Some uniqueness results for constant mean curvature graphs

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Abstract

The aim of this paper is to give two uniqueness results for the Dirichlet problem associated to the constant mean curvature equation. We study constant mean curvature graphs over strips of $\mathbb{R}^2$. The proofs are based on height estimates and the study of the asymptotic behaviour of solutions to the Dirichlet problem.

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Introduction

The surfaces with constant mean curvature are the mathematical modelling of soap films. These surfaces appear as the interfaces in isoperimetric problems. There exist different points of view on constant mean curvature surfaces, one is to consider them as graphs.

Let Ω be a domain of $\mathbb{R}^2$. The graph of a function u over Ω has constant mean curvature $H > 0$ if it satisfies the following partial differential equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 2H \tag{CMC}$$

The graph of such a solution is called a H-graph and has a upward pointing mean curvature vector.

Thanks to the work of J. Serrin [Se1, Se2] and J. Spruck [Sp], we can build a lot of H-graphs over bounded domains of $\mathbb{R}^2$. Over unbounded domains, the Dirichlet problem associated to (CMC) is more complicated. Existence results are known and due to P. Collin [Co] and R. López [Lo1, Lo2]. The question we ask in this paper is the uniqueness of solutions for the boundary data that these authors study.

In [Lo1, Lo2], R. López proves existence results for vanishing boundary data. In the case of bounded boundary data, the uniqueness is known. In [Co] and [Lo1], the authors build solutions of (CMC) over strips with boundary data that can be unbounded. In this paper, we prove that, for these boundary data, we have uniqueness (see Theorems 10 and 12).

There are two major steps to prove these results. First, if there are two solutions for a same boundary data, the difference between these solutions can not stay bounded. Thus, this gives us an information on the asymptotic behaviour of the boundary data. The second step consists in seeing the consequences of this asymptotic behaviour on the asymptotic behaviour of a solution. In this second step, we use the notion of line of divergence that the author has defined in [Ma1].

The paper is devided as follows. In the first section, we give the two existence results of P. Collin and R. López that we are interested in. We also precise some definitions and notations. In the second part, we give a set of upper and under-bounds for the H-graph that we study. These results are important to prepare the proofs of our uniqueness results. Section 3 is devoted to the proof of the uniqueness of Collin's solutions. In Section 4, we prove the uniqueness of López's solutions. The two proofs are very similar.

1 Two existence results

In this first section, we recall two existence results for Dirichlet problem; these results were proved by P. Collin in [Co] and R. López in [Lo1].

In both cases, the author studies the Dirichlet problem associated to the constant mean curvature equation (CMC) on a strip $\Omega = \mathbb{R} \times (-l, l)$ of width $2l$. It was proven in [Lo2], that the width $2l$ needs to be less than $1/H$ for having a solution.

1.1 The existence results

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous real function. We define φ_f on $\partial\Omega$ by $\varphi_f(x, \pm l) = f(x)$. P. Collin and R. López have looked for a solution u of the constant mean curvature equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 2H \tag{CMC}$$

such that $u|_{\partial\Omega} = \varphi_f$.

The result of P. Collin concerns the limiting case $2l = 1/H$.

Theorem (P. Collin, [Co]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex continuous function. There exists a solution u of* (CMC) *on $\Omega = \mathbb{R} \times (-1/(2H), 1/(2H))$ which takes φ_f as boundary value.*

This result was independently proved by A. N. Wang for the convex function $x \mapsto x^2$ (see [Wa]).

Let Ω be a domain in $\mathbb{R}^2$, we say that Ω satisfies a *uniform exterior R-circle condition* if at each point $p \in \partial\Omega$ there exists a disk D with radius R such that $\overline{D} \cap \overline{\Omega} = \{p\}$. This tells us that a circle of radius R can "roll" outside Ω along $\partial\Omega$ touching each point of $\partial\Omega$ along its deplacement.

Let $f : \mathbb{R} \to \mathbb{R}$ be a countinuous function. f satisfies a *uniform R-circle under condition* if the domain $\{(x, y) \in \mathbb{R}^2 \,|\, y \geq f(x)\}$ satisfies a uniform exterior R-circle condition. It says that a circle of radius R can "roll" under the graph of f touching each point of the graph along its deplacement.

The result of R. López deals with the case where $2l < 1/H$. We give it now using some notations that we shall introduce in the following subsection.

Theorem (R. López, [Lo1]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $t \in \mathbb{R}_+^*$. We assume that f satisfies a uniform $\rho_t(H)$-circle under condition. Then, there exists a solution u of* (CMC) *on the strip $\Omega = \mathbb{R} \times (-h_t(H), h_t(H))$ which takes φ_f as boundary value.*

The technique used by P. Collin and R. Lopez is the Perron technique. They build their solutions as the supremum of under-solutions. The difficulty is to have good barrier functions to ensure the boundary value.

The aim of this paper is to prove that the solutions build by P. Collin and R. López are unique for the boundary data φ_f.

Theorem (Theorems 10 and 12). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We consider one of the two cases below:*

1. *$\Omega = \mathbb{R} \times (-\frac{1}{2H}, \frac{1}{2H})$ and f is convex.*
2. *$\Omega = \mathbb{R} \times (-h_t(H), h_t(H))$ and f satisfies a uniform $\rho_t(H)$-circle under condition.*

Let u and v be two solutions of (CMC) *on Ω with φ_f as boundary value. Then $u = v$.*

1.2 The 1-parameter family of nodoids

In this subsection, we recall many classical results on nodoids, the reader can refer to [De, Ee, Lo1] for more explanations.

The constant mean curvature surfaces of revolution are the Delaunay surfaces. This set of surfaces splits into two 1-parameter families. One is composed of embedded surfaces which are called unduloids. When the parameter moves, the family of the unduloids deforms the cylinder of radius $1/(2H)$ into a stack of tangent sphere of radius $1/H$.

The second 1-parameter family is composed of non-embedded surfaces: these are the nodoids. The interest in considering the nodoids is that, since they have self-intersections, each one contains a piece that looks like a catenoidal neck with mean curvature vector pointing outward.

