Existence and uniqueness of the solution of a space-time periodic reaction-diffusion equation

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Abstract

This paper is concerned with the study of the periodic solutions and the entire solutions of the equation:

$$\partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u = f(t,x,u) \tag{1}$$

where the diffusion matrix A, the advection term q and the reaction term f are periodic in t and x. We prove that the sign of the periodic principal eigenvalue associated with the linearized problem determines the existence and the uniqueness of the periodic solution. Introducing another eigenvalue, we are able to state uniqueness conditions for the entire solution and to derive the asymptotic behavior of the solutions of the associated Cauchy problem.

Key-words: Parabolic periodic operators; Reaction-diffusion equations; Maximum principles; Liouville type results.

1 Introduction and main results

1.1 Introduction

We are concerned with the equation:

$$\partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u = f(t,x,u) \tag{2}$$

with a periodic dependance in t and x. This equation arises in population genetics, combustion and population dynamics models (see [1, 13, 25] for example). It is a generalization of the homogeneous equation $\partial_t u - \Delta u = u(1-u)$.

The homogeneous equation was first introduced in 1937 in the pioneering papers of Fisher [13] and Kolmogorov, Petrovsky and Piskunov [18]. In these papers, they proved that there exists *travelling front solutions*, namely solutions of the form $u(t,x) = U(x \cdot e + ct)$, where $-e$ is the propagation direction and c is the propagation speed. Using these travelling fronts,

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they showed that starting from a nonnegative initial datum $u_0 \neq 0$ with compact support then $u(t,x) \to 1$ as $t \to +\infty$ and the set where $u(t,x)$ is close to 1 expands at a given *spreading speed*, which is exactly the minimal speed of the travelling fronts. These results were generalized by Aronson and Weinberger [1] and Fife and Mc Leod [12] to more general classes of reaction terms f and to multidimensional spaces.

In 1979, Freidlin and Gartner were the first to investigate the *heterogeneous* equation in [15]. They studied the case of space periodic coefficients and generalized the spreading properties. The study of the effect of this heterogeneity was the main subject of many works in the case of a flame propagation model, for which the reaction term is always positive and the asymptotic state is constant (see [2, 7, 14, 22, 23, 24, 28]).

The case where the sign of f can change has a particular importance in population dynamics models. In such models, the reaction term $f(t,x,u)$ represents an intrinsic growth rate which can depend on the environment. This growth rate may be negative in very unfavorable areas, when the death rate is higher than the birth rate. This model has been studied by Cantrell and Cosner [10, 11], in the case of a bounded domain in x. The case of a time periodic environment has been studied by Hess [16] and Hutson, Mischaikow and Polacik [17]. The case of a space periodic domain with no drift has been investigated by Berestycki, Hamel and Roques [4]. In all these papers, the authors proved that the existence of a positive periodic state, its uniqueness, its stability and the large-time behavior of the solutions of the associated Cauchy problems were all determined by the sign of the principal eigenvalue associated with the linearized equation in the neighborhood of the homogeneous solution 0. In other terms, the instability of the null state yields the existence of a positive time periodic state and its global attractivity.

Recently, the case of a general unbounded domain has been investigated by Berestycki, Hamel and Rossi [6] and Berestycki and Rossi [9]. It has been proved that one can define some generalized eigenvalues and that their signs determine the existence of a positive space-periodic state and its attractivity.

Berestycki, Hamel and Roques [5] proved that in the case of a space periodic domain with a sign-changing reaction term, if the conditions for the existence and the uniqueness of the positive space periodic state are fulfilled, then there exists some *pulsating travelling fronts* that link the instable solution 0 to the stable one. A pulsating front of speed c is a solution of the form $u(t,x) = U(x \cdot e + ct, x)$ where U is periodic in its second variable. This result has also been proved by Weinberger [27] in a discrete framework, which includes the case of a time periodic environment.

In the present paper, we study the case of a space-time periodic environment with drift. This is a preliminary work to [21]. In this next paper, we prove that there exists pulsating travelling fronts that link 0 to the positive periodic solution when it exists.

First of all, we extend the results of [4], that is, we prove that the existence and uniqueness of a periodic positive solution are fully determined by the signs of generalized principal eigenvalues. Adding a general advection term in the equation leads to some strange phenomenons. In particular, we need to use two generalized eigenvalues instead of one. In the case when one of this eigenvalue is negative and the other one is nonnegative, the periodic positive solution exists but is not globally attractive. The investigation of this particular

issue is totally new as far as we know.

This article is coupled with [20], which investigates more precisely the properties of the two generalized principal eigenvalues that will be central in our study.

1.2 Hypotheses

The function $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is supposed to be of class $\mathcal{C}^{\frac{\alpha}{2},\alpha}$ in (t,x) locally in u for a given $0 < \alpha < 1$, locally Lipschitz-continuous in u and of class $\mathcal{C}^1$ on $\mathbb{R} \times \mathbb{R}^N \times [0,\beta]$ for a given $\beta > 0$. We set $\mu(t,x) = f'_u(t,x,0)$. We assume that $\forall x, \forall t, f(t,x,0) = 0$.

The matrix field $A : \mathbb{R} \times \mathbb{R}^N \to S_N(\mathbb{R})$ is supposed to be of class $\mathcal{C}^{\frac{\alpha}{2},1+\alpha}$. We suppose furthermore that A is uniformly elliptic and continuous: there exist some positive constants γ and Γ such that for all $\xi \in \mathbb{R}^N, (t,x) \in \mathbb{R} \times \mathbb{R}^N$ one has:

$$\gamma\|\xi\|^2 \leq \sum_{1 \leq i,j \leq N} a_{i,j}(t,x)\xi_i\xi_j \leq \Gamma\|\xi\|^2 \tag{3}$$

where $\|\xi\| = (|\xi_1|^2 + ... + |\xi_N|^2)^{1/2}$ and $a_{i,j}$ is the coefficient (i,j) of the matrix A. The drift term $q : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is of class $\mathcal{C}^{\frac{\alpha}{2},\alpha}$.

We assume that f, A and q are periodic in t and x, that is, there exist $T, L_1, ..., L_N > 0$ such that for all t, x, s, i, one has:

$$
\begin{array}{llllll}
A(t+T,x) & = & A(t,x) & , & q(t+T,x) & = & q(t,x) & , & f(t+T,x,s) & = & f(t,x,s) \\
A(t,x+L_ie_i) & = & A(t,x) & , & q(t,x+L_ie_i) & = & q(t,x) & , & f(t,x+L_ie_i,s) & = & f(t,x,s),
\end{array}
\tag{4}
$$

where $(e_1, ..., e_n)$ is an orthonormal basis of $\mathbb{R}^N$. We set $C = (0,L_1) \times ... \times (0,L_N)$ the periodicity cell. In the sequel the periodicity will always refer to the periods $(T, L_1, ..., L_N)$.

This periodicity allows us to search for particular solutions of (2), the solutions that are both periodic in time and space:

$$
\begin{cases}
\partial_t p - \nabla \cdot (A(t,x)\nabla p) + q(t,x) \cdot \nabla p = f(t,x,p), \\
p \text{ periodic in } x \text{ and in } t.
\end{cases}
\tag{5}
$$

In the results below, we will refer to the two following additional hypotheses on f:

$$\forall x \in \mathbb{R}^N, \forall t \in \mathbb{R}, \ s \to f(t,x,s)/s \ \text{ is decreasing in } s > 0, \tag{6}$$

$$\exists\, M > 0 \mid \forall x \in \mathbb{R}^N, \forall t \in \mathbb{R}, \forall s \geq M, f(t,x,s) \leq 0. \tag{7}$$

These hypotheses both have a biological meaning. The first hypothesis means that the intrinsic growth rate decreases when the population density is increasing. This is due to the intraspecific competition for resources effect. The second hypothesis means that there is a saturation effect: when the population density is very large, the death rate is higher than the birth rate and the population decreases.

