Saddle towers with infinitely many ends

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1 Introduction

The topic of minimal surfaces in flat 3-manifolds, with finite genus but infinite total curvature, has recently attracted some attention [1, 2]. In the complete flat 3-manifold $\mathbb{R}^2 \times \mathbb{S}^1$, the only known examples of properly embedded minimal surfaces with infinite total curvature come from doubly or triply periodic minimal surfaces in $\mathbb{R}^3$. In particular, they are all periodic in $\mathbb{R}^2 \times \mathbb{S}^1$. In this paper, we point out an application of the theorem of Jenkins and Serrin [3] to construct properly embedded minimal surfaces in $\mathbb{R}^2 \times \mathbb{S}^1$ with genus zero and infinite total curvature. We prove:

Theorem 1 *There exists a properly embedded singly periodic minimal surface in $\mathbb{R}^3$, whose quotient by all its periods has genus zero, infinitely many ends and exactly one limit end.*

Recall that Scherk's singly periodic surface can be constructed as follows: consider the unit square, and mark its two horizontal edges with $+\infty$ and its two vertical edges with $-\infty$. By the theorem of Jenkins and Serrin [3], there exists a function u which solves the Jenkins-Serrin problem on the square, namely, whose graph is minimal in the interior of the square, and which goes to $\pm\infty$ on the edges, as indicated by the marking. The graph of u is bounded by four vertical lines above the vertices of the square and is a fundamental piece for Scherk's doubly periodic surface. The conjugate minimal surface of the graph of u is bounded by four horizontal symmetry curves, lying in two horizontal planes at distance 1 from each other. By reflecting about one of the two symmetry planes, we obtain a fundamental domain for Scherk's singly periodic surface, which has period $T = (0, 0, 2)$, and four ends in the quotient.

H. Karcher [4] has generalized this construction by replacing the unit square by any convex polygonal domain Ω with $2k$ edges of length one, $k \geq 2$. To satisfy the hypothesis of the theorem of Jenkins and Serrin, the domain Ω must be assumed to be non-special, see definition 1 below (this known fact does not seem to have been written yet, so we provide a proof in the appendix). Solving the Jenkins-Serrin problem on Ω, taking the conjugate and reflecting, one obtains a properly embedded singly periodic minimal surface with period $T = (0, 0, 2)$, $2k$ Scherk-type ends and genus zero in the quotient. These surfaces are now called Karcher's Saddle Towers, and have recently been classified as the only properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ with genus zero and finitely many Scherk-type ends in the quotient [10].

Definition 1 *We say a convex polygonal domain with $2k$ unitary edges is special if $k \geq 3$ and its boundary is a parallelogram with two sides of length one and two sides of length $k - 1$.*

In this paper we follow the same strategy except that we start with an unbounded convex domain Ω with infinitely many edges, so we end up with a minimal surface with infinitely many ends, as desired. More precisely, we consider an unbounded convex domain $\Omega \subset \mathbb{R}^2$ such that:

1. The boundary $\partial\Omega$ of Ω is a polygonal curve with an infinite number of edges, all of length one.

2. Ω is not the plane, nor a half plane, nor a strip, nor an infinite special domain, see definition 2 below.

Definition 2 *An unbounded convex polygonal domain is said to be special when its boundary is made of two parallel half lines and one edge of length one (such a domain may be seen as a limit of special domains with $2k$ edges, when $k \to \infty$).*

Given a domain Ω as above, mark the edges on its boundary alternately by $+\infty$ and $-\infty$. In section 3, we solve the Jenkins-Serrin problem for Ω. In order to do this, we consider an exhaustion of Ω by bounded convex domains Ω_n and solve the Jenkins-Serrin problem on each Ω_n, obtaining a solution u_n in Ω_n. Then we prove that the sequence $\{u_n\}$ has a limit u. Such a function u, which is defined on Ω, has the required behavior on the boundary and its graph M is minimal. Taking the conjugate minimal surface of M and extending by symmetry, we obtain the desired minimal surface. Such a surface can be seen as a limit, when $k \to \infty$, of a sequence of Karcher's Saddle Towers with $2k$ ends. In section 4, we study the asymptotic behavior of this surface.

