RIGIDITY OF RIEMANNIAN MANIFOLDS CONTAINING AN EQUATOR

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Abstract. In this paper, we prove that a Riemannian n-manifold M with sectional curvature bounded above by 1 that contains a minimal 2-sphere of area 4π which has index at least $n-2$ has constant sectional curvature 1. The proof uses the construction of ancient mean curvature flows that flow out of a minimal submanifold. As a consequence we also prove a rigidity result for the Simon-Smith minimal spheres.

1. Introduction

Let g be a complete Riemannian metric on the 2-sphere $\mathbb{S}^2$. If its sectional curvature is between 0 and 1, it is known that any closed geodesic on $(\mathbb{S}^2, g)$ has length at least 2π [20]. Moreover if such a closed geodesic has length 2π, $(\mathbb{S}^2, g)$ is isometric to the unit 2-sphere $\mathbb{S}^2_1 = \{p \in \mathbb{R}^3 \mid \|p\| = 1\}$ with the induced metric. The proof of this result is given in [2] where the authors attribute the theorem to E. Calabi.

So a question is what happens in higher dimension. In dimension 3, one can replace geodesics by minimal 2-sphere. Actually, using Gauss equation and Gauss-Bonnet theorem, one can prove that, if the sectional curvature is bounded above by 1, any minimal 2-sphere has area at least 4π (see computations in Theorem 6 proof). In [15], H. Rosenberg and the author study the equality case. If (M, g) is a Riemannian 3-manifold with sectional curvature $0 \le K \le 1$ that contains a minimal 2-sphere of area 4π, they prove that the universal cover of M is isometric to the unit 3-sphere $\mathbb{S}^3_1$ or the product $\mathbb{S}^2_1 \times \mathbb{R}$.

One purpose of this paper is to investigate generalizations of this result to higher dimensions. Actually if (M, g) is a Riemannian n-manifold with sectional curvature $K \le 1$, we still have that the area of a minimal 2-sphere is at least 4π. So what can be said in the equality case ?

A model of this situation is an equatorial 2-sphere in the unit n-sphere $\mathbb{S}^n_1$. So one could expect that under some extra hypotheses this is the only example.

If Σ is a minimal m-submanifold in M, Σ is critical for the volume functional. The stability of this critical point is given by the Jacobi operator which is a self-adjoint second order elliptic operator that acts on sections of the normal bundle to Σ. As a critical point, the index of Σ is given by the

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number of negative eigenvalues of this operator. In the case of an equatorial 2-sphere S in $\mathbb{S}^n_1$, the index of S is $n-2$.

The first main result of the paper is a rigidity result under such an instability hypothesis.

Theorem. *Let M be a Riemannian $n \geq 3$-manifold whose sectional curvature is bounded above by 1. Let us assume that M contains an immersed minimal 2-sphere of area 4π which has index at least $n-2$. Then the universal cover of M is isometric to the unit sphere $\mathbb{S}^n_1$.*

Let us notice that the instability hypothesis can be replaced by an other version.

Definition 1. *Let Σ be a minimal submanifold in (M,g). We say that Σ is unstable in any parallel directions if the restriction of the Jacobi operator to any parallel sub-bundle of the normal bundle to Σ has index at least 1.*

The above theorem gives an answer to a question that arises from a result in [2]. In [2, Corollary 5.11], L. Andersson and R. Howard prove that a Riemannian n-manifold M $(n \leq 3)$ with sectional curvature below 1 containing isometrically a neighborhood of the equator $\mathbb{S}^{n-1}_1$ in $\mathbb{S}^n_1$ is isometric to $\mathbb{S}^n_1$. The hypothesis that a whole neighborhood of the equator belongs to M seems strong and the question is to find a weaker hypothesis. Actually our main result gives an infinitesimal version of Andersson-Howard result. If M, with $K \leq 1$, contains a totally geodesic hypersurface isometric to $\mathbb{S}^{n-1}_1$ that is unstable as a minimal hypersurface, then M is isometric to $\mathbb{S}^n_1$. The idea is that the totally geodesic $\mathbb{S}^{n-1}_1$ contains a minimal 2-sphere of area 4π and index at least $n-2$. Actually in the same spirit as Andersson-Howard theorem, there is a result by D. Panov and A. Petrunin [19, Theorem 1.4] with a weaker hypothesis: if S is an equatorial 2-sphere in $\mathbb{S}^n_1$ and S^+ denotes an hemisphere of S, Panov and Petrunin need just that M contains isometrically a neighborhood of S^+ in $\mathbb{S}^n_1$.

The proof of the main theorem uses ideas that already appear in [15]: if S is an immersed 2-sphere we define the F functional by $F(S) = \mathcal{A}(S) + \int_S \|\vec{H}\|^2 - 4\pi$ where $\mathcal{A}(S)$ is the area of S and $\vec{H}$ is the mean curvature vector of S. Under the curvature assumption $K \leq 1$, F is non-negative. Besides if $F(S)$ vanishes, S is totally umbilical and we obtain some information on the sectional curvature of M along S. So if S_0 is the minimal 2-sphere given by the statement of the theorem $F(S_0) = 0$. The idea is to explore the geometry of M by computing $F(S_t)$ along a deformation $\{S_t\}_t$ of S_0. One of the novelties is the construction of the family $\{S_t\}_t$. Actually we produce $\{S_t\}$ as a mean curvature flow that flows out of S_0. More precisely, we construct non trivial ancient solutions $\{S_t\}_{t\in(-\infty,b)}$ of the mean curvature flow such that, as $t \to -\infty$, S_t converges to S_0.

The idea is that the eigen-sections of the Jacobi operator associated to the first eigenvalue give directions in which such an ancient mean curvature flow can be initiated. A similar idea appear in the work of K. Choi and

C. Mantoulidis [7] where they construct ancient mean curvature flows "tangent" to the eigenspaces with negative eigenvalues. Then they prove several uniqueness results for ancient mean curvature flow in $\mathbb{S}^n_1$. An other example is [17] where A. Mramor and A. Payne produce an eternal solution of the mean curvature flow that flows out of the catenoid. Let us notice that good introductions to the study of high codimension mean curvature flow can be found in the paper of K. Smoczyk [23] and the PhD thesis of C. Baker [3].

In the case of lower bound on the scalar curvature, rigidity theorems were obtained under the existence of area minimizing surfaces. The first result in this direction is due to Cai and Galloway [6] for nonnegative scalar curvature then we have the results by Bray, Brendle, Eichmair and Neves [4, 5] for positive lower bounds and Nunes [18] for negative lower bounds.

Here the index hypothesis in the above theorem can appear very particular. Actually there are situations where it is quite natural. As critical points of the area functional, minimal hypersurfaces can be produced by a minimization process. However one have to consider a non-trivial class of hypersurfaces to produce a non-trivial critical point. So in order to solve this difficulty, a Morse theoretical approach has been developed. In [22], F. Smith is able to construct minimal 2-spheres in any Riemannian $(\mathbb{S}^3, g)$. Its proof is based on the following ideas. Let Λ be the set of paths $\{\sigma_t\}_{t\in[-1,1]}$ in the space of 2-spheres in $\mathbb{S}^3$ that sweeps out $\mathbb{S}^3$ (see precise definitions and statements in Section 5). Then he considers the quantity

$$W(\mathbb{S}^3, g) = \inf_{\{\sigma_t\}\in\Lambda} \max_{t\in[-1,1]} \mathcal{A}(\sigma_t)$$

called the Simon-Smith width of $(\mathbb{S}^3, g)$.

First this quantity is positive and Smith proves that it is realized by the area of a finite collection of minimal spheres. Besides it is reasonable to think that the index of these collection of minimal spheres is 1. F. C. Marques and A. Neves [13] proved the upper-bound by 1. So the second main result of this paper is

Theorem. *Let $(\mathbb{S}^3, g)$ be a Riemannian 3-sphere whose sectional curvature is bounded above by 1. Then $W(\mathbb{S}^3, g) \geq 4\pi$ and, if $W(\mathbb{S}^3, g) = 4\pi$, then $(\mathbb{S}^3, g)$ is isometric to $\mathbb{S}^3_1$.*

If one knows that $W(\mathbb{S}^3, g)$ is realized by an index 1 minimal 2-sphere the above theorem is a direct consequence of our first rigidity result. So the difficulty is to deal with the case where $W(\mathbb{S}^3, g)$ is realized by a 2-sphere of index 0. Actually one can think about the following example: the cylinder $\mathbb{S}^2_1 \times [-1, 1]$ capped by two hemispheres $\mathbb{S}^{3+}_1$ (see Figure 1). This defines a $C^{1,1}$ Riemannian metric $\bar{g}$ on $\mathbb{S}^3$ whose sectional curvature is bounded above by 1 in any reasonable weak sense. Its Simon-Smith width is 4π so this implies that the above result is false for a weak sense of sectional curvature. Actually the above example is exactly the type of situation we have to consider in the proof of the above theorem: we prove that the Simon-Smith

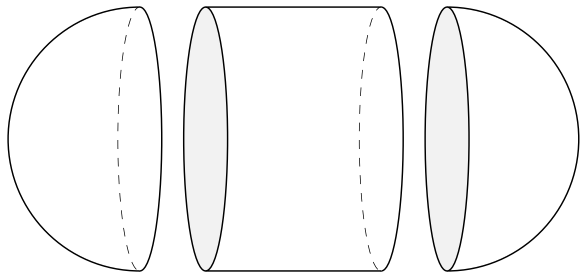

FIGURE 1. a $C^{1,1}$ Riemannian metric with width 4π

width is realized by a minimal 2-sphere which is not almost stable. Moreover $\bar{g}$ can be smoothed to produce sequences (g_n) of smooth Riemannian metrics with sectional curvature bounded above by 1, $W(\mathbb{S}^3, g_n) \to 4\pi$ and $g_n \to \bar{g}$. So the g_n are far of the round metric of $\mathbb{S}^3_1$. This implies that the above rigidity result is not stable.

The upper bound on the sectional curvature seems to be a strong hypothesis in the above result. However it is not clear if one can expect a similar result with a weaker upper bound on the curvature tensor.

Let us also notice that studying the Simon-Smith width for reversed curvature inequalities has been done by F. C. Marques and A. Neves. This time just a control on the Ricci and scalar curvature is assumed.

Theorem (Marques, Neves [12])**.** *Let $(\mathbb{S}^3, g)$ be a Riemannian 3-sphere with positive Ricci curvature and scalar curvature $R \geq 6$. Then $W(\mathbb{S}^3, g) \leq 4\pi$ and, if $W(\mathbb{S}^3, g) = 4\pi$, then $(\mathbb{S}^3, g)$ is isometric to $\mathbb{S}^3_1$.*

The paper is organized as follows. In Section 2, we recall some basic formulas and definitions of submanifold geometry. Section 3 is devoted to the construction of ancient solutions of the mean curvature flow (Theorem 1). In Section 4, we prove our first rigidity result (Theorem 6) and its Corollary concerning manifolds containing an equator of $\mathbb{S}^n_1$. Section 5 is devoted to the study of the Simon-Smith width and the proof of the second rigidity result (Theorem 8). In Appendix A, we prove a Schauder type estimate used in the proof of Theorem 1.

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2. Geometry of submanifolds

In this section we recall some classical notations and formulas concerning the geometry of submanifolds.

Let (M, g) be a Riemannian manifold of dimension n and Σ a manifold of dimension m. If $F_0 : \Sigma \to M$ is an immersion, we can consider the induced Riemannian metric $g_0 = F_0^* g$ on Σ making F_0 a local isometry. In the paper, we often identify Σ with its image $\Sigma_0 = F_0(\Sigma)$ at least locally where F_0 is an embedding: for example, we often identify $T_p\Sigma$ with $(F_0)_*(T_p\Sigma) \subset T_{F_0(p)}M$.

If ∇ and ∇^0 are respectively the Levi-Civita connections on M and Σ, we can define the second fundamental form on Σ by

$$B_p(v, w) = \nabla_v w - \nabla^0_v w \in N_p\Sigma$$

where $v, w \in T_p\Sigma$ and $N_p\Sigma$ is the normal subspace to Σ at p.

The mean curvature vector of Σ is then

$$\vec{H}(p) = \frac{1}{m} \operatorname{tr}_{T_p\Sigma} B_p \in N_p\Sigma$$

where tr_P denotes the trace operator on the subspace P. We define $\mathring{B}_p = B_p - \vec{H}(p)g_0$ the traceless part of the second fundamental form. We recall that the normal bundle $N\Sigma$ inherits from g and ∇ a normal connection $\nabla^\perp$.

Let $(F_t)_t$ be a smooth family of immersion of Σ and define the vectorfield $X = \frac{d}{dt}F_{t|t=0}$ along F_0 and let $\Sigma_t = F_t(\Sigma)$. We have a family of metrics $g_t = F_t^* g$ defined on Σ with associated volume measure $d\sigma_t$. If, X is orthogonal to Σ, it is well known that, for any function f on Σ:

$$\frac{d}{dt}_{|t=0} \int_\Sigma f d\sigma_{g_t} = -\int_\Sigma (X, m\vec{H}) f d\sigma_0$$

So if Σ is critical with respect to the m-volume functional $\mathcal{A}$, we have $\vec{H} = 0$ along Σ: Σ is minimal.

We are interested in understanding how the mean curvature vector $\vec{H}$ is deformed along the family F_t. So let us denote by $\vec{H}_t(p)$ the mean curvature vector of Σ_t at $F_t(p)$.

Lemma 1. *If X is normal to Σ, we have*

$$\frac{D}{dt} m\vec{H}_{t|t=0} = \Delta^\perp X + (R(e_i, X)e_i)^\perp + (X, B(e_i, e_j))B(e_i, e_j) - (m\vec{H}_0, \nabla_{e_i} X)e_i$$

with the convention that summations are made over repeated indices, $(e_1, \dots, e_m)$ is an orthonormal frame of Σ_0, R is the Riemann curvature tensor associated to g with the convention $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$ and $\Delta^\perp$ denotes the Laplacian operator acting on normal sections: $\Delta^\perp X = \operatorname{tr} {\nabla^\perp}^2 X$.

Proof. Let $E_1, \dots, E_m$ be an orthonormal frame on (Σ, g_0) and consider at $F_t(p)$ the tangent frame $e_i = (F_t)_*(E_i)$ to $T\Sigma_t$. We assume that $\nabla^0_{E_i} E_j = 0$ for any i, j at $\bar{p}$ where the computation is made. Let us denote $g_{ij} = (e_i, e_j)$ and (g^{ij}) the inverse matrix. We have

$$mH_t = g^{ij}(\nabla_{e_i} e_j)^\perp$$

where $Y^\perp$ denotes the orthogonal projection to $N\Sigma_t$. At $t = 0$, we have $\frac{D}{dt}e_i = \nabla_{e_i} X$.