Let us recall the construction of nodoids and fix notations. Let $r(u)$ be a positive smooth function defined on an open interval I and consider the surface of revolution parametrized by:

$$X(u,\theta) = (r(u)\cos(\theta), r(u)\sin(\theta), u)$$

We fix the normal to the surface to be

$$N(u,\theta) = \frac{1}{\sqrt{1+r'^2}}\big(\cos(\theta), \sin(\theta), -r'\big)$$

The surface has constant mean curvature H if r satisfies:

$$2H = -\frac{1}{r\sqrt{1+r'^2}} + \frac{r''}{(1+r'^2)^{3/2}}$$

After a first integration, this equation implies that there exists $c \in \mathbb{R}$ such that:

$$Hr^2 = -\frac{r}{\sqrt{1+r'^2}} + c \tag{1}$$

Since Hr^2 is positive, c needs to be positive. When $c > 0$, there exist h, ρ and $r : [-h, h] \to [0, \rho]$ a solution to (1) such that r is even and the initial value $r(0) = t > 0$ is the minimum of r. Besides $r(h) = \rho$ and $r'(h) = +\infty$, then $H\rho^2 = c$. The associated surface X is a nodoid.

For $u = 0$, we have $Ht^2 + t = c$ then:

$$t = \frac{-1 + \sqrt{1+4Hc}}{2H}$$

t is an increasing function of c with $t = 0$ for $c = 0$ and $\lim_{c\to+\infty} t = +\infty$. In the following, t is then used as parameter for the family of nodoids. We have:

$$\rho = \rho_t(H) = \sqrt{\frac{Ht^2+t}{H}}$$

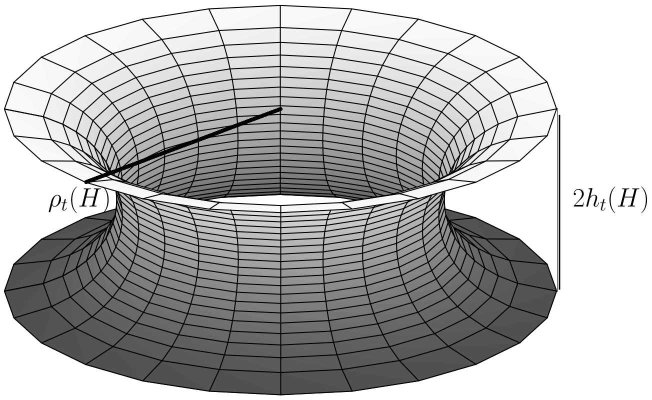

Figure 1:

Moreover, we have:

$$h = h_t(H) = \int_t^{\rho} \frac{H(\rho^2 - x^2)}{\sqrt{x^2 - H^2(\rho^2 - x^2)^2}} \mathrm{d}x$$

We can summarize the properties in the following propostion and in Figure 1.

Proposition 1. *There exists a* 1*-parameter family of nodoids* $\{\mathcal{N}_t, t > 0\}$ *with constant mean curvature* H *given by the rotation of a curve* γ_t *around the* z*-axis and with the following properties:*

1. *The curve* γ_t *is a graph on* $[h_t(H), h_t(H)]$ *of an even function.*

2. *The curve* γ_t *has horizontal tangents at* $\pm h_t(H)$*. Then* $\mathcal{N}_t$ *is included in the slab* $\mathcal{S}_t : |z| \leq h_t(H)$ *and is tangent to it.*

3. *The mean curvature vector points outside the bounded domain determined by* $\mathcal{N}_t$ *in the slab* $\mathcal{S}_t$*.*

4. *The circle* C_t *of* $\mathcal{N}_t$ *with smallest radius is given by* $x^2+y^2 = t^2,\ z = 0$*.*

5. *The function* $h_t(H)$ *is strictly increasing on* t *and*

$$\lim_{t\to 0} h_t(H) = 0 \qquad\qquad \lim_{t\to+\infty} h_t(H) = \frac{1}{2H}$$

6. *The function $\rho_t(H)$ is strictly increasing and*

$$\lim_{t\to 0}\rho_t(H)=0 \qquad \lim_{t\to+\infty}\rho_t(H)=+\infty \qquad \lim_{t\to+\infty}\rho_t(H)-t=\frac{1}{2H}$$

The two limits of $h_t(H)$ and $\rho_t(H)$ as $t\to+\infty$ allow us to consider P. Collin result as a limiting case of R. López theorem. Actually, when R goes to $+\infty$, the uniform R-circle under condition for f becomes the convexity since the circle becomes a straight-line.

2 Preliminaries

2.1 The maximal and minimal solutions

To prove the uniqueness of the solutions built by P. Collin and R. López, we need some control on solutions of these Dirichlet problem. We have a first result.

Lemma 2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. On $\Omega=\mathbb{R}\times(-l,l)$, there exists a solution w of the minimal surface equation*

$$\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right)=0 \tag{MSE}$$

with $w=\varphi_f$ on the boundary of Ω. Besides, we have $w\geq u$ for every solution u of (CMC) *on Ω with $u=\varphi_f$ on $\partial\Omega$.*

Proof. Let us consider $n\in\mathbb{N}^*$. Because of a result by H. Jenkins and J. Serrin [JS], if n is big enough, there exists two solutions w_n^+ and w_n^- of (MSE) on $(-n,n)\times(-l,l)$ with $w_n^\pm=\varphi_f$ on $(-n,n)\times\{-l,l\}$ and $w_n^\pm=\pm\infty$ on $\{-n,n\}\times(-l,l)$.

By maximum principle, for every n and m, we have $w_n^+\geq w_m^-$ and (w_n^+) is a decreasing sequence. This implies that (w_n^+) converges to a solution w of (MSE) on Ω with φ_f as boundary value.

Let us consider now a solution u of (CMC) on Ω with φ_f as boundary value. By maximum principle, for every n, we have $w_n^+\geq u$; then in the limit, $w\geq u$. $\square$

This lemma gives an upper-bound to a solution of (CMC) without any hypothesis on the function f. To get an under-bound, we need such hypotheses.

Let us consider the function c which is defined on $\Omega=\mathbb{R}\times(-1/(2H),1/(2H))$ by:

$$c(x,y)=-\frac{1}{\cos\theta}\sqrt{\frac{1}{4H^2}-y^2}+(x-x_0)\tan\theta+z_0$$

c is a solution of (CMC): its graph is in fact the half-cylinder with the two straight-lines of equation $z=(x-x_0)\tan\theta+z_0$ over $\partial\Omega$ as boundary.

