}}\right) \\ &= \mathrm{div}_{\mathbb{H}^2}\left(\int_0^1 \frac{\nabla v_k - \chi_{w_{k,s}}(\nabla v_k, \chi_{w_{k,s}})}{\sqrt{1+\|Gw_{k,s}\|^2}}\mathrm{d}s\right) \end{aligned} \tag{9}$$

with $w_{k,s} = u_{\bar{t}} + s v_k$. Let us denote by $P_{k,s}$ the linear operator

$$P_{k,s}(X) = \frac{X - \chi_{w_{k,s}}(X, \chi_{w_{k,s}})}{\sqrt{1+\|Gw_{k,s}\|^2}}.$$

Actually, $P_{k,s}$ is a smooth section of the endomorphisms on $T\mathbb{H}^2$ defined on A.

Let $n\in\mathbb{N}^*$, we know that the functions u_{t_k} and $u_{\bar t}$ are uniformly bounded in $C^{2,\alpha}$ norm over A_n. Thus the family $(w_{k,s})_{k\in\mathbb{N},s\in[0,1]}$ is uniformly bounded in $C^{2,\alpha}$ norm over A_n. This implies first that the family $(P_{k,s})$ is uniformly bounded in $C^{1,\alpha}$ norm over A_n and that there exist a constant $c_n>0$ such that, for any $X\in T\mathbb{H}^2$,

$$(P_{k,s}X,X)=\frac{\|X\|-(X,\chi_{w_{k,s}})^2}{\sqrt{1+\|Gw_{k,s}\|^2}}\geq\frac{(1-\|\chi_{w_{k,s}}\|^2)\|X\|^2}{\sqrt{1+\|Gw_{k,s}\|^2}}\geq c_n\|X\|^2$$

Thus we have proved that the operators

$$L_k(w)=\operatorname{div}_{\mathbb{H}^2}\int_0^1 P_{k,s}(\nabla w)\mathrm{d}s$$

are uniformly elliptic with uniformly bounded $C^{0,\alpha}$ coefficients in A_n. So, by Schauder estimates, there is a constant c_n such that

$$\|v_k\|_{C^{2,\alpha}(A_n)}\leq c_n(\|v_k\|_{C^0(A_{n+1})}+\|v_k\|_{C^{2,\alpha}(\{r=r_0\})})\leq 2c_n|t_k-\bar t|$$

The last inequality comes from the fact that $|v_k|=|u_{t_k}-u_{\bar t}|\leq|t_k-\bar t|$ and $v_k=t_k-\bar t$ along $\{r=r_0\}$. Thus $v_k/(t_k-\bar t)$ is uniformly bounded in $C^{2,\alpha}$ norm over A_n. So, by a diagonal process, there is a subsequence $v_{k'}/(t_{k'}-\bar t)$ that converges smoothly to a function v on any compact subset of A.

Besides, since $v_k\to 0$ smoothly, the operator $P_{k,s}$ converges uniformly to the operator P defined by:

$$P(X)=\frac{X-\chi_{u_{\bar t}}(X,\chi_{u_{\bar t}})}{\sqrt{1+\|Gu_{\bar t}\|^2}}.$$

So v is a solution of (8). □

Since $u_t\leq u_{t'}\leq u_t+(t'-t)$ if $t\leq t'$, the function v satisfies $0\leq v\leq 1$. Moreover, on ∂A, we have $v=1$. The constant function 1 is an obvious solution to (8) that satisfies to the same properties; in the following we will indeed prove that 1 is the only such solution.

5.3 The associate surface in $\mathbb{E}(0,\tau')$

Let $\bar t$ be in $[0,\delta]$ and Σ be the graph of $u_{\bar t}$. We will consider several metrics on the annulus Σ, so in the computations, we will always explain to which metric the computation is made. The function v constructed by Lemma 6 can be viewed as a function on Σ. From Appendix A, if ν is the angle function (N, ξ), $v\nu$ is a Jacobi function on (Σ, g) where g is the induced metric from $\mathbb{E}(-1, \tau)$. In this section, we use the associated family of constant mean curvature surfaces introduced by Daniel [2] and the work of Daniel and Hauswirth [3] to prove the following result.