1.3 Existence and uniqueness results

The existence and uniqueness results are directly linked to the signs of the following *generalized principal eigenvalues*:

$$\lambda'_1 = \inf\{\lambda \in \mathbb{R} \quad, \quad \exists \phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R} \times \mathbb{R}^N), \ \phi > 0, \ \phi \text{ is T-periodic,} \atop (\mathcal{L} - \lambda)\phi \leq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N\}, \tag{8}$$

$$\lambda_1 = \sup\{\lambda \in \mathbb{R}, \ \exists \phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N), \ \phi > 0, \ \phi \text{ is T-periodic}, (\mathcal{L} - \lambda)\phi \geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N\},$$

where $\mathcal{L}$ is the operator defined by:

$$\mathcal{L}\phi = \partial_t \phi - \nabla \cdot (A(t,x)\nabla\phi) + q(t,x) \cdot \nabla\phi - \mu(t,x)\phi.$$

Such formulas were used by Berestycki, Nirenberg and Varadhan in [8] and by Berestycki, Hamel and Rossi in [6] in order to generalize the definition of the principal eigenvalue of an elliptic operator to general domains. Here, we use these formulas to define a generalized principal eigenvalue for a parabolic operator in $\mathbb{R}^N$. In [20], the author proved that these quantities were well-defined. Many results that are stated in the following are still true in the case of a general domain which is not necessarily periodic in x.

In section 2, we give a few results that are proved in [20] and we characterize these eigenvalues in the case of space-time periodic coefficients. In particular, we give a few conditions which ensure that $\lambda_1 = \lambda'_1$. One always has $\lambda'_1 \leq \lambda_1$ but the other inequality is not always true, as noticed by Berestycki, Hamel and Rossi in [6]. For example, take:

$$\mathcal{L}\phi = -\phi'' + \phi' - \frac{1}{8}\phi.$$

Taking $\phi \equiv 1$, one easily gets $\lambda'_1 \leq -\frac{1}{8}$ (in fact these two quantities are equal). One can remark that:

$$e^{-\frac{x}{2}}\mathcal{L}(e^{\frac{x}{2}}\psi) = \mathcal{L}'\psi = -\psi'' + \frac{1}{8}\psi.$$

This modified operator is self-adjoint, thus one has $\lambda_1(\mathcal{L}') = \lambda_1(\mathcal{L}) = \frac{1}{8}$. We conclude that $\lambda'_1 = -\frac{1}{8}$ and $\lambda_1 = \frac{1}{8}$. The case $\lambda'_1 < 0 < \lambda_1$ is thus possible and will be discussed in the sequel.

Existence and uniqueness of periodic solutions and uniformly positive entire solutions

We first state the following existence result for the positive solutions of equation (5):

Theorem 1.1 (Existence of a periodic solution)
   If $\lambda'_1 < 0$ and if hypothesis (7) is satisfied, then there exists a positive periodic solution $p \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ of equation (5).
   If $\lambda'_1 \geq 0$ and if hypothesis (6) is satisfied, then the only nonnegative bounded and entire solution of (2) is 0.

This means that the existence of a positive periodic solution only depends on the stability of the solution 0. We will prove that such a solution is unique in the class of periodic solutions, but it is in fact possible to prove a uniqueness result in the larger class of entire solutions, that are defined as follows:

Definition 1.2 *We say that u is an entire solution of equation* (2) *if u is a classical solution of* (5) *on the whole space $\mathbb{R} \times \mathbb{R}^N$.*

Our uniqueness result only holds for entire solutions with a positive infimum.

Theorem 1.3 (Uniqueness in the class of uniformly positive entire solutions) *Assume that* (6) *is satisfied. Then if u and p are two bounded entire solutions of* (2) *such that $\inf_{\mathbb{R} \times \mathbb{R}^N} u > 0$ and $\inf_{\mathbb{R}^N \times \mathbb{R}} p > 0$, one has $u \equiv p$.*

Gathering Theorems 1.1 and 1.3, we get the uniqueness in the class of periodic functions for the solution of equation (5).

Corollary 1.4 (Uniqueness of the periodic solution) *If* (6) *and* (7) *are satisfied, there is a positive periodic solution of equation* (5) *if and only if $\lambda_1' < 0$. If it exists, it is unique.*

Uniqueness of positive entire solutions

It is not true in general that a positive entire solution u of (5) satisfies $\inf_{\mathbb{R} \times \mathbb{R}^N} u > 0$. As a counterexample, consider the equation:

$$\partial_t u - \partial_{xx} u + c \partial_x u = u(1 - u) \text{ in } \mathbb{R} \times \mathbb{R}^N. \tag{9}$$

One can check that $\lambda_1' = -1 < 0$. This equation has constant coefficients and thus the periodic solution given by Theorem 1.1 is the constant function 1. It has been proved in [18] that if $c \geq 2$, then there exists a positive entire solution U of (9) which does not depend on t and such that $U(-\infty) = 0$ and $U(+\infty) = 1$. Of course $U \not\equiv 1$.

Thus, one cannot hope to prove a general uniqueness result for positive entire solutions when $\lambda_1' < 0$. Some more hypotheses are needed.

Theorem 1.5 (Liouville type result for positive entire solutions) *Assume that $\lambda_1 < 0$ and that* (6) *is satisfied. Then if u is a positive bounded entire solution of* (2) *that satisfies:*

$$\exists x_0 \in \mathbb{R}^N | \inf_{t \in \mathbb{R}} u(t, x_0) > 0, \tag{10}$$

one has $\inf_{t \in \mathbb{R}, x \in \mathbb{R}^N} u(t, x) > 0$.

Hence, if (7) is also satisfied, *all* the positive entire stationary solutions have a positive infimum using Theorem 1.5 and thus the periodic stationary solution is unique in the class of positive entire solutions which satisfy (10) using Theorem 1.3.

In [4], Berestycki, Hamel and Roques proved such a result for stationary solutions when the coefficients do not depend on t and $q \equiv 0$. These two hypotheses simplify the investigation of the uniqueness since:

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- Stationnary solutions that are not 0 always satisfy (10).

  - If the coefficients do not depend on t and $q \equiv 0$, then $\lambda_1 = \lambda_1'$ (see [4] or Proposition 2.5 below).

If $q \not\equiv 0$ and the coefficients depend on t, the periodic solution might not be the unique positive entire solution if $\lambda_1' < 0 < \lambda_1$. We will investigate this issue in section 2.

Lastly, we underline that the hypotheses $\lambda_1 < 0$ and (10) are both needed to get a uniqueness result. Consider first equation (9), with $c \geq 2$. We have already emphasize that there is no general uniqueness result for positive entire solutions. The solution U described above satisfies (10) since it does not depend on t, but we can compute $\lambda_1 = -1 + \frac{c^2}{4} \geq 0$. Thus Theorem 1.5 cannot be applied.

In the other hand, consider the homogeneous equation $\partial_t u - \partial_{xx} u = u(1 - u)$. Then $\lambda_1 = -1$. For all $c \geq 2$, this equation admits travelling fronts of speed c, that is, positive entire solutions u that can be written $u(t, x) = U(x - ct)$, where $U(-\infty) = 0$ and $U(+\infty) = 1$. Then for all $x_0 \in \mathbb{R}$, one has $\inf_{t \in \mathbb{R}} u(t, x_0) = 0$.