Lemma 3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function and u a solution of* (CMC) *on $\Omega=\mathbb{R}\times(-1/(2H),1/(2H))$ with φ_f as boundary value. Let x_0 be in $\mathbb{R}$ and $z=(x-x_0)\tan\theta_0+f(x_0)$ be a straight-line which is below the graph of f (such a line exists because of the convexity). Let c denote the half-cylinder associated to this line. Then we have $u\geq c$ on Ω.*

Proof. Let h denote the function defined on Ω by $h(x,y)=(x-x_0)\tan\theta_0+f(x_0)$. On the boundary $u\geq h$.

If the function f is affine, *i.e.* $f(x)=(x-x_0)\tan\theta_0+f(x_0)$, it is known that c is the only constant mean curvature extension for φ_f (see Theorem 8 in [Ma2]), then $u=c$.

If the function f is non-affine, the set of θ such that there exists $x_1\in\mathbb{R}$ with $z=(x-x_1)\tan\theta+f(x_1)$ is below the graph of f is an interval $I\subset\mathbb{R}$. We assume that θ_0 is in the interior of this interval. If θ_0 is an end-point of this interval, the property is proved by continuity.

Since θ_0 is in the interior of I, there exist $x_1<x_0<x_2$ and $\theta_1<\theta_0<\theta_2$ such that $(x-x_1)\tan\theta_1+f(x_1)\leq f$ and $(x-x_2)\tan\theta_2+f(x_2)\leq f$. By Proposition 3 in [Ma2], there exists $K\in\mathbb{R}_+$ such that:

$$\begin{aligned}u(x,y)&\geq(x-x_1)\tan\theta_1+f(x_1)-K\\ u(x,y)&\geq(x-x_2)\tan\theta_2+f(x_2)-K\end{aligned}$$

Since $\theta_1<\theta_0<\theta_2$, these two equations imply that $u(x,y)\geq h(x,y)$ if $|x|$ is big enough. We have $h\geq c$ on Ω (we recall that c is the half-cylinder associated to $z=(x-x_0)\tan\theta_0+f(x_0)$); then $u\geq c$ on $\partial\Omega$ and outside a compact of $\overline{\Omega}$. Finally, by maximum principle, $u\geq c$ in Ω. □

In the case of López solutions, we get the following under-bound.

Lemma 4. *Let $f:\mathbb{R}\to\mathbb{R}$ a continuous function that satisfies a uniform $\rho_t(H)$-circle under condition. Let x be in $\mathbb{R}$ and $\mathcal{C}$ a circle of radius $\rho_t(H)$ that established the uniform $\rho_t(H)$-circle under condition at the point $(x,f(x))$. Let u be a solution of* (CMC) *on $\Omega=\mathbb{R}\times(-h_t(H),h_t(H))$ with φ_f as boundary value. Then, the graph of u is above the nodoid $\mathcal{N}_t$ which have* *horizontal axis and is bounded by the two parallel circles $\mathcal{C}$ in the vertical plane $y = -h_t(H)$ and $y = h_t(H)$.*

Proof. Let e_z denote the vertical unit vector $(0,0,1)$. For s in $\mathbb{R}$, let us translate by se_z the nodoid $\mathcal{N}_t$ bounded by the two parallel circles $\mathcal{C}$. For enough negative s, $\mathcal{N}_t + se_z$ is below the graph of u. Let s grow until the first contact. The mean curvature of the graph is upward pointing and the mean curvature of $\mathcal{N}_t$ points outside. So by maximum principle, the first contact can not be an interior point. Then, because of the hypothesis on f, the first contact is at $s = 0$ and the lemma is proved. $\square$

These estimates have a lot of important consequences in our study of the uniqueness. First, it gives us a technical lemma.

Lemma 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We consider one of the two cases below:*

1. $\Omega = \mathbb{R} \times (-\frac{1}{2H}, \frac{1}{2H})$ *and f is convex.*
2. $\Omega = \mathbb{R} \times (-h_t(H), h_t(H))$ *and f satisfies a uniform $\rho_t(H)$-circle under condition.*

Let $\mathcal{D}$ denote the set of all solutions u of (CMC) *on Ω with φ_f as boundary value. Let u_1 and u_2 be in $\mathcal{D}$ then there exist v^+ and v^- in $\mathcal{D}$ such that:*

$$v^+ \geq \max(u_1, u_2) \qquad\qquad v^- \leq \min(u_1, u_2)$$

Proof. Let n be in $\mathbb{N}$; we define $\Omega_n = \{(x,y) \in \Omega \mid -n - \sqrt{1/(2H)^2 - y^2} \leq x \leq n + \sqrt{1/(2H)^2 - y^2}\}$. The boundary of Ω is composed of two segments and two circle-arcs of curvature $2H$.

Using Perron process (see [CH, GT]), we can build

- the solution v_n^+ of (CMC) on Ω_n with $\max(u_1, u_2)$ on the boundary and
- the solution v_n^- of (CMC) on Ω_n with $\min(u_1, u_2)$ on the boundary.

To build v_n^+, we use subsolutions (let us observe that $\max(u_1, u_2)$ is a subsolution). By maximum principle, every subsolution is less than w the solution of (MSE) given by Lemma 2. Then, we can define v_n^+ as the supremum over all subsolutions. v_n^+ takes the good boundary values on the two segments because $\max(u_1, u_2) = w$ on it. For the two circle-arcs, we use the barrier functions built by J. Serrin in [Se1]. For v_n^-, we use supersolutions

($\min(u_1, u_2)$ is one). By maximum principle, every supersolution satisfies to the under-bound of Lemmas 3 or 4. Then we define v_n^- as the infimum of all supersolutions. The half-circles and nodoids of Lemmas 3 and 4 are used as barrier functions and give us the boundary value of v_n^- on the two segments. For the two circle arcs, we use J. Serrin arguments.