2 Preliminaries

Let $u = u(x_1, x_2)$ be a solution of the minimal graph equation:

$$(1 + u_2^2)u_{11} - 2u_1u_2u_{12} + (1 + u_1^2)u_{22} = 0, \tag{1}$$

defined on a simply-connected domain $D \subset \mathbb{R}^2$. By an elementary computation,

$$d\psi_u := \frac{u_1}{\sqrt{1 + |\nabla u|^2}}\, dx_2 - \frac{u_2}{\sqrt{1 + |\nabla u|^2}}\, dx_1 \tag{2}$$

is an exact form in D. Hence there exists a function $\psi_u = \psi_u(x_1, x_2)$, called *conjugate function of* u, whose differential is given by (2). Note that ψ_u is well defined up to an additive constant. In fact, if we write $X(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$ and call $X^* = X^*(x_1, x_2)$ its conjugate minimal immersion, then the third coordinate function of X^* can be written as $X_3^*(x_1, x_2) = \psi_u(x_1, x_2)$ (although the conjugate surface is not the graph of ψ_u).

Since $|\nabla\psi_u| = \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} < 1$, ψ_u is a Lipschitz function, so it can be extended continuously to ∂D.

Next we expose some results related to the convergence of a sequence $\{u_n\}_n$ of minimal graphs defined on D. They are based on the theory developed by L. Mazet [5, 6], following the ideas of Jenkins and Serrin (the main improvement over the work of Jenkins and Serrin is that we do not require monotonicity of the sequence $\{u_n\}_n$).

Given a sequence $\{u_n\}_n$ of solutions for the minimal graph equation in D, define the *convergence domain* of the sequence $\{u_n\}_n$ as

$$\mathcal{B}(u_n) = \{p \in D \mid \{|\nabla u_n|(p)\}_n \text{ is bounded}\}.$$

For each component D' of $\mathcal{B}(u_n)$, there is a subsequence of $\{u_n - u_n(Q)\}_n$ converging uniformly on compact sets of D' to a solution of (1), where Q is some fixed point of D'. This fact justifies the name for $\mathcal{B}(u_n)$. Moreover, $D - \mathcal{B}(u_n)$ consists of a union of straight lines:

$$D - \mathcal{B}(u_n) = \cup_{i \in I} L_i,$$

where each $L_i \subset D$ is a component of the intersection of a straight line with D, for each $i \in I$. The straight lines L_i are called *divergence lines*.

Clearly, to ensure the convergence of a subsequence of $\{u_n\}_n$ on D, it suffices to prove there are no divergence lines. The following lemmas 1 and 2 can be useful to conclude this.

Lemma 1 ([6]) *If $T \subset \partial D$ is an open straight segment such that each u_n diverges to $+\infty$ when we approach T, then a divergence line cannot end in T.*

Lemma 2 ([5]) *Given a segment T contained in a divergence line, it holds $\int_T d\psi_n \to \pm|T|$.*

Once we have ensured the convergence of the sequence $\{u_n\}_n$ to a solution u of the minimal graph equation, the next natural step is to understand the behavior of u on the boundary of D.

Lemma 3 ([3, 6]) *Let u be a solution of (1) on D, and $T \subset \partial D$ be an open straight segment oriented as ∂D. Then $\int_T d\psi_u = |T|$ if and only if u diverges to $+\infty$ on T.*

Finally, we have the following uniqueness result for the limit u under some constraints.

Lemma 4 ([7]) *Let u_1 and u_2 be two solutions of (1) in a connected domain D, whose conjugate functions ψ_{u_1}, ψ_{u_2} are bounded in D, and such that $\psi_{u_1} = \psi_{u_2}$ on ∂D. Then $u_1 - u_2$ is constant in D.*

3 Solving the Jenkins-Serrin problem on Ω

Let Ω be an unbounded convex domain as in the introduction. We choose a vertex p_0 such that the inner angle at p_0 is less than π. We label the vertices p_i, $i \in \mathbb{Z}$, in the order that we meet them when traveling along the boundary of Ω with its natural orientation. We mark the edge $[p_i, p_{i+1}]$ with $+\infty$ if i is even and $-\infty$ if i is odd.