These two examples prove that our conditions $\lambda_1 < 0$ and (10) are optimal in order to get a general uniqueness result for positive entire solutions.

Asymptotic behaviour

The uniqueness Theorem 1.3 for entire solutions enables us to prove the following asymptotic convergence when $t \to +\infty$:

Theorem 1.6 (Attractivity of the periodic solution) *Let $u_0 \in \mathcal{C}^0(\mathbb{R}^N)$ be a nonnegative, bounded and non-null initial datum. Set u the solution of the associated Cauchy problem:*

$$\begin{cases} \partial_t u - \nabla \cdot (A(t, x)\nabla u) + q(t, x) \cdot \nabla u = f(t, x, u), \\ u(0, x) = u_0(x). \end{cases} \tag{11}$$

If hypotheses (6) and (7) are satisfied, then:

1) If $\lambda_1 < 0$, then $u(t + s, x) - p(t + s, x) \to 0$ as $s \to +\infty$ in $\mathcal{C}_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$, where p is the unique positive solution of (5).

2) If $\lambda_1' \geq 0$, then $u(s, x) \to 0$ as $s \to +\infty$ uniformly with respect to $x \in \mathbb{R}^N$. .

If $\lambda_1' < 0 < \lambda_1$, then the asymptotic behavior of the solutions of the Cauchy problem depends on the properties of the initial datum. We will prove sharp results in Propositions 2.7 and 2.8 of section 2. As some more properties of the generalized principal eigenvalues are needed to state these general results, we only give a simple corollary of these Propositions now:

Corollary 1.7 *Assume that $\lambda_1' < 0 < \lambda_1$ and that hypotheses (6) and (7) are satisfied. Consider u_0 as in Theorem 1.6.*

1) If u_0 is compactly supported, then $u(s, x) \to 0$ as $s \to +\infty$ locally uniformly with respect to $x \in \mathbb{R}^N$.

2) If $\inf_{\mathbb{R}^N} u_0 > 0$, then $u(t + s, x) - p(t + s, x) \to 0$ as $s \to +\infty$ in $\mathcal{C}_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$, where p is the unique positive solution of (5).

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The case $\lambda_1 = 0 > \lambda_1'$ stays open. We underline that the convergence in 2) is locally uniform and not uniform over $\mathbb{R}^N$ as in Theorem 1.6. We cannot get a better convergence if $\lambda_1' < 0$. For example, consider the equation (9) with $c > 2$ so that $\lambda_1' = -1 < 0 < \lambda_1 = -1 + \frac{c^2}{4}$. Assume that u_0 satisfies the hypotheses of 1) of Proposition 1.7. Set u the solution of the associated Cauchy problem and $v(t, x) = u(t, x + ct)$. We know from Proposition 1.7 that $u(t, x) \to 0$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$. The function v satisfies

$$\partial_t v - \partial_{xx} v = v(1 - v),$$

and $v(0, x) = u_0(x)$. It is well-known that $v(t, x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$. Thus u cannot uniformly converge to 0. In fact, the function u is blown away at speed c but converges to 1 in a moving frame.

This property can be generalized to space-time periodic media. Berestycki, Hamel and the author have proved (Theorem 1.13 in [3]) that in dimension $N = 1$, as soon as $\lambda_1' < 0$, there always exists a range of speeds $(w^*(-e), w^*(e))$ so that $u(t, x + wte) - p(t, x + wte) \to 0$ (resp. $u(t, x + wte) \to 0$) as $t \to +\infty$ locally uniformly with respect to $x \in \mathbb{R}^N$ for all $w \in (w^*(-e), w^*(e))$ (resp. for all $w \notin (w^*(-e), w^*(e))$). If $\lambda_1 > 0$, this property is still true but $0 \notin [w^*(-e), w^*(e)]$.

Using some properties of these two generalized principal eigenvalues, we will be able to give some more results on the asymptotic behavior of the solutions of the Cauchy problem in section 2.

2 Preliminaries: the associated eigenvalue problem

In this section, we give a characterization of the generalized principal eigenvalues λ_1 and λ_1' using the periodic principal eigenvalues of a modified operator. This characterization allows us to get a concavity result, an approximation of the generalized principal eigenvalue λ_1 and a determination of the large-time behavior of the solutions of the Cauchy problem associated with particular initial data. These results have been proved in [20].

Definition of the periodic principal eigenvalue

We define the following modified operator for all $\alpha \in \mathbb{R}^N$:

$$L_\alpha \phi = e^{-\alpha \cdot x} \mathcal{L}(e^{\alpha \cdot x} \phi) = \partial_t \phi - \nabla \cdot (A \nabla \phi) - 2\alpha A \nabla \phi + q \cdot \nabla \phi - (\alpha A \alpha + \nabla \cdot (A\alpha) - q \cdot \alpha + \mu)\phi.$$

In this section, we assume that A and q satisfy the same regularity and ellipticity hypotheses as in section 1. The function μ is in $\mathcal{C}_{per}^{\frac{\alpha}{2}, \alpha}(\mathbb{R} \times \mathbb{R}^N)$. We recall the following definition:

Definition 2.1 *A periodic principal eigenfunction of the operator L_α is a function $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ such that it exists a constant k so that:*

$$\begin{cases} L_\alpha \phi &= k\phi, \\ \phi &> 0, \\ \phi(., . + T) &= \phi, \\ \phi(. + L_i e_i, .) &= \phi \; for \; i = 1...N. \end{cases} \tag{12}$$

Such a k is called a periodic principal eigenvalue.

Theorem 2.2 *[20] There exists a couple (k, ϕ) that satisfies (12). Furthermore, k is unique and ϕ is unique up to multiplication by a positive constant.*

The proof of this theorem includes the proof of the following proposition, that we will need in the sequel:

Proposition 2.3 *[20] There exists $\beta_0 \in \mathbb{R}$ such that for all $g \in \mathcal{C}^0_{per}(\mathbb{R} \times \mathbb{R}^N)$, $\beta > \beta_0$ and $\alpha \in \mathbb{R}^N$, there exists a unique function $u \in \mathcal{C}^{1,2}_{per}(\mathbb{R} \times \mathbb{R}^N)$ that satisfies:*

$$L_\alpha u + \beta u = g$$

We define $k_\alpha = k$ and $\phi_\alpha = \phi$ the eigenelements associated with L_α and normalized by $\|\phi_\alpha\|_\infty = 1$.

Characterization of the generalized principal eigenvalues

The periodic principal eigenvalues family $(k_\alpha)_{\alpha \in \mathbb{R}^N}$ enables us to give the following characterizations for the generalized principal eigenvalues λ'_1 and λ_1:

Theorem 2.4 *[20]*

$$\lambda'_1 = k_0 \ and \ \lambda_1 = \max_{\alpha \in \mathbb{R}^N} k_\alpha.$$

These characterizations are useful to try to characterize the cases when $\lambda'_1 = \lambda_1$. Namely, one can prove that:

Proposition 2.5 *[20] If A and μ have a common symmetry axis in t or in x, that is:*

$$\exists x_0 \mid \forall t, x, A(t, x_0+x) = A(t, x_0-x), q(t, x_0+x) = q(t, x_0-x) \ and \ \mu(t, x_0+x) = \mu(t, x_0-x)$$

or $\exists t_0 \mid \forall t, x, A(t_0+t, x) = A(t_0-t, x), q(t_0+t, x) = q(t_0-t, x) \ and \ \mu(t_0+t, x) = \mu(t_0-t, x)$

and if q can be written $q = A\nabla Q$ where $Q \in \mathcal{C}^{\alpha/2, 1+\alpha}(\mathbb{R} \times \mathbb{R}^N)$ with $\int_{(0,T) \times C} A^{-1}q = 0$, then $\lambda'_1 = \lambda_1$.