Notice that $g_{ij|t=0} = \delta_i^j$, so at $t = 0$:

$$\begin{aligned}\frac{d}{dt}g^{ij} = -\frac{d}{dt}g_{ij} &= -(\frac{D}{dt}e_i, e_j) - (e_i, \frac{D}{dt}e_j)\\ &= -(\nabla_{e_i}X, e_j) - (e_i, \nabla_{e_j}X) = 2(X, B(e_i, e_j))\end{aligned}$$

Let Y be a vector field along $t \mapsto F_t(p)$. We have $Y^\perp = Y - g^{ij}(Y, e_j)e_i$, so at $t = 0$:

$$\begin{aligned}\frac{D}{dt}Y^\perp &= \left(\frac{D}{dt}Y\right)^\perp + ((\nabla_{e_i}X, e_j) + (e_i, \nabla_{e_j}X))(Y, e_j)e_i\\ &\quad - (Y, \frac{D}{dt}e_i)e_i - (Y, e_i)\frac{D}{dt}e_i\\ &= \left(\frac{D}{dt}Y\right)^\perp + (\nabla_{e_i}X, Y^\top)e_i + (Y, e_j)(\nabla_{e_j}X)^\top\\ &\quad - (Y, \nabla_{e_i}X)e_i - (Y, e_i)\nabla_{e_i}X\\ &= \left(\frac{D}{dt}Y\right)^\perp - \sum_i (Y^\perp, \nabla_{e_i}X)e_i - (Y, e_i)(\nabla_{e_i}X)^\perp\end{aligned}$$

where $Z^\top$ denotes the tangential part of Z.

We also have

$$\begin{aligned}\frac{D}{dt}\nabla_{e_i}e_i &= R(e_i, X)e_i + \nabla_{e_i}\frac{D}{dt}e_i\\ &= R(e_i, X)e_i + \nabla_{e_i}\nabla_{e_i}X\end{aligned}$$

So combining all the above computations at $\bar{p}$, we obtain

$$\frac{D}{dt}mH_t = 2(X, B(e_i, e_j))B(e_i, e_j) + (R(e_i, X)e_i)^\perp + (\nabla_{e_i}\nabla_{e_i}X)^\perp - (mH_0, \nabla_{e_i}X)e_i$$

Since $(\nabla_{e_i}\nabla_{e_i}X)^\perp = \nabla^\perp_{e_i}\nabla^\perp_{e_i}X - (X, B(e_i, e_j))B(e_i, e_j)$ we finally have

$$\frac{D}{dt}mH_t = \Delta^\perp X + (X, B(e_i, e_j))B(e_i, e_j) + (R(e_i, X)e_i)^\perp - (mH_0, \nabla_{e_i}X)e_i$$

$\square$

As a consequence, if Σ_0 is minimal, the second derivative of the m-volume of $\Sigma_t = F_t(\Sigma)$ is given by

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{A}(\Sigma_t)_{|t=0} &= -\int_\Sigma (X, \Delta^\perp X + (X, B(e_i, e_j))B(e_i, e_j) + (R(e_i, X)e_i))d\sigma_0\\ &= \int_\Sigma \|\nabla^\perp X\|^2 - (R(e_i, X)e_i, X) - (X, B(e_i, e_j))^2 d\sigma_0^2\\ &= Q_\Sigma(X, X)\end{aligned}$$

So Q_Σ is a quadratic form acting on sections of the normal bundle $N\Sigma$. It is attached to the Jacobi operator acting on normal sections:

$$LX = \Delta^\perp X + (R(e_i, X)e_i)^\perp + (X, B(e_i, e_j))B(e_i, e_j)$$

This operator is elliptic and self-adjoint. It has a spectrum $\lambda_0 \le \lambda_1 \le \cdots$. If $\lambda_0 < 0$, Σ is called unstable. The index of L (the number of negative eigenvalues) is called the index of Σ.

3. Ancient solutions of the mean curvature flow

3.1. **The mean curvature flow.** First let us recall some basics of the mean curvature flow and state our main result. For a good introduction to the high co-dimension case, one can have a look to Smoczyk's paper [23].

Let (M, g) be a Riemannian manifold and Σ a m-manifold. Let $F : \Sigma \times I \to M$ (I an interval) be a smooth map such that $F_t = F(\cdot, t)$ is an immersion for any t. We say that $F_t(\Sigma)$ evolves by mean curvature flow if for any $p \in \Sigma$ and $t \in I$

$$\text{(MCF)} \qquad \frac{dF}{dt}(p,t) = m\vec{H}(p,t)$$

where $\vec{H} \in T_{F_t(p)}M$ is the mean curvature vector of $F_t(\Sigma)$ at $F_t(p)$.

For example, if $F_0(\Sigma)$ is a minimal submanifold, then $F_t = F_0$ for $t \in I$ is a solution of the mean curvature flow: minimal submanifolds are fixed points of the mean curvature flow.

Our aim is to produce solutions that flow out of a minimal surface. More precisely, we construct non constant ancient solutions of the mean curvature flow (*i.e.* defined on a time interval $(-\infty, b)$) such that, as $t \to -\infty$, $F_t(\Sigma)$ converges to a minimal surface.

It is well known that one difficulty in the solvability of (MCF) is the invariance under the diffeomorphism group which causes a lack of parabolicity of the system. One solution to this difficulty consists in adding a tangential component to the time derivative of F which has no impact on the geometric evolution.

Let us explain such a solution. Let Σ be an immersed closed submanifold in M. Let $N\Sigma$ denote the normal bundle to Σ. Then we can consider the map

$$\Phi : \begin{array}{ccc} N\Sigma & \longrightarrow & M \\ (p,v) & \longmapsto & \exp_p(v) \end{array}$$

For $\varepsilon > 0$, let us denote $N\Sigma^\varepsilon = \{(p,v) \in N\Sigma \mid \|v\| < \varepsilon\}$. If ε is small enough, the restriction of Φ to $N\Sigma^\varepsilon$ is an immersion so the metric g can be lifted to $h = \Phi^* g$ on $N\Sigma^\varepsilon$. Now studying immersed submanifolds close to Σ consists in looking at sections of $N\Sigma^\varepsilon$ close to 0. Actually one can extend the Riemannian metric h to the whole $N\Sigma$ and just look at sections close to 0.

So the general setting we have to consider is the following. Let E be a vector bundle over a closed manifold Σ and consider g a Riemannian metric on the manifold E. We say that E is a normal bundle if the fibers are orthogonal to Σ_0 the image of the 0 section. If E is a normal bundle there is a natural identification between E and the normal bundle to Σ_0. So as a normal bundle, E inherits a bundle metric $g^\perp$ and a connection $\nabla^\perp$.

If U is a section of E, $U(\Sigma)$ is a submanifold in E. Then sections are a particular way to parametrize submanifolds in E. Let U be a section of E and $p \in \Sigma$. Since U is a section, the tangent space $T_{U(p)}E$ splits as $T_pU \oplus T_{U(p)}E_p$ where E_p is the fiber of E over p. Moreover there is a natural identification of $T_{U(p)}E_p$ with E_p. So for any $Y \in T_{U(p)}E$, one can define $Y^\sharp$ the projection of Y to E_p parallel to T_pU.

With this type of notation, we can define the bundle mean curvature flow in the following way: let $U : \Sigma \times I \to E$ a smooth map such that $U_t = U(\cdot, t)$ is a section of E, we say that $(U_t)_{t\in I}$ evolves by bundle mean curvature flow if for any $p \in \Sigma$ and $t \in I$

$$\frac{dU}{dt}(p,t) = (m\vec{H}(U_t, p))^\sharp \tag{1}$$

where $\vec{H} \in T_{U_t(p)}E$ is the mean curvature vector of the graph of U_t at $U_t(p)$.

$(\vec{H}(U_t, p))^\sharp$ is equal to $\vec{H}(U_t, p)$ plus a tangent vector to $U_t(\Sigma)$. So solutions to (1) give rise to solutions to the mean curvature flow (MCF) after a reparametrization.

Let us define the operator $\mathbf{H} : \Gamma(E) \to \Gamma(E)$ by $\mathbf{H}(U)(p) = (m\vec{H}(U_t, p))^\sharp$. $\mathbf{H}$ is a smooth quasilinear elliptic differential operator of order 2.

Let us assume E is a normal bundle and Σ_0 is minimal, *i.e.* $\mathbf{H}(0) = 0$. We can compute the differential of $\mathbf{H}$ with respect to U at 0, Lemma 1 gives

$$D\mathbf{H}(0)(V) = \mathbf{L}(V) = \Delta^\perp V + (R(e_i, V)e_i)^\perp + (V, B(e_i, e_j))B(e_i, e_j)$$

which is an elliptic self-adjoint operator on $\Gamma(E)$. So $\mathbf{L}$ has a discrete spectrum $\lambda_0 \le \lambda_1 \le \cdots$. Let us notice that Σ_0 is unstable if $\lambda_0 < 0$. So the main theorem of the section is the following

Theorem 1. *Let $E \to \Sigma$ be as above. Assume that the first eigenvalue λ_0 of $\mathbf{L}$ is negative. Then for any section V in the first eigenspace, i.e. $\mathbf{L}V = -\lambda_0 V$, there is U an ancient solution of* (1) *defined on $(-\infty, b)$ such that*

$$\lim_{t\to-\infty} e^{\lambda_0 t} U_t = V$$

One can compare this result with [7, Theorem 1.6 and Theorem 3.3] by Choi and Mantoulidis. The main difference is that here the ancient solution is parametrized by its asymptotic as $t \to -\infty$ while Choi and Mantoulidis parametrized it by its value at time $t = 0$. Moreover this allows them to obtain a family of flows tangent to the space of negative eigensections.

3.2. **The functional spaces.** In order to prove the above result we need to introduce some functional spaces. Following Solonnikov [24], we recall the definition of the Hölder spaces.

Let $\Omega \subset \mathbb{R}^m$ be a smooth domain and $P = \Omega \times [a,b]$. Then for $u : P \to \mathbb{R}^N$ and $\beta \in (0,1)$, we define the Hölder semi-norms

$$[u]_{\beta,P,x} = \sup_{(x,t)\neq(y,t)\in P} \frac{|u(x,t)-u(y,t)|}{|x-y|^\beta}$$

$$[u]_{\beta,P,t} = \sup_{(x,t)\neq(x,s)\in P} \frac{|u(x,t)-u(x,s)|}{|t-s|^\beta}$$

and the uniform norm

$$\|u\|_{0,P} = \sup_{X\in P} |u(X)|$$

For $\alpha \in (0,1)$ we define the combined Hölder semi-norms

$$[u]_{2,\alpha,P,x} = [\partial_x^2 u]_{\alpha,P,x} + [\partial_t u]_{\alpha,P,x}$$

$$[u]_{2,\alpha,P,t} = [\partial_x u]_{(1+\alpha)/2,P,t} + [\partial_x^2 u]_{\alpha/2,P,t} + [\partial_t u]_{\alpha/2,P,t}$$

Finally we have the Hölder norms

$$\|u\|_{0,\alpha,P} = \|u\|_{0,P} + [u]_{\alpha,P,x} + [u]_{\alpha/2,P,t}$$

$$\|u\|_{2,\alpha,P} = \sum_{i=0}^{2} \|\partial_x^i u\|_{0,P} + \|\partial_t u\|_{0,P} + [u]_{2,\alpha,P,x} + [u]_{2,\alpha,P,t}$$

When u is defined on Ω, u does not depend on t so all the terms corresponding to the t variable disappear and we have the specific notations:

$$|u|_{0,\alpha,\Omega} = \|u\|_{0,\Omega} + [u]_{\alpha,\Omega,x}$$

$$|u|_{2,\alpha,\Omega} = \sum_{i=0}^{2} \|\partial_x^i u\|_{0,\Omega} + [u]_{2,\alpha,\Omega,x}$$

We then have the associated Hölder spaces $C^{0,\alpha}(P)$, $C^{2,\alpha}(P)$, $C^{0,\alpha}(\Omega)$, $C^{2,\alpha}(\Omega)$ made of applications u such that the above norms are well defined and finite.

This Hölder spaces can be analogously defined on a closed Riemannian manifold (Σ, g) and for sections of a vector bundle E over Σ where E is equipped with a bundle metric h and a metric connection $\overline{\nabla}$. If $I \subset \mathbb{R}$ is an interval, the vector bundle E can be extended as a vector bundle denoted by E_I over $\Sigma \times I$. So if $P = \Sigma \times [a,b]$ and $U : P \to E_{[a,b]}$ is a section, we can define the Hölder semi-norms

$$[U]_{\beta,E_{[a,b]},x} = \sup_{\substack{(x,t)\neq(y,t)\in P\\ d_g(x,y)<i_g}} \frac{|U(x,t)-P_{y,x}U(y,t)|}{|x-y|^\beta}$$

$$[U]_{\beta,E_{[a,b]},t} = \sup_{(x,t)\neq(x,s)\in P} \frac{|U(x,t)-U(x,s)|}{|t-s|^\beta}$$

where i_g denotes the injectivity radius of Σ and $P_{y,x}$ is the parallel transport operator from y to x. Once this is defined we can construct the Hölder norms similarly to the Euclidean case. The uniform norm:

$$\|U\|_{0,E_{[a,b]}} = \sup_{X\in P} |U(X)|$$

For $\alpha \in (0,1)$ we define the combined Hölder semi-norms

$$[U]_{2,\alpha,E_{[a,b]},x} = [\overline{\nabla}^2_x U]_{\alpha,E_{[a,b]},x} + [\partial_t U]_{\alpha,E_{[a,b]},x}$$

$$[U]_{2,\alpha,E_{[a,b]},t} = [\overline{\nabla}_x U]_{(1+\alpha)/2,E_{[a,b]},t} + [\overline{\nabla}^2_x U]_{\alpha/2,E_{[a,b]},t} + [\partial_t U]_{\alpha/2,E_{[a,b]},t}$$

Finally we have the Hölder norms

$$\|U\|_{0,\alpha,E_{[a,b]}} = \|U\|_{0,E_{[a,b]}} + [U]_{\alpha,E_{[a,b]},x} + [U]_{\alpha/2,E_{[a,b]},t}$$

$$\|U\|_{2,\alpha,E_{[a,b]}} = \sum_{i=0}^{2} \|\overline{\nabla}^i_x U\|_{0,E_{[a,b]}} + \|\partial_t U\|_{0,E_{[a,b]}} + [U]_{2,\alpha,E_{[a,b]},x} + [U]_{2,\alpha,E_{[a,b]},t}$$

When U is defined on Σ, we have the specific notations:

$$|U|_{0,\alpha,E} = \|U\|_{0,E} + [U]_{\alpha,E,x}$$

$$|U|_{2,\alpha,E} = \sum_{i=0}^{2} \|\partial^i_x U\|_{0,E} + [U]_{2,\alpha,E,x}$$

We then have the associated Hölder spaces $C^{0,\alpha}(E_{[a,b]})$, $C^{2,\alpha}(E_{[a,b]})$, $C^{0,\alpha}(E)$, $C^{2,\alpha}(E)$. In the sequel we will also use the L^2 norms $\|\cdot\|_{L^2(E_{[a,b]})}$ and $|\cdot|_{L^2(E)}$. For a section U defined over $\Sigma\times\mathbb{R}$, we denote $U_t(\cdot) = U(\cdot,t)$.