Lemma 7. *There exist a flat metric g_0 on Σ and a vector field G on Σ such that any function w such that $w\nu$ is a Jacobi function of (Σ, g) is a solution to*

$$\operatorname{div}_0\left(\frac{\nabla^0 w - \chi(\nabla^0 w, \chi)}{\sqrt{1 + \|G\|_0^2}}\right) = 0 \tag{10}$$

with $\chi = \frac{G}{\sqrt{1+\|G\|_0^2}}$ (the sub or superscript 0 means that the computation are made with respect to the g_0 metric).

Proof. Let $\widetilde{\Sigma}$ be the universal cover of Σ. The immersion X of $\widetilde{\Sigma}$ in $\mathbb{E}(-1, \tau)$ is encoded in the induced metric g, the shape operator S, the function ν and the orthogonal projection T of the vector field ξ. As in Section 2.3, let θ be such that $\tau + \frac{i}{2} = e^{i\theta}\sqrt{\frac{1}{4} + \tau^2}$. Then, the data

$$\begin{aligned} g' &= g \\ S' &= e^{\theta J}(S - \frac{1}{2}I) \\ \nu' &= \nu \\ T' &= e^{\theta J}T \end{aligned}$$

encode a minimal immersion X' of $\widetilde{\Sigma}$ in $\mathbb{E}(0, \sqrt{\frac{1}{4} + \tau^2})$ (J denotes the rotation by $\pi/2$ in the tangent space to $\widetilde{\Sigma}$) (see [2]).

The Jacobi operator of $X'(\widetilde{\Sigma})$ is:

$$\begin{aligned} \Delta_{g'} + (-2\tau'^2 + 4\tau'^2\nu'^2 + \|S'\|^2) &= \Delta_g + (-2(\frac{1}{4} + \tau^2) + 4(\frac{1}{4} + \tau^2)\nu^2 + \|S\|^2 - \frac{1}{2}) \\ &= \Delta_g + (-\frac{1}{2} - 2\tau^2) + (1 + 4\tau^2)\nu^2 + \|S\|^2) \end{aligned}$$

where the computation are made with respect to the metric $g = g'$. Thus the Jacobi operator of $X'(\widetilde{\Sigma})$ is the same that the one of $X(\widetilde{\Sigma})$. If w is viewed as a function on $\widetilde{\Sigma}$, the function $w\nu$ is a Jacobi function on $X(\widetilde{\Sigma})$ so it is a Jacobi function on $X'(\widetilde{\Sigma})$.

Let π be the submersion from $\mathbb{E}(0,\tau')$ to $\mathbb{R}^2$. Since $\nu' = \nu > 0$, the map $\pi \circ X'$ is a local diffeomorphism from $\widetilde{\Sigma}$ to $\mathbb{R}^2$, so we can lift to $\widetilde{\Sigma}$ the flat metric of $\mathbb{R}^2$. Let g_0 denote this flat metric on $\widetilde{\Sigma}$. Besides locally, we can describe $X'(\widetilde{\Sigma})$ has the graph of a section s. By Appendix A, since $w\nu'$ is a Jacobi function on $X'(\widetilde{\Sigma})$, the function w, viewed as a function on $\mathbb{R}^2$, is a solution of

$$\operatorname{div}_{\mathbb{R}^2}\left(\frac{\nabla^{\mathbb{R}^2} w - \chi_s(\nabla^{\mathbb{R}^2} w, \chi_s)}{\sqrt{1+\|Gs\|^2_{\mathbb{R}^2}}}\right) = 0$$