If $q \equiv 0$, μ and A do not depend on t, then it has been proved by Berestycki, Hamel and Roques [4] that $\lambda'_1 = \lambda_1$. This property is not true in general.

Approximation of a generalized principal eigenvalue

We define the eigenelements $(\phi_R^{t_0, x_0}, \lambda_R^{t_0, x_0})$ for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N$ by (see [16] for example):

$$\begin{cases} \partial_t \phi_R^{t_0, x_0} - \nabla \cdot (A(t+t_0, x+x_0)\nabla \phi_R^{t_0, x_0}) + q(t+t_0, x+x_0) \cdot \nabla \phi_R^{t_0, x_0} \\ -\mu(t+t_0, x+x_0)\phi_R^{t_0, x_0} = \lambda_R^{t_0, x_0} \phi_R^{t_0, x_0} \text{ in } \mathbb{R} \times B_R(0), \\ \phi_R^{t_0, x_0} \text{ is periodic in t}, \ \phi^{t_0, x_0} > 0, \ \text{in } \mathbb{R} \times B_R(0), \ \phi^{t_0, x_0} = 0 \text{ in } \mathbb{R} \times \partial B_R(0), \\ \phi_R^{t_0, x_0}(0, 0) = 1, \end{cases}$$

where $B_R(0)$ is the open ball of center 0 and radius R. The following approximation result holds:

Theorem 2.6 *One has $\lambda_R^{t_0,x_0} \to \lambda_1$ uniformly in (t_0,x_0) as $R \to +\infty$.*

The pointwise convergence has been proved by the author in [20]. We need in the present paper a uniform convergence in (t_0,x_0).

Some more results about the large-time behavior

We are now able to give two more results which prove that, in the case where $\lambda_1' < 0 < \lambda_1$, one cannot expect a general conclusion to this issue. The case $\lambda_1' < 0 = \lambda_1$ remains open.

Proposition 2.7 *Let u_0 be a continuous nonnegative function, $u_0 \neq 0$. Assume that (6) and (7) are satisfied and $\lambda_1 > 0$. If there exists $\alpha \in \mathbb{R}^N$ such that $k_\alpha > 0$ and:*

$$\exists\, R > 0 \mid \forall x \in \mathbb{R}^N, |u_0(x)| \leq Re^{\alpha \cdot x},$$

then the solution of the associated Cauchy problem (11) has the following asymptotic behavior:

$$u(s,x) \to 0 \text{ as } s \to +\infty \text{ uniformly on the compact subsets of } \mathbb{R} \times \mathbb{R}^N.$$

Proposition 2.8 *Assume that (6) and (7) are satisfied and that $\lambda_1' < 0$. Take, u_0 a continuous bounded function such that:*

$$\exists\, B \in \mathbb{R}, \kappa > 0 \mid \alpha \cdot x < B \Rightarrow u_0(x) \geq \kappa e^{\alpha \cdot x}$$

where $\alpha \in \mathbb{R}^N$ is such that $k_\alpha < 0$ and $t \mapsto k_{t\alpha}$ is increasing in the neighboorhood of 1. Then:

$$u(s+t,x) - p(s,x) \to 0 \text{ as } s \to +\infty \text{ in } \mathcal{C}_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}^N).$$

An initial datum u_0 with compact support satisfies the hypotheses of Proposition 2.7. A function that satisfies $\inf_{\mathbb{R}^N} u_0 > 0$ obviously checks the hypotheses of Proposition 2.8. Thus Coroallry 1.7 is an immediate consequence of Propositions 2.7 and 2.8.

The increasing condition on k_α is necessary. First, if $k_\beta = \lambda_1 \geq 0$, then the function $t \mapsto k_{t\beta}$ is increasing for t small enough since it reaches its maximum in 1 and it is concave. Thus, this condition does have a meaning. Secondly, assume that the proposition is true for all α such that $k_\alpha < 0$. For $|\alpha|$ large enough, one has $k_\alpha < 0$ (see [5] for a proof in the time-homogeneous case, but this property is also true in our case). Then if $\beta \neq 0$ satisfies $k_\beta = \lambda_1 \geq 0$, one can choose a function u_0 such that $e^{t\beta \cdot x} \leq u_0 \leq e^{\beta \cdot x}$ when $\beta \cdot x < 0$, where $t > 1$ is large enough so that $k_{t\beta} < 0$. Proposition 2.7 then yields that $u \to 0$ which is a contradiction.

As a conclusion, we investigate more precisely the case of dimension 1. Assume that $\lambda_1' < 0 < \lambda_1$ and take β such that $k_\beta = \lambda_1$. Up to a change of variable $x \mapsto -x$, one can assume that $\beta > 0$. The concavity of the function $\alpha \mapsto k_\alpha$ yields that there exists a unique $\alpha_0 \in (0,\beta]$ such that $k_{\alpha_0} = 0$. Then it is easy to see that the two previous Propositions can be stated as: if there exists $C > 0$ and $\gamma > \alpha_0$ (resp. $\gamma < \alpha_0$) such that $u_0 \leq Ce^{\gamma \cdot x}$ (resp. $u_0 \geq Ce^{\gamma \cdot x}$) for x small enough, then $u \to 0$ (resp. $u - p \to 0$). Thus the cases covered by Propositions 2.7 and 2.8 are quite general.

Lastly, we can wonder what happens if $\lambda_1 = 0 > \lambda_1'$. In this case, there exists a pulsating travelling front ϕ_0 with speed 0 (see [21]). This standing front is another entire solution of equation (2), which might attract some solutions.

9

3 Existence of a periodic solution

This section is dedicated to the proof of Theorem 1.1.

Lemma 3.1 *If there exists a strict subsolution u and a strict supersolution w of equation (5) such that $u(t,x) < w(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, then there exists a solution v of equation (5) and one has: $u(t,x) < v(t,x) < w(t,x)$, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.*

Proof. Set $m = max\{\|u\|_\infty, \|w\|_\infty\}$, then f is Lipschitz-continuous in u on $[0,T] \times \overline{C} \times [-m,m]$.

Let P be the operator defined by:

$$Pz = \partial_t z - \nabla \cdot (A(t,x)\nabla z) + q(t,x) \cdot \nabla z + Mz$$

Proposition 2.3 yields that for M large enough, for all $g \in \mathcal{C}^0_{per}(\mathbb{R} \times \mathbb{R}^N)$ there exists a unique $z \in \mathcal{C}^{1,2}_{per}(\mathbb{R} \times \mathbb{R}^N)$ so that $Pz = g$. We take $M > Lip(f)$ and we call such a solution $z = P^{-1}g$. Set $g(t,x,s) = f(t,x,s) + Ms$, obviously, g is increasing in s.

Set $Tz = P^{-1}(g(t,x,z))$ and define (u_n) by:

$$\begin{cases} u_0 &= u, \\ u_{n+1} &= T(u_n). \end{cases}$$

We will prove that (u_n) is nondecreasing by induction. For $n = 0$, one has:

$$P(u_1 - u)(t,x) = g(t,x,u(t,x)) - Pu(t,x) > 0$$

since u is a subsolution of (5). Therefore, as $M > 0$ and as $u_1 - u$ is periodic in t and x, the weak maximum principle yields that $u_1(t,x) \geq u_0(t,x)$ for all t,x.