On Ω_n, we have $\max(u_1, u_2) \leq v_n^+ \leq w$ then a subsequence converges to v^+ on Ω and $v^+ \in \mathcal{D}$. Clearly $\max(u_1, u_2) \leq v^+$. The sequence v_n^- is upper-bounded by $\min(u_1, u_2)$ and satisfies the under-bounds of Lemmas 3 or 4. Then a subsequence converges to v^- a solution of (CMC) with φ_f as boundary value. Besides, $\min(u_1, u_2) \geq v^-$ □

With this Lemma, we can prove:

Proposition 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We consider one of the two cases below:*

1. $\Omega = \mathbb{R} \times (-\frac{1}{2H}, \frac{1}{2H})$ *and f is convex.*
2. $\Omega = \mathbb{R} \times (-h_t(H), h_t(H))$ *and f satisfies a uniform $\rho_t(H)$-circle under condition.*

There exist u_{max} and u_{min} two solutions of (CMC) *on Ω with φ_f as boundary value such that, for every solution u of* (CMC) *on Ω with φ_f as boundary value, we have:*

$$u_{min} \leq u \leq u_{max}$$

Proof. Let us denote $\mathcal{D}$ the set of all solutions u of (CMC) on Ω with φ_f as boundary value; thanks to P. Collin and R. López, $\mathcal{D}$ is non-empty. So we define u_{max} and u_{min} at $p \in \Omega$ by:

$$u_{max}(p) = \sup_{u \in \mathcal{D}} u(p)$$
$$u_{min}(p) = \inf_{u \in \mathcal{D}} u(p)$$

By Lemma 2, u_{max} is well defined ; Lemmas 3 and 4 ensure that $u_{min} > -\infty$. As in the classiccal Perron process, it can be proved that u_{max} and u_{min} are two solutions of (CMC) on Ω: in fact the argument we need is that for every u_1 and u_2 in $\mathcal{D}$ there exist $u_3 \in \mathcal{D}$ that upper-bounds $\max(u_1, u_2)$ and $u_4 \in \mathcal{D}$ that under-bounds $\min(u_1, u_2)$ (this is Lemma 5).

Using the solution w of (MSE) built in Lemma 2, the half-cylinders of Lemma 3 or the nodoids of Lemma 4 as barrier functions, we finally prove that u_{max} and u_{min} have φ_f as boundary value. Besides the construction gives , for every $u \in \mathcal{D}$:

$$u_{min} \leq u \leq u_{max}$$

□

We have an important remark on these two solutions. For every $(x, y) \in \Omega$, they satisfy :

$$u_{max}(x, y) = u_{max}(x, -y) \tag{2}$$

$$u_{min}(x, y) = u_{min}(x, -y) \tag{3}$$

This is due to the fact that both functions $(x, y) \mapsto u_{max}(x, -y)$ and $(x, y) \mapsto u_{min}(x, -y)$ are in $\mathcal{D}$.

2.2 Upper-bounds

In this subsection, we look for explicit upper-bounds for solutions of (CMC). First, we have the following upper-bound:

Proposition 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a contiuous function and $x_0 \in \mathbb{R}$. We assume that f is monotonous on $[x_0, +\infty)$. Let u be a solution of* (CMC) *on $\Omega = \mathbb{R} \times (-a, a)$ with φ_f as boundary value. Then for $x \geq x_0 + 1/H$, we have:*

$$u(x, y) \leq f(x) + \frac{1}{2H}$$

Proof. We only consider the case where f is increasing on $[x_0, +\infty)$. We consider $a \geq x_0 + 1/H$ and denote by $C((a - 1/(2H), s), 1/(2H))$ the horizontal cylinder of axis $\{x = a - 1/(2H)\} \cap \{z = s\}$ and radius $1/(2H)$. For big s the cylinder $C((a - 1/(2H), s), 1/(2H))$ is above the graph of u. Let s decrease until s_0 where the first contact happens. By maximum principle, this first contact point is on the boundary at a point of first coordinate $a' \in [a - 1/(2H), a]$. We have $f(a') \geq s_0 - 1/(2H)$.

Since for every $s \geq s_0$, $C((a - 1/(2H), s), 1/(2H))$ is above the graph of u, we have $u(a, y) \leq s$. Then $u(a, y) \leq s_0 \leq f(a') + 1/(2H)$. Since $a' < a$ and f is increasing, $u(a, y) \leq f(a) + 1/(2H)$. □

Let us introduce a definition. Let $f : \mathbb{R} \to \mathbb{R}$ be a continous function. f satifies *a R-circle upper condition at* $a \in \mathbb{R}$ if there exists in $\{(x, y) \in \mathbb{R}^2 |\, y \geq f(x)\}$ a disk D with radius R such that $(a, f(a)) \in \partial D$.

Remark. Let $D((a,s),R)$ denote the disk with center (a,s) and radius R. For big s, $D((a,s),R)$ is included in $\{(x,y)\in\mathbb{R}^2|\, y\geq f(x)\}$. Let s decrases until the first contact with the graph of f, f then satifies a R-circle upper condition at the first coordinates of the contact points. In changing a, we get all the abscissas where f satifies a R-circle upper condition. This implies that for every $a\in\mathbb{R}$ there is $a'\in[a-R,a+R]$ where f satisfies a R-circle upper condition.

Proposition 8. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Let u be a solution of* (CMC) *on $\Omega=\mathbb{R}\times(-l,l)$ with φ_f as boundary value. We assume the f satisfies a $1/(2H)$-circle upper condition at $x_0\in\mathbb{R}$. Then, for every $y\in[-l,l]$, $u(x_0,y)\leq f(x_0)$.*

Proof. Let $\Gamma(a,b)$ denote the circle of center (a,b) and radius $1/(2H)$ which belongs to $\{z\geq f(x)\}$ and such that $(x_0,f(x_0))\in\Gamma(a,b)$. Let us denote by $C((a,b+s),1/(2H))$ the horizontal cylinder of axis $\{x=a\}\cap\{z=b+s\}$ and radius $1/(2H)$. For big s the cylinder $C((a-1/(2H),s),1/(2H))$ is above the graph of u. Let s decrease until the first contact happens. Because of maximum principle and the existence of $\Gamma(a,b)$, this first contact happens for $s=0$. Then, on the segment $I_{x_0}=\{x_0\}\times[-l,l]$, u is upper-bounded by $f(x_0)$. □

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function that satisfies a uniform R-circle under condition. Let $a\in\mathbb{R}$ denote a point where f satifies a R'-circle upper condition. Since at a, there is a circle below and over the graph of f, the graph of f has a tangent. Then either $f'(a)$ exists or $f'(a)=\pm\infty$. In all the cases, we can deal with the sign of the derivative of f at a. We then have a kind of Rolle's Theorem.