The computation are made with respect to the Euclidean metric of $\mathbb{R}^2$. But the vector field $G = (\pi \circ X')^*(Gs)$ is globally well defined on $\widetilde{\Sigma}$ and then v is a solution on $\widetilde{\Sigma}$ of:

$$\operatorname{div}_0\left(\frac{\nabla^0 w - \chi(\nabla^0 w, \chi)}{\sqrt{1+\|G\|^2_0}}\right) = 0$$

where $\chi = \frac{G}{\sqrt{1+\|G\|^2_0}}$. Let us now see that this description passes to the quotient surface Σ. Let us consider γ a generator of $\pi_1(\Sigma)$. The element γ acts on $\widetilde{\Sigma}$ as a diffeomorphism without any fixed point. Moreover, from the uniqueness part of Theorem 4.3 in [2], there exists M an isometry of $\mathbb{E}(0,\tau')$ such that for any $p \in \widetilde{\Sigma}$ we have $X'(\gamma \cdot p) = M(X'(p))$. Besides there is an isometry m of $\mathbb{R}^2$ such that $\pi(M(q)) = m(\pi(q))$ for any $q \in \mathbb{E}(0,\tau')$. This implies that $\pi(X'(\gamma \cdot p)) = m(\pi(X'(p)))$ so $\gamma^* g_0 = g_0$ and $\gamma^* G = G$. This implies that the metric and the vector field pass to the quotient surface Σ and w on Σ is a solution of (10). □

In fact, the metric g_0 on the surface Σ which has a compact boundary satisfies the following property.

Lemma 8. *The flat metric g_0 is complete.*

Proof. It suffices to prove that every curve in Σ starting from a point in the boundary of Σ which is proper in Σ has infinite length with respect to g_0. On Σ, we lift the function r which is the radial coordinate on D_{-1}; $r : \Sigma \to [0,1)$ is proper. r can also be lifted to $\widetilde{\Sigma}$.

Actually, it suffices to prove that every curve in $\widetilde{\Sigma}$ starting from a point in the boundary of $\widetilde{\Sigma}$ such that $r \circ \gamma \to 1$ has infinite length with respect to g_0.

So let γ be such a curve in $\widetilde{\Sigma}$. We have the following estimate:

$$\ell(\gamma, g_0) = \ell(\pi(X'(\gamma)), \mathbb{R}^2) \geq \int_{X'(\gamma)} \nu' \mathrm{d}\ell_{g'} = \int_{X(\gamma)} \nu \mathrm{d}\ell_g \geq \int_{\pi(X(\gamma))} \nu \mathrm{d}\ell_{\mathbb{H}^2}$$

In $\{r_0 \le r \le r_1\}$, we know that there is $\nu_0 > 0$ such that $\nu \ge \nu_0$. So for a curve γ with $r \circ \gamma$ proper, we have:

$$\ell(\gamma, g_0) \ge 2\nu_0(\operatorname{argth} r_1 - \operatorname{argth} r_0) = \ell_0 > 0$$

This estimate implies that we can only consider points in $\widetilde{\Sigma}$ which are at a distance larger than ℓ_0 from $\partial\widetilde{\Sigma}$ in the g_0 metric. Using this, we can apply the work of Daniel and Hauswirth [3] to prove that, if $(\widetilde{\Sigma}, g_0)$ does not satisfied to the expected property, then $X'(\widetilde{\Sigma})$ has a subset which is a graph over a strip $S \in \mathbb{R}^2$ isometric to $(0, \varepsilon) \times \mathbb{R}$. Moreover this graph goes to $+\infty$ on one of its boundary component. But the existence of such a minimal graph is impossible by Theorem 6.3 in [3]. □

5.4 A uniqueness result for Jacobi function

In this subsection, we prove a uniqueness result for solution of a certain partial derivative equation over a flat surface.