Proposition 1 *The Jenkins-Serrin problem on Ω has a solution u. Moreover, $0 \leq \psi_u \leq 1$ on Ω.*

Proof. Given $n \geq 1$, the chord $[p_{-n}, p_n]$ divides Ω into two components. Call U_n the bounded one. Let Ω_n be the union of U_n and its symmetric image about the midpoint of $[p_{-n}, p_n]$. We also extend by symmetry the marking on the edges. Since Ω is an unbounded convex domain, the sum of the inner angles of Ω_n at p_{-n} and p_n is at most π. This implies that Ω_n is a (bounded) convex domain.

Let us prove that Ω_n is non-special. If $n = 1$ then this is true by definition (a special domain has at least six edges). Assume that $n \geq 2$. If Ω_n were special, then p_0 would be a corner of the parallelogram Ω_n because of the way we chose it. Then either $p_{-2}p_{-1}p_0p_1$ or $p_{-1}p_0p_1p_2$ would be a rhombus. Since Ω is an unbounded convex domain, it follows that Ω would be an infinite special domain, a contradiction.

Hence Ω_n is non-special, so by proposition 3 that will be proven in the appendix, it satisfies the hypotheses of Jenkins and Serrin. Let u_n be the solution to the Jenkins-Serrin problem on Ω_n normalized by $u_n(Q) = 0$, where Q is some fixed point in Ω_1. Denote by ψ_n the conjugate function associated to u_n, normalized so that $\psi_n(p_0) = 0$. From lemma 3 we have $\int_{p_i}^{p_{i+1}} d\psi_n = (-1)^i$, which implies that $\psi_n(p_i)$ is equal to 0 if i is even, and equal to 1 if i is odd. Moreover, ψ_n is an affine function on each edge, so $0 \leq \psi_n \leq 1$ on $\partial\Omega_n$. Since the domain Ω_n is bounded, the maximum principle implies that $0 \leq \psi_n \leq 1$ in Ω_n.

Next we are going to prove that $\{u_n\}_n$ converges uniformly on compact sets of Ω. Let D be a bounded subdomain of Ω. For n large enough, we have $D \subset \Omega_n$ so we can restrict u_n to D and apply the results exposed in Section 2.

In our case, this means that every divergence line has to end at vertices.

Firstly, we are going to prove there are no divergence lines. Suppose that there is a divergence line L. Since $0 \leq \psi_n \leq 1$, we deduce from lemma 2 that L must have length no bigger than one. Thus taking a larger domain D if necessary and using lemma 1, we obtain L has to be a segment $[p_i, p_j]$. If i and j have the same parity, then $\psi_n(p_i) = \psi_n(p_j)$ so L has length zero (again using lemma 2), which is absurd. If i and j have different parity, then $|\psi_n(p_i) - \psi_n(p_j)| = 1$, so L is a chord of length one between p_i and p_j. However, this is impossible on a non-special domain (see proposition 3).

Hence there exists a subsequence of $\{u_n\}_n$ which converges on compact subsets of D. Taking an exhaustion of Ω by bounded subdomains and using a diagonal process, we obtain a subsequence of $\{u_n\}_n$ converging on compact subsets of Ω to a solution u of the minimal graph equation. By lemma 3, u takes the marked values $\pm\infty$ on $\partial\Omega$. Its conjugate function ψ_u is the limit of $\{\psi_n\}_n$, hence $0 \leq \psi_u \leq 1$. By lemma 4, we know that u is the unique solution to the Jenkins-Serrin problem with bounded conjugate function. In particular, we deduce that the whole sequence $\{u_n\}_n$ converges to u.