Assume that: $u_0(t,x) \leq u_1(t,x) \leq ... \leq u_n(t,x)$ for all t,x. Then:

$$\forall t,x, P(u_{n+1})(t,x) - P(u_n)(t,x) = g(t,x,u_n(t,x)) - g(t,x,u_{n-1}(t,x)) \geq 0$$

since g is increasing. Using the weak maximum principle, one gets $u_{n+1}(t,x) - u_n(t,x) \geq 0$ for all t,x. One concludes by induction. Similarly, one can show that for all n, $u_n \leq w$.

Let $v(t,x)$ be the simple limit of the sequence $(u_n(t,x))$ for all (t,x). The Sobolev injections and the Schauder parabolic estimates yield that (u_n) is bounded in $\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}_{per}(\mathbb{R} \times \mathbb{R}^N)$. Then one can extract a sequence $(u_{n'})$ that converges in $\mathcal{C}^{1,2}_{per}(\mathbb{R} \times \mathbb{R}^N)$. Thus v belongs to $\mathcal{C}^{1,2}_{per}(\mathbb{R} \times \mathbb{R}^N)$ and it is a solution of the equation $v = Tv$. This yields that v solves (5). $\square$

Proof of Theorem 1.1. First case: $\lambda'_1 < 0$ and hypothesis (7) is satisfied.

The constant M given by hypothesis (7) is a supersolution of (5). Since f is of class $\mathcal{C}^1$ in $\mathbb{R} \times \mathbb{R}^N \times [0,\beta]$ (with $\beta > 0$), for κ small enough, as λ'_1 is negative, one gets:

$$f(t,x,\kappa\phi_0) \geq \mu(t,x)\kappa\phi_0 + \frac{\lambda'_1}{2}\kappa\phi_0 \text{ in } \mathbb{R} \times \mathbb{R}^N.$$

Therefore, it follows that:

$$\partial_t(\kappa\phi_0) - \nabla \cdot (A(t,x)\nabla\kappa\phi_0) + q(t,x) \cdot \nabla\kappa\phi_0 - f(t,x,\phi_0) \leq \frac{\lambda_1'}{2}(\kappa\phi_0) < 0.$$

As $\kappa\phi_0$ is periodic in t and x, it is a subsolution of (5). The previous theorem leads to the existence of a positive solution of (5).

Second case: $\lambda_1' \geq 0$ and hypothesis (6) is satisfied.

Assume that p is a nonnegative bounded entire solution of (2). Let ϕ_0 be the positive eigenvalue associated to $\mathcal{L} = L_0$ such that $\|\phi_0\|_\infty = 1$ and set:

$$\gamma^* = \inf\{\gamma > 0, \gamma\phi_0 > p \text{ in } \mathbb{R} \times \mathbb{R}^N\} \geq 0.$$

Since p is bounded and ϕ_0 is a positive, periodic, continuous function, γ^* is well-defined. Assume that $\gamma^* > 0$ and set $z = \gamma^*\phi_0 - p$. Then, $z \geq 0$ and there exists a sequence $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^N$ such that $z(t_n, x_n) \to 0$ as $n \to 0$.

There exists a sequence $(\overline{t_n}, \overline{x_n}) \in [0, T] \times \overline{C}$ such that $t_n - \overline{t_n} \in T\mathbb{Z}$ and $x_n - \overline{x_n} \in \Pi_{i=1}^N L_i\mathbb{Z}$. Up to the extraction of some subsequence, we can assume that $\overline{t_n} \to \overline{t_\infty}$ and $\overline{x_n} \to \overline{x_\infty}$ as $n \to \infty$.

Next, set $\phi_n(t,x) = \phi_0(t + t_n, x + x_n)$ and $p_n(t,x) = p(t + t_n, x + x_n)$. From hypothesis (6) and from the periodicity of f and A, one gets:

$$\partial_t p_n - \nabla \cdot (A(t + \overline{t_n}, x + \overline{x_n})\nabla p_n) + q(t + \overline{t_n}, x + \overline{x_n}) \cdot \nabla p_n$$
$$= f(t + \overline{t_n}, x + \overline{x_n}, p_n) \text{ in } \mathbb{R} \times \mathbb{R}^N,$$

$$\partial_t(\gamma^*\phi_n) - \nabla \cdot (A(t + \overline{t_n}, x + \overline{x_n})\nabla(\gamma^*\phi_n)) + q(t + \overline{t_n}, x + \overline{x_n}) \cdot \nabla(\gamma^*\phi_n)$$
$$- f(t + \overline{t_n}, x + \overline{x_n}, \gamma^*\phi_n) > \lambda_1'\gamma^*\phi_n \geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N.$$

From the standard parabolic estimates, it follows that, up to the extraction of a subsequence, p_n converge to a function p_∞ that satisfies:

$$\partial_t p_\infty - \nabla \cdot (A(t + \overline{t_\infty}, x + \overline{x_\infty})\nabla p_\infty) + q(t + \overline{t_\infty}, x + \overline{x_\infty}) \cdot \nabla p_\infty$$
$$= f(t + \overline{t_\infty}, x + \overline{x_\infty}, p_\infty) \text{ in } \mathbb{R} \times \mathbb{R}^N.$$

On the other hand, ϕ_n converges to the function $\phi_\infty = \phi(. + \overline{t_\infty}, . + \overline{x_\infty})$, that satisfies:

$$\partial_t(\gamma^*\phi_\infty) - \nabla \cdot (A(t + \overline{t_\infty}, x + \overline{x_\infty})\nabla(\gamma^*\phi_\infty)) + q(t + \overline{t_\infty}, x + \overline{x_\infty}) \cdot \nabla(\gamma^*\phi_\infty)$$
$$- f(t + \overline{t_\infty}, x + \overline{x_\infty}, \gamma^*\phi_\infty) > \lambda_1'\phi_\infty \geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N.$$

Set $z_\infty = \gamma^*\phi_\infty - p$. The definition of γ^* yields that $z_\infty \geq 0$ and $z_\infty(0,0) = 0$. It follows from the strong parabolic maximum principle that $z_\infty(t,x) = 0$ for all $t \leq 0, x \in \mathbb{R}^N$, which contradicts the previous strict inequality. Finally, $\gamma^* = 0$ and thus $p = 0$. This ends the proof.□

4 Uniqueness of the solution

4.1 Uniqueness of the periodic solution

Proof of Theorem 1.3. Set: $\gamma^* := \sup\{\gamma > 0, u(t, x) > \gamma p(t, x), \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N\}$. One has $\gamma^* > 0$ since $\inf_{\mathbb{R} \times \mathbb{R}^N} u > 0$. If $\gamma^* \geq 1$, then one can conclude using a symmetry argument. Assume that $\gamma^* < 1$.

Let $z = u - \gamma^* p$, then $\inf_{\mathbb{R} \times \mathbb{R}^N} z = 0$. There exists $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^N$ such that $z(t_n, x_n) \to 0$. Let $(s_n, y_n) \in [0, T] \times \overline{C}$ such that: $\forall n, \ x_n - y_n \in \prod_i L_i \mathbb{Z}$ and $t_n - s_n \in T\mathbb{Z}$. Up to extraction, one may assume that $y_n \to y_\infty$ and $s_n \to s_\infty$.