Lemma 9. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function that satisfies a uniform R-circle under condition. Let $a<b$ be two points where f satifies a R'-circle upper condition. We assume that $f'(a)>0$ and $f'(b)<0$. Then there exists $c\in[a,b]$ such that:*

1. *f satifies a R'-circle upper condition at c.*
2. *$f'(c)=0$.*

Proof. Let g denote the function defined by $g(x)=R'-\sqrt{R'^2-x^2}$ on $[-R',R']$; its graph is a half-circle of radius R'. Since f satifies a R'-circle upper condition at a and $f'(a)>0$, f is upper bounded by $f(a)+g(x-a)$ on $[a-R',a]$. In the same way, f is upper-bounded by $f(b)+g(x-b)$ on

$[b, b+R']$. Let $c \in [a,b]$ denote a point where $f(c) = \max_{[a,b]} f$. Then $f(x)$ is upper-bounded by $m(x)$ on $[a-R', b+R']$ where $m(x)$ is defined by:

$$m(x) = \begin{cases} f(c) + g(x-a) & \text{for } x \in [a-R', a] \\ f(c) & \text{for } x \in [a,b] \\ f(c) + g(x-b) & \text{for } x \in [b, b+R'] \end{cases}$$

This implies that f satisfies a R'-circle upper condition at c and then $f'(c) = 0$. □

3 The uniqueness of Collin's solutions

The aim of this section is to prove the uniqueness of the solutions for the Dirichlet problem studied by P. Collin in [Co]. More precisely, we have the following result.

Theorem 10. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Let u and v be two solutions of* (CMC) *on $\Omega = \mathbb{R} \times (-1/(2H), 1/(2H))$ with φ_f as boundary value. Then $u = v$.*

The proof of Theorem 10 is long, so the rest of the section is devoted to it. In this proof, we shall use the differential 1-form ω_u. If u is a function on a domaine of $\mathbb{R}^2$, ω_u is defined by :

$$\omega_u = \frac{u_x}{\sqrt{1+|\nabla u|^2}}\mathrm{d}y - \frac{u_y}{\sqrt{1+|\nabla u|^2}}\mathrm{d}x$$

with u_x and u_y the two first derivatives of u. When u is a solution of (CMC), ω_u satisfies $\mathrm{d}\omega_u = 2H\mathrm{d}x \wedge \mathrm{d}y$ (see [Sp]).

3.1 Preliminaries

By Proposition 6, there are two solutions u_{min} and u_{max} of (CMC) on Ω with φ_f as boundary value such that, for every solution u of the same Dirichlet problem, $u_{min} \le u \le u_{max}$. Then to prove the uniqueness, it is sufficient to prove : $u_{min} = u_{max}$.

So let us assume that $u_{min} \neq u_{max}$; it is then known that $u_{max} - u_{min}$ is unbounded on Ω [CK]. By exchanging x with $-x$, we can assume that:

$$\lim_{x\to+\infty} \max_{I_x}(u_{max} - u_{min}) = +\infty \tag{4}$$

where $I_x = \{x\} \times [-1/(2H), 1/(2H)]$. Let c denote $\max_{I_0} u_{max} - u_{min}$. Then there exists $\mathcal{D}$ a connected component of $\{u_{max} \geq u_{min} + 2c\}$ that is included in $\mathbb{R}_+ \times [-1/(2H), 1/(2H)]$. $\mathcal{D}$ is unbounded.

Since f is convex, f has a left derivative f'_l and a right derivative f'_r at every point. These two functions increase and have the same limit at $+\infty$. If $\lim_{+\infty} f'_l = \lim_{+\infty} f'_r < +\infty$, f is lipschitz continuous on $\mathbb{R}_+$. Then Theorem 5 in [Ma2] contradicts (4).

Then f must satisfy

$$\lim_{+\infty} f'_l = \lim_{+\infty} f'_r = +\infty \tag{5}$$

3.2 Asymptotic behaviour of u_{min}

Let (x_n) be a real sequence with $\lim x_n = +\infty$. Let us define u_n on Ω by $u_n(x, y) = u_{min}(x + x_n, y)$. For $a \in \mathbb{R}$, let us denote by $C^+(a)$ the circle arc $\{x \geq a\} \cap \{(x - a))^2 + y^2 = 1/(4H^2)\}$. This circle-arc has $(a, -1/(2H))$ and $(a, 1/(2H))$ as end-points. Besides $C^+(a)$ contains the point $(a + 1/(2H), 0)$. We then have the following result.

Lemma 11. *There exists (x_n) a real increasing sequence with $\lim x_n = +\infty$ such that (u_n) has $C^+(0)$ as line of divergence.*

Before the proof, let us recall what is a line of divergence. We refer to [Ma1] for the details. Let (v_n) be a sequence of solutions of (CMC) and N_n denote the upward pointing normal to the graph of v_n. Let us assume that $N_n(P)$ tends to a horizontal unit vector $(\nu, 0)$ ($\nu \in \mathbb{S}^1$). Let C denote the circle-arc in the xy-plane with radius $1/(2H)$ such that $P \in C$ and $2H\nu$ is the curvature vector of C at P. C is then a line of divergence of the sequence (v_n). Let us extend the definition of ν along C by $2H\nu(Q)$ is the curvature vector of C at $Q \in C$ ($\nu(Q) \in \mathbb{S}^1$). Then, for every $Q \in C$, $N_n(Q) \to (\nu(Q), 0)$. This implies that for every C' a subarc of C:

$$\lim_{n \to +\infty} \int_{C'} \omega_u = \ell(C')$$

with $\ell(C')$ the length of C'. C' is oriented such that ν is left-hand side pointing along C'.