Remark 1 *In general, if u is a solution to the Jenkins-Serrin problem on Ω, then its conjugate function ψ_u satisfies $0 \leq \psi_u \leq 1$ on the boundary of Ω. However, if Ω is not contained in a strip, ψ_u might very well be unbounded, in which case we could not use the maximum principle to guarantee that $0 \leq \psi_u \leq 1$ in Ω. This is why we took special care to construct Ω_n and u_n in such a way that ψ_n is bounded.*

Let M be the graph of u on Ω. It is a minimal surface bounded by infinitely many vertical straight lines above the vertices of Ω. Let n_i be the normal to the edge $[p_i, p_{i+1}]$, pointing outwards of Ω. Along the edge $]p_i, p_{i+1}[$, the downward pointing normal to M converges to $(-1)^i n_i$. Let M^* be the conjugate minimal surface of M. Since $0 \leq \psi_u \leq 1$, M^* is included in the slab $\{0 \leq x_3 \leq 1\}$. Moreover, $\psi_u = 0$ (resp. 1) at p_i when i is even (resp. odd), so the vertical line above p_i on M corresponds in the conjugate surface M^* to an infinite horizontal symmetry curve lying on the plane $\{x_3 = 0\}$ (resp. $\{x_3 = 1\}$). The normal along this curve rotates from $(-1)^{i-1} n_{i-1}$ to $(-1)^i n_i$. Finally, M is a graph on a convex domain, thus M^* is also a graph on a (non convex) domain by the theorem of R. Krust.

Extending M^* by symmetry with respect to the horizontal planes at integer heights, we obtain a properly embedded singly periodic minimal surface

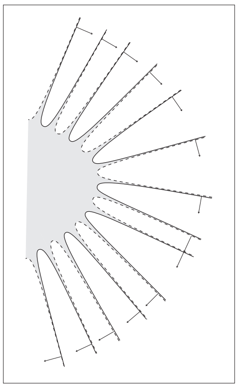

Figure 1: The domain on which the conjugate surface M^* is a graph

with period $(0,0,2)$. It is easy to check that he quotient of M^* by its period has genus 0, infinitely many ends and one limit end. This concludes the proof of theorem 1.

4 Asymptotic behavior

Assume that Ω is not contained in a strip. We will prove that the surface M we constructed in the previous section is asymptotic to two Scherk's doubly periodic surfaces. When Ω is contained in a strip, it may be proven that the surface is asymptotic to a KMR example [4, 8, 9], which is a doubly periodic minimal surface with parallel Scherk type ends (they have been classified in [9] as the only properly embedded doubly periodic minimal surfaces with genus 1 and a finite number of ends in the quotient). The argument is similar, although a little more involved. Thus we will only consider here the case where Ω is not contained in a strip.

Let $a_n = p_{n+1} - p_n \in \mathbb{S}^1$. Since Ω is convex, the limits $a_\infty = \lim_{n\to\infty} a_n$ and $a_{-\infty} = \lim_{n\to-\infty} a_n$ exist. Let $\widetilde{\Omega}_n = \Omega - p_{2n}$, by which we mean the domain Ω translated by $-p_{2n}$. When $n \to \infty$, $\widetilde{\Omega}_n$ converges (on compact subsets of $\mathbb{R}^2$) to a half plane $\widetilde{\Omega}$ bounded by the line $\mathrm{Span}(a_\infty)$, and a similar statement holds for $n \to -\infty$. Hence it is natural to study the Jenkins-Serrin problem on this half plane.