Set: $u_n(t, x) = u(t + t_n, x + x_n)$ and $p_n(t, x) = p(t + t_n, x + x_n)$, one has:

$$\partial_t u_n - \nabla \cdot (A(t + s_n, x + y_n)\nabla u_n) + q(t + s_n, x + y_n) \cdot \nabla u_n = f(t + s_n, x + y_n, u_n),$$

$$\partial_t p_n - \nabla \cdot (A(t + s_n, x + y_n)\nabla p_n) + q(t + s_n, x + y_n) \cdot \nabla p_n = f(t + s_n, x + y_n, p_n).$$

Using the classical parabolic estimates, up to extraction, one may suppose that $u_n \to u_\infty$ and $p_n \to p_\infty$ in $\mathcal{C}^{1,2}_{loc}(\mathbb{R} \times \mathbb{R}^N)$. One has $\inf_{\mathbb{R} \times \mathbb{R}^N} u_\infty > 0$ and $\inf_{\mathbb{R} \times \mathbb{R}^N} p_\infty > 0$. If $z_\infty = u_\infty - \gamma^* p_\infty$, then $z_\infty \geq 0$ and $z_\infty(0, 0) = 0$. Moreover:

$$\begin{aligned}
&\partial_t z_\infty - \nabla \cdot (A(t + s_\infty, x + y_\infty)\nabla z_\infty) + q(t + s_\infty, x + y_\infty) \cdot \nabla z_\infty \\
&= f(t + s_\infty, x + y_\infty, u_\infty) - \gamma^* f(t + s_\infty, x + y_\infty, p_\infty) \\
&> f(t + s_\infty, x + y_\infty, u_\infty) - f(t + s_\infty, x + y_\infty, \gamma^* p_\infty).
\end{aligned}$$

since $\gamma^* < 1$ and hypothesis (6) is satisfied. On the other hand, f is Lipschitz-continuous, and one can define a bounded function b that satisfies:

$$\partial_t z_\infty - \nabla \cdot (A(t + s_\infty, x + y_\infty)\nabla z_\infty) + q(t + s_\infty, x + y_\infty)\nabla z_\infty - b(t, x)z_\infty > 0.$$

As $z_\infty(0, 0) = 0$ and $z_\infty \geq 0$, the strong parabolic maximum principle leads to:

$$\forall x \in \mathbb{R}^N, \ t \leq 0, \ z_\infty(t, x) = 0.$$

Using the previous partial differential inequation, one gets a contradiction. $\square$

4.2 Uniqueness of the entire solution

Proof of Theorem 2.6. First, using a translation of the origin in time, one easily sees that $\lambda_R^{t_0, x_0}$ does not depend on t_0. It has been proved by the author in [20] that $\lambda_R^{0, x_0} \searrow \lambda_1$ as $R \to +\infty$. But the function $x_0 \to \lambda_R^{0, x_0}$ is periodic and continuous. The Dini's lemma then yields that the convergence is uniform on every compact subset K of $\mathbb{R}^N$. Taking $K = \overline{C}$, we end the proof. $\square$

We first prove the following lemma:

Lemma 4.1 *If u satisfies the same hypotheses as in the statement of Theorem 1.5, then for all compact subset $K \subset \mathbb{R}^N$, one has:* $\inf_{x \in K, t \in \mathbb{R}} u(t, x) > 0$.

Proof. Assume that there exists $x_n \in K, t_n \in \mathbb{R}$ such that $u(t_n, x_n) \to 0$. For all n, take $s_n \in [0, T)$ such that $t_n - s_n \in \mathbb{Z}T$. Up to extraction, the compactness yields that we can assume that $s_n \to s_\infty$ and $x_n \to x_\infty$.

Set $u_n(t, x) = u(t + t_n, x)$. This function satisfies:

$$\partial_t u_n - \nabla \cdot (A(t + s_n, x)\nabla u_n) + q(t + s_n, x) \cdot \nabla u_n = f(t + s_n, x, u_n)$$

The classical Schauder estimates yield that one may assume the convergence of the sequence $(u_n)_n$ to a function u_∞ in $\mathcal{C}^{1,2}_{loc}$ such that:

$$\partial_t u_\infty - \nabla \cdot (A(t + s_\infty, x)\nabla u_\infty) + q(t + s_\infty, x) \cdot \nabla u_\infty = f(t + s_\infty, x, u_\infty)$$

and $u_\infty(0, x_\infty) = 0$. As u_∞ is nonnegative, the strong maximum principle yields $\forall t \leq 0, \forall x, u_\infty(t, x) = 0$.

In the other hand, set $\varepsilon = \inf_{t \in \mathbb{R}} u(t, x_0) > 0$. Then for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, one has $u_n(t, x_0) \geq \varepsilon$ and then for all $t \in \mathbb{R}$, $u_\infty(t, x_0) \geq \varepsilon > 0$, which is a contradiction. $\square$

Proof of Theorem 1.5. The previous approximation Theorem 2.6 yields that it exists $R_0 > 0$ such that:

$$\forall R \geq R_0, \ \forall (s, y) \in \mathbb{R} \times \mathbb{R}^N, \ \lambda_R^{s,y} < \frac{\lambda_1}{2} < 0.$$

In the following, we fix $R \geq R_0$.

We know that there exists $\beta > 0$ such that $f \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^N \times [0, \beta])$, furthermore $f(t, x, 0) = 0$ and f is periodic in t and x. Thus, it exists κ_0 such that

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, (s, y) \in \mathbb{R} \times \mathbb{R}^N, 0 < \kappa < \kappa_0, \ f(t + s, x + y, \kappa) \geq \kappa\mu(t + s, x + y) + \frac{\lambda_1}{2}\kappa.$$

We define $u^{s,y}(t, x) = u(t + s, x + y)$, $u^{s,y}$ satisfies:

$$\partial_t u^{s,y} - \nabla \cdot (A(t + s, x + y)\nabla u^{s,y}) + q(t + s, x + y) \cdot \nabla u^{s,y} = f(t + s, x + y, u^{s,y}) \text{ in } \mathbb{R} \times \mathbb{R}^N.$$

Furthermore for all $0 < \kappa < \kappa_0$:

$$\begin{aligned}
&\partial_t(\kappa\phi_R^{s,y}) - \nabla \cdot (A(t + s, x + y)\nabla(\kappa\phi_R^{s,y})) + q(t + s, x + y) \cdot \nabla(\phi_R^{s,y}) \\
&\quad - f(t + s, x + y, \kappa\phi_R^{s,y})
\end{aligned}$$

$$\begin{aligned}
&\leq \kappa\Big\{\partial_t(\phi_R^{s,y}) - \nabla \cdot (A(t + s, x + y)\nabla(\phi_R^{s,y})) \\
&\quad + q(t + s, x + y) \cdot \nabla(\phi_R^{s,y}) - \mu(t + s, x + y)\phi_R^{s,y} - \frac{\lambda_1}{2}\phi_R^{s,y}\Big\}
\end{aligned}$$

$$\leq (\lambda_R^{s,y} - \frac{\lambda_1}{2})\kappa\phi_R^{s,y}$$

$$< 0 \text{ in } \mathbb{R} \times B_R(0).$$

Thus $\kappa\phi_R^{s,y}$ is a subsolution of the equation satisfied by $u^{s,y}$. We want to prove that $u^{s,y} \geq \kappa_0\phi_R^{s,y}$ in $\mathbb{R} \times \overline{B_R(0)}$. If this was not true, we set:

$$\kappa^* := \sup\{\kappa > 0, \kappa\phi_R^{s,y} < u^{s,y} \text{ in } \mathbb{R} \times \overline{B_R(0)}\}.$$

This quantity is well-defined since $\inf_{x \in \overline{B_R(0)}, t \in \mathbb{R}} u(t+s, x+y) > 0$ because of Lemma 4.1. Assume that $\kappa^* < \kappa_0$.

