

# Propagation phenomena in various reaction-diffusion models

## Habilitation à diriger des recherches

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*To Joachim, Léo and Mathilde.*



## Résumé

### Phénomènes de propagation pour différents modèles de réaction-diffusion

Les équations de réaction-diffusion engendrent des phénomènes de propagation, qu'on peut quantifier mathématiquement à travers deux notions : en localisant les lignes de niveau des solutions (vitesses d'expansion) ou à l'aide de solutions de type traveling waves. Le but de ce manuscrit est de discuter de ces deux notions pour différentes variantes de l'équation de Fisher-KPP.

Dans le premier et le second chapitres, nous étudierons ces phénomènes pour des équations de Fisher-KPP hétérogènes dépendant de façon générale de  $t$  et de  $x$ . Des bornes sur les vitesses d'expansion dans chaque direction peuvent être obtenues à l'aide de valeurs propres généralisées, donnant des vitesses d'expansion exactes notamment pour des coefficients presque périodiques, uniquement ergodiques ou encore constant à l'infini dans des secteurs angulaires. Nous discuterons ensuite de la notion de front de transition généralisé pour cette équation, et montrerons leur existence quand les coefficients ne dépendent que de  $t$  ou bien sont presque périodiques en  $x$ .

Le troisième chapitre traite des équations de Fisher-KPP non-locales ou à retard. Ces équations n'admettant plus de principe de comparaison, le comportement asymptotique des traveling waves est encore mal compris. Une nouvelle méthode numérique appuie la conjecture de la convergence d'un état homogène vers l'autre pour des noyaux symétriques. Pour des noyaux asymétriques, la convergence des traveling waves dépend de la position de l'asymétrie par rapport à la direction de propagation.

Une équation de réaction-diffusion cinétique, avec un noyau mesurant la probabilité de changer de vitesse, sera étudiée dans le quatrième chapitre. Si ce noyau est à support compact, on retrouve l'existence de traveling waves, dont on peut caractériser les vitesses. Dans le cas d'un noyau somme de deux Dirac, on retrouve une équation de Fisher-KPP avec un retard dans la loi de diffusion, dont on discutera une application en archéologie. Dans le cas d'un noyau Gaussien, les lignes de niveau se propagent en  $t^{3/2}$ , et on cherchera à déterminer ce taux de propagation de deux façons différentes.

Enfin, dans la cinquième partie, nous présenterons plusieurs résultats sur la dépendance entre les coefficients de l'équation et la vitesse de propagation dans le cas d'une équation périodique en espace. On peut en particulier montrer que le réarrangement de Schwarz du terme de croissance accélère la propagation en dimension 1, mais que la situation est beaucoup moins tranchée en dimension supérieure.

**Mots-clés:** équations de réaction-diffusion, traveling waves, vitesse d'expansion, équations de Hamilton-Jacobi, équations nonlocales et à retard, valeurs propres principales, réarrangement de Schwarz.

# Abstract

## Propagation phenomena in various reaction-diffusion models

Reaction-diffusion equations generate propagation phenomena, which can be quantified mathematically through two notions : by locating the level lines of the solutions (expansion speeds) or using traveling waves. The purpose of this manuscript is to discuss these two concepts for different variants of the Fisher-KPP equation.

In the first and the second chapters, we will study these phenomena for heterogeneous Fisher-KPP equations depending in a general way on  $t$  and  $x$ . Bounds on the expansion speeds in each direction can be obtained with the aid of generalized eigenvalues, giving exact expansion speeds in particular for coefficients that are almost periodic, uniquely ergodic or constant at infinity in angular sectors. We will then discuss the notion of generalized transition waves for this equation and show their existence when the coefficients depend only on  $t$  or else are almost periodic in  $x$ .

The third chapter deals with non-local and delayed Fisher-KPP equations. Since these equations no longer admit a comparison principle, the asymptotic behavior of traveling waves is still poorly understood. A new numerical method supports the conjecture of the convergence of one homogeneous state towards the other for symmetric kernels. For asymmetric kernels, the convergence of the traveling waves depends on the position of the asymmetry with respect to the direction of propagation.

A kinetic reaction-diffusion equation, with a probability kernel measuring the velocity changes, will be studied in the fourth chapter. If this kernel is compactly supported, the existence of traveling waves can be proved and their velocities can be characterized. In the case where the kernel is a sum of two Dirac masses, we find a Fisher-KPP equation with a delay in the diffusion law, of which an application in archaeology will be discussed. In the case of a Gaussian kernel, the level lines propagate in  $t^{3/2}$ , and we will seek to determine this propagation rate in two different ways.