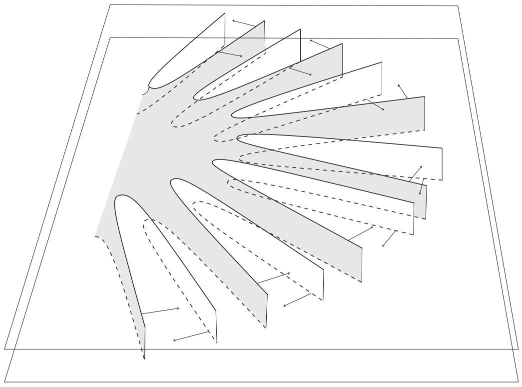

Figure 2: A sketch of the conjugate surface M^*

Without loss of generality we may assume that $\widetilde{\Omega}$ is the half plane $x_2 \geq 0$ with the boundary data $+\infty$ on $[\widetilde{p}_i, \widetilde{p}_{i+1}]$ if i is even and $-\infty$ if i is odd, where $\widetilde{p}_i = (i, 0)$. The Jenkins-Serrin problem on $\widetilde{\Omega}$ has the following explicit solution : Let U be the half band $0 \leq x_1 \leq 1$, $x_2 \geq 0$, with boundary data $+\infty$ on the horizontal segment and 0 on the vertical half lines. A piece of Scherk's singly periodic surface, rotated so that its period is $(2, 0, 0)$, solves the Jenkins-Serrin problem on U. Extending by symmetry, we obtain a solution to the Jenkins-Serrin problem on $\widetilde{\Omega}$. Let us call u_S this solution and S its graph. The conjugate minimal surface S^* of S is a piece of Scherk's doubly periodic surface, rotated so that its periods are $(0, 2, 0)$ and $(0, 0, 2)$, lying in the slab $0 \leq x_3 \leq 1$. Hence the conjugate function ψ_S of u_S is bounded. By lemma 4, u_S is the unique solution to the Jenkins-Serrin problem on $\widetilde{\Omega}$ with bounded conjugate function.

Proposition 2 *Let $\widetilde{u}_n(p) = u(p - p_{2n})$. Then $\widetilde{u}_n - \widetilde{u}_n(Q)$ converges, on compact subsets of $\widetilde{\Omega}$, to u_S, where $Q = (0, 1)$.*

Proof. The situation here is slightly more complicated than in the previous section because the domain is moving. If D is a bounded subdomain of $\widetilde{\Omega}$, we would like to say that $D \subset \widetilde{\Omega}_n$ for n large enough, in order to restrict $\widetilde{u}_n$ to D and study its convergence. This is true when the closure of D is included in $\widetilde{\Omega}$ but it could be false if D contains part of the boundary of $\widetilde{\Omega}$.

This means that we cannot use directly the results in section 2 about the boundary of the domain.

Let D be a bounded subdomain of $\widetilde{\Omega}$. Consider the restriction of $\widetilde{u}_n$ to D (for n large enough). Suppose that this sequence has a line of divergence L. By taking larger and larger domains D, we can extend L as far as we want in at least one direction (because $\widetilde{\Omega}$ is a half plane). However, since $0 \leq \widetilde{\psi}_n \leq 1$, lemma 2 says L has length at most one, a contradiction. Hence a subsequence of $\widetilde{u}_n - \widetilde{u}_n(Q)$ converges, on compact subsets of (the interior) of $\widetilde{\Omega}$, to a function $\widetilde{u}$ solution of the minimal graph equation on $\widetilde{\Omega}$.

Let us prove that $\widetilde{u}$ has the expected boundary behavior, so it solves the Jenkins-Serrin problem on $\widetilde{\Omega}$. Denote by $\widetilde{\psi}$ the conjugate function of $\widetilde{u}$. Let $\widetilde{p}_i^n = p_i - p_{2n}$ be the vertices of $\widetilde{\Omega}_n$, so $\widetilde{p}_i^n \to \widetilde{p}_i = (i, 0)$ when $n \to \infty$. Given $\varepsilon > 0$, call $q_i = (i, \varepsilon)$. For even i and n large enough, we have $q_i \in \widetilde{\Omega}_n$ and:

- $\int_{\widetilde{p}_i}^{\widetilde{p}_{i+1}} d\widetilde{\psi} = \int_{\widetilde{p}_i}^{q_i} d\widetilde{\psi} + \int_{q_i}^{q_{i+1}} d\widetilde{\psi} + \int_{q_{i+1}}^{\widetilde{p}_{i+1}} d\widetilde{\psi} \geq \int_{q_i}^{q_{i+1}} d\widetilde{\psi} - 2\varepsilon$;
- $\int_{q_i}^{q_{i+1}} d\widetilde{\psi} \geq \int_{q_i}^{q_{i+1}} d\widetilde{\psi}_n - \varepsilon$;
- $\int_{q_i}^{q_{i+1}} d\widetilde{\psi}_n = \int_{q_i}^{\widetilde{p}_i^n} d\widetilde{\psi}_n + \int_{\widetilde{p}_i^n}^{\widetilde{p}_{i+1}^n} d\widetilde{\psi}_n + \int_{\widetilde{p}_{i+1}^n}^{q_{i+1}} d\widetilde{\psi}_n \geq \int_{\widetilde{p}_i^n}^{\widetilde{p}_{i+1}^n} d\widetilde{\psi}_n - 4\varepsilon$;
- $\int_{\widetilde{p}_i^n}^{\widetilde{p}_{i+1}^n} d\widetilde{\psi}_n = 1$.