Proof of Lemma 11. Let v_n be defined on Ω by $v_n(x, y) = u_{min}(x + n, y)$. The boundary value of v_n is φ_{f_n} with $f_n(x) = f(x + n)$. Because of (5), f_n is increasing on $[-1/H, +\infty)$ for big n. Then, by Proposition 7, $v_n(0, 0) \leq$

$f_n(0)+1/(2H)$. Now, let $\theta_n \in [0,\pi/2)$ such that ${f_n}'_l(0) \le \tan\theta_n \le {f_n}'_r(0)$. By Lemma 3:

$$v_n(\frac{1}{H},0) \ge -\frac{1}{\cos\theta_n}\sqrt{\frac{1}{4H^2}}+\frac{1}{H}\tan\theta_n + f_n(0)$$

Because of (5), $\theta_n \to \pi/2$. Then:

$$v_n(\frac{1}{H},0)-v_n(0,0) \ge \frac{1}{H\cos\theta_n}(\sin\theta_n-\frac{1}{2})-\frac{1}{2H} \xrightarrow[n\to+\infty]{} +\infty$$

Then the sequence of derivatives $\frac{\partial v_n}{\partial x}$ can not stay upper-bounded on $[0,1/H]\times\{0\}$. Then there exists a sequence (a_n) in $[0,1/H]$ such that:

$$\lim \frac{\partial v_n}{\partial x}(a_n,0)=+\infty \tag{6}$$

Let x_n be defined by $n+a_n-1/(2H)$, we remak that $\lim x_n=+\infty$. We consider (u_n) the sequence of solution of (CMC) associated to (x_n). (6) becomes:

$$\lim \frac{\partial u_n}{\partial x}(\frac{1}{2H},0)=+\infty$$

Since $\frac{\partial u_n}{\partial y}(1/(2H),0)=0$ by (3), the limit normal to the sequence of graphs over $(1/(2H),0)$ is $(-1,0,0)$. Then $C^+(0)$ is a line of divergence for (u_n). In considering a subsequence of (x_n), we can assume that it is increasing; this ends the proof. $\square$

3.3 End of Theorem 10 proof

Let (x_n) be a sequence given by Lemma 11. Let $\mathcal{D}_n$ denote the following intersection:

$$\mathcal{D}_n=\mathcal{D}\bigcap\left\{(x,y)\in\Omega \,|\, x\le x_n+\sqrt{\frac{1}{4H^2}-y^2}\right\}$$

The boundary of $\mathcal{D}_n$ is composed of $\partial\mathcal{D}\cap\mathcal{D}_n$ and Γ_n which is the part included in the circle-arc $C^+(x_n)$ (see Figure 2). Let $\widetilde{\omega}$ denote $\omega_{u_{max}}-\omega_{u_{min}}$; we then have:

$$0=\int_{\partial\mathcal{D}_n}\widetilde{\omega}=\int_{\partial\mathcal{D}\cap\mathcal{D}_n}\widetilde{\omega}+\int_{\Gamma_n}\widetilde{\omega}$$

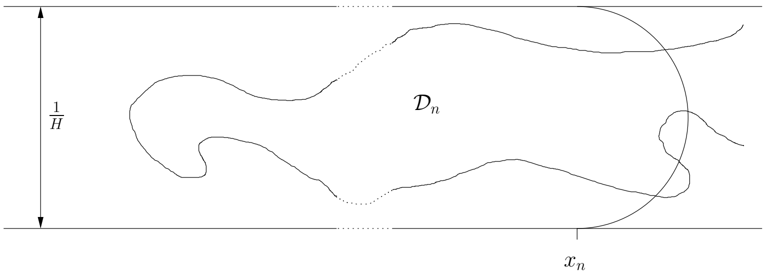

Figure 2:

Thanks to Lemma 2 in [CK], the integral on $\partial\mathcal{D}\cap\mathcal{D}_n$ is negative; besides, since (x_n) is increasing, it decreases when n is increasing. Besides we have:

$$0 < -\int_{\partial\mathcal{D}\cap\mathcal{D}_n} \widetilde{\omega} = \int_{\Gamma_n} \widetilde{\omega} \leq 2\ell(\Gamma_n)$$

where $\ell(\Gamma_n)$ denote the length of Γ_n. Then $\ell(\Gamma_n)$ is far from 0 uniformaly under-bounded. Because of Lemma 11 and since $\Gamma_n \subset C^+(x_n)$, there exists (α_n) a sequence in $[0,1]$ such that $\lim \alpha_n = 1$ and

$$\int_{\Gamma_n} \omega_{u_{min}} \geq \alpha_n \ell(\Gamma_n)$$

Finally, for $n \geq n_0 > 0$, we have:

$$\begin{aligned} 0 < -\int_{\partial\mathcal{D}\cap\mathcal{D}_{n_0}} \widetilde{\omega} \leq -\int_{\partial\mathcal{D}\cap\mathcal{D}_n} \widetilde{\omega} &= \int_{\Gamma_n} \omega_{u_{max}} - \int_{\Gamma_n} \omega_{u_{min}} \\ &\leq \ell(\Gamma_n) - \alpha_n\ell(\Gamma_n) \\ &\leq (1-\alpha_n)\ell(\Gamma_n) \xrightarrow[n\to+\infty]{} 0 \end{aligned}$$

Then we have a contradiction and Theorem 10 is proved. □

4 The uniqueness of López's solutions

In this section, we prove the uniqueness of the solutions for the Dirichlet problem studied by R. López in [Lo1]. More precisely, we have the following theorem.

Theorem 12. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies a uniform $\rho_t(H)$-circle under condition. Let u and v be two solutions of* (CMC) *on $\Omega = \mathbb{R} \times (-h_t(H), h_t(H))$ with φ_f as boundary value. Then $u = v$.*

The proof of this theorem is very similar to the one of Theorem 10. The following of the section is devoted to it.

4.1 Preliminaries

By Proposition 6, there are two solutions u_{min} and u_{max} of (CMC) on Ω with φ_f as boundary value such that, for every solution u of the same Dirichlet problem, $u_{min} \leq u \leq u_{max}$. Then to prove the uniqueness, it is sufficient to prove : $u_{min} = u_{max}$.