Set $z = u^{s,y} - \kappa^* \phi_R^{s,y}$. Then $z \geq 0$ and it exists $(t_n, x_n) \in \mathbb{R} \times \overline{B_R(0)}$ such that $z(t_n, x_n) \to 0$. Set $s_n \in [0, T]$ such that $t_n - s_n \in T\mathbb{Z}$. One may assume that $(s_n, x_n) \to (s_\infty, x_\infty)$. We define $z_n(t, x) = z(t + t_n, x)$. Then, up to extraction, we can assume that $z_n \to z_\infty$ in $\mathcal{C}_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$. Since f is Lipschitz-continuous in u, it exists a bounded function b such that z_∞ satisfies:

$$\partial_t z_\infty - \nabla \cdot (A(t+s+s_\infty, x+y) \nabla z_\infty) + q(t+s+s_\infty, x+y) \cdot \nabla z_\infty - b(t, x) z_\infty$$
$$\geq 0 \text{ in } \mathbb{R} \times B_R(0).$$

In the other hand, $z_\infty \geq 0$ and this function vanishes at the point $(0, x_\infty)$. Since $\phi_R^{s,y}(t + s_\infty, x)(t, x) = 0$ if $(t, x) \in \mathbb{R} \times \partial B_R(0)$, one has $x_\infty \in B_R(0)$. Then, the strong parabolic maximum principle yields that, $z_\infty(t, x) \equiv 0$ for all $t \leq 0$ and $x \in \overline{B_R(0)}$. This contradicts the conditions on $\partial B_R(0)$.

Thus $u^{s,y} \geq \kappa_0 \phi_R^{s,y}$ in $\mathbb{R} \times \overline{B_R(0)}$. In particular $u^{s,y}(0,0) = u(s, y) \geq \kappa_0 \phi_R^{s,y}(0,0)$ for all $(s, y) \in \mathbb{R} \times \mathbb{R}^N$. Moreover $(s, y) \to \phi_R^{s,y}(0,0)$ is periodic in s and y, thus its minimum is positive. This concludes the proof.$\square$

4.3 Large time behavior

Proof of Theorem 1.6

First case: $\lambda_1 < 0$. Take $s_n \to +\infty$ and set $u_n(t, x) = u(x, t + s_n)$, u_n is defined over $[-s_n, +\infty[$. For all n, set $\overline{s_n} \in [0, T]$ such that $s_n - \overline{s_n} \in T\mathbb{Z}$, then, up to extraction, one may assume that $\overline{s_n} \to s_\infty$.

The function u_n satisfies the equation:

$$\partial_t u_n - \nabla \cdot (A(t + \overline{s_n}, x) \nabla u_n) + q(t + \overline{s_n}, x) \cdot \nabla u_n = f(t + \overline{s_n}, x, u_n)$$

Take K a compact subset of $\mathbb{R}^N$ and $[T_1, T_2]$ a compact subset of $\mathbb{R}$. It exists a rank n_0 such that for all $n \geq n_0$, u_n is well defined over $[T_1, T_2]$. The classical parabolic estimates yield that it exists $M > 0$ such that for all $n \geq n_0$: $\|u_n\|_{\mathcal{C}^{1+\frac{\alpha}{2}, 2+\alpha}([T_1, T_2] \times K)} \leq M$.

Thus, one can extract a subsequence (that we still call $(u_n)_n$) that converges in $\mathcal{C}^{1,2}([T_1, T_2] \times K)$. Using a diagonal method, one can extract a subsequence such that $u_n \to u_\infty$ in $\mathcal{C}_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$.

The function u_∞ satisfies:

$$\partial_t u_\infty - \nabla \cdot (A(t + s_\infty, x) \nabla u_\infty) + q(t + s_\infty, x) \cdot \nabla u_\infty = f(t + s_\infty, x, u_\infty), \qquad (13)$$

and $u_\infty \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$, u_∞ is nonnegative.

In order to conclude, as $\lambda_1 < 0$, the uniqueness Theorem 1.5 yields that one only has to prove that $\exists x_0 | \inf_{t \in \mathbb{R}} u_\infty(t, x_0) > 0$ and that u is bounded.

As $u_0 \neq 0$ is positive, the parabolic Harnack inequalities yield that $\inf_{x \in B_R(0)} u(1, x) > 0$, for all $R > 0$. One can choose $\kappa > 0$ such that for all $x \in B_R(0), u(1, x) > \kappa \phi_R(1, x)$, where, using the same notation as in the proof of Theorem 1.5, $R > R_0$ is fixed and $\phi_R = \phi_R^{0,0}$. We set:

$$v_1(t, x) = \begin{cases} \kappa \phi_R(t, x) \text{ if } x \in B_R(0), \\ 0 \text{ otherwise.} \end{cases}$$

We have already prove that for κ small enough, v_1 is a subsolution of (2). Set $z(t,x) = u(1+t,x) - v_1(1+t,x)$, then $z(0,x) \geq 0$ and it exists a bounded function b such that z satisfies:

$$\partial_t z - \nabla \cdot (A(1+t,x)\nabla z) + q(1+t,x) \cdot \nabla z + b(t,x)z \geq 0, \text{ in } \mathbb{R}^+ \times B_R(0).$$

Using the parabolic maximum principle, one gets $z(t,x) \geq 0$, for all $t \geq 0, x \in B_R(0)$. In other words: $\forall t \geq 1, x \in B_R(0), u(t,x) \geq \kappa \phi_R(t,x)$.

Next, take n_0 large enough in order to have $n \geq n_0$, $t + s_n \geq 1$. Then for all $n \geq n_0, u_n(t,x) \geq \kappa \phi_R(t + \overline{s_n}, x)$, thus: $u_\infty(t,x) \geq \kappa \phi_R(t + s_\infty, x)$. Finally, taking $x = 0$, one gets:

$$\inf_{t \in \mathbb{R}} u_\infty(t,0) \geq \kappa \inf_{t \in \mathbb{R}} \phi_R(t + s_\infty, 0) > 0.$$

Hypothesis (7) yields that it exists $M > 0$ such that:

$$\forall s \geq M, \forall t, x, f(t,x,s) \leq 0.$$

The initial datum u_0 is bounded and thus one can take M large enough such that: $\|u_0\|_\infty \leq M$. We already know that M is a supersolution of (5). Thus, using the same construction as previously, we can show that $u(t,x) \leq M$ for all (t,x).

This yields that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, $u_\infty(t,x) \leq M$. Thus u_∞ is bounded. The uniqueness Theorem 1.3 gives the conclusion.

Second case: $\lambda_1' \geq 0$. Consider M as above and set v_2 the solution of the Cauchy problem (11) with $v_2(0,x) = M$ for all $x \in \mathbb{R}^N$. We know that $u(t,x) \leq v_2(t,x)$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$. Moreover, as carried above, for all sequence $s_n \to +\infty$, up to extraction, $v_2(t + s_n, x) \to v_\infty(t,x)$ in $\mathcal{C}_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ as $n \to +\infty$ where v_∞ is nonnegative, bounded and satisfies (13). But, as $\lambda_1' \geq 0$, Theorem 1.1 yields $v_\infty \equiv 0$. Thus the standard arguments give $v_2(t,x) \to 0$ as $t \to +\infty$ locally uniformly with respect to $x \in \mathbb{R}^N$. Moreover, as the constant function M is space periodic, the function $v_2(t, \cdot)$ stays periodic in x for all $t \geq 0$. Thus its convergence to 0 is uniform in x. Finally, one gets $u(t,x) \to 0$ as $t \to +\infty$ uniformly in x. $\square$

Proof of Proposition 2.7. Take $\phi_\alpha > 0$ the principal eigenfunction associated with k_α such that

$$\min_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \phi_\alpha(t,x) = 1.$$

We define $v(t,x) = R\phi_\alpha(t,x)e^{\alpha \cdot x - k_\alpha t}$.

