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Partial Differential Equations

Pulsating traveling fronts in space–time periodic media

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Abstract

This Note deals with the existence of pulsating traveling fronts for some reaction–diffusion equation in space–time periodic media. Under some hypotheses, there exist two speeds c^* and c^{**} such that there exist some pulsating traveling fronts of speed c for all $c \geqslant c^{**}$ and that there exists no such front of speed $c < c^*$. In the case of a KPP-type reaction term, we characterize this speed with the help of a family of eigenvalues associated with the equation. Lastly, we study the dependence between this minimal speed and the coefficients of the equation. ***To cite this article: G. Nadin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Fronts pulsatoires en milieu périodique en temps et en espace. Cette Note traite de l'existence de fronts pulsatoires pour une équation de réaction–diffusion en milieu périodique en temps et en espace. Sous certaines hypothèses, il existe deux vitesses c^* et c^{**} telles qu'il existe des fronts pulsatoires de vitesse c pour tout $c \geqslant c^{**}$ et qu'il n'existe pas de tel front de vitesse $c < c^*$. Dans le cas d'un terme de réaction de type KPP, nous caractérisons cette vitesse à l'aide d'une famille de valeurs propres associée à l'équation. Enfin, nous étudions la dépendance entre cette vitesse minimale et les coefficients de l'équation. ***Pour citer cet article : G. Nadin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Cette Note étudie l'équation :

$$\partial_t u - \nabla \cdot \big(A(t,x)\nabla u\big) + q(t,x) \cdot \nabla u = f(t,x,u), \tag{1}$$

où la matrice de diffusion A, le terme d'advection q et le terme de réaction f sont périodiques en t et en x. Cette équation intervient dans des modèles de dynamique des populations, de combustion et de génétique.

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Le terme de réaction f représente le taux d'accroissement global. On suppose que $f(t,x,0)=0$ et que

$$\sup_{(t,x,s)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{+*}} \frac{f(t,x,s)}{s} < +\infty.$$

On suppose que l'Éq. (1) admet une solution p continue, positive et périodique en temps et en espace, et que si q est une autre solution périodique en espace telle que $q \leqslant p$ et $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} q(t,x) > 0$, alors $u \equiv p$.

L'opérateur linéarisé au voisinage de 0 est défini par

$$\mathcal{L}\varphi = \partial_t\varphi - \nabla\cdot\big(A(t,x)\nabla\varphi\big) + q(t,x)\cdot\nabla\varphi - d_u f(t,x,0)\varphi.$$

Il a été montré dans [9] que pour tout $\lambda\in\mathbb{R}^N$, il existe une valeur propre principale périodique k_λ définie par l'existence d'une fonction positive φ périodique en temps et en espace telle que $\mathcal{L}(e^{\lambda\cdot x}\varphi) = k_\lambda e^{\lambda\cdot x}\varphi$. On note $\lambda_1' = k_0$ et on suppose que $\lambda_1' < 0$, ce qui veut dire que l'état 0 est linéairement instable par rapport aux perturbations périodiques. Si f ne dépend pas de t et x, celà revient à supposer que $f'(0) < 0$.

On dit qu'une fonction u est un *front pulsatoire* de vitesse c dans la direction $-e$ reliant 0 à p si on peut écrire $u(t,x) = \phi(x\cdot e + ct, t, x)$, où le *profil* $\phi \in L^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N)$ est périodique dans sa deuxième et sa troisième variable, où $\phi(z,t,x)$ converge uniformément vers 0 quand $z\to-\infty$ et vers p quand $z\to+\infty$ et où pour presque tout $y\in\mathbb{R}^N$, la fonction $(t,x)\mapsto\phi(y+x\cdot e+ct,t,x)$ est une solution de l'Éq. (1).