3.3. **Linear operators.** If the fiber of E has dimension k, sections of E can locally be written has maps: $u = (u^a)_{1\le a\le k} : \Omega \to \mathbb{R}^k$. In the sequel, we consider families $(L_t)_t$ of linear differential operators of order 2 acting on sections of E which in coordinates takes the form

$$(L_t u)^a = \sum_{|I|\le 2, b\le k} A^{aI}_b(x,t)\partial_I u^b \tag{2}$$

where I denote a multi-index and ∂_I is the partial derivative associated to I. L_t will be elliptic in the following sense: there is a constant $\lambda > 0$ such that for any $\xi = (\xi_1,\dots,\xi_n)$ and $v = (v_1,\dots,v_k)$ we have

$$\sum_{i,j=1}^{n}\sum_{a,b=1}^{k} A^{aij}_b \xi_i\xi_j v_b v_a \ge \lambda|\xi|^2|v|^2$$

Moreover we say that L_t has C^α coefficients if the functions A^{aI}_b are in $C^{0,\alpha}$. We denote by Λ the maximum of the $C^{0,\alpha}$ norms of these coefficients.

An important result for us is the following Schauder estimate for solutions of parabolic systems associated to such operators L_t

Theorem 2 ([24, Theorem 4.11])**.** *Let $\Omega' \subset \Omega \subset \mathbb{R}^n$ be smooth bounded domains with $\overline{\Omega'} \subset \Omega$. Let $P = \Omega\times[0,T]$ and $P' = \Omega'\times[0,T]$. Let L_t be elliptic differential operators of order 2 as in (2) with C^α coefficients in $\overline{\Omega}$.*

Then there is a constant C depending on Ω, Ω', λ, Λ, α and T such that for any $u \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^k)$ and $f \in C^{0,\alpha}(\overline{P}, \mathbb{R}^k)$ satisfying $\partial_t u - L_t u = f$ we have

$$\|u\|_{2,\alpha,P'} \leq C(\|f\|_{0,\alpha,P} + |u_0|_{2,\alpha,\Omega} + \|u\|_{L^2(P)})$$

where $u_0(\cdot) = u(\cdot, 0)$.

Similar estimates can also be found in Friedman's paper [9].

Using finitely many local charts for a vector bundle $E \to \Sigma$ (Σ is closed), we can obtain an equivalent version for operators acting on sections of E.

Theorem 3 ([24, Theorem 4.11])**.** *Let E be a vector bundle over a closed manifold Σ. Let L_t be elliptic differential operators of order 2 as in (2) in any local charts with C^α coefficients. Then there is a constant C such that, for any $U \in C^{2,\alpha}(E_{[0,T]})$ and $F \in C^{0,\alpha}(E_{[0,T]})$ satisfying $\partial_t U - L_t U = F$, we have*

$$\|U\|_{2,\alpha,E_{[0,T]}} \leq C(\|F\|_{0,\alpha,E_{[0,T]}} + |U_0|_{2,\alpha,E} + \|U\|_{L^2(E_{[0,T]})})$$

A consequence is the following solution to the Cauchy problem

Theorem 4. *Let E be a vector bundle over a closed manifold Σ. Let L_t be elliptic differential operators of order 2 as in (2) in any local charts with C^α coefficients. Then, for any $U_0 \in C^{2,\alpha}(E)$ and $F \in C^{0,\alpha}(E_{[0,T]})$, there is a unique $U \in C^{2,\alpha}(E_{[0,T]})$ such that*

$$\begin{cases} \partial_t U - L_t U(\cdot, t) = F(\cdot, t) \\ U(\cdot, 0) = U_0 \end{cases}$$

For a proof see [11, Theorem 2.4].

In Theorem 3, the constant C depends on the length T of the time interval: actually it is uniformly bounded as $T \to 0$ but not as $T \to \infty$. However the proof can be adapted in order to obtain the following result where the constant is time independent. This is important for our arguments.

Proposition 1. *Let E be a vector bundle over a closed manifold Σ. Let L be a time independent elliptic differential operator of order 2 as in (2) in any local charts with C^α coefficients . Then there is a constant C (independent of T) such that for any $U \in C^{2,\alpha}(E_{[0,T]})$ and $F \in C^{0,\alpha}(E_{[0,T]})$ satisfying $\partial_t U - LU = F$ we have*

$$\|U\|_{2,\alpha,E_{[0,T]}} \leq C(\|F\|_{0,\alpha,E_{[0,T]}} + |U_0|_{2,\alpha,E} + \|U\|_{L^2(E_{[0,T]})})$$

See the proof in Appendix A

3.4. **The ancient flow.** In this section we prove Theorem 1. So $E \to \Sigma$ is a vector bundle as in Theorem 1 and we use the notations introduced in the preceding sections. We start by giving a result that ensures the existence of solutions to (1).

Theorem 5. *Let $E \to \Sigma$ be as above. There is δ_0 such that for any $\delta < \delta_0$ there is $\varepsilon > 0$ such that, for any $W \in C^{2,\alpha}(E)$ with $|W|_{2,\alpha,E} \leq \varepsilon$, there is a unique solution $U \in C^{2,\alpha}(E_{[0,1]})$ of*

$$\begin{cases} \partial_t U = \mathbf{H}(U) \\ U(\cdot,0) = W \end{cases}$$

with $\|U\|_{2,\alpha,E_{[0,1]}} < \delta$.

Proof. Let us consider the map

$$F: \begin{array}{ccc} C^{2,\alpha}(E) \times C^{2,\alpha}(E_{[0,1]}) & \to & C^{2,\alpha}(E) \times C^{0,\alpha}(E_{[0,1]}) \\ (W,U) & \mapsto & (U(\cdot,0) - W, \partial_t U - \mathbf{H}(U)) \end{array}$$

F is a C^1 map and $F(0,0) = (0,0)$ since Σ_0 is minimal. If we compute the differential of F with respect to U at $(0,0)$ we have

$$D_U F(0,0): \begin{array}{ccc} C^{2,\alpha}(E_{[0,1]}) & \to & C^{2,\alpha}(E) \times C^{0,\alpha}(E_{[0,1]}) \\ Z & \mapsto & (Z(\cdot,0), \partial_t Z - \mathbf{L}Z) \end{array}$$

So the invertibility of this differential is given by the solution to the Cauchy problem (Theorem 4). Hence the implicit function theorem solves $F(W,U) = (0,0)$ for any W with $|W|_{2,\alpha,E}$ small. □

The above theorem produces solutions to the bundle MCF (1). Let $\varepsilon(\delta)$ be given by Theorem 5 for $\delta < \delta_0$. Actually it allows you to extend a solution U as long as $|U_t|_{2,\alpha,E} < \varepsilon(\delta)$.

Proposition 2. *Let $\delta < \delta_0$. Let U be a solution of the bundle MCF defined on $\Sigma \times [a,b]$ with $\|U\|_{2,\alpha,E_{[a,b]}} \leq \delta$. Let $\bar{t} \in (b-1,b)$ and assume that $|U_{\bar{t}}|_{2,\alpha,E} \leq \varepsilon(\delta)$ then U can be extended as a solution of the bundle MCF defined on $\Sigma \times [a,\bar{t}+1]$*

Proof. Let Z be the solution of (1) defined on $\Sigma \times [\bar{t},\bar{t}+1]$ with $Z(\cdot,\bar{t}) = U(\cdot,\bar{t})$ given by Theorem 5. It suffices to prove that $Z = U$ on $\Sigma \times [\bar{t},b]$ to conclude. This uniqueness is given by the following remark: we have

$$\begin{aligned} \partial_t(Z-U) = \mathbf{H}(Z_t) - \mathbf{H}(U_t) &= \int_0^1 \frac{d}{ds}\mathbf{H}(sZ_t + (1-s)U_t)ds \\ &= \int_0^1 D\mathbf{H}(sZ_t + (1-s)U_t)(Z_t - U_t)ds \\ &= L_t(Z_t - U_t) \end{aligned}$$

where L_t are elliptic linear differential operators of order 2 acting on sections of E with coefficient in C^α. Then by the uniqueness part of Theorem 4 and since $(Z-U)_{\bar{t}} = 0$ we have $Z - U = 0$ on $\Sigma \times [\bar{t},b]$. □

To prove Theorem 1, we consider V an eigen-section associated to $\lambda_0 < 0$. We chose $\delta > 0$ as in Theorem 5. Let a_δ be such that $e^{-\lambda_0 a_\delta}|V|_{2,\alpha,E} = \varepsilon =$

$\varepsilon(\delta)$. Then for any $a < a_\delta$, $e^{-\lambda_0 a}|V|_{2,\alpha,E} < \varepsilon$ so we can consider the section $U^{(a)}$ solution to the problem

$$\begin{cases} \partial_t U = \mathbf{H}(U) \\ U(\cdot, a) = e^{-\lambda_0 a} V \end{cases}$$

on $\Sigma \times [a, b]$ where b is chosen the largest possible such that $\|U^{(a)}\|_{2,\alpha,E_{[a,b]}} \le \delta$, $|U_t^{(a)}|_{2,\alpha,\Sigma} \le \varepsilon$ and $\|e^{-\lambda_0 t} V\|_{2,\alpha,E_{[a,b]}} \le \delta$. So the proof consists in estimating the norm of $U^{(a)}$ in order to control b and prove that, as $a \to -\infty$, $U^{(a)}$ converges to the desired solutions of (1).

Let us introduce $Z^{(a)} = U^{(a)} - e^{-\lambda_0 t} V$ for $t \in [a, b]$. We have the following result.

Lemma 2. *There is $\delta > 0$ and $b_0 \in \mathbb{R}$ such that for any $a < \min(a_\delta, b_0)$, $U^{(a)}$ is defined on $[a, b_0]$. Moreover for any $a \le b \le b_0$*

$$\|Z^{(a)}\|_{L^2(E_{[a,b]})} \le e^{-3\lambda_0 b/2}$$

Proof. Let us choose $\delta \in (0, \delta_0)$ as in Theorem 5 that will be fixed below. The operator $\mathbf{H}$ is a smooth operator on $C^{2,\alpha}(E)$ so we can write $\mathbf{H} = \mathbf{L} + G$ where G satisfies $|G(U)|_{0,\alpha,E} \le C|U|^2_{2,\alpha,E}$ for any section U of E with $|U|_{2,\alpha,E} \le \delta_0$. Actually G satisfies $\|G(U)\|_{0,\alpha,E_{[a,b]}} \le C\|U\|^2_{2,\alpha,E_{[a,b]}}$ for any section U of $E_{[a,b]}$ with $\|U\|_{2,\alpha,E_{[a,b]}} \le \delta_0$ and C independent of a, b. In the computation below, the constant C will change line to line but independently of a.

Let b_δ be such that $\|e^{-\lambda_0 t} V\|_{2,\alpha,E_{(-\infty,b_\delta]}} \le \delta$. Then for $a < \min(a_\delta, b_\delta)$, we consider the solution $U^{(a)}$ defined on $[a, b]$, then $Z^{(a)}$ satisfies

(3)
$$\partial_t Z^{(a)} = \partial_t U^{(a)} + \lambda_0 e^{-\lambda_0 t} V = \mathbf{L}(Z^{(a)}) + G(U^{(a)}) = \mathbf{L}(Z^{(a)}) + G(Z^{(a)} + e^{-\lambda_0 t} V)$$

Since $\|Z^{(a)}\|_{2,\alpha,E_{[a,b]}} \le 2\delta$, for $c \in [a, b]$, Solonnikov's estimate (Proposition 1) gives:

$$\begin{aligned} \|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} &\le C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + \|G(Z^{(a)} + e^{-\lambda_0 t} V)\|_{0,\alpha,E_{[a,c]}}) \\ &\le C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + C(\|Z^{(a)}\|^2_{2,\alpha,E_{[a,c]}} + e^{-2\lambda_0 c})) \\ &\le C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + C(\delta\|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} + e^{-2\lambda_0 c})) \end{aligned}$$

So we can choose and fix δ small enough such that $C\delta < 1$ to obtain:

$$\|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} \le C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + e^{-2\lambda_0 c}) \tag{4}$$

So if $\|Z^{(a)}\|_{L^2(E_{[a,c]})} \le e^{-3\lambda_0 c/2}$, we obtain $\|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} \le Ce^{-3\lambda_0 c/2}$ and $\|U^{(a)}\|_{2,\alpha,E_{[a,c]}} \le Ce^{-\lambda_0 c} \le \min(\delta, \varepsilon)$ if c is less than some $\bar{c}$ (we restrict the definition of $U^{(a)}$ to $(-\infty, \bar{c}]$). So as long as the estimate $\|Z^{(a)}\|_{L^2(E_{[a,c]})} \le e^{-3\lambda_0 c/2}$ is true the solution $U^{(a)}$ is well defined. Let us now prove the estimate.

Since $Z^{(a)}(\cdot,a)=0$ the estimate is true at $c=a$. So let c_1 denote the first time where the estimate fails. Because of (3), we have the expression

$$Z_t^{(a)}=\int_a^t e^{(t-s)\mathbf{L}}G(Z_s^{(a)}+e^{-\lambda_0 s}V)ds$$

Since λ_0 is the first eigenvalue of $\mathbf{L}$ we have

$$\begin{aligned}
|Z_t^{(a)}|_{L^2(E)} &\le \int_a^t e^{-\lambda_0(t-s)}|G(Z_s^{(a)}+e^{-\lambda_0 s}V)|_{L^2(E)}ds\\
&\le \int_a^t e^{-\lambda_0(t-s)}C|G(Z_s^{(a)}+e^{-\lambda_0 s}V)|_{0,E}ds\\
&\le \int_a^t e^{-\lambda_0(t-s)}C(|Z_s^{(a)}|^2_{2,\alpha,E}+e^{-2\lambda_0 s})ds\\
&\le \int_a^t e^{-\lambda_0(t-s)}C(\|Z^{(a)}\|^2_{L^2(E_{[a,s]})}+e^{-2\lambda_0 s})ds\\
&\le \int_a^t e^{-\lambda_0(t-s)}C(e^{-3\lambda_0 s}+e^{-2\lambda_0 s})ds\\
&\le Ce^{-\lambda_0 t}\int_a^t e^{-\lambda_0 s}ds \le Ce^{-2\lambda_0 t}
\end{aligned}$$

Then $\|Z^{(a)}\|_{L^2(E_{[a,c_1]})}\le Ce^{-2\lambda c_1}$. So we see that c_1 must satisfies $Ce^{-\lambda_0 c_1/2}\ge 1$; *i.e.* c_1 is bounded below by some universal constant c_0. So the Lemma is proved with $b_0=\min(c_0,\bar{c},b_\delta)$. □

Let b_0 be given by Lemma 2. By (4), we have $\|Z^{(a)}\|_{2,\alpha,E_{[a,b_0]}}\le Ce^{-3\lambda_0 b_0/2}$. So by Arzela-Ascoli theorem, there is $Z\in C^{2,\alpha}(E_{(-\infty,b_0]})$, such that $Z^{(a)}$ subconverge in C^2 to Z. Moreover, Z satisfies $\partial_t Z=\mathbf{L}Z+G(Z+e^{-\lambda_0 t}V)$, *i.e.* $U=Z+e^{-\lambda_0 t}V$ is a solution of (1). Since $\|Z^{(a)}\|_{2,\alpha,E_{[a,t]}}\le Ce^{-3\lambda_0 t/2}$ for $t\le b_0$, we have $\|Z\|_{2,\alpha,E_{(-\infty,t]}}\le Ce^{-3\lambda_0 t/2}$ and then $\lim_{t\to-\infty}e^{\lambda_0 t}U_t=V$ in $C^{2,\alpha}$.