So let us assume that $u_{min} \neq u_{max}$; it is then known that $u_{max} - u_{min}$ is unbounded on Ω. By exchanging x with $-x$, we can assume that:

$$\lim_{x\to+\infty} \max_{I_x}(u_{max} - u_{min}) = +\infty \tag{7}$$

where $I_x = \{x\} \times [-h_t(H), h_t(H)]$. Let c denote $\max_{I_0} u_{max} - u_{min}$. Then there exists $\mathcal{D}$ a connected component of $\{u_{max} \geq u_{min}+2c\}$ that is included in $\mathbb{R}_+ \times [-h_t(H), h_t(H)]$. $\mathcal{D}$ is unbounded.

Equation (7) has consequences. First, the existence of two different solutions implies:

Lemma 13. *There exists $x_0 \in \mathbb{R}_+$ such that f is monotonous on $[x_0, +\infty)$.*

Proof. Let us consider the set $\mathcal{S}$ of points where f satisfies a $1/(2H)$-circle upper condition. From a remark in Section 2.2, $\mathcal{S}$ is non-empty and is unbounded. Let us recall that, for every point in $\mathcal{S}$, we can deal with the sign of the derivative of f. First, we prove that there exists $x_1 \in \mathbb{R}^+$ such that, for every $x \in \mathcal{S} \cap [x_1, +\infty)$, the sign of $f'(x)$ is constant. If it is not true there exists two sequences (a_n) and (b_n) in $\mathcal{S}$ such that:

- $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to+\infty} b_n = +\infty$
- $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \cdots$
- for every n, $f'(a_n)$ is positive and $f'(b_n)$ is negative.

Because of Lemma 9, there exists a sequence (c_n) in $\mathcal{S}$ such that $a_n < c_n < b_n$ and $f'(c_n) = 0$. Let u denote a solution of (CMC) on Ω with φ_f as boundary value. By Proposition 8, $\max_{I_{c_n}} u \leq f(c_n)$. Besides, since $f'(c_n) = 0$, Lemma 4 implies that $\min_{I_{c_n}} u \geq f(c_n) - (\rho_t(H) - t)$.

So this implies that

$$\max_{I_{c_n}}(u_{max} - u_{min}) \leq (\rho_t(H) - t)$$

Since $\lim c_n = +\infty$, this contradicts (7). Then there exists $x_1 \in \mathbb{R}^+$ such that, for every $x \in \mathcal{S} \cap [x_1, +\infty)$, the sign of $f'(x)$ is constant. We assume in the following that these derivatives are positive.

If there is no x_0 such that f increases on $[x_0, +\infty)$, there is a sequence $a_n \in [x_1, +\infty)$ such that:

- (a_n) increases and $\lim a_n = +\infty$
- for every a_n, $f(a_n)$ is a local maximum of f.

Since f satisfies a $\rho_t(H)$-circle under condition, we remark that f is differentiable at every a_n. Let $\Gamma((a_n - 1/(2H), s)$ denote the circle of center $(a_n - 1/(2H), s)$ and radius $1/(2H)$. For big s, $\Gamma((a_n - 1/(2H), s)$ is above the graph of f. Let s decrease until s_0 where the first contact happens. We get a point x where f satisfies a $1/(2H)$-circle upper condition. By what we proved above, $f'(x) > 0$ then x belongs to $[a_n - 1/(2H), a_n]$. Let $b_n \in [x, a_n]$ denote a point where $f(b_n) = \max_{[x,a_n]} f$. Since $f'(x) > 0$, $b_n \in (x, a_n]$ then $f'(b_n) = 0$. Using horizontal cylinders with $\Gamma((a_n - 1/(2H), s)$ as vertical section, we prove that $\max_{I_{b_n}} u \leq f(x) + 1/(2H) \leq f(b_n) + 1/(2H)$ with u a solution of (CMC) on Ω with φ_f as boundary value. Besides since $f'(b_n) = 0$, $\min_{I_{b_n}} u \geq f(b_n) - (\rho_t(H) - t)$. This implies that:

$$\max_{I_{b_n}} (u_{max} - u_{min}) \leq 1/(2H) + (\rho_t(H) - t)$$

As $\lim b_n = +\infty$, the above inequation contradicts (7). The lemma is then proved. $\square$

As in the above proof, we assume in the following of Theorem 12 proof that f is increasing on some $[x_0, +\infty)$. If f decreases the argument are similar to the one we are going to give.

From Theorem 5 in [Ma2], we know that $f(x+4/H) - f(x)$ can not stay bounded when x goes to $+\infty$. We even know that:

$$\lim_{x \to +\infty} f(x + 4/H) - f(x) = +\infty \tag{8}$$

In this proof, this indentity plays the same role as (5) in the proof of Theorem 10.

4.2 The asymptotic behaviour of u_{min}

Let (x_n) be a real sequence with $\lim x_n = +\infty$. Let us define u_n on Ω by $u_n(x,y) = u_{min}(x+x_n, y)$. For $a \in \mathbb{R}$, let us denote by $C^+(a)$ the circle arc:

$$\left\{ \left(x - \left(a - \sqrt{\frac{1}{4H^2} - h_t(H)^2} \right) \right)^2 + y^2 = \frac{1}{4H^2} \right\} \bigcap \{x \geq a\}$$

This circle-arc has $(a, -h_t(H))$ and $(a, +h_t(H))$ as end-points. Besides $C^+(a)$ contains the point $(a+K, 0)$ with $K = \frac{1}{2H} - \sqrt{\frac{1}{4H^2} - h_t(H)^2}$. We then have the following result.

Lemma 14. *There exists (x_n) a real increasing sequence with $\lim x_n = +\infty$ such that (u_n) has $C^+(0)$ as line of divergence.*

Proof. Let v_n be defined on Ω by $v_n(x,y) = u_{min}(x+n, y)$. The boundary value of v_n is φ_{f_n} with $f_n(x) = f(x+n)$. For n big enough, f_n is increasing on $[1/H, +\infty)$; so, using Proposition 7, $v_n(0,0) \leq f_n(0) + 1/(2H)$. Now let us apply Lemma 4, we get that $v_n(4/H + \rho_t(H), 0) \geq f_n(4/H) - (\rho_t(H) - t)$. To get this under-bound, Lemma 4 is applied at $4/H$; the graph of v_n is then above a nodoid $\mathcal{N}_t$ with horizontal axis in the vertical plane $x = 4/H + A$ ($0 \leq A \leq \rho_t(H)$ since f increases). Since $\mathcal{N}_t$ is below the graph $v_n(4/H + A, 0) \geq f_n(4/H) - (\rho_t(H) - t)$ (see Figure 3). Now let us translate $\mathcal{N}_t$ by the horizontal vector $e_x = (1,0,0)$; since f_n is increasing, the nodoid $\mathcal{N}_t + se_x$ stays under the graph since it does not cross its boundary. Then for $s = \rho_t(H) - A$ we get $v_n(4/H + \rho_t(H), 0) \geq f_n(4/H) - (\rho_t(H) - t)$.