In the first line, we have used that $|d\widetilde{\psi}| \leq 1$ and $d(\widetilde{p}_i, q_i) = \varepsilon$. In the second line, we have used that $d\widetilde{\psi}_n \to d\widetilde{\psi}$ uniformly on compact subset of $\widetilde{\Omega}$. In the third line, we have used that $|d\widetilde{\psi}_n| \leq 1$ and $d(\widetilde{p}_i^n, q_i) \leq 2\varepsilon$ if n is large enough. In the last line we have used lemma 3, since $\widetilde{u}_n$ diverges to ∞ when we approach the segment $[\widetilde{p}_i^n, \widetilde{p}_{i+1}^n]$. All this gives

$$1 \geq \int_{\widetilde{p}_i}^{\widetilde{p}_{i+1}} d\widetilde{\psi} \geq 1 - 7\varepsilon.$$

Since this holds for any $\varepsilon > 0$, we conclude that the integral is one, so by lemma 3, $\widetilde{u} = +\infty$ on the edge $[\widetilde{p}_i, \widetilde{p}_{i+1}]$ when i is even. In the same way, $\widetilde{u} = -\infty$ on the edge $[\widetilde{p}_i, \widetilde{p}_{i+1}]$ when i is odd. Since $0 \leq \widetilde{\psi} \leq 1$, we conclude that $\widetilde{u}$ is the unique solution to the Jenkins-Serrin problem on $\widetilde{\Omega}$ with bounded conjugate function, so $\widetilde{u} = u_S$.

Let us return to the minimal surface M that we constructed in the last section. Let $P_n = (p_n, 0) \in M$ be the point at height 0 on the vertical line through p_n, and P_n^* be the corresponding point on the conjugate minimal surface M^*. Then $M - P_{2n}$ converges when $n \to \infty$ to S and consequently $M^* - P_{2n}^*$ converges to S^*. The above convergence is only on compact subsets of $\mathbb{R}^3$. When $n \to -\infty$, $M^* - P_{2n}^*$ also converges to Scherk's doubly periodic surface.

5 Appendix

For completeness, we prove in this section that, among all the bounded convex unitary polygonal domains, the ones that fail to satisfy the hypothesis of the theorem of Jenkins and Serrin are the special domains.

Let Ω be a bounded convex polygonal domain, with sides marked alternately $+\infty$ and $-\infty$, and $\mathcal{P}$ be any polygonal subdomain of Ω (this means that its vertices are vertices of Ω). Denote by α (resp. β) the total length of the edges of $\mathcal{P}$ which are edges of Ω with mark $+\infty$ (resp. $-\infty$), and call γ the perimeter of $\mathcal{P}$. The domain Ω satisfies the hypothesis of the theorem of Jenkins and Serrin (and so one can solve the Jenkins-Serrin problem on Ω) if and only if $2\alpha < \gamma$ and $2\beta < \gamma$ for each strict subpolygon $\mathcal{P}$ of Ω, and $\alpha = \beta$ when $\mathcal{P} = \Omega$.

Consider a convex polygonal domain Ω as above, and suppose all its edges have length one. Label its vertices $p_1, \cdots, p_{2n}$ so that $[p_1, p_2]$ is marked with $-\infty$ (so $[p_i, p_{i+1}]$ is marked with $+\infty$ if i is even and with $-\infty$ if i is odd, with the convention $p_{2n+1} = p_1$). We say p_i is an even vertex if i is even and an odd vertex if i is odd. We will refer as a *chord* to a straight segment that joints two different non consecutive vertices of Ω.