Lemma 9. *Let $(S, \mathrm{d}s^2)$ be a complete flat surface with a compact boundary. Let G be a smooth vector field on S. We consider w and w' two smooth functions on S with the same boundary value which solve the following partial derivatives equation:*

$$\operatorname{div}\left(\frac{\nabla u - \chi(\nabla u, \chi)}{\sqrt{1 + \|G\|^2}}\right) = 0$$

with $\chi = \frac{G}{\sqrt{1+\|G\|^2}}$. If $w - w'$ is bounded then $w = w'$.

Proof. The first step of the proof is to estimate the growth of the surface S. Let d be the distance function from ∂S. d is a Lipschitz function. If n is the inward unit normal vector to ∂S, the set $\{d = d_0\}$ is included in the image of the map $\varphi_{d_0} : \partial S \to S; p \mapsto \exp_p(d_0 N(p))$ (actually this map is well defined only on a subset of ∂S). With κ the geodesic curvature of ∂S, we obtain:

$$\ell(\{d = d_0\} \le \ell(\operatorname{im} \varphi_{d_0}) \le \int_{\partial S} |1 + \kappa d_0| \mathrm{d}\ell \le (1 + \bar{\kappa} d_0)\ell(\partial S)$$

where $\bar{\kappa} = \max |\kappa|$. Thus $\ell(\{d = d_0\}$ has at most a linear growth.

Let P be the linear map defined on TS by:

$$P(X) = \frac{X - \chi(X, \chi)}{\sqrt{1 + \|G\|^2}}$$

We have

$$
\begin{aligned}
\|P(X)\|^2 &= \frac{\|X\|^2 - (X,\chi)^2}{1+\|G\|^2} - \frac{(1-\|\chi\|^2)(X,\chi)^2}{1+\|G\|^2} \\
&\leq \frac{\|X\|^2 - (X,\chi)^2}{1+\|G\|^2} = \frac{(X,P(X))}{\sqrt{1+\|G\|^2}} \\
&\leq (X,P(X))
\end{aligned}
$$

The function d is Lipschitz continuous and the set $\{0 \leq d \leq d_0\}$ is compact with rectifiable boundary. So let us define

$$
I(d_0) = \int_{\partial\{0\leq d\leq d_0\}} (w-w')(P(\nabla w - \nabla w'),\eta)\mathrm{d}\mathcal{H}^1
$$

and $\mu(d_0) = \displaystyle\int_{\{d=d_0\}} \|P(\nabla w - \nabla w')\|\mathrm{d}\mathcal{H}^1$. Since $w - w' = 0$ on $\{d = 0\}$, $|w-w'| \leq M$ and $\partial\{0 \leq d \leq d_0\} \subset \{d = 0\} \cup \{d = d_0\}$, we have $|I(d_0)| \leq M\mu(d_0)$. Using Stokes and coarea formulas and $\|\nabla d\| = 1$ a.e., we also have for $d_0 > d_1$:

$$
\begin{aligned}
I(d_0) &= \int_{\{d\leq d_0\}} (\nabla w - \nabla w', P(\nabla w - \nabla w')) \\
&\geq I(d_1) + \int_{\{d_1\leq d\leq d_0\}} \|P(\nabla w - \nabla w')\|^2 \\
&\geq I(d_1) + \int_{d_1}^{d_0}\int_{\{d=s\}} \|P(\nabla w - \nabla w')\|^2\mathrm{d}\mathcal{H}^1\mathrm{d}s
\end{aligned}
$$

Since $\ell(\{d = s\})$ is at most linear, there is a $c > 0$ such that for $s \geq d_1$:

$$
\mu(s)^2 \leq \ell(\{d=s\})\int_{\{d=s\}} \|P(\nabla w - \nabla w')\|^2\mathrm{d}\mathcal{H}^1 \leq cs\int_{\{d=s\}} \|P(\nabla w - \nabla w')\|^2\mathrm{d}\mathcal{H}^1
$$