Finally, in the fifth part, we present several results on the dependence between the coefficients of the equation and the speed of propagation in the case of a periodic equation in space. In particular, one can show that the Schwarz rearrangement of the growth term accelerates the propagation in dimension 1, but that the situation is much less understood in higher dimensions.

**Keywords:** reaction-diffusion equations, traveling waves, expansion speeds, Hamilton-Jacobi equations, nonlocal and delayed equations, principal eigenvalues, Schwarz rearrangement.



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# Introduction

This manuscript is devoted to the investigation of reaction-diffusion equations, which read :

$$\partial_t u = \underbrace{A[u]}_{\text{Diffusion term}} + \underbrace{F[u]}_{\text{Reaction term}},$$

where the unknown  $u$  is a scalar or vectorial function depending on time  $t$  and location  $x$ .

The investigation of this type of equations goes back to the 30's, with the parallel works of Kolmogorov, Petrovsky and Piskunov [63] and Fisher [39] on the scalar equation, today called the *Fisher-KPP* equation :

$$\partial_t u = D\Delta u + ru(1 - u). \quad (1)$$

This type of model arise naturally in population dynamics, where  $u$  is a population density,  $D$  is a dispersal rate and  $r$  is a growth rate at low density. The greater the population density, the greater the intensity of the competition between individuals for access to the resources, which are assumed to be limited, until it reaches a maximum carrying capacity of the media, which can always be assumed to be equal to  $u = 1$  up to renormalization.

The aim of this manuscript is to understand and to describe the propagation phenomena emerging from the reaction-diffusion equations. We can quantify these phenomena using two different but nested mathematical notions.

## 1 First notion : expansion speed

The utility of this model was confirmed empirically in Skellam's work in 1951 [97]. In order to understand the invasion of the muskrat in Eastern Europe, he plotted the area occupied by the population of muskrats as a function of the year of observation. The result, reproduced Figure 0.1, shows a linear growth of the root of the area occupied as a function of time. Skellam found this growth rate heuristically via the equation (1), and this result was rigorously formalized later by Aronson and Weinberger [6] : if  $u(0, \cdot)$  is compactly supported, then the function  $u$  "occupies", after a sufficiently long time  $t$ , a ball with a radius  $w^*t$ , which area  $\sqrt{\pi}w^*t$  is linear in  $t$ . More precisely, the level lines of  $u(t, \cdot)$  are located approximately around the sphere of radius  $w^*t$ .

This result can be mathematically formalized in the following way.

**Theorem 0.1.** [6] *Let  $u_0 = u(0, \cdot) \not\equiv 0$  be an initial datum with compact support, positive, bounded, and non-zero. Let  $u$  be the solution of the Cauchy problem associated with (1) and  $u_0$ . So :*

$$\begin{cases} \lim_{t \rightarrow +\infty} u(t, x) = 1 & \text{unif. on } |x| \leq ct & \text{if } 0 \leq w < 2\sqrt{Dr}, \\ \lim_{t \rightarrow +\infty} u(t, x) = 0 & \text{unif. on } |x| \geq ct & \text{if } w > 2\sqrt{Dr}, \end{cases}$$

We call  $w^* = 2\sqrt{Dr}$  the speed of expansion.