Then we have:

$$v_n(4/H + \rho_t(H), 0) - v_n(0,0) \geq f_n(4/H) - f_n(0) - \frac{1}{2H} - \rho_t(H) + t$$

By (8), $\lim v_n(4/H + \rho_t(H), 0) - v_n(0,0) = +\infty$. Then the sequence of derivatives $\frac{\partial v_n}{\partial x}$ can not stay upper-bounded on $[0, 4/H + \rho_t(H)] \times \{0\}$. Then there exists a sequence (a_n) in $[0, 4/H + \rho_t(H)]$ such that:

$$\lim \frac{\partial v_n}{\partial x}(a_n, 0) = +\infty \tag{9}$$

Let us recall that K denote $\frac{1}{2H} - \sqrt{\frac{1}{4H^2} - h_t(H)^2}$. Let x_n be defined by $n + a_n - K$, we remak that $\lim x_n = +\infty$. We consider (u_n) the sequence

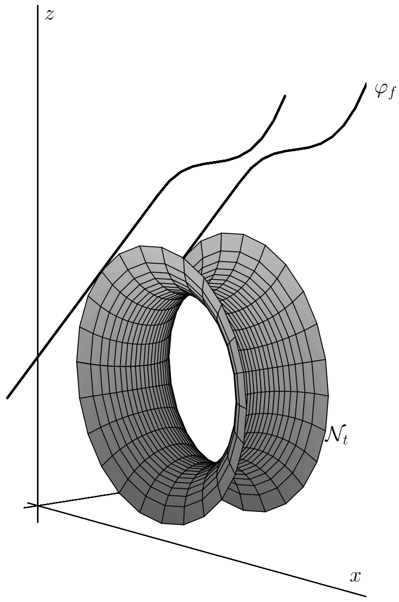

Figure 3:

of solution of (CMC) associated to (x_n). (9) becomes:

$$\lim \frac{\partial u_n}{\partial x}(K,0) = +\infty$$

Since $\dfrac{\partial u_n}{\partial y}(K,0) = 0$ by (3), the limiting normal to the sequence of graphs over $(K,0)$ is $(-1,0,0)$. Then $C^+(0)$ is a line of divergence for (u_n). In considering a subsequence of (x_n), we can assume that it is increasing; this ends the proof. $\square$

4.3 End of Theorem 12 proof

Let (x_n) be the sequence given by Lemma 14. Let $\mathcal{D}_n$ denote the following intersection:

$$\mathcal{D}_n = \mathcal{D} \cap \left\{ (x,y) \in \Omega \,|\, x \leq x_n + \sqrt{\frac{1}{4H^2} - y^2} - \sqrt{\frac{1}{4H^2} - h_t(H)^2} \right\}$$

The boundary of $\mathcal{D}_n$ is composed of $\partial\mathcal{D} \cap \mathcal{D}_n$ and Γ_n which is the part included in the circle-arc $C^+(x_n)$. Let $\widetilde{\omega}$ denote $\omega_{u_{max}} - \omega_{u_{min}}$; we then have:

$$0 = \int_{\partial\mathcal{D}_n} \widetilde{\omega} = \int_{\partial\mathcal{D}\cap\mathcal{D}_n} \widetilde{\omega} + \int_{\Gamma_n} \widetilde{\omega}$$

On $\partial\mathcal{D} \cap \mathcal{D}_n$, the integral is negative; besides, since (x_n) is increasing, it decreases when n is increasing (Lemma 2 in [CK]). Besides, we have:

$$0 < -\int_{\partial\mathcal{D}\cap\mathcal{D}_n} \widetilde{\omega} = \int_{\Gamma_n} \widetilde{\omega} \leq 2\ell(\Gamma_n)$$

where $\ell(\Gamma_n)$ denote the length of Γ_n. Then $\ell(\Gamma_n)$ is far from 0 uniformaly under-bounded. Because of Lemma 14 and since $\Gamma_n \subset C^+(x_n)$, there exists (α_n) a sequence in $[0,1]$ such that $\lim \alpha_n = 1$ and

$$\int_{\Gamma_n} \omega_{u_{min}} \geq \alpha_n \ell(\Gamma_n)$$

Finally, for $n \geq n_0 > 0$, we have:

$$\begin{aligned} 0 < -\int_{\partial\mathcal{D}\cap\mathcal{D}_{n_0}} \widetilde{\omega} \leq -\int_{\partial\mathcal{D}\cap\mathcal{D}_n} \widetilde{\omega} &= \int_{\Gamma_n} \omega_{u_{max}} - \int_{\Gamma_n} \omega_{u_{min}} \\ &\leq \ell(\Gamma_n) - \alpha_n \ell(\Gamma_n) \\ &\leq (1-\alpha_n)\ell(\Gamma_n) \xrightarrow[n\to+\infty]{} 0 \end{aligned}$$

Then we have a contradiction and Theorem 12 is proved. □

Let us explain what are the differences if we assume that f is decreasing and not increasing. In this case, we have to study the asymptotic behaviour of u_{max}. We prove that there exists a sequence (x_n) with $\lim x_n = +\infty$ such that $C^-(0)$ is line of divergence of (u_n). Here u_n is defined by $u_n(x,y) = u_{max}(x + x_n, y)$ and $C^-(a)$ denotes the circle-arc:

$$\left\{ \left(x - \left(a + \sqrt{\frac{1}{4H^2} - h_t(H)^2} \right) \right)^2 + y^2 = \frac{1}{4H^2} \right\} \bigcap \{x \leq a\}$$

With this result, we can make the computations of the end of the proof.

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