So we have:

$$
\frac{I(d_1)}{M} + \int_{d_1}^{d_0} \frac{\mu^2(s)}{cMs}\mathrm{d}s \leq \mu(d_0)
$$

We notice that μ is *a priori* just a locally bounded measurable function, but we can integrate the above differential inequality. Indeed let $g(d_0)$ be the left-hand side of the above inequality. Since μ is locally bounded, g is locally Lipschitz. Moreover

$$
g(d_0) = \frac{I(d_1)}{M} + \int_{d_1}^{d_0} \frac{\mu^2(s)}{cMs}\mathrm{d}s \geq \frac{I(d_1)}{M} + \int_{d_1}^{d_0} \frac{g^2(s)}{cMs}\mathrm{d}s
$$

Now if $f(d_0)$ is the right-hand side of the above inequality, f is C^1, satisfies the same differential inequality and we have :

$$f'(s) = \frac{g^2(s)}{cMs} \geq \frac{f^2(s)}{cMs}$$

So

$$f(d_0) \geq \frac{I(d_1)}{1 - \frac{I(d_1)}{M^2 c} \log(\frac{d_0}{d_1})}$$

Now since $f(d_0) \leq g(d_0) \leq \mu(d_0)$

$$\mu(d_0) \geq \frac{I(d_1)}{1 - \frac{I(d_1)}{M^2 c} \log(\frac{d_0}{d_1})}$$

But this lower bound blows up at $d_0 = d_1 \exp(\frac{M^2 c}{I(d_1)})$, so we get a contradiction and this finishes the proof of Lemma 9. □

5.5 End of the proof

Let us now use all the preceding lemmas to finish the proof of Proposition 5.

So let u and $\tilde{u}$ be the function introduced at the beginning of the section. Let $(u_t)_{t\in[0,\delta]}$ be the family of function constructed in Section 5.1.

Let $\bar{t} \in [0, \delta]$ and (t_k) be a sequence converging to $\bar{t}$. From Lemma 6, there are a solution v of (8) and a subsequence such that

$$\frac{u_{t_{k'}} - u_{\bar{t}}}{t_{k'} - \bar{t}} \longrightarrow v$$

By construction $0 \leq v \leq 1$ and $v = 1$ on ∂A. $w \equiv 1$ is an other solution of (8) with the same properties. Let us see v and w as functions on the graph Σ of $u_{\bar{t}}$. From Lemmas 7 and 8, there is a complete flat metric g_0 on Σ and a vector field G such that v and w are solutions of (10). Thus, by Lemma 9, $v = w \equiv 1$. The uniqueness of the possible limit implies that

$$\frac{u_{t_k} - u_{\bar{t}}}{t_k - \bar{t}} \longrightarrow 1$$

for every sequence (t_k). So u_t is differentiable with respect to t and $\dfrac{\partial u_t}{\partial t} = 1$. This implies that $\tilde{u} \geq u_\delta = u_0 + \delta = u + \delta$. So we have a contradiction with $\inf_A(\tilde{u} - u) = 0$ and it finishes the proof of Proposition 5

A Some computations in Killing Riemannian submersions

In this appendix we recall some definitions about Killing Riemannian submersions and make some computations concerning graphs in such an ambient space (see [10, 5]).

Let $(M^{n+1}, \bar{g})$ and $(B^n, (\cdot,\cdot))$ be two complete Riemannian manifolds. Let $\pi : M \to B$ be a submersion. The tangent space T_pM at p then splits in $\ker d\pi \oplus (\ker d\pi)^\perp$ where $\ker d\pi$ is the 1-dimensional space of *vertical* vectors and $(\ker d\pi)^\perp$ is the space of *horizontal* vectors. The submersion π is called *Riemannian* if $d\pi$ is an isometry from $(\ker d\pi)^\perp$ to $T_{\pi(p)}B$.