Then v satisfies $u_0 \leq v(0, \cdot)$ and:

$$\partial_t v - \nabla \cdot (A(t,x)\nabla v) + q(t,x) \cdot \nabla v = \mu(t,x)v \geq f(t,x,v).$$

Thus, the function v is a supersolution of the Cauchy problem (11) satisfied by u. The parabolic maximum principle yields that for all $t \geq 0, x \in \mathbb{R}^N, u(t,x) \leq v(t,x)$.

Since $k_\alpha > 0$, one has $v(t + s, x) \to 0$ as $s \to +\infty$ uniformly on every compact subset of $\mathbb{R} \times \mathbb{R}^N$. We conclude that the same convergence holds for u. This ends the proof. $\square$

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Proof of Proposition 2.8. As $k_\alpha < 0$, by continuity, one might assume that $k_{(1+\varepsilon)\alpha} < 0$. For all β, take $\phi_\beta > 0$ the principal eigenfunction associated with k_β such that $\|\phi_\beta\|_\infty = 1$. We define $w(t,x) = \kappa\phi_\alpha(t,x)e^{\alpha \cdot x} - R\phi_{(1+\varepsilon)\alpha}(t,x)e^{(1+\varepsilon)\alpha \cdot x}$, where R is such that $w(0,x) \leq u_0(x)$ for all $x \in \mathbb{R}^N$.

Take $\delta > 0$ such that $\delta < |k_{(1+\varepsilon)\alpha}|$. There exists $\eta > 0$ such that :

$$\forall (t,x,s) \in \mathbb{R} \times \mathbb{R}^N \times [0,\eta], \ (\mu(t,x) - \delta)s \leq f(t,x,s).$$

One can fix A such that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, one has $w(t,x) \leq \eta$. Next, set $\Omega = \{(t,x) \in \mathbb{R} \times \mathbb{R}^N, w(t,x) > 0\}$. One can compute in Ω:

$$\partial_t w - \nabla \cdot (A\nabla w) + q \cdot \nabla w = (\mu - \delta)w + \kappa(k_\alpha + \delta)\phi_\alpha e^{\alpha \cdot x} - R(k_{(1+\varepsilon)\alpha} + \delta)\phi_{(1+\varepsilon)\alpha}e^{(1+\varepsilon)\alpha \cdot x}$$

$$\leq f(t,x,w) - \kappa(|k_\alpha| - \delta)\phi_\alpha e^{\alpha \cdot x} + R(|k_{(1+\varepsilon)\alpha}| - \delta)\phi_{(1+\varepsilon)\alpha}e^{(1+\varepsilon)\alpha \cdot x}$$

$$\leq f(t,x,w) - \kappa(|k_\alpha| - \delta)\phi_\alpha e^{\alpha \cdot x} + \kappa(|k_{(1+\varepsilon)\alpha}| - \delta)\phi_\alpha e^{\alpha \cdot x} \leq f(t,x,w),$$

(14)

since $w > 0$ in Ω and $|k_\alpha| \geq |k_{(1+\varepsilon)\alpha}|$. Thus, the function w is a subsolution of the equation satisfied by u in Ω. Moreover, for all $(t,x) \in \partial\Omega$, one has $w(t,x) = 0 \leq u(t,x)$. The parabolic strong maximum principle yields that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, one has $w(t,x) \leq u(t,x)$.

Set

$$r = \frac{1}{\varepsilon}\ln\Big(\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \frac{\kappa\phi_\alpha(t,x)}{R\phi_{(1+\varepsilon)\alpha}(t,x)}\Big)$$

and take an arbitrary x_0 such that $\alpha \cdot x_0 < r$. Then for all $t \in \mathbb{R}$, one has $u(t,x_0) > w(t,x_0) > 0$ and, as w is periodic in t, there exists a positive constant $m > 0$ such that: $\inf_{t\in\mathbb{R}^+} u(t,x_0) \geq m > 0$.

Lastly, as in the prove of Theorem 1.6, we take an arbitrary sequence $s_n \to +\infty$ and we consider the sequence $u_n(t,x) = u(t + s_n, x)$. Then one can extract a subsequence $(u_{n'})$ that converges to a positive bounded entire solution u_∞ of (2). As $\inf_{t\in\mathbb{R}} u_\infty(t,x_0) \geq m > 0$, Theorem 1.5 yields that $u_\infty \equiv p$. This concludes the proof. $\square$

5 Conclusion

In this article, we proved that the existence of a positive periodic solution p of equation (2) is determined by the sign of a generalized principal eigenvalue λ_1' and that, when it exists, it is the unique possible positive periodic solution. Similarly, we proved that the only possible positive entire solution that satisfies $\exists x_0 \mid \inf_{t\in\mathbb{R}} u(t,x_0) > 0$ is the positive periodic solution p if the other generalized principal eigenvalue λ_1 is negative.

This uniqueness result for entire solutions allowed us to prove that all the solutions of the Cauchy problem associated with (2) go to zero if $\lambda_1 \geq 0$ and to p if $\lambda_1' < 0$. In the case where $\lambda_1' < 0 \leq \lambda_1$, there is no possible general conclusion to this problem. If the initial datum can be compared with an exponential function at infinity, then we obtained an optimal result that states if the solution of the Cauchy problem goes to 0 or p.

We underline that the uniqueness and the attractivity of the space-time periodic solution in the class of periodic solutions might be proved using other methods. The main difficulties in the present paper arise when one consider general entire solutions or initial data. For example, the uniqueness and the attractivity in the class of periodic solutions could be proved through the method of the part metric, described in [19, 26, 29]. If $u, v \in \mathcal{C}^0(\mathbb{R}^N)$ are two positive periodic functions, there exists $\alpha \geq 1$ such that $\frac{1}{\alpha}v \leq u \leq \alpha v$. The part metric is then defined as

$$d(u, v) = \inf\{\ln \alpha, \ \alpha \geq 1 \text{ satisfies } \frac{1}{\alpha}v \leq u \leq \alpha v\}.$$

It has been proved in [19, 26, 29] that under hypothesis (6), if u, v are the associated solutions of the Cauchy problem (11), then

$$t \mapsto d(u(t, \cdot), v(t, \cdot)) \text{ decreases with respect to } t \in \mathbb{R}^+.$$

This gives the uniqueness of the space-time periodic solution of (5) and its attractivity.

We underline that this method does not work at all when one considers, for example, compactly supported initial data since the part metric is not defined anymore in this case.

These results mean that the species survival only depends on the stability of the null state. There are two possible generalized principal eigenvalues that can characterize this stability. In [20], we studied the influence of the coefficients (A, q, μ) on these two generalized principal eigenvalues. If one has $\lambda_1(A, q, \mu) < \lambda_1(A', q', \mu')$ and $\lambda_1'(A, q, \mu) < \lambda_1'(A', q', \mu')$, that is, if changing an environment (A, q, μ) into an environment (A', q', μ') increases the two generalized principal eigenvalues, then we can say that the environment (A, q, μ) is better for the species survival than the environment (A', q', μ') since the case where there is extinction of the species in the environment (A', q', μ') and not in the environment (A, q, μ) can arise. But this comparison does not hold true if the eigenvalues associated with the two environment are both negative. In this case, we do not have a mean to compare the behavior of the species in these two environments.

In [21], the author proves that in this case, there exists pulsating travelling fronts solutions. In [3], Berestycki, Hamel and the author prove spreading behaviors and compute the spreading speed of a solution with compactly supported initial datum with the help of the periodic principal eigenvalues $(k_\alpha)_\alpha$. This gives another way to compare the effect of different environments on the species behavior.

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