Théorème 0.1. *Sous les hypothèses précédentes, pour tout vecteur unité e, il existe une vitesse c_e^{**} telle que pour tout $c \geqslant c_e^{**}$, il existe un front pulsatoire croissant en z de vitesse c. De plus, il existe une vitesse minimale c_e^* telle qu'il existe un front pulsatoire u de vitesse c_e^*, et pour toute vitesse $c < c_e^*$, il n'existe pas de tel front.*

Enfin, si $f(t,x,s) \leqslant d_u f(t,x,0)s$ pour tout $(t,x,s)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^+$, alors

$$c_e^* = c_e^{**} = \min_{\lambda>0}\frac{-k_{\lambda e}}{\lambda}. \tag{2}$$

La caractérisation (2) permet d'étudier en détail la dépendance entre les coefficients (A,q,f) et la vitesse minimale c_e^* (voir [10]) quand $f(t,x,s) \leqslant d_u f(t,x,0)s$ pour tout $(t,x,s)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^+$. Par exemple, le résultat suivant donne la réponse à des questions laissées en suspens dans [4] :

Théorème 0.2. *On suppose que A est une matrice constante.*

(1) *Si $\int_{(0,T)\times C}\mu(t,x)\,\mathrm{d}t\,\mathrm{d}x \geqslant 0$, on a* :

$$\frac{c^*(\gamma A, 0, d_u f(t,x,0))}{\sqrt{\gamma}} \to 2\sqrt{\frac{eAe}{T|C|}\int_{(0,T)\times C}\mu(t,x)\,\mathrm{d}t\,\mathrm{d}x} \quad \textit{quand } \gamma\to+\infty.$$

(2) *Si $\max_{x\in\mathbb{R}^N}(\int_0^T d_u f(t,x,0)\,\mathrm{d}t) \geqslant 0$, alors* :

$$\frac{c^*(\gamma A, 0, d_u f(t,x,0))}{\sqrt{\gamma}} \to 2\sqrt{\frac{eAe}{T}\max_{x\in\mathbb{R}^N}\left(\int_0^T d_u f(t,x,0)\,\mathrm{d}t\right)} \quad \textit{quand } \gamma\to 0.$$

(3) *On peut trouver une fonction f et deux constantes $\gamma > \gamma' > 0$ telles que* :

$$c^*(\gamma I_N, 0, \mu) < c^*(\gamma' I_N, 0, \mu).$$

Ceci signifie que lorsque la diffusion est très grande, la population se disperse trop vite et ne voit plus l'hétérogénéité. Au contraire, quand la diffusion est faible, la vitesse de propagation de la population ne dépend que du maximum du taux de reproduction. La plupart de ces résultats sont prouvés dans [10].

1. Hypotheses

This Note is concerned with the equation:

$$\partial_t u - \nabla \cdot \big(A(t,x)\nabla u\big) + q(t,x)\cdot \nabla u = f(t,x,u). \tag{3}$$

We assume that there exists $T, L_1, \dots, L_N > 0$ such that for all i:

$$\begin{aligned} &A(t+T,x) = A(t,x), \qquad q(t+T,x) = q(t,x), \qquad f(t+T,x,s) = f(t,x,s),\\ &A(t,x+L_ie_i) = A(t,x), \qquad q(t,x+L_ie_i) = q(t,x), \qquad f(t,x+L_ie_i,s) = f(t,x,s), \end{aligned} \tag{4}$$

where $(e_1,\dots,e_N)$ is some orthonormal basis of $\mathbb{R}^N$.

The function $f:\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^+\to\mathbb{R}$ is supposed to be of class $C^{\frac{\delta}{2},\delta}$ in (t,x) locally in u for a given $0<\delta<1$ and locally Lipschitz-continuous in u and of class C^2 on $\mathbb{R}\times\mathbb{R}^N\times[0,\beta]$ for some given $\beta>0$. We assume that 0 is a state of equilibrium: $\forall x, \forall t,\ f(t,x,0)=0$. We also ask that the growth rate be bounded:

$$\forall t,x, \quad \eta(t,x) = \sup_{s>0}\frac{f(t,x,s)}{s} < +\infty. \tag{5}$$

The matrix field A is supposed to be of class $C^{\frac{\delta}{2},1+\delta}$ and is uniformly elliptic and continuous: it exist some positive constants γ and Γ such that for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, $\gamma I_N \leqslant A(t,x) \leqslant \Gamma I_N$. The drift term $q:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is supposed to be of class $C^{\frac{\delta}{2},\delta}$.