4. The rigidity result

In this section we prove a rigidity result concerning $\mathbb{S}_1^n=\{p\in\mathbb{R}^{n+1}\mid \|p\|=1\}$ endowed with the induced metric $g_{\mathbb{S}_1^n}$. For $0\le k\le n-1$, let us consider the map:

$$\Psi:\begin{array}{ccc}\mathbb{S}_1^k\times\mathbb{R}\times\mathbb{S}_1^{n-k-1} & \longrightarrow & \mathbb{S}_1^n\\ (p,s,q) & \longmapsto & ((\cos s)p,(\sin s)q)\end{array}$$

We notice that $\Psi(\mathbb{S}_1^k\times[0,\frac{\pi}{2}]\times\mathbb{S}_1^{n-k-1})=\mathbb{S}_1^n$, Ψ is injective on $\mathbb{S}_1^k\times(0,\frac{\pi}{2})\times\mathbb{S}_1^{n-k-1}$, $\Psi(p,0,q)=(p,0)$ and $\Psi(p,\frac{\pi}{2},q)=(0,q)$. So $\mathbb{S}_1^n$ can be seen as the joint of $\mathbb{S}_1^k$ and $\mathbb{S}_1^{n-k-1}$. Moreover $\Psi^*(g_{\mathbb{S}_1^n})=\cos^2 s g_{\mathbb{S}_1^k}+ds^2+\sin^2 s g_{\mathbb{S}_1^{n-k-1}}$. The curves $s\mapsto\Psi(p,s,q)$ are geodesics of $\mathbb{S}_1^n$.

For $k = 2$, we see that $\Psi(\mathbb{S}_1^2, 0, q)$ is an immersed minimal sphere in $\mathbb{S}_1^n$ which is isometric to $\mathbb{S}_1^2$. Actually it is a totally geodesic equatorial 2-sphere in $\mathbb{S}_1^n$. As a minimal surface its index is $n-2$. Our rigidity result looks at such an immersed sphere in a Riemannian manifold.

Theorem 6. *Let M be a Riemannian $n \geq 3$-manifold whose sectional curvature is bounded above by 1. Then any immersed minimal 2-sphere has area at least 4π. Besides, let us assume that M contains an immersed minimal 2-sphere of area 4π which is*

- *either of index at least $n-2$*
- *or unstable in any parallel directions.*

Then the universal cover of M is isometric to the sphere $\mathbb{S}_1^n$.

Moreover, it will come from the proof a minimal 2-sphere of area 4π is totally geodesic.

Proof. Let S be an immersed 2-sphere in M, using Gauss and Gauss-Bonnet formulas we have

$$\begin{aligned}4\pi = \int_S K_S &= \int_S K_{TS} + (B(e_1,e_1), B(e_2,e_2)) - \|B(e_1,e_2)\|^2\\ &= \int_S K_{TS} + \|\vec{H}\|^2 - \|\mathring{B}(e_1,e_1)\|^2 - \|B(e_1,e_2)\|^2\\ &\leq \mathcal{A}(S) + \int_S \|\vec{H}\|^2 - \frac{1}{2}\|\mathring{B}\|^2\end{aligned}$$

where (e_1, e_2) is an orthonormal basis of TS, K_S denotes the sectional curvature of S, K_{TS} the sectional curvature of M on the 2-plane TS and $\mathring{B}$ is the traceless part of B. As a consequence

$$F(S) = \mathcal{A}(S) + \int_S \|\vec{H}\|^2 - 4\pi \geq \int_S \frac{1}{2}\|\mathring{B}\|^2 \geq 0 \tag{5}$$

Hence $F(S) = 0$ implies that S is totally umbilic, $K_{TS} = 1$ and $K_S = 1 + \|\vec{H}\|^2$. If S is minimal, $F(S) \geq 0$ implies that $\mathcal{A}(S) \geq 4\pi$.

An immersed minimal 2-sphere in M lifts to its universal cover with the same instability property. So we assume that M is simply connected and $X : \mathbb{S}^2 \to M$ an immersed minimal 2-sphere as in the statement of the theorem. Let us notice that since $\Sigma = X(\mathbb{S}^2)$ is a minimal surface of area 4π, $F(\Sigma) = 0$ and Σ is totally geodesic and $K_\Sigma = 1$ so Σ is isometric to $\mathbb{S}_1^2$: we can choose X such that X is an isometry between $\mathbb{S}_1^2$ and $\Sigma = X(\mathbb{S}^2)$. Let us denote by NX the normal vector bundle $\{(p,v) \in X^*TM \mid v \in T_pX^\perp\}$ and consider the map

$$\Phi : \begin{array}{ccc} NX & \to & M \\ (p,v) & \mapsto & \exp_{X(p)}(v) \end{array}$$

We want to study the pull-back metric $h = \Phi^* g$ on NX in order to control when Φ is an immersion.

The first step of the proof is

Lemma 3. *The normal bundle NX is parallelizable. Moreover for any $p \in \mathbb{S}^2_1$ and unit vectors $e \in T_pX$ and $v \in T_pX^\perp$, $(R(e,v)e,v) = 1$.*

Proof of Lemma 3. By hypothesis Σ is unstable so the Jacobi operator L has eigen-sections with negative eigenvalue (notice that if Σ is unstable in any parallel directions, Σ is unstable since the whole normal bundle is a parallel sub-bundle). Let V be such an eigen-section with negative eigenvalue λ. Let us prove that $\lambda = -2$ and V is a parallel section of NX.

We have $LV = -\lambda V$. For small t, we consider the immersed sphere $\Sigma_t = \{\Phi(p, tV(p)); p \in \mathbb{S}^2\}$. We then have $F(\Sigma_t) \ge 0$ for any t. $F(\Sigma_0) = 0$, so the first derivative of $t \mapsto F(\Sigma_t)$ must vanish at $t = 0$: it is confirmed by the computation

$$\frac{d}{dt}F(\Sigma_t)_{|t=0} = \int_\Sigma (-2\vec{H}, V) + \int_\Sigma \|\vec{H}\|^2(-2\vec{H}, V) + \int_\Sigma (LV, \vec{H}) = 0$$

Now the second derivative has to be non-negative and we have by Lemma 1

$$\begin{aligned}\frac{d^2}{dt^2}F(\Sigma_t)_{|t=0} &= \int_\Sigma -(LV,V) + \int_\Sigma \frac{1}{2}(LV,LV)\\ &= \int_\Sigma (\lambda + \frac{1}{2}\lambda^2)\|V\|^2\end{aligned}$$

So $\lambda^2 + 2\lambda \ge 0$: $\lambda \le -2$. Since Σ is totally geodesic, we also have

$$\begin{aligned}\lambda\int_\Sigma \|V\|^2 &= \int_\Sigma (-LV, V)\\ &= \int_\Sigma \|\nabla^\perp V\|^2 - (R(e_i,V)e_i,V) - (V, B(e_i,e_j))^2\\ &= \int_\Sigma \|\nabla^\perp V\|^2 - (R(e_i,V)e_i,V) \ge \int_\Sigma -2\|V\|^2\end{aligned}$$

where we used $K \le 1$ and Σ has dimension 2 in the last inequality. So $\lambda \ge -2$. This gives $\lambda = -2$.

The above computation shows also that V is a parallel normal vector field to Σ and $(R(e,V)e,V) = \|V\|^2$ for any vector $e \in T\Sigma$.

Let $V_1, \dots, V_d$ be a basis of the eigenspace associated to the eigen-value -2. Let $\mathcal{B}$ be the sub-bundle of NX generated by $V_1, \dots, V_d$: $\mathcal{B} = \{(p,v) \in NX \mid v \in \mathrm{span}(V_1(p), \dots, V_d(p))\}$. $\mathcal{B}$ is parallelizable and, on $\mathcal{B}$, the stability operator is $L = -\Delta^\perp - 2$. So the index of L restricted to $\mathcal{B}$ is precisely d. If $d < n-2$, both hypotheses on Σ implies that the restriction of L to $\mathcal{B}^\perp$ must have a negative eigenvalue. Thus there is an eigensection of eigenvalue -2 in $\mathcal{B}^\perp$ contradicting the definition of $\mathcal{B}$. So $d = n-2$ and $\mathcal{B} = NX$ which ends the proof. $\square$

The sequel of the proof is a generalization of the above argument.

Let us fix V an eigen-section of L associated to the eigenvalue -2. By Theorem 1, let $(\Sigma_t)_{t\in(-\infty,b)}$ be the ancient solution of the mean curvature flow flowing out of Σ in the direction V. $(-\infty, b)$ is a maximal time interval of existence.

We look at the evolution of $F(\Sigma_t)$. We know that $\lim_{t\to-\infty} F(\Sigma_t) = 0$. Computing its derivative, we obtain

$$\frac{d}{dt}F(\Sigma_t) = \int_{\Sigma_t} -4\|\vec{H}\|^2 + \int_{\Sigma_t} -4\|\vec{H}\|^4 + \int_{\Sigma_t} -2\|\nabla^\perp \vec{H}\|^2 \tag{6}$$

$$2(R(e_i, \vec{H})e_i, \vec{H}) + 2(\vec{H}, B(e_i, e_j))^2 \tag{7}$$

$$\leq \int_{\Sigma_t} 2[(R(e_i, \vec{H})e_i, \vec{H}) - 2\|\vec{H}\|^2)] + \int_{\Sigma_t} 2(\vec{H}, \mathring{B}(e_i, e_j))^2 \tag{8}$$

$$\leq \int_{\Sigma_t} 2\|\vec{H}\|^2\|\mathring{B}\|^2 \leq 4\sup_{\Sigma_t}\|\vec{H}\|^2 F(\Sigma_t) \tag{9}$$

where we use (5) in the last inequality.

By construction of Σ_t, we know that close to $-\infty$, $\sup_{\Sigma_t}\|\vec{H}\|^2 \leq Ce^{4t}$. So by Gronwall lemma we have for $s \leq t$

$$F(\Sigma_t) \leq F(\Sigma_s)\exp(C(e^{4t} - e^{4s}))$$

Using $\lim_{s\to-\infty} F(\Sigma_s) = 0$ and letting $s \to -\infty$, this gives $F(\Sigma_t) \leq 0$ for any t and then $F(\Sigma_t) = 0$ for any t. So $\mathring{B} = 0$ on Σ_t, $K_{T\Sigma_t} = 1$ (see below Equation (5)).

This also implies that the derivative of $F(\Sigma_t)$ vanishes so equality in (8) and (9) gives: $\vec{H}$ is a parallel section of the normal bundle to Σ_t and $(R(e, \vec{H})e, \vec{H}) = \|\vec{H}\|^2$ for any unit vector $e \in T\Sigma_t$. Since $\vec{H}$ is a parallel section, we have $\|H\|$ is constant along Σ_t (notice that $\|\vec{H}\| \neq 0$ by construction). So we can write $\vec{H} = H_t\nu$ ($H_t = \|\vec{H}\|(t)$) where ν is a unit normal vector field to Σ_t. Moreover ν is a parallel section of the normal bundle to Σ_t.

Let us define a new time parameter $s = s(t) = \int_{-\infty}^t H_u du$, so that $\frac{ds}{dt} = H_t$. Hence the derivative of Σ_s with respect to s is given by ν.

If $q \in \Sigma_s$, the map $(a, b) \mapsto (R(a, b)a, b)$ defined for unit vectors a, $b \in T_qM$ is bounded above by 1 (since the sectional curvature is bounded above by 1) and is equal to 1 at $(a, b) = (f, \nu)$ where $f \in T_q\Sigma_s$. So computing the derivatives with respect to a and b,we have $(R(f, \nu)f, v) = 0$ for any $v \in \nu^\perp$ and $(R(f, \nu)v, \nu) = 0$ for any $v \in f^\perp$.

We can compute $\frac{\overline{D}}{ds}2\vec{H}_s$ in two ways:

$$\frac{\overline{D}}{ds}2\vec{H}_s = 2\frac{d}{ds}H_s\nu + 2H_s\frac{\overline{D}}{ds}\nu \qquad \text{and}$$

$$\begin{aligned}\frac{\overline{D}}{ds}2\vec{H}_s &= \Delta^\perp\nu + (R(e_i, \nu)e_i)^\perp + (\nu, B(e_i, e_j))B(e_i, e_j) - (\vec{H}, \nabla_{e_i}\nu)e_i \\ &= 2\nu + 2H_s^2\nu\end{aligned}$$

Since $\frac{\overline{D}}{ds}\nu$ is orthogonal to ν, the comparison of the above computations gives $\frac{\overline{D}}{ds}\nu = 0$: the evolution follows geodesics and $\Sigma_s = \{\Phi(p, sV(p)); p \in \Sigma\}$. Besides $\frac{d}{ds}H_s = 1 + H_s^2$, so $H_s = \tan s$.

Let γ_p be the geodesic $s \mapsto \Phi(p, sV(p))$. We are going to study some Jacobi fields along γ_p. Let $f_1, \dots, f_{n-1}$ be parallel orthonormal fields along γ_p such that $f_1, \dots, f_{n-1}, \nu$ is orthonormal and, at $s = 0$, f_1, f_2 is a basis of T_pX. For $i \in \{1, 2\}$, we define ∂_i the Jacobi field along γ_p such that $\partial_i(0) = f_i$ and $\frac{\overline{D}}{ds}\partial_i(0) = 0$. Actually if $e_i \in T_p\mathbb{S}^2$ is such that $X_*(e_i) = f_i$, we have $\partial_i = D_{e_i}(\Phi(\cdot, sV(\cdot))$ so $\partial_i(s)$ is tangent to Σ_s as long as Σ_s is well defined. ∂_i is a Jacobi field so because of the above computations of the Riemann tensor

$$0 = \frac{\overline{D}^2}{ds^2}\partial_i + R(\nu, \partial_i)\nu = \frac{\overline{D}^2}{ds^2}\partial_i + \partial_i$$

Decomposing this equation in $(f_1, \cdots, f_{n-1})$, we obtain that $\partial_i = \cos s f_i$. Hence (∂_1, ∂_2) is an orthogonal basis of $T\Sigma_s$: Σ_s is an immersion for $s \in [0, \pi/2)$.

As a consequence the orthogonal of $T\Sigma_s$ is generated by $(\nu, f_3, \cdots, f_{n-1})$. Let ∂_j ($j \geq 3$) be the Jacobi fields along γ_p with $\partial_j(0) = 0$ and $\frac{\overline{D}}{ds}\partial_j(0) = f_j$. We have $(R(\nu, \partial_j)\nu, f_i) = (\partial_j, f_i)$ for $i \in \{1, 2\}$ and $s \in [0, \frac{\pi}{2}]$. So (∂_j, f_i) is solution of the ODE $y'' + y = 0$ with vanishing initial value and derivative. Thus $(\partial_j, f_i) = 0$ and ∂_j belongs to $\mathrm{span}(f_3, \cdots, f_{n-1})$. Moreover Rauch comparison theorem implies that ∂_j, $j \in \{3, \dots, n-1\}$, are non vanishing on $(0, \pi)$.