Definition 1. *A Riemannian submersion $\pi : M \to B$ is a Killing submersion if it admits a complete vertical unit Killing vector field.*

If $\pi : M \to B$ is a Killing Riemannian submersion, we denote by ξ this unit Killing vector field. Besides, if X is a vector field in B, we denote by $\widetilde{X}$ its horizontal lift by π.

Using this notation there exists a 2-form ω on B such that for any vector fields X, Y on B we have:

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]} + \omega(X, Y)\xi$$

We notice that $[\widetilde{X}, \xi] = 0$. When X is a tangent vector to B, we denote X^ω the vector such that $(X^\omega, Y) = \omega(X, Y)$ for any Y. With this notation the Levi-Civita connection of M and B are relied by

$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}\omega(X, Y)\xi \tag{11}$$

$$\overline{\nabla}_{\widetilde{X}}\xi = -\frac{1}{2}\widetilde{X^\omega} \tag{12}$$

$$\overline{\nabla}_\xi \xi = 0 \tag{13}$$

In a Killing Riemannian submersion, we are interested by surfaces that are the image of sections σ. These surfaces are called vertical graphs in M and the image of σ is also called the graph of σ. If σ is a section defined over $\Omega \subset B$, we define on Ω a vector field $G\sigma$ by the following property:

$$(G\sigma, X) = \bar{g}(\mathrm{d}\sigma(X), \xi)$$

for any X tangent to B. This vector field $G\sigma$ plays the role of the gradient of a function.

First, the upward pointing unit normal to the graph of σ is given by the following expression:

$$N = \frac{-\widetilde{G\sigma} + \xi}{\sqrt{1 + \|G\sigma\|^2}}$$

In the following, we denote $\sqrt{1+\|G\sigma\|^2}$ by W. In fact the expression of N is defined in the whole $\pi^{-1}(\Omega)$ and the mean curvature of the graph of σ is given by

$$nH = -\operatorname{div}_{\bar{g}}\left(\frac{-\widetilde{G\sigma}+\xi}{W}\right)$$

So if (E_i) is an orthonormal frame of T_pB we have

$$\begin{aligned}
nH &= -\sum_i \bar{g}(\overline{\nabla}_{\widetilde{E}_i}\frac{-\widetilde{G\sigma}+\xi}{W}, \widetilde{E}_i) - \bar{g}(\overline{\nabla}_{\xi}\frac{-\widetilde{G\sigma}+\xi}{W}, \xi) \\
&= \sum_i \bar{g}(\widetilde{\nabla_{E_i}\frac{G\sigma}{W}}, \widetilde{E_i}) + \frac{1}{2W}\bar{g}(\widetilde{E_i^{\omega}}, \widetilde{E_i}) \\
&= \operatorname{div}\left(\frac{G\sigma}{W}\right) + \sum_i \frac{1}{2W}\omega(E_i, E_i) \\
&= \operatorname{div}\left(\frac{G\sigma}{W}\right)
\end{aligned}$$

where div denote the divergence operator on B. So the mean curvature is given by

$$nH = \operatorname{div}\left(\frac{G\sigma}{\sqrt{1+\|G\sigma\|^2}}\right) \tag{14}$$

Let us denote by Σ the graph of σ. The map $\sigma : \Omega \to \Sigma$ is a chart, so let us make some computation using this system of coordinates. We have:

$$\mathrm{d}\sigma(X) = \widetilde{X} + (G\sigma, X)\xi$$

so the induced metric is $g(X,Y) = (X,Y) + (G\sigma, X)(G\sigma, Y)$. If u is a function on Ω, we get

$$(\nabla u, X) = \mathrm{d}u(X) = g(\nabla_g u, X) = (\nabla_g u, X) + (G\sigma, \nabla_g u)(G\sigma, X)$$