Take any periodic parabolic operator $\mathcal{L}$ defined by:

$$\mathcal{L}\varphi = \partial_t\varphi - \nabla\cdot\big(A(t,x)\nabla\varphi\big) + q(t,x)\cdot\nabla\varphi - \zeta(t,x)\varphi.$$

It has been proved in [9] that for all $\lambda\in\mathbb{R}^N$, there exists a periodic principal eigenvalue $k_\lambda = k_\lambda(A,q,\zeta)$ which is defined by the existence of a space–time periodic positive eigenfunction φ_λ such that $\mathcal{L}(e^{\lambda\cdot x}\varphi_\lambda) = k_\lambda e^{\lambda\cdot x}\varphi_\lambda$. We associate to these families the quantity:

$$c_e^*(A,q,\zeta) = \min_{\lambda>0}\frac{-k_{\lambda e}(A,q,\zeta)}{\lambda}.$$

In the sequel, the quantities $c_e^*(A,q,\mu)$ and $c_e^*(A,q,\eta)$, where $\mu(t,x) = d_u f(t,x,0)$ is the derivative of f with respect to s in 0 and η is defined by (5), will be involved.

We denote $\lambda_1' = k_0(A,q,\mu)$ and we assume that $\lambda_1' < 0$, which means that the state 0 is linearly unstable with respect to periodic perturbations. If f does not depend on (t,x), this means that $f'(0)>0$.

We only make one strong hypothesis. Namely, we assume that Eq. (3) admits a positive continuous space–time periodic solution p, and that if q is a space periodic solution of Eq. (3) such that $q\leqslant p$ and $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} q(t,x)>0$, then $u\equiv p$. It is not easy to check that this hypothesis is fulfilled.

In [8], we proved that if $\lambda_1'<0$ and

$$\forall (t,x)\in\mathbb{R}\times\mathbb{R}^N, \quad s\mapsto\frac{f(t,x,s)}{s} \text{ is decreasing,} \tag{6}$$

$$\exists M>0, \forall x\in\mathbb{R}^N, \forall t\in\mathbb{R}, \forall s\geqslant M, \quad f(t,x,s)\leqslant 0, \tag{7}$$

then there exists a space–time periodic solution p of (3) which is the unique bounded solution such that $\inf_{\mathbb{R}\times\mathbb{R}^N} p>0$. Thus our uniqueness hypothesis is satisfied in this case.

Similarly, if $f(t,x,1)=0$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and if f is positive between 0 and 1, then it can also be proved that the uniqueness hypothesis is true.

2. Notion of fronts in space–time periodic media

The simplest example of such an equation is the Fisher-KPP equation: $\partial_t u - \Delta u = u(1-u)$, which has been first investigated in the pioneering articles of Kolmogorov, Petrovsky and Piskunov [7] and Fisher [6]. The behavior of the solutions of this homogeneous equation is interesting. First, there exist *planar fronts*, that is to say solutions of the form $u(t,x) = U(x\cdot e + ct)$, where e is an unitary vector and c is the speed of propagation in the direction $-e$. Next, beginning with a positive initial data $u_0 \neq 0$, with compact support, we get $u(t,x) \to 1$ when $t \to +\infty$, locally in x, moreover, the set where u is close to 1 spreads with a speed which is equal to the minimal speed of the planar fronts (see [1]).

Eq. (3) arises in population genetics, combustion and population dynamics models. The existence of fronts and the spreading properties have useful interpretations. In population dynamics models, it is very relevant to consider heterogeneous environments and to study the effect of the heterogeneity on the propagation properties.

The study of propagation phenomena in periodic environments is a first step in order to understand the effect of some heterogeneity on the propagation. In [12], Shigesada, Kawasaki and Teramoto defined the notion of *pulsating traveling fronts*, which is a generalization of the notion of planar fronts to space periodic environments. Namely, a pulsating traveling front is a solution u that satisfies: $u(t,x) \to 0$ as $x \cdot e \to -\infty$, $u(t,x) \to 1$ as $x \cdot e \to -\infty$ and $\forall i \in [1,N], x \in \mathbb{R}^N, t \in \mathbb{R}, u(t + \frac{L_i \cdot e}{c}, x) = u(t, x + L_i)$. Equivalently, a pulsating traveling front is a solution u which can be written $u(t,x) = \phi(x \cdot e + ct, x)$, where ϕ is periodic in its second variable, $\phi(-\infty, x) = 0$ and $\phi(+\infty, x) = 1$. It has been proved in [2,4] that it is possible to define a minimal speed c^* such there exist some pulsating traveling fronts of speed c if and only if $c \geqslant c^*$. This result has been extended to time periodic media in [5].