We know that NX is parallelizable so we can fix an isometric parametrization by $NX \simeq \mathbb{S}_1^2 \times \mathbb{R}^{n-2}$ and we have a map

$$\Phi : \mathbb{S}_1^2 \times \mathbb{R}^{n-2} \to M$$

Using a polar decomposition of $\mathbb{R}^{n-2}$ as $\mathbb{R}_+ \times \mathbb{S}^{n-3}$ and coordinates $(p, s, q) \in \mathbb{S}^2 \times \mathbb{R}_+ \times \mathbb{S}^{n-3}$, this gives a map $\Psi : \mathbb{S}^2 \times \mathbb{R}_+ \times \mathbb{S}^{n-3} \to M$ defined by $\Psi(p, s, q) = \Phi(p, sq)$. The above study of the Jacobi fields along the geodesic gives that the lift of the metric is given by

$$\Psi^* g = \cos^2 s g_{\mathbb{S}_1^2} + ds^2 + g_{p,s}$$

for $s \in [0, \pi/2]$ and $g_{p,s}$ is a smooth family of metrics on $\mathbb{S}^{n-3}$ depending on $(p, s) \in \mathbb{S}^2 \times (0, \pi/2)$. Let us notice that $g_{p,s} = s^2 g_{\mathbb{S}_1^{n-3}} + o_0(s^2)$ and $g_{p,\frac{\pi}{2}}$ is a well defined metric on $\mathbb{S}^{n-3}$.

As a consequence $\Psi(p, \frac{\pi}{2}, q)$ is a point $Q \in M$ that does not depend on p. If $\bar{p}$ is fixed $\Sigma' = \Psi(\bar{p}, \frac{\pi}{2}, \mathbb{S}^{n-3})$ is then an immersed submanifold of M given by the immersion $X'(\cdot) = \Psi(\bar{p}, \frac{\pi}{2}, \cdot)$. Let us study the geometry near Σ'. The geodesics $s \mapsto \Psi(p, s, q)$ arrive orthogonally to Σ' when $s = \frac{\pi}{2}$. Let us fix $\bar{q} \in \mathbb{S}^{n-3}$ and define a map

$$G : \begin{array}{ccc} \mathbb{S}^2 & \to & U_{\bar{q}}X'^{\perp} \\ p & \mapsto & \frac{d}{ds}\Psi(p, s, \bar{q})_{|s=\frac{\pi}{2}} \end{array}$$

where $U_{\bar{q}}X'^{\perp}$ is the unit sphere in the normal bundle $TX'^{\perp}$ at $\bar{q}$.

For $r \in [0, \pi/2]$ let us define $F_r : \mathbb{S}^2 \to M; p \mapsto \exp_{\overline{Q}}(-rG(p))$. We have $F_r(p) = \Psi(p, \frac{\pi}{2} - r, \bar{q})$ so ${F_r}^* g = \sin^2 r g_{\mathbb{S}_1^2}$. So $G^* g_{\overline{Q}} = \lim_{r \to 0} \frac{1}{r^2} {F_r}^* g = g_{\mathbb{S}_1^2}$. So G is a linear isometry between $\mathbb{S}^2$ and $U_{\bar{q}}X'^{\perp}$. As a consequence $G(-p) = -G(p)$. Thus $\Psi(p, s + \frac{\pi}{2}, \bar{q}) = \exp_{\overline{Q}}(sG(p)) = \exp_{\overline{Q}}(-sG(-p)) = \Psi(-p, \frac{\pi}{2} - s, \bar{q})$ for $s \in [0, \frac{\pi}{2}]$. So $\Psi(p, \pi, q) = \Psi(-p, 0, q) = X(-p)$. This implies that π is a conjugate time for the Jacobi fields ∂_j $(3 \leq j \leq n-1)$, *i.e.* ∂_j vanishes at time 0 and π as in the constant curvature 1 case. So we are in the equality case of Rauch comparison theorem, $|\partial_j(s)| = |\sin s|$ and $(R(\partial_j, \nu)\partial_j, \nu) = \sin^2 s$. This implies that $(R(\partial_j, \nu)f, \nu) = 0$ for any f orthogonal to ∂_j. So $\partial_j = \sin(s) f_j$ and $R(f_j, \nu)f_j, \nu) = 1$ for any j. So $\Psi^* g = \cos^2 s g_{\mathbb{S}_1^2} + ds^2 + \sin^2 s g_{\mathbb{S}_1^{n-3}}$. As a consequence, Ψ generates a local isometry Ψ' from the joint of $\mathbb{S}^2$ with $\mathbb{S}^{n-3}$ endowed with the metric $\cos^2 s g_{\mathbb{S}_1^2} + ds^2 + \sin^2 s g_{\mathbb{S}_1^{n-3}}$ to M. So Ψ' is a local isometry and thus a covering map from $\mathbb{S}_1^n$ to M. Since M is simply connected Ψ' is a global isometry. $\square$

The above result has a corollary concerning manifold containing an "equator". This is an infinitesimal version of [2, Corollary 5.11].

Corollary 1. *Let M be a Riemannian $n \geq 3$-manifold whose sectional curvature is bounded above by 1. Let us assume that there is a totally geodesic isometric immersion $f : \mathbb{S}_1^{n-1} \to M$. Moreover, assume that f is unstable as a minimal hypersurface. Then the universal cover of M is isometric to $\mathbb{S}_1^n$.*

Proof. Let $G(2, n-1)$ be the set of totally geodesic 2 spheres in $\mathbb{S}_1^{n-1}$: the intersections of $\mathbb{S}_1^{n-1}$ with any 3-dimensional subspace of $\mathbb{R}^n$. For any S in $G(2, n-1)$, $f(S)$ is totally geodesic in M. Moreover $f(S)$ has index at least $n-3$. So it is enough to prove that one of these S has index at least $n-2$ to conclude by Theorem 6. For any $S \in G(2, n-1)$, let Q_S be the quadratic form associated to the Jacobi operator on $f(S)$.

Let ν be the unit normal to $f(\mathbb{S}_1^{n-1})$. For $S \in G(2, n-1)$, $\mathrm{span}(\nu)$ is a parallel normal bundle along $f(S)$. Let $\lambda_0 < 0$ be the first eigenvalue of the Jacobi operator on $f(\mathbb{S}_1^{n-1})$ and u a first eigenfunction. On $\mathbb{S}_1^{n-1}$, we can define a quadratic form $q = du \otimes du - (R(\nu, \cdot)\nu, \cdot)$. Since f is totally geodesic, the quadratic form $Q_{\mathbb{S}_1^{n-1}}$ associated to the Jacobi operator on $f(\mathbb{S}_1^{n-1})$ satisfies to

$$Q_{\mathbb{S}_1^{n-1}}(u, u) = \int_{\mathbb{S}_1^{n-1}} \mathrm{tr}_{T\mathbb{S}_1^{n-1}}\, q = \lambda_0 \int_{S_1^{n-1}} u^2 < 0$$

Now there is a dimensional constant c_n such that

$$c_n \int_{\mathbb{S}_1^{n-1}} \mathrm{tr}_{T\mathbb{S}_1^{n-1}} q = \int_{G(2,n-1)} dS \int_S \mathrm{tr}_{TS}\, q$$
$$= \int_{G(2,n-1)} Q_S(u\nu, u\nu) dS$$

where tr_P denotes the trace operator on the subspace P and dS is the Haar measure on $G(2, n-1)$ coming from the Haar measure on the Grassmannian of 3-planes in $\mathbb{R}^{n+1}$. Thus there is S such that $Q_S(u\nu, u\nu) < 0$. So the restriction to $\mathrm{span}(\nu)$ of the stability operator for S is unstable: S has index at least $n-2$. $\square$

5. Rigidity of Simon-Smith width

Here we state Smith theorem concerning the existence of a minimal sphere in any Riemannian 3-sphere $(\mathbb{S}^3, g)$ and our second rigidity result.

We start with the standard sweep-out of the sphere $\mathbb{S}^3$ given by horizontal spheres

$$S_t = \{(x_1, x_2, x_3, x_4) \in \mathbb{S}^3 \mid x_4 = t\} \text{ for } t \in [-1, 1]$$

If $F_t : \mathbb{S}^3 \to \mathbb{S}^3$ ($t \in [-1, 1]$) is a smooth family of diffeomorphisms isotopic to the identity map, we can defined a general sweep-out of $\mathbb{S}^3$ by spheres as the family $\sigma_t = F_t(S_t)$ ($t \in [-1, 1]$). Let Λ denote the set of all general sweep-outs of $\mathbb{S}^3$ by spheres. The Simon-Smith width of a Riemannian 3-sphere $(\mathbb{S}^3, g)$ is then defined by

$$W(\mathbb{S}^3, g) = \inf_{\{\sigma_t\}\in\Lambda} (\max_{t\in[-1,1]} \mathcal{A}(\sigma_t))$$

For $\sigma \in \Lambda$, we denote $L(\sigma) = \max_{t\in[-1,1]} \mathcal{A}(\sigma_t)$. So a minimizing sequence for the Simon-Smith width W is a sequence (σ^n) in Λ such that $\lim L(\sigma^n) = W$. For such a minimizing sequence, one can consider a sequence $(t_n) \in [-1, 1]$ such that $\lim \mathcal{A}(\sigma^n_{t_n}) = W$. Such a sequence is called a min-max sequence. The main result in [22] is

Theorem 7 (Smith)**.** *There is a min-max sequence that converges in the sense of varifolds to a disjoint union of embedded minimal spheres (possibly with multiplicity) whose area is $W(\mathbb{S}^3, g)$.*

Moreover the quantity $W(\mathbb{S}^3, g)$ is positive so the collection of minimal 2-spheres is not empty.

If the sectional curvature of $(\mathbb{S}^3, g)$ is bounded above by 1 the area of each sphere in the collection is at least 4π so $W(\mathbb{S}^3, g) \geq 4\pi$. We notice that for the Euclidean sphere $\mathbb{S}^3_1$ we have $W(\mathbb{S}^3_1) = 4\pi$. The main result of this section is a rigidity result for the equality case.

Theorem 8. *Let $(\mathbb{S}^3, g)$ be a Riemannian 3-sphere whose sectional curvature is bounded above by 1. Then $W(\mathbb{S}^3, g) \geq 4\pi$ and, if $W(\mathbb{S}^3, g) = 4\pi$, then $(\mathbb{S}^3, g)$ is isometric to $\mathbb{S}^3_1$.*

In order to give the proof of this result we need to understand the index property of the minimal spheres that realize $W(\mathbb{S}^3, g)$. Actually we need to prove that the minimal spheres given by Theorem 7 are not local minima of the area functional. In order to do that, we first prove that the convergence in Theorem 7 is also true in the space of flat cycles (see Section 5.1). In Section 5.2, we introduce de notion of almost stable minimal hypersurfaces which concerns minimal hypersurfaces with vanishing first Jacobi eigenvalue. In Section 5.3, we prove that almost stable implies that we are a local minimum. We then use this to prove Theorem 8 in Section 5.4. We will use concepts from geometric measure theory, for the notations we refer to [8] and [13].

5.1. **The Simon-Smith min-max surface.** In this section we recall some aspect of the min-max construction by Smith in [22]. Actually we mainly refer to the statements contained in the paper by Colding and De Lellis [8]. The aim of the section is to prove that the convergence in Theorem 7 works also as current.

If the varifold V is a varifold limit of some min-max-sequence we say that V is a min-max varifold. So the proof of Theorem 7 consists in finding a min-max varifold which is smooth and has the right topological type.

The first step in this proof is that there is a minimizing sequence (σ^n) such that any min-max varifold coming from (σ^n) is stationary (see [8, Proposition 4.1]). The second step toward regularity use the notion of almost-minimizing surface.

Definition 2. *Given $\varepsilon > 0$, an open set $U \in \mathbb{S}^3$ and a surface Σ. Σ is ε-almost minimizing in U if there is no isotopy ψ supported in U such that*

$$\mathcal{A}(\psi_t(\Sigma)) \leq \mathcal{A}(\Sigma) + \varepsilon/8 \text{ for all } t$$
$$\mathcal{A}(\psi_1(\Sigma)) \leq \mathcal{A}(\Sigma) - \varepsilon$$

A sequence (Σ_n) is said to be almost minimizing in U if Σ_n is ε_n-almost minimizing in U for some sequence $\varepsilon_n \to 0$

For p and $r > 0$, we define the set of annuli centered at p of outer radius less that r $\mathcal{AN}_r(p) = \{B_t(p) \setminus \overline{B}_s(p), 0 < s < t < r\}$. Then one can select a positive function $p \mapsto r(p)$ on $\mathbb{S}^3$ and a min-max sequence $(\sigma^n_{t_n})$ which is almost minimizing in any small annuli: *i.e.* in any $A \in \mathcal{AN}_{r(p)}(p)$ for all $p \in \mathbb{S}^3$ (see [8, Proposition 5.1]). Then the author proves that if $\sigma^n_{t_n} \to V^*$ as varifold, V^* has the expected properties. Concerning smoothness, the authors introduce the notion of replacement.

Definition 3. *Let V be a stationary varifold and U is an open subset of $\mathbb{S}^3$. A stationary varifold V' is said to be a replacement of V in U if*

- *$V' = V$ on $G(M \setminus \overline{U})$ and $\|V'\|(M) = \|V\|(M)$*
- *$V' \llcorner U$ is supported by a stable minimal surface Σ with $\overline{\Sigma} \setminus \Sigma \subset \partial U$.*

The proof of the smoothness V^* consist basically in proving that V^* has replacements in any small annuli and that they coincide with V^* (see [8, Theorem 7.1]).

Let $(\Sigma_n) = (\sigma_{t_n}^n)$, we know that $\Sigma_n \to V^*$ as varifold. Moreover, viewing Σ_n as a flat cycle, we can assume that $\Sigma_n \to T \in \mathcal{Z}_2(M, \mathbb{Z}_2)$ as current where $\mathcal{Z}_2(M, \mathbb{Z}_2)$ denotes the space of 2-dimensional flat cycles in M with coefficient in $\mathbb{Z}_2$. So one can ask the relation between T and V^*. Since V^* is made of smooth surface with integer multiplicities, there is a corresponding element $[V^*] \in \mathcal{Z}_2(M, \mathbb{Z}_2)$ by reducing each multiplicities mod 2. So one can suspect that $T = [V^*]$. This is confirmed by the result below (we are inspired by a similar result for the Almgren-Pitts min-max theory by Marques and Neves [14, Proposition 4.10]). Actually the support of T has to be contained in $\operatorname{spt} V^*$. So if $\{S_i\}_{1\le i\le N}$ is the collection of connected components of $\operatorname{spt} V^*$, the Constancy Theorem implies that $T = n_1 S_1 + \cdots + n_N S_N$ for some $n_i \in \{0, 1\}$.

Proposition 3. *Let (Σ_n) be a min-max sequence which is almost-minimizing in any small annuli and such that $\Sigma_n \to V$ as varifold and*

$$V = m_1 S_1 + \cdots + m_N S_N$$

where each S_i is a smooth embedded minimal surface. Let us assume that $\Sigma_n \to T \in \mathcal{Z}_2(M, \mathbb{Z}_2)$ as current and

$$T = n_1 S_1 + \cdots + n_N S_N$$

with $n_i \in \{0, 1\}$. Then $m_i = n_i \mod 2$.

In [27], White studies limits of sequence of integral varifolds as varifolds and flat chains. He proves that they coincide as flat chains under some regularity assumption. Here this assumption is replaced by the almost-minimizing property. Before giving the proof of the above proposition, let us give some preliminary results.

One tool in the proof of the smoothness of V^* is the following result of Meeks, Simon and Yau.