Thus $\nabla u = \nabla_g u + (G\sigma, \nabla_g u)G\sigma$; this implies that

$$\nabla_g u = \nabla u - (\chi_\sigma, \nabla u)\chi_\sigma$$

where $\chi_\sigma = \frac{G\sigma}{\sqrt{1+\|G\sigma\|^2}}$. As a consequence, we have:

$$\begin{aligned} g(\nabla_g v, \nabla_g w) &= (\nabla v - (\chi_\sigma, \nabla v)\chi_\sigma, \nabla w - (\chi_\sigma, \nabla w)\chi_\sigma) \\ &\quad + (G\sigma, \nabla v - (\chi_\sigma, \nabla v)\chi_\sigma)(G\sigma, \nabla w - (\chi_\sigma, \nabla w)\chi_\sigma) \\ &= (\nabla v, \nabla w) - (2 - \|\chi_\sigma\|^2)(\chi_\sigma, \nabla v)(\chi_\sigma, \nabla w) \\ &\quad + W^2(1 - \|\chi_\sigma\|^2)^2(\chi_\sigma, \nabla v)(\chi_\sigma, \nabla w) \\ &= (\nabla v, \nabla w) - (\chi_\sigma, \nabla v)(\chi_\sigma, \nabla w) \end{aligned}$$

Besides the divergence operator for the metric g have the following expression:

$$\operatorname{div}_g X = \frac{1}{\sqrt{1+\|G\sigma\|^2}} \operatorname{div}\big(\sqrt{1+\|G\sigma\|^2} X\big)$$

If Σ has constant mean curvature, the function $\nu = (N, \xi)$ is a Jacobi function so $0 = \Delta_\Sigma \nu + (Ric_{\bar g}(N,N) + \|S\|^2)\nu$ with S the shape operator of Σ. Let v be a function on Σ, we then have

$$\begin{aligned} \Delta_\Sigma(\nu v) + (Ric_{\bar g}(N,N) + \|A\|^2)(\nu v) &= (\Delta_\Sigma \nu)v + 2(\nabla_\Sigma \nu, \nabla_\Sigma v) + \nu(\Delta_\Sigma v) \\ &\quad + (Ric_{\bar g}(N,N) + \|S\|^2)(\nu v) \\ &= \nu\big(\Delta_\Sigma v - 2\nu(\nabla_\Sigma \frac{1}{\nu}, \nabla_\Sigma v)\big) \end{aligned}$$

Thus νv is a Jacobi function if and only if $\Delta_\Sigma v - 2\nu(\nabla_\Sigma \frac{1}{\nu}, \nabla_\Sigma v) = 0$. Looking at v as a function defined on Ω, since $\nu = W^{-1}$, it gives:

$$\begin{aligned} 0 &= \Delta_g v - 2W^{-1} g(\nabla_g W, \nabla_g v) \\ &= W^{-1}\big(\operatorname{div}(W(\nabla v - (\chi_\sigma, \nabla v)\chi_\sigma)) - 2((\nabla W, \nabla v) - (\chi_\sigma, \nabla W)(\chi_\sigma, \nabla v))\big) \\ &= W^{-1}\big(\operatorname{div}(W^2 \frac{\nabla v - (\chi_\sigma, \nabla v)\chi_\sigma}{W}) - (\nabla W^2, \frac{\nabla v - (\chi_\sigma, \nabla v)\chi_\sigma}{W})\big) \\ &= W\big(\operatorname{div}(\frac{\nabla v - (\chi_\sigma, \nabla v)\chi_\sigma}{W})\big) \end{aligned}$$

So $v\nu$ is a Jacobi function if and only if

$$0 = \operatorname{div}(\frac{\nabla v - (\chi_\sigma, \nabla v)\chi_\sigma}{W}) \tag{15}$$

B Some barriers

In this appendix, we construct some barriers from above and below on the exterior boundary component of an annulus for (3).

We use the model for $\mathbb{E}(-1,\tau)$ but we consider hyperbolic polar coordinates on D_{-1}, so $x=\tanh(\rho/2)\cos\theta$ and $y=\tanh(\rho/2)\sin\theta$.