It is not easy to extend this definition to space–time periodic media. One can try to define a pulsating traveling front as a solution u which can be written in the form $u(t,x) = \phi(x \cdot e + ct, t, x)$, where ϕ is periodic in its second and third variables. Here, the periodicity is equivalent to some formula $u(t, x + pL) = u(t + pL/c, x)$, where L is the spatial period of the medium and $p \in \mathbb{Z}$, if and only if $c \in \frac{L}{T}\mathbb{Q}$ where T is the temporal period. As we are expecting to find a half-line of speeds associated with pulsating traveling fronts, this is not satisfying.

As ϕ satisfies a degenerate equation, there is a lack of regularity and this function may only be measurable. Thus, setting $u(t,x) = \phi(x \cdot e + ct, t, x)$ may not be relevant since the hyperplane $z = x \cdot e + ct$ is of measure 0. This issue does not occur in space periodic media since one can use the regularity of Eq. (3), which is not possible here because of the extra-variable $z = x \cdot e + ct$. Thus, we use the following weakened definition:

Definition 2.1. We say that a function u is a *pulsating traveling front* of speed c in the direction $-e$ that connects 0 to p if it can be written $u(t,x) = \phi(x \cdot e + ct, t, x)$, where $\phi \in L^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N)$ is such that for almost every $y \in \mathbb{R}$, the function $(t,x) \mapsto \phi(y + x \cdot e + ct, t, x)$ satisfies Eq. (3). We ask the function ϕ to be periodic in its second and third variables and to satisfy:

$$\begin{cases} \phi(z,t,x) \to 0 & \text{as } z \to -\infty \text{ in } L^\infty(\mathbb{R} \times \mathbb{R}^N), \\ \phi(z,t,x) - p(t,x) \to 0 & \text{as } z \to +\infty \text{ in } L^\infty(\mathbb{R} \times \mathbb{R}^N). \end{cases} \tag{8}$$

Of course if ϕ is continuous, there is no such issue: one can define $u(t,x) = \phi(x \cdot e + ct, t, x)$. We say that a solution u of (3) is a *continuous* pulsating traveling front of speed c if it can be written $u(t,x) = \phi(x \cdot e + ct, t, x)$, where ϕ is continuous, periodic in its second and third variables and satisfies (8). It is in fact possible to get such ϕ under some strong KPP-type assumption.

In 2002, Weinberger proved the existence of pulsating traveling fronts in discrete space–time periodic media (see [13]) and left the investigation of continuous space–time periodic media as an open problem. In 2006, Nolen, Rudd and Xin investigated the case of a space–time periodic drift in [11], with a positive homogeneous nonlinearity. Using an equivalent definition, they proved that such fronts do exist for at least *one* speed c^* and that there is no front of speed $c < c^*$ in the KPP case. This speed c^* is a good candidate for the minimal speed but they did not prove that there exists a front of speed c for all $c > c^*$. One can also wonder if there exist some pulsating traveling fronts of speed $c < c^*$ and if it is possible to consider some heterogeneity in A and in f.

3. Existence of a minimal speed

The main result of this Note is the following one:

Theorem 3.1. *Assume that the previous hypotheses are satisfied. Then for all unit vector e, for all speed $c \geqslant c_e^*(A,q,\eta)$, there exists a pulsating traveling front u of speed c in direction $-e$ that connects 0 to p and which is nondecreasing with respect to z. If $c < c_e^*(A,q,\mu)$, then there exists no pulsating traveling front of speed c in direction $-e$.*

We managed to obtain this result by using a new method, which is inspired of [3]. The main improvement with respect to [11] is that pulsating traveling fronts exist for at least a *half-line* of speeds. Furthermore, we proved this result for heterogeneous diffusion A and reaction term f, the function f is not supposed to be positive and we were able to treat compressible drifts q.