Proposition 3 *The following statements are equivalent :*

(i) Ω *is not a special domain.*

(ii) Every chord from an even vertex to an odd vertex has length greater than 1.

(iii) Ω *satisfies the hypotheses of the theorem of Jenkins and Serrin.*

Before proving proposition 3, let us recall the following elementary result proven in [10], lemma 5.2.

Lemma 5 *Let $ABCD$ be a convex quadrilateral such that $|BC| = |AD|$ and $\widehat{A}+\widehat{B} \le \pi$, where $\widehat{A}$ means the interior angle at A. Then $|CD| \le |AB|$, with equality if and only if $ABCD$ is a parallelogram.*

Proof. Let us see that $(i) \Rightarrow (ii)$. Arguing by contraposition, we must prove that if Ω has a chord of length ≤ 1 from an even vertex to an odd vertex, then Ω is special. Let C be such a chord. It divides Ω into two convex domains, Ω_1 and Ω_2. For one of them, let us say Ω_1, the sum of the inner angles at the endpoints of C is $\le \pi$. We may rename the vertices of Ω, without changing their parity, so that C is the segment $[p_1, p_{2r}]$ and the vertices on the boundary of Ω_1 are $p_1 \cdots, p_{2r}$. Lemma 5 assures $|p_2 p_{2r-1}| \le |p_1 p_{2r}| \le 1$; and by induction, we obtain $|p_i p_{2r+1-i}| \le 1$ for all $1 \le i \le r$ (here we use that the sum of the inner angles remains $\le \pi$ because Ω is convex). But $[p_r, p_{r+1}]$ is an edge on the boundary of Ω, so $|p_r p_{r+1}| = 1$. Hence equality holds everywhere, and all quadrilaterals $p_i p_{i+1} p_{2r-i} p_{2r+1-i}$ are parallelograms. Since Ω_1 is convex, $p_1 p_r p_{r+1} p_{2r}$ is a parallelogram (with two sides of length 1 and two sides of length $r-1$). Hence the sum of the inner angles of Ω_1 at the endpoints of C is π, so the sum of the inner angles of Ω_2 at the same points is $\le \pi$. Applying the same argument to Ω_2, we obtain that Ω_2 is also a parallelogram, so Ω is a special domain.

Let us see that $(ii) \Rightarrow (iii)$. Let $\mathcal{P}$ be a strict subpolygon of Ω. Let us orient $\partial\Omega$ and $\partial\mathcal{P}$ as boundaries of Ω and $\mathcal{P}$. Note that for an edge in $\partial\mathcal{P} \cap \partial\Omega$, both orientations are the same. Let us prove that $2\alpha < \gamma$. This is clearly true if $\partial\mathcal{P}$ contains no edge marked $+\infty$. Let $[p_{2i}, p_{2i+1}]$ be an edge on the boundary of $\mathcal{P}$ marked $+\infty$. Let $[p_{2j}, p_{2j+1}]$ be the next edge on the boundary of $\mathcal{P}$ marked $+\infty$, when traveling along the boundary in the direction given by its orientation. Let C be the part of $\partial\mathcal{P}$ between p_{2i+1} and p_{2j}. If C contains an edge marked $-\infty$ then $|C| \ge 1$. Else C contains only chords. Since C connects an odd vertex with an even vertex, at least one of its chords goes from an odd vertex to an even vertex, so $|C| > 1$. Hence the part of the boundary of $\mathcal{P}$ between two edges marked $+\infty$ always has length ≥ 1, with strict inequality for at least one of them (else $\mathcal{P} = \Omega$). Hence $2\alpha < \gamma$. The proof of $2\beta < \gamma$ is exactly the same, exchanging the roles of $+\infty$ and $-\infty$.

Finally, $(iii) \Rightarrow (i)$ is obvious : a special domain Ω does not satisfy the hypothesis of Jenkins and Serrin, because if $\mathcal{P}$ is a rhombus then $2\alpha = \gamma$ (or $2\beta = \gamma$).

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