Let $0<\rho_1<\rho_0$ be two radii and f be a smooth function on $\{\rho=\rho_0\}$. For M a constant, we construct a smooth function h on $\{\rho_1\le\rho\le\rho_0\}$ such that

- $h=f$ on $\{\rho=\rho_0\}$, $h\ge M$ in $\{\rho=\rho_1\}$ and
- $\operatorname{div}_{\mathbb{H}^2}\left(\frac{Gh}{\sqrt{1+\|Gh\|^2}}\right)\le 1$

We see f as a function of θ and we define on $\{\rho_1\le\rho\le\rho_0\}$ the function $h(\rho,\theta)=f(\theta)+\alpha(\rho_0-\rho)$. Let us prove that for α sufficiently large h satisfies the expected properties. We have $h(\rho_0,\theta)=f(\theta)$ and $h(\rho_1,\theta)=f(\theta)+\alpha(\rho_0-\rho_1)\ge M$ if $\alpha\ge(M-\min f)/(\rho_0-\rho_1)$. In the polar coordinates, we have

$$Gh=-\alpha\partial_\rho+\frac{1}{\sinh\rho}\Big(\frac{f'(\theta)}{\sinh\rho}-2\tau\tanh\frac{\rho}{2}\Big)\partial_\theta$$

Thus

$$\begin{aligned}\operatorname{div}_{\mathbb{H}^2}\left(\frac{Gh}{\sqrt{1+\|Gh\|^2}}\right)=&-\frac{\cosh\rho}{\sinh\rho}\frac{\alpha}{W}+\alpha\frac{\partial_\rho\big(\frac{f'(\theta)}{\sinh\rho}-2\tau\tanh\frac{\rho}{2}\big)^2}{2W^3}+\frac{1}{\sinh\rho}\frac{\partial_\theta\big(\frac{f'(\theta)}{\sinh\rho}\big)}{W}\\&-\frac{1}{\sinh\rho}\frac{\big(\frac{f'(\theta)}{\sinh\rho}-2\tau\tanh\frac{\rho}{2}\big)\partial_\theta\big(\frac{f'(\theta)}{\sinh\rho}-2\tau\tanh\frac{\rho}{2}\big)^2}{2W^3}\end{aligned}$$

where

$$W=\sqrt{1+\alpha^2+\Big(\frac{f'(\theta)}{\sinh\rho}-2\tau\tanh\frac{\rho}{2}\Big)^2}$$

With $\alpha>0$, there exists a positive constant m that only depends on f, ρ_0 and ρ_1 such that:

$$\operatorname{div}_{\mathbb{H}^2}\left(\frac{Gh}{\sqrt{1+\|Gh\|^2}}\right)\le\frac{m}{\alpha^2}+\frac{m}{\alpha}+\frac{m}{\alpha^3}$$

So when α is sufficently large, the mean curvature of the graph of h satisfies the expected estimate.

Now let us define $k(\rho,\theta)=f(\theta)+\alpha(\rho_0-\rho)$ with $\alpha<0$ By choosing α small, we can ensure that $k(\rho_1,\theta)\le M$. Moreover, because of the above computation, there is a $m>0$ such that

$$\operatorname{div}_{\mathbb{H}^2}\left(\frac{Gk}{\sqrt{1+\|Gk\|^2}}\right)\ge\frac{\cosh\rho_0}{\sinh\rho_0}\frac{-\alpha}{\sqrt{\alpha^2+m}}-\frac{m}{\alpha^2}-\frac{m}{|\alpha|}-\frac{m}{|\alpha|^3}$$

So choosing α sufficiently close from $-\infty$, k satisfies to

- $k = f$ on $\{\rho = \rho_0\}$, $k \leq M$ in $\{\rho = \rho_1\}$ and
- $\mathrm{div}_{\mathbb{H}^2}\left(\dfrac{Gk}{\sqrt{1 + \|Gk\|^2}}\right) \geq 1$

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