Theorem 9 (Meeks-Simon-Yau [16]). *Let Σ be a surface in M and U an open subset of M. Let (Σ_k) be a minimizing sequence for the Problem $(\Sigma, \mathfrak{Is}(U))$, i.e. $\Sigma_k = \psi(\Sigma)$ for some $\psi \in \mathfrak{Is}(U)$ ($\mathfrak{Is}(U)$ is the set of isotopies supported in U) and*

$$\mathcal{A}(\Sigma_k) \to \inf_{\psi \in \mathfrak{Is}(U)} \mathcal{A}(\psi(\Sigma)),$$

which converges to a varifold V. Then $V \llcorner U$ is an integer rectifiable varifold whose support is a stable minimal surface Γ with $\overline{\Gamma} \setminus \Gamma \subset \partial U$. Moreover as current $\Sigma_k \llcorner U \to [V \llcorner U]$ in $\mathbf{I}_2(U, \mathbb{Z}_2)$.

The proof of the regularity of $V \llcorner U$ is local so is contained in [16]. As above this regularity implies the existence of flat chain mod 2, $[V \llcorner U] \in \mathbf{I}_2(U, \mathbb{Z}_2)$.

The proof of the convergence as current is a byproduct of the regularity proof as noticed by Almgren and Simon in [1, Remark 5.20].

As mentioned above the smoothness proof uses the fact that replacement of V^* coincide with V^*. This is a natural property of stationary varifold with smooth support.

Lemma 4. *Let V be a integer stationary varifold whose support is a smooth minimal surface S and $p \in S$. Then there is $\rho > 0$ such the following is true: if V' is a stationary varifold such that $V' \llcorner B_\rho(p)^c = V \llcorner B_\rho(p)^c$, then $V' = V$.*

Proof. Let $r > 0$ be chosen such that $\Delta = S \cap B_r$ is connected, stable and with non-empty smooth boundary. Let u be a non negative eigenfunction for the first eigenvalue of the Jacobi operator on Δ with $u = 0$ on $\partial\Delta$. Let $\Delta_t = \{\exp_p(tu(p)\nu(p)), p \in \Delta\}$ where ν is the unit normal to Δ. There is $\varepsilon > 0$ such that, for $|t| \le \varepsilon$, Δ_t is a smooth surface with mean curvature pointing to $\Delta = \Delta_0$. Let ρ such that $B_\rho(p) \subset \cup_{|t|<\varepsilon}\Delta_t$.

Let V' be as in the Lemma, we notice that $\operatorname{spt} V' \subset S \cup B_\rho(p)$. Using the surfaces Δ_t as barriers and the maximum principle [28], we have that $\operatorname{spt} V' \subset S$. Then the Constancy theorem and $V' \llcorner B_\rho(p)^c = V \llcorner B_\rho(p)^c$ implies $V' = V$. □

We then have

Proof of Proposition 3. Let $p \in S_1$ and ρ be given by Lemma 4. Let $A \subset B_\rho(p)$ be an annulus centered at p such that Σ_n is ε_n-almost minimizing in A.

Let

$$\mathfrak{Is}_n(A) = \{\psi \in \mathfrak{Is}(A) \mid \mathcal{A}(\psi_t(\Sigma_n)) \le \mathcal{A}(\Sigma_n) + \varepsilon_n/8\}$$

and $\psi^k \in \mathfrak{Is}_n(A)$ such that

$$\mathcal{A}(\psi_1^k(\Sigma_n)) \to \inf_{\psi \in \mathfrak{Is}_n(A)} \mathcal{A}(\psi_1(\Sigma_n))$$

Let $q \in S_1 \cap A$ and $\varepsilon > 0$ such that $B_\varepsilon(q) \subset A$. We first fix n and k. Let $\varphi^j \in \mathfrak{Is}(B_\varepsilon(q))$ be a sequence such that

$$\mathcal{A}(\varphi_1^j(\Sigma_n^k)) \to \inf_{\varphi \in \mathfrak{Is}(B_\varepsilon(q))} \mathcal{A}(\varphi_1(\Sigma_n^k))$$

Let W_n^k be the varifold limit of $\varphi_1^j(\Sigma_n^k)$ and $R_n^k \in \mathcal{Z}_2(M, \mathbb{Z}_2)$ be the limit of $\varphi_1^j(\Sigma_n^k)$ for the flat convergence. By Theorem 9, $R_n^k \llcorner B_\varepsilon(q) = [W_n^k \llcorner B_\varepsilon(q)] \in \mathbf{I}_2(B_\varepsilon(q), \mathbb{Z}_2)$. Besides we have $R_n^k \llcorner \overline{B}_\varepsilon(q)^c = \Sigma_n^k \llcorner \overline{B}_\varepsilon(q)^c$.

Letting $k \to \infty$ we consider W_n a varifold limit of W_n^k and R_n a limit of R_n^k for the flat convergence. Since $\operatorname{spt}(W_n^k) \cap B_\varepsilon(q)$ is a stable minimal surface, we have curvature estimates and the convergence of $\operatorname{spt}(W_n^k) \cap B_\varepsilon(q)$ to $\operatorname{spt}(W_n)$ is locally smoothly graphical. So taking multiplicities of these graphical leaves into account, varifold and flat convergences give $R_n \llcorner B_\varepsilon(q) = [W_n \llcorner B_\varepsilon(q)]$.

There is a subsequence $(\varphi_1^{j(k)}(\Sigma_n^k))$ which converges to W_n as varifold and R_n as current. By [8, Lemma 7.6] there is $\Phi^{j(k)} \in \mathfrak{Is}(B_\varepsilon(q))$ such that $\Phi_1^{j(k)} = \varphi_1^{j(k)}$ and $\mathcal{A}(\Phi_t^{j(k)}(\Sigma_n^k)) \le \mathcal{A}(\Sigma_n^k) + \varepsilon_n/8$. This implies that $\varphi_1^{j(k)}(\Sigma_n^k)$ can be constructed from Σ_n by an isotopy in $\mathfrak{Is}_n(A)$. Then $(\varphi_1^{j(k)}(\Sigma_n^k))$ is a minimizing sequence in $\mathfrak{Is}_n(A)$. So $\operatorname{spt}(W_n) \cap A$ is a stable minimal surface [8, Lemma 7.4]. Moreover a varifold limit W of W_n is a replacement for V in A [8, Proposition 7.5] and then $V = W$ because of Lemma 4 and $A \subset B_\rho(p)$. Let R be a current limit of R_n.

As above since $\operatorname{spt}(W_n) \cap A$ is a stable minimal surface, the varifold and flat convergence give $R \llcorner A = [W \llcorner A] = [V \llcorner A]$. Actually $\operatorname{spt} R \subset \operatorname{spt} W = \operatorname{spt} V$ and $R \llcorner \overline{A}^c = T \llcorner \overline{A}^c$. So by the Constancy Theorem, $R = T$. So $T \llcorner A = [V \llcorner A]$, this implies that the multiplicity of T on S_1 is equal to the one of V mod 2. Then $T = [V]$ by considering p on other connected components. $\square$

5.2. **Almost stable minimal hypersurfaces.** If Σ is a stable minimal hypersurface in a Riemannian manifold M, then Σ is a local minimum for the area functional (see [26]). In this section we introduce almost stable minimal hypersurfaces as minimal hypersurfaces with vanishing first Jacobi function with an extra property. In the next section we will prove that such a minimal hypersurface has the local minimum property. We can give a sense to stability of a minimal hypersurface Σ whose first Jacobi eigenvalue vanishes, *i.e.* degenerate stable minimal hypersurface.

Because of the use of curvature estimates, in Sections 5.2 and 5.3, we restrict ourselves to an ambient space M of dimension n at most 7.

In order to quickly explain our notion of almost stability, let us consider the simple case of a function $a : \mathbb{R} \to \mathbb{R}$ with $a'(0) = 0$ and $a''(0) = 0$. Then there is a maximal interval $[s^-, s^+] \subset \mathbb{R}$ containing 0 such that $a(s) = a(0)$ for $s \in [s^-, s^+]$ (may be $s^- = 0 = s^+$). Then if $a(s) \ge a(0)$ for s close to s^- and s^+, we say that 0 is almost stable. Let us remark that, even if 0 is almost stable, there could be points s outside $[s^-, s^+]$ close to the endpoints such that $a(s) = a(0)$, they are critical points of a. Let us go back to minimal hypersurfaces.

Actually we focus only on 2-sided minimal hypersurfaces, the 1-sided case can be considered similarly. So let Σ be a connected 2-sided minimal hypersurface and parameterize the ε-neighborhood of Σ by $\Phi : \Sigma \times (-\varepsilon, \varepsilon) \to M; (p,t) = \exp_p(t\nu(p))$. This defines a vectorfield ∂_t on the neighborhood. If u is a smooth function on Σ, we define $\Phi_u : \Sigma \to M$ by $\Phi_u(p) = \Phi(p, u(p))$ and $\Sigma_u = \Phi_u(\Sigma)$. If u is small, Σ_u is embedded and we define $\nu_u(p)$ the unit normal to Σ_u at $\Phi_u(p)$ such that $(\nu_u, \partial_t) \ge 0$ and $H_u(p) \in \mathbb{R}$ such that the mean curvature vector of Σ_u is $H_u(p)\nu_u(p)$ at $\Phi_u(p)$. Let W_u be the Jacobian of the map Φ_u. Then the area of Σ_u can be computed as

$\mathcal{A}(\Sigma_u) = A(u) = \int_\Sigma W_u$. For u and v two functions we then have

$$DA(u)(v) = \int_\Sigma -nH_u(\nu_u, \partial_t)vW_u = \int_\Sigma h_u v \tag{10}$$

where $h_u = -nH_u(\nu_u, \partial_t)W_u$. The map $u \mapsto h_u$ is a smooth operator of order 2.

Lemma 5. *Let Σ be a degenerate stable minimal 2-sided hypersurface Σ with first Jacobi eigen-function u_0. There is $\varepsilon > 0$ and a smooth map $v : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}; (t,p) \mapsto v_t(p)$ with the following properties.*

- *$v_0 = 0$, $\partial_t v_{t|t=0} = 0$ and $\int_\Sigma u_0 v_t = 0$.*
- *for each $t \in (-\varepsilon, \varepsilon)$, $h_{tu_0+v_t}$ is a multiple of u_0.*

A similar result can be found in [5].

Proof. Let V be the L^2 orthogonal of u_0 and π_V the orthogonal projection on V. Let $V^{k,\alpha} = V \cap C^{k,\alpha}(\Sigma)$. Let us define the map: $G : \mathbb{R} \times V^{2,\alpha} \to V^{0,\alpha}; (t,v) \mapsto \pi_V(h_{tu_0+v})$. We notice that $G(0,0) = 0$ and $D_vG(0,0)(w) = \pi_V(Lw)$ where L is the Jacobi operator. Since L is invertible from $V^{2,\alpha}$ to $V^{0,\alpha}$, $D_vG(0,0)$ is invertible and the implicit function theorem gives the map v. □

Once the family $\{v_t\}$ is constructed we then define a foliation of a neighborhood of Σ by $S_t = \Sigma_{tu_0+v_t}$, this foliation is called the canonical foliation given by Σ. Let $a(t) = \mathcal{A}(S_t)$. We notice that by construction there is $c_t \in \mathbb{R}$ such that $h_{tu_0+v_t} = c_t u_0$ so by (10)

$$a'(t) = \int_\Sigma h_{tu_0+v_t}(u_0 + \partial_t v_t) = c_t \int_\Sigma u_0^2$$

So $a'(t) = 0$ if and only if $c_t = 0$ *i.e.* S_t is a minimal surface. Moreover, any small u can be written $u = tu_0 + v$ with t small and $v \in V$ small, so if Σ_u is a minimal surface $h_u = 0$ and then $G(t,v) = 0$. So the implicit function theorem implies that, if Σ_u is minimal (u small), $\Sigma_u = S_t$ for some small t.

It is possible that $a(t) = \mathcal{A}(\Sigma)$ for $t \in (-\varepsilon, \varepsilon)$, $t \in [0, \varepsilon)$ or $t \in (-\varepsilon, 0]$ (if it is not the case we can skip the discussion and we are in case 1 below). In the first case, a whole neighborhood of Σ is foliated by minimal hypersurfaces, we say that Σ is *minimally foliating on both side.* In both remaining cases, only one side of the neighborhood is foliated by minimal hypersurfaces, we say that Σ is *minimally foliating only on one side* (we refer to Song's work [25] for a similar discussion).

If $\{S_t\}_t$ is such a smooth family of minimal surfaces with $\Sigma = S_0$, we notice that the derivative at t of the family is given by a Jacobi field on S_t. So up to a change of parametrization $\{S_s\}_s$, we can assume that this Jacobi field as unit L^2-norm (we use this unit speed parametrization in order to control the case where the family is defined on an unbounded interval).

One can try to extend such a family. Let $\{S_s\}_{s\in(s^-,s^+)}$ be such a family of minimal hypersurfaces with $S_0 = \Sigma$ (possibly $s^- = 0$ and $s \in [0, s^+)$). First we notice that the hypersurfaces S_s are disjoint for s close to 0. So if $S_s \cap S_{s'} \neq \emptyset$ for some $0 \leq s' < s$, there is a larger σ such that all the hypersurfaces in $\{S_s\}_{0\leq s<\sigma}$ are disjoint and S_σ intersect Σ. Then the maximum principle implies that $\Sigma = S_\sigma$ and then $S_s = S_{s+\sigma}$ and the family is periodic and defined on $\mathbb{R}$ (here we recall that we assume Σ to be connected).

Assume now that the family is made of disjoint minimal hypersurfaces. Each S_s is an index 0 minimal hypersurface of area $\mathcal{A}(\Sigma)$. So by compactness result (see [21]) , as $s \to s^+$, a subsequence converges in the varifold sense to a minimal hypersurface Σ^+ : if Σ^+ is two-sided the convergence is smooth, if Σ^+ is one sided the convergence is with mulitplicity 2 and smooth in the double cover of the tubular neighborhood of Σ^+. Actually this implies that as $s \to s^+$ the whole family $\{S_s\}$ converges to Σ^+. The same can be done as $s \to s^-$ to produce Σ^-. If Σ^+ is one sided, then the whole tubular neighborhood of Σ^+ is foliated by $\{S_s\}_{s\in(s^-,s^+]}$ where $S_{s^+} = \Sigma^+$. If Σ^+ is two sided, the family extends for $s = s^+$ and $S_{s^+} = \Sigma^+$ is either minimally foliating only on one side and the family $\{S_t\}_{t\in(s^-,s^+]}$ can't be extended across s^+ or Σ^+ is minimally foliating on both sides and the family extends to $t \in (s^-, s')$ with $s^+ < s'$. As a consequence we are in one of the following cases

(1) the family extends to $\{S_s\}_{s\in[s^-,s^+]}$ and $S_{s^\pm}$ are minimally foliating only on one side or, in the case $s^- = 0 = s^+$, Σ is not minimally foliating on any side
(2) the family is periodic and gives a global foliation of M
(3) the family extends to $\{S_s\}_{s\in(s^-,s^+]}$ such that $S_s \to \Sigma^-$ a 1-sided minimal hypersurface as $s \to s^-$ and S_{s^+} is minimally foliating only on one side.
(4) the family extends to $\{S_s\}_{s\in(s^-,s^+)}$ such that $S_s \to \Sigma^\pm$ two 1-sided minimal hypersurfaces as $s \to s^\pm$. In that case $\{S_s\}_{s\in[s^-,s^+]}$ with $S_{s_\pm} = \Sigma^\pm$ gives a global foliation of M.