The non-existence result for sufficiently small speed is in fact a corollary of the next *spreading* result, which generalizes a result of [11] using a slightly different method:

Proposition 3.2. *Take $u_0 \leqslant p$ a nonnegative continuous initial data and an interval $[a_1, a_2] \subset \mathbb{R}$ such that $\inf_{x\in\mathbb{R}^N, e\cdot x\in[a_1,a_2]} u_0(x) > 0$. Then for all $-c^*_{-e}(A, q, \mu) < c < c^*_e(A, q, \mu)$, the solution u of Eq.* (3) *associated with the initial datum u_0 satisfies*:

$$u(t, x - cte) - p(t, x - cte) \to 0 \quad \text{as } t \to +\infty,$$

locally uniformly in $x \in \mathbb{R}^N$.

If f is of KPP type, that is to say that $f(t, x, s) \leqslant d_u f(t, x, 0)s$ for all $(t, x, s) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^+$, then one has $\eta \equiv \mu$ and thus the speed $c^*_e(A, q, \mu)$ is minimal in the sense that there exists a pulsating traveling front of speed c if and only if $c \geqslant c^*_e(A, q, \mu)$.

If f is not of KPP-type, it is not clear if such a minimal speed exists, but we can at least state:

Proposition 3.1. *Assume that the previous hypotheses are satisfied. Then for all unit vector e, there exists a minimal speed c^*_e such that there exists a pulsating traveling front u of speed c^*_e, while there exists no such front of speed $c < c^*_e$.*

Lastly, under some particular hypothesis, we are able to construct *continuous* pulsating traveling fronts:

Theorem 3.3. *Assume that the previous hypotheses are satisfied and that $s \mapsto f(t, x, s)/s$ is nonincreasing for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Then for all speed $c \geqslant c^*_e(A, q, \mu)$, there exists a continuous pulsating traveling front u of speed c which is nondecreasing with respect to z.*

4. Dependence with respect to the diffusion

In the KPP case, Theorem 3.1 gives that the minimal speed of propagation is equal to:

$$c^*_e = \min_{\lambda>0} \frac{-k_{\lambda e}(A, q, \mu)}{\lambda}. \tag{9}$$

This characterization yields that the investigation of the dependence relations between the minimal speed $c^*(A, q, \mu)$ and the coefficients reduces to an optimization problem for the eigenvalue family $(k_{\lambda e})_{\lambda>0}$ in this case. This enabled us to prove many dependence results (see [10]).

We only state in this Note a dependence relation with respect to the diffusion amplitude, which answers to some open questions stated in [4]:

Theorem 4.1. *Assume that A is a constant matrix.*

(1) *If $\int_{(0,T)\times C} \mu \geqslant 0$, then*:

$$\frac{c^*_e(\gamma A, 0, \mu)}{\sqrt{\gamma}} \to 2\sqrt{\frac{eAe}{T|C|} \int_{(0,T)\times C} \mu(t, x)\, \mathrm{d}t\, \mathrm{d}x} \quad \text{as } \gamma \to +\infty.$$

(2) *If $\max_{x\in\mathbb{R}^N}(\int_0^T \mu(t, x)\, \mathrm{d}t) \geqslant 0$, then*:

$$\frac{c^*_e(\gamma A, 0, \mu)}{\sqrt{\gamma}} \to 2\sqrt{\frac{eAe}{T} \max_{x\in\mathbb{R}^N}\left(\int_0^T \mu(t, x)\, \mathrm{d}t\right)} \quad \text{as } \gamma \to 0.$$

(3) *There exists some space–time periodic function μ and some $\gamma > \gamma' > 0$ such that*:

$$c_e^*(\gamma I_N, 0, \mu) < c^*(\gamma' I_N, 0, \mu).$$

This means that if the diffusion is very large, then the population spreads very fast and does not see the heterogeneity but only the averaged media. On the contrary, if the diffusion is small, the population will gather in favorable areas.

Most of the results of this Note are proved in [10].

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