We notice that the family can't be defined on an unbounded interval without being periodic because of compactness result and the fact that the family is parameterized at unit speed.

We then say that Σ is *almost stable* if we are in cases (2), (4) or in cases (1) and (3) with the extra property: the area function $a^\pm$ associated to the canonical foliation given by $S_{s^\pm}$ also satisfies locally $a^\pm \geq \mathcal{A}(S_{s^\pm}) = \mathcal{A}(\Sigma)$. In the sequel we will focus on case (1) . For example in case (1), if $s^- = 0 = s^+$ we just ask that the area function a associated to Σ has a local minimum at 0. We notice that, in case (1) and (3), $S_{s^\pm}$ is minimally foliating only on one side so there are value of t close to 0 such that $a^\pm(t) > \mathcal{A}(\Sigma)$.

5.3. **Almost stable minimal hypersurfaces are local minima.** We are going to apply the above classification to an ambient space M which is $\mathbb{S}^3$. So we are interested in type 1 almost stable minimal hypersurface.

So we have a family of minimal hypersurface $\{S_s\}_{s\in[s^-,s^+]}$ that we can view as a subset $\mathcal{M}$ of currents in $\mathcal{Z}_n(M,\mathbb{Z}_2)$. We notice that $\mathcal{M}$ is compact for the flat topology. In $\{S_s\}$, there are two particular elements $S^-=S_{s^-}$ and $S^+=S_{s^-}$ that are minimally foliating only on one side; if $s^-=0=s^+$, $S^-=\Sigma=S^+$ and Σ is not minimally foliating on any side. In the non minimally foliated side, the leaves of the canonical foliation given by $S^\pm$ have area at least $\mathcal{A}(\Sigma)$. However, it may contain minimal hypersurfaces of area $\mathcal{A}(\Sigma)$, we add to $\mathcal{M}$ all these minimal hypersurfaces that are sufficiently close to $S^\pm$ to define $\mathcal{M}'$. $\mathcal{M}'$ is still compact for the flat topology.

We have the following result which is a first version of the fact that a type 1 almost stable minimal hypersurface is a local minimum for the area functional.

Proposition 4. *Let Σ be a type 1 almost stable minimal hypersurface and $\mathcal{M}'$ be the set of minimal hypersurface of area $\mathcal{A}(\Sigma)$ in the canonical foliation as defined above. Then there is an open set U containing Σ such that $\mathcal{A}(\Sigma)\le\mathbf{M}(T)$ for any $T\in\mathcal{Z}_n(M,\mathbb{Z}_2)$ homologous to Σ with support in U and, if $\mathcal{A}(\Sigma)=\mathbf{M}(T)$, then $T\in\mathcal{M}'$.*

Proof. The proof is similar to the one of [26, Theorem 2] for the stable case. So we consider the neighborhoods K_r, $0\le r\le\varepsilon$ of Σ given by [26, Theorem 1] (these K_r has the property that the mass among currents with support in K_r will produce current satisfying a certain almost minimizing property in M, see definitions in [26]). Let T_r minimize the mass $\mathbf{M}$ among all currents homologous to Σ in K_r. As $r\to 0$, the currents T_r must converge to Σ.

As explained by White [26], since the T_r are uniformly almost minimizing, the convergence $T_r\to\Sigma$ and regularity theory implies that T_r can be described as the normal graph of a function u_r on Σ with $u_r\to 0$ in $C^{1,\alpha}$. As u_r is close to 0, one can write $u_r=w_{t_r}+f_r$ where $w_t=tu_0+v_t$ is given by Lemma 5 and $f_r\in V^{1,\alpha}$ (indeed the map $\mathbb{R}\times V^{1,\alpha}\to C^{1,\alpha};(t,f)\mapsto w_t+f$ is a local diffeomorphism at $(0,0)$ since $\partial_t w_{t|t=0}=u_0$). So one can estimate the mass of T_r by

$$\begin{aligned}\mathbf{M}(T_r)&=A(u_r)\\&=A(w_{t_r}+f_r)\\&=A(w_{t_r})+\int_0^1 DA(w_{t_r}+tf_r)(f_r)dt\\&=A(w_{t_r})+DA(w_{t_r})(f_r)+\int_0^1\int_0^1 tD^2A(w_{t_r}+stf_r)(f_r,f_r)dsdt\end{aligned}$$

By Lemma 5 and (10), $DA(w_{t_r})(f_r)=\int_\Sigma h_{w_{t_r}}f_r=0$ since $h_{w_{t_r}}$ is a multiple of u_0 and $f_r\in V$. Moreover, for r close to 0, there is $c>0$ such that $D^2A(w_{t_r}+stf_r)(f_r,f_r)\ge c\|f_r\|_2^2$ since $D^2A(0)$ is given by the Jacobi operator. So $\mathbf{M}(T_r)\ge A(w_{t_r})+\frac{c}{2}\|f_r\|_2^2\ge\mathcal{A}(\Sigma)+\frac{c}{2}\|f_r\|_2^2$. Since T_r is minimizing

$\mathbf{M}(T_r) \le \mathcal{A}(\Sigma)$. So $f_r = 0$ and $A(w_{t_r}) = \mathcal{A}(\Sigma) : T_r = \Sigma_{w_{t_r}}$ which is minimal. So $T_r \in \mathcal{M}'$ for r close to 0 and K_r is the expected neighborhood. □

The following proposition states a second version of the local minimum property.

Proposition 5. *Let Σ, $\mathcal{M}$ and $\mathcal{M}'$ be as above. There is $\varepsilon > 0$ so that for every $T \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ with $\mathcal{F}(T, \mathcal{M}) < \varepsilon$, we have $\mathbf{M}(T) > \mathcal{A}(\Sigma)$ unless $T \in \mathcal{M}'$ where $\mathbf{M}(T) = \mathcal{A}(\Sigma)$.*

Proof. The proof follows the lines of the proof of [14, Proposition 6.2].

First let $(T_i) \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ be a sequence converging to some S in $\mathcal{M}$ in the flat topology with $\mathbf{M}(T_i) \le \mathcal{A}(\Sigma)$. From Proposition 4, we obtain a neighborhood U of S such that elements in $\mathcal{M}'$ are the only minimizers of $\mathbf{M}$ among cycles contained in U homologous to S. Let us consider a smaller neighborhood V with $\overline{V} \subset U$. In [14], Marques and Neves then construct a sequence of cycles R_i satisfying the following properties

- $\mathbf{M}(R_i) \le \mathbf{M}(T_i) \le \mathcal{A}(\Sigma)$
- $R_i \to S$ in the flat topology
- the support of $R_i - T_i$ is contained in $M \setminus V$
- the support of R_i is in U for large i

Since $R_i \to S$, R_i is homologous to S for large i. So because of Proposition 4, $R_i = S_i$ for some $S_i \in \mathcal{M}'$. Moreover, since $R_i \to \Sigma$, $S_i \subset V$ for large i then $\mathbf{M}(T_i) = \mathbf{M}(S_i + T_i \llcorner (M \setminus V)) = \mathcal{A}(S_i) + \mathbf{M}(T_i \llcorner (M \setminus V))$ so $T_i \llcorner (M \setminus V) = 0$ and $T_i = R_i = S_i$ *i.e.* $T_i \in \mathcal{M}'$. □

So the consequence of this is the following result: a type 1 almost stable minimal hypersurface is at the bottom of a basin of the area functional.

Proposition 6. *Let Σ and $\mathcal{M}$ be as above. For any $\varepsilon > 0$, there is $\delta > 0$ and a neighborhood $\mathcal{N} \subset \mathbb{Z}_n(M, \mathbb{Z}_2)$ of $\mathcal{M}$, such that $\mathcal{N} \subset \mathcal{N}_\varepsilon(\mathcal{M}) = \{T \in \mathcal{Z}_n(M, \mathbb{Z}_2) \mid \mathcal{F}(T, \mathcal{M}) < \varepsilon\}$ and, for any $T \in \partial\mathcal{N}$, we have $\mathbf{M}(T) \ge \mathcal{A}(\Sigma) + \delta$.*

In the case of a stable minimal hypersurface Σ (positive first Jacobi eigenvalue), the same result is true with $\mathcal{M} = \{\Sigma\}$ by [14, Proposition 6.2].

Proof. First we assume that ε is such that Proposition 5 applies. Let $S_\pm$ denote both extremal hypersurfaces of the family (possibly $S^+ = \Sigma = S^-$). Let $\{w_t^\pm\} = \{tu_0^\pm + v_t^\pm\}$ be the families of functions given by Lemma 5 associated to $S^\pm$. Since Σ is almost minimizing and $S^\pm$ are minimally foliating only on one side, there is $t^+ > 0$ (resp. $t^- < 0$) such that $S^\pm_{w_s^\pm} \in \mathcal{N}_\varepsilon(\mathcal{M})$ for $0 \le s \le t^+$ (resp. $t^- \le s \le 0$) and $A(w^\pm_{t^\pm}) > \mathcal{A}(\Sigma)$.

Let $\widetilde{\mathcal{M}} = \mathcal{M} \cup \{S^\pm_{w_s^\pm}, 0 \le \pm s \le t^\pm\}$. We can notice that any element in $\mathcal{M}' \setminus \widetilde{\mathcal{M}}$ is at a positive $\mathcal{F}$ distance from $\widetilde{\mathcal{M}}$. Let $\eta > 0$ such that $\mathcal{N}_\eta(\widetilde{\mathcal{M}}) \subset \mathcal{N}_\varepsilon(\mathcal{M})$ and $\overline{\mathcal{N}_\eta(\widetilde{\mathcal{M}})} \cap \mathcal{M}' = \widetilde{\mathcal{M}} \cap \mathcal{M}'$. If T_i is a sequence in $\partial\mathcal{N}_\eta(\widetilde{\mathcal{M}})$ such that $\mathbf{M}(T_i) \to \inf\{\mathbf{M}(T), T \in \partial\mathcal{N}_\eta(\widetilde{\mathcal{M}})\}$. By compactness we can assume that $T_i \to T$ and, by lower-semicontinuity of the mass, $\mathbf{M}(T) =$

$\inf\{\mathbf{M}(T), T \in \partial\mathcal{N}_\eta(\widetilde{\mathcal{M}})\}$. Moreover $\mathcal{F}(T, \widetilde{\mathcal{M}}) = \eta$ and $\mathcal{F}(T, \mathcal{M}) \le \varepsilon$. So by Proposition 5, $\mathbf{M}(T) \ge \mathcal{A}(\Sigma)$ with equality iff $T \in \mathcal{M}'$ which would implies $T \in \widetilde{\mathcal{M}}$ a contradiction. So $\mathbf{M}(T) > \mathcal{A}(\Sigma)$ and the result is proved. □

5.4. **The rigidity result.** In this section we finally prove Theorem 8. So let us fix a Riemannian 3-sphere $(\mathbb{S}^3, g)$ with sectional curvature bounded above by 1. Its Simon-Smith width is realized by a collection of minimal spheres whose areas are at least 4π so the width is at least 4π. If the width is 4π, the width is then realized by a minimal 2-sphere Σ with multiplicity 1.

If Σ has index at least 1, the rigidity comes from Theorem 6. So we need to prove that index 0 cannot occur.

If Σ has vanishing first Jacobi eigenvalue then Σ may belong to a family of minimal spheres $\{S_s\}$. Since we are in $\mathbb{S}^3$, this family is of type 1. Let $\mathcal{M} = \{S_s, s \in [s_-, s_+]\}$ be the associated compact subset in $\mathcal{Z}_2(M, \mathbb{Z}_2)$ ($\mathcal{M} = \{\Sigma\}$ if Σ is stable).

If Σ is almost-stable (or Σ stable), let ε be less that the $\mathcal{F}$ distance from $\mathcal{M}$ to the 0 cycle and let $\mathcal{N}$ be the neighborhood of $\mathcal{M}$ given by Proposition 6.

Since Σ realize the width, there are sequences $\{\sigma^n\}$ in Λ and (t_n) in $[-1, 1]$ such that $\sigma^n_{t_n} \to \Sigma$ in the varifold sense and $(\sigma^n_{t_n})$ is a min-max sequence which is almost minimizing in small annuli. Because of Proposition 3, we have $\sigma^n_{t_n} \to \Sigma \in \mathcal{Z}_2(M, \mathbb{Z}_2)$ for the $\mathcal{F}$ metric. So for n large enough, the continuous path $t \mapsto \sigma^n_t$ in $\mathcal{Z}_2(M, \mathbb{Z}_2)$ enters into $\mathcal{N}$. As this path starts and ends at the 0 cycle, it must cross $\partial\mathcal{N}$ so

$$\max_{t\in[-1,1]} \mathcal{A}(\sigma^n_t) \ge 4\pi + \delta$$

where δ is given by Proposition 6. So $\{\sigma^n\}$ can not be a minimizing sequence.

Thus the width is realized by a type 1 minimal sphere Σ in M of area 4π with vanishing first Jacobi eigenvalue and which is not almost-stable. Σ belongs to a local foliation by minimal surfaces (it may contain only one leaf) which contains one minimal sphere $\overline{S}$ of area 4π whose canonical foliation $\{S_t\}$ contains leaves such that $\mathcal{A}(S_t) < \mathcal{A}(\Sigma)$ for $t > 0$ arbitrarily close to 0. Notice that $\overline{S}$ has also vanishing first Jacobi eigenvalue. Let $a(t) = \mathcal{A}(S_t)$ be the associated area function. We notice that, if $a'(t) = 0$, S_t is a minimal sphere and then $a(t) \ge 4\pi$. This implies that

- either $a(t) \ge 4\pi$ for $t \in [0, \varepsilon)$ (ε small),
- or $a(t)$ is decreasing on $[0, \varepsilon)$ (ε small).

Since $\overline{S}$ is not almost-stable, we are in the second situation. In order to exploit this situation we need to introduce a slightly different local foliation near $\overline{S}$.

Lemma 6. *Let Σ be a degenerate stable minimal 2-sided hypersurface with first Jacobi eigen-function u_0. There is $\varepsilon > 0$ and a smooth map $\tilde{v} : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}$ with the following properties.*

- $\tilde{v}_0 = 0$, $\partial_t \tilde{v}_{t|t=0} = 0$ *and* $\int_\Sigma u_0 \tilde{v}_t = 0$.

- *for each $t \in (-\varepsilon, \varepsilon)$, $H_{tu_0+\tilde v_t}$ is a multiple of u_0.*

The proof is the same as Lemma 5 proof and gives the same consequence: if Σ_w is a minimal hypersurface with w small then $w = tu_0 + \tilde v_t$ for some t.

So let $\{\widetilde S_t\} = \{\overline{S}_{tu_0+\tilde v_t}\}$ be the family given by applying the above lemma to $\overline{S}$. Let $\tilde a(t) = \mathcal{A}(\widetilde S_t)$ be the associated area function. If $\tilde a'(t) = 0$, $\widetilde S_t$ is a minimal sphere and then $\tilde a(t) \ge 4\pi$. So we also have the alternative

- either $\tilde a(t) \ge 4\pi$ for $t \in [0, \varepsilon)$,
- or $\tilde a(t)$ is decreasing on $[0, \varepsilon)$.

Let us see that we are in the second case. If we are in the first one, there is $t_0 > 0$ such that $\tilde a'(t_0) \ge 0$, this implies that the mean curvature vector of $\widetilde S_{t_0}$ points to $\overline{S}$. For $t > 0$ small, S_t is between $\bar S$ and $\widetilde S_{t_0}$, let t_1 be the first t such that S_t intersect $\widetilde S_{t_0}$, then the mean curvature vector of S_{t_1} must points to $\overline{S}$ which contradict $a'(t_1) < 0$. So we are in the second case.

If $\tilde w_t = tu_0 + \tilde v_t$, we define on the neighborhood of $\overline{S}$ the vectorfield X by $X(\Phi_{\tilde w_t}(p)) = (\partial_t \tilde w_t)\partial_t$ such that $\widetilde S_t$ is the image of $\overline{S}$ by the flow given by X. We also consider $\tilde\nu_t$ the unit normal to $\widetilde S_t$ (with the convention $(\tilde\nu_t, \partial_t) \ge 0$) and $Y = (X, \tilde\nu_t)\tilde\nu_t$. This vectorfield is normal to $\widetilde S_t$ and $\widetilde S_t$ is the image of $\overline{S}$ by the flow given by Y. Now we apply ideas similar to the proof of Theorem 6. Looking at the F functional on $\widetilde S_t$ we have:

$$
\begin{aligned}
\frac{d}{dt} F(\widetilde S_t) = &-\int_{\widetilde S_t} 2\widetilde H_t (Y, \tilde\nu_t) d\sigma - \int_{\widetilde S_t} 2\widetilde H_t^3 (Y, \tilde\nu_t) d\sigma \\
&+ \int_{\widetilde S_t} (\Delta^\perp Y, \widetilde H_t \tilde\nu_t) + (Y, B(e_i, e_j))(B(e_i, e_j), \widetilde H_t \tilde\nu_t) d\sigma \\
&+ \int_{\widetilde S_t} \widetilde H_t R(e_i, Y)e_i, \tilde\nu_t) d\sigma \\
= &\int_{\widetilde S_t} \widetilde H_t (X, \tilde\nu_t)(\mathrm{Ric}(\tilde\nu_t) - 2) d\sigma + \int_{\widetilde S_t} \widetilde H_t (X, \tilde\nu_t) \|\mathring{B}\|^2 d\sigma \\
&- \int_{\widetilde S_t} (\nabla (X, \tilde\nu_t), \nabla \widetilde H_t) d\sigma
\end{aligned} \tag{11}
$$

where $\Delta^\perp$ and ∇ are operators on $\widetilde S_t$ and $\widetilde H_t$ is the mean curvature of $\widetilde S_t$.

Let $\tilde u_t$ be the positive function defined on $\widetilde S_t$ by $\tilde u_t(\Phi_{\tilde w_t}(p)) = u_0(p)$. By construction there is $\tilde c_t \in \mathbb{R}$ such that $\widetilde H_t = \tilde c_t \tilde u_t$. Moreover since we are in the second case, $\tilde c_t > 0$ and $\widetilde H_t > 0$. This also implies $\widetilde H_t (X, \tilde\nu_t) \ge 0$ and $(\nabla (X, \tilde\nu_t), \nabla \widetilde H_t) = c_t (\nabla (X, \tilde\nu_t), \nabla \tilde u_t)$. At $t = 0$, $(\nabla(X, \tilde\nu_t), \nabla \tilde u_t) = \|\nabla u_0\|^2 \ge 0$. So if u_0 is not constant $\int_{\widetilde S_t} (\nabla(X, \tilde\nu_t), \nabla \widetilde H_t) d\sigma \ge 0$ for small t and, if u_0 is constant, $\widetilde H_t = \tilde c_t \tilde u_t$ is constant and $\int_{\widetilde S_t} (\nabla(Y, \tilde\nu_t), \nabla \widetilde H_t) d\sigma$ vanishes. In both cases, the last term in (11) is non positive and

$$
\frac{d}{dt} F(\widetilde S_t) \le C F(\widetilde S_t)
$$

for some constant C as in Theorem 6 proof. This implies that $F(\widetilde{S}_t) = 0$ for any t. So we are in the equality case: $\mathrm{Ric}(\tilde{\nu}_t) = 2$ since $\tilde{c}_t > 0$. So $\mathrm{Ric}(\tilde{\nu}_0) = 2$ and the Jacobi operator on $\overline{S}$ is $\Delta + 2$ which contradict that 0 is its first eigenvalue.

Appendix A. A Schauder estimate

In this appendix, we prove Proposition 1. To lighten notations, we denote $E_T = E_{[0,T]}$ and $P_T = \mathbb{R}^n \times [0,T]$. First, we give some complementary notations for Hölder norms. For a map $u : D \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$, we define

$$
\begin{gathered}
[u]_{2,\alpha,D} = [u]_{2,\alpha,D,x} + [u]_{2,\alpha,D,t} \\
[u]_{1,\alpha,D,x} = [\partial_x u]_{0,\alpha,D,x} \\
[u]_{1,\alpha,D,t} = [u]_{(1+\alpha)/2,D,t} + [\partial_x u]_{\alpha/2,D,t} \\
[u]_{1,\alpha,D} = [u]_{1,\alpha,D,x} + [u]_{1,\alpha,P,t} \\
\|u\|_{1,\alpha,D} = \|u\|_{0,D} + \|\partial_x u\|_{0,D} + [u]_{1,\alpha,D}
\end{gathered}
$$

Associated to these norms, we have the $C^{l,\alpha}$ Hölder space. We also define the space $C_0^{l,\alpha}(P_T)$ as maps $u \in C^{l,\alpha}(P_T)$ satisfying $\partial_t^i u_{|t=0} = 0$ for $0 \leq i \leq l/2$.

We have some classical interpolation inequalities.

Lemma 7. *Let $l, m \in \{0,1,2\}$, $\alpha, \beta \in [0,1]$ such that $l + \alpha > m + \beta$ and let $T > 0$. Then for any $\varepsilon > 0$ there is a constant C such that, for any $u : P_T \to \mathbb{R}^k$*

$$\|u\|_{m,\beta,P_T} \leq C\|u\|_{L^2(P_T)} + \varepsilon [u]_{l,\alpha,P_T}$$

Proof. Let us explain the proof when $l = m = 0$ and $\beta = 0 < \alpha$. Let us notice that its sufficient to look only at one component u^a of u. Let $X \in P_T$ and consider $\delta > 0$. For δ small, we can consider a box B which is a translate of $[0,\delta]^n \times [0,\delta^2]$ such that $B \subset P_T$ and $X \in B$ (δ can be chosen independently of X). Then there is $\overline{X} \in B$ such that $\int_B u^a = \delta^{n+2} u^a(\bar{X})$. So

$$
\begin{aligned}
|u^a(X)| &\leq |u^a(\overline{X})| + |u^a(\overline{X}) - u^a(X)| \\
&\leq \frac{1}{\delta^{n+2}} \int_B |u^a| + 2\delta^\alpha [u^a]_{0,\alpha,B} \\
&\leq \frac{1}{\delta^{n/2+1}} \|u^a\|_{L^2(B)} + 2\delta^\alpha [u^a]_{0,\alpha,B} \\
&\leq \frac{1}{\delta^{n/2+1}} \|u^a\|_{L^2(P_T)} + 2\delta^\alpha [u^a]_{0,\alpha,P_T}
\end{aligned}
$$

So choosing δ small enough we have the result.

Once this first estimate is established, the other ones can be obtained by similar arguments (for example, see Section 6.8 in [10]). $\square$

The second result that we shall need is a Schauder type estimate for solution of

$$\partial_t u - \mathcal{L}u = f \tag{12}$$

over $\mathbb{R}^n \times [0,T]$ where $\mathcal{L}$ is an operator as in (2) with constant coefficients and only second order terms.

Lemma 8. *Let $\mathcal{L}$ be an operator as in* (2) *with constant coefficients and only second order terms. Then there is a constant C such that the following statement is true. If $u \in C_0^{2,\alpha}(P_T)$ is a solution of* (12) *with $f \in C_0^{0,\alpha}(P_T)$ such that u_t has compact support for any t then*

$$[u]_{2,\alpha,P_T} \le C[f]_{0,\alpha,P_T} \tag{13}$$

This result is established in Section 15 of [24] (see Theorem 4.1 and Equation (4.43))

We now want a similar result when L depends on the variable x and have terms of any order.

Lemma 9. *Let L be an operator as in* (2) *with C^α coefficients independent of t. Then there is a constant C such that the following statement is true. If $u \in C_0^{2,\alpha}(P_T)$ is a solution of* (12) *with $f \in C_0^{0,\alpha}(P_T)$ such that u_t has compact support for any t then*

$$[u]_{2,\alpha,P_T} \le C(\|f\|_{0,\alpha,P_T} + \|u\|_{L^2(P_T)}$$
$$\|u\|_{2,\alpha,P_T} \le C(\|f\|_{0,\alpha,P_T} + \|u\|_{L^2(P_T)})$$

Proof. Let p be a point in $\mathbb{R}^n$, we are going to prove the estimate near p. Let $\delta > 0$ and consider φ_δ be a non-negative C^∞ function on $\mathbb{R}^n$ with support on the ball B centered at p and radius 2δ, equal to 1 on the ball B' of radius δ and such that $[\varphi_\delta]_{i,E} \le \frac{C}{\delta^i}$. We are going to estimate $\varphi_\delta u$.

Let L_p denote the operator $L(p)$ and $\mathcal{L}_p$ the part of L_p with only second order terms. Since $\partial_t u - Lu = f$ we have

$$\partial_t(\varphi_\delta u) - \mathcal{L}_p(\varphi_\delta u) = \varphi_\delta f + \varphi_\delta Lu - L(\varphi_\delta u) + L(\varphi_\delta u) - L_p(\varphi_\delta u) + L_p(\varphi_\delta u) - \mathcal{L}_p(\varphi_\delta u)$$

So the estimate (13) gives

$$\begin{aligned}[\varphi_\delta u]_{2,\alpha,B_T} \le & C([\varphi_\delta f]_{0,\alpha,B_T} + [\varphi_\delta Lu - L(\varphi_\delta u)]_{0,\alpha,B_T} \\ & + [L(\varphi_\delta u) - L_p(\varphi_\delta u)]_{0,\alpha,B_T} + [L_p(\varphi_\delta u) - \mathcal{L}_p(\varphi_\delta u)]_{0,\alpha,B_T})\end{aligned}$$

where B_T denotes $B \times [0,T]$.

In the right hand side of the above estimate, the first term can be estimated by $[\varphi_\delta f]_{0,\alpha,B_T} \le C_\delta \|f\|_{0,\alpha,B_T}$ (in the sequel C_δ will denote a constant that depends on δ). In $\varphi_\delta Lu - L(\varphi_\delta u)$, the terms where the second derivatives of u appears cancel, so $[\varphi_\delta Lu - L(\varphi_\delta u)]_{0,\alpha,B_T} \le C_\delta \|u\|_{1,\alpha,B_T}$. Similarly for the last term, $[L_p(\varphi_\delta u) - \mathcal{L}_p(\varphi_\delta u)]_{0,\alpha,B_T} \le C_\delta \|u\|_{1,\alpha,B_T}$. For the third term, if Λ bounds the $C^{0,\alpha}$ norm of the coefficients of L_p, we have

$$[L(\varphi_\delta u) - L_p(\varphi_\delta u)]_{0,\alpha,B_T} \le \Lambda\delta^\alpha[\varphi_\delta u]_{2,\alpha,B_T} + 2\Lambda\|\varphi_\delta u\|_{2,B_T}$$

So fixing δ such that $C\Lambda\delta^\alpha < 1$ we obtain

$$[\varphi_\delta u]_{2,\alpha,B_T} \leq C_\delta(\|f\|_{0,\alpha,B_T} + \|u\|_{1,\alpha,B_T} + \|\varphi_\delta u\|_{2,B_T})$$

Since $\varphi = 1$ on B' we obtain

$$[u]_{2,\alpha,B'_T} \leq C_\delta(\|f\|_{0,\alpha,P_T} + \|u\|_{1,\alpha,P_T} + \|u\|_{2,P_T})$$

Since we can consider any point p in $\mathbb{R}^n$, we have

$$[u]_{2,\alpha,P_T} \leq C_\delta(\|f\|_{0,\alpha,P_T} + \|u\|_{1,\alpha,P_T} + \|u\|_{2,P_T})$$

By interpolation inequalities, we have $\|u\|_{1,\alpha,P_T} + \|u\|_{2,P_T} \leq C_\varepsilon\|u\|_{L^2(P_T)} + \varepsilon[u]_{2,\alpha,P_T}$. So choosing ε such that $C_\delta\varepsilon < 1$, we obtain

$$[u]_{2,\alpha,P_T} \leq C_\delta(\|f\|_{0,\alpha,P_T} + \|u\|_{L^2(P_T)})$$

Using interpolation inequality again we finally have

$$\|u\|_{2,\alpha,P_T} \leq C_\delta(\|f\|_{0,\alpha,P_T} + \|u\|_{L^2(P_T)})$$

□

We can now give the proof of Proposition 1.

Proof of Proposition 1. Let U and F be as in the proposition. If $T \leq 1$ the result is given by Theorem 3, so let us assume that $T \geq 1$ and let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\eta = 1$ in a neighborhood of 0 and $\eta = 0$ on $[1/2, +\infty)$. So we can write $U = \eta(t)U + (1 - \eta(t))U$. By Theorem 3, we have $\|\eta(t)U\|_{2,\alpha,E_T} \leq C\|U\|_{2,\alpha,E_1} \leq C(\|F\|_{0,\alpha,E_T} + |U_0|_{2,\alpha,E} + \|U\|_{L^2(E_T)})$. The section $W = (1-\eta)U \in C_0^{2,\alpha}(E_T)$ is then a solution of

$$\partial_t W - LW = (1-\eta)F - \eta' U$$

Let $(\varphi_i)_{1\leq i\leq m}$ be a partition of the unity such that the φ_i has support in local charts Ω^i of E. Then each $W_i = \varphi_i W$ is a solution of $\partial_t W_i - LW_i = G_i$. In the chart, W_i can be written as a section w_i over $\mathbb{R}^n$ with compact support which is a solution of $\partial_t w_i - Lw_i = g_i$. So, by Lemma 9

$$\|w_i\|_{2,\alpha,P_T} \leq C(\|g_i\|_{0,\alpha,P_T} + \|w_i\|_{L^2(P_T)})$$

This gives

$$\|W_i\|_{2,\alpha,\Omega^i{}_T} \leq C(\|g_i\|_{0,\alpha,\Omega^i{}_T} + \|w_i\|_{L^2(\Omega^i{}_T)})$$

where $\Omega^i{}_T$ denotes the bundle over $\Omega^i \times [0,T]$. Summing these estimates and using that a finite number of W_i is sufficient we obtain

$$\begin{aligned}\|W\|_{2,\alpha,1,E_T} &\leq C(\|F\|_{0,\alpha,E_T} + \|U\|_{0,\alpha,E_1} + \|U\|_{L^2(E_T)})\\ &\leq C(\|F\|_{0,\alpha,E_T} + |U_0|_{2,\alpha,E} + \|U\|_{L^2(E_T)})\end{aligned}$$

where we have used $\eta' = 0$ outside $[0,1]$ and Solonnikov's estimate on E_1. So adding both estimates for ηU and $(1-\eta)U$ gives the desired result. □

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