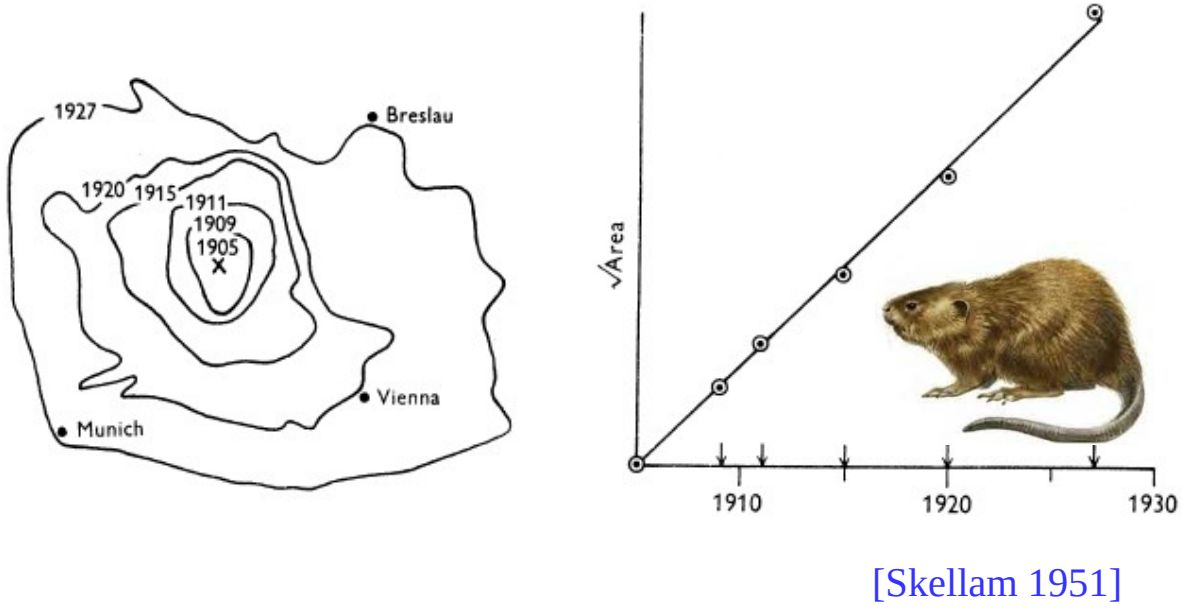


FIGURE 1 – Muskrat invasion in Eastern Europe.

## 2 Second notion : traveling waves

We know that solutions propagate with speed  $w^*$ . One can then ask what is the asymptotic behavior of the solutions around the transition between the unstable state 0 and the stable state 1. This is where the traveling waves are useful.

**Definition 0.2.** A *traveling wave of speed  $c$  and direction  $e \in \mathbb{S}^{N-1}$*  is a solution of (1)  $U(t, x) = U(x - cte)$ , with  $U > 0$ ,  $U(-\infty) = 1$ ,  $U(+\infty) = 0$ .

For the Fisher-KPP equation, traveling waves exist for a half-line of speeds.

**Theorem 0.3.** [6, 63] *There is a traveling wave of speed  $c$  if and only if  $c \geq c^* := 2\sqrt{Dr}$ .*

One immediately notices that the minimum speed of existence of traveling waves is equal to the expansion speed  $w^*$ . We can precise this link between the two notions : in dimension 1, the solution of the Cauchy problem associated with a Heaviside [63] or a compactly supported [29] initial datum converges to the minimal speed traveling wave along the level lines, that is, if we define  $u(t, m(t)) = \theta$  for  $\theta \in (0, 1)$ , then  $u(t, x + m(t)) \rightarrow U(x)$  is the traveling wave of speed  $c^*$  normalized by  $U(0) = \theta$ . The translation  $m(t)$  behaves at first order as  $c^*t$  but its development is followed by a logarithmic correction [29, 49].

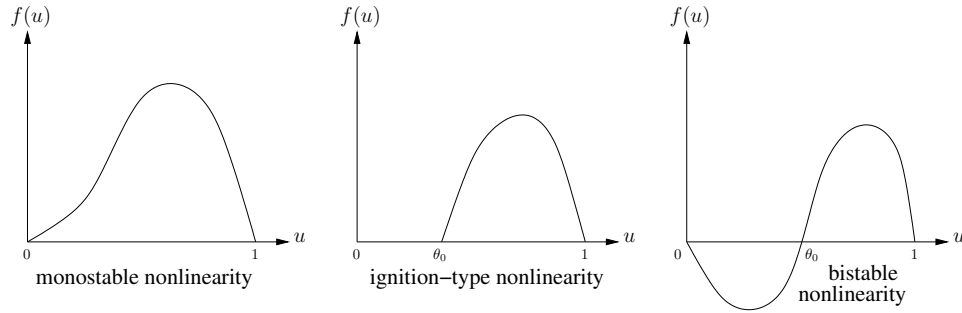
There is therefore a clear link between the two notions in the case of the homogeneous Fisher-KPP equation. But the two notions can diverge for more complex equations, and we will be led to consider one or the other according to the questions we consider.

## 3 Areas of application and other types of nonlinearities

This model of a particularly simple principle thus gives rise to non-trivial mathematical phenomena. It has since been defeated and generalized in many fields of applications, among which :

- population dynamics
- chemistry, for which it is rather reaction-diffusion systems that are used (see, for example, the pioneering article of Turing [100]).
- combustion theory [59, 104], for which the nonlinearities are of the ignition type (see Figure 0.3),
- genetics [6, 9],
- humanities [4].

Nous considérerons dans ce manuscript essentiellement des nonlinearities du type Fisher-KPP. Mais nous serons amenés à mentionner les equations monostables, bistables and de type ignition afin de discuter des results, pour lesquelles la nonlinearity  $ru(1 - u)$  est remplacée par une fonction  $f(u)$  de la forme décrite Figure 0.3.



Cette *habilitation à diriger des recherches* is in the field of applied mathematics. Our aim is therefore to understand the propagation phenomena for more complex reaction-diffusion equations than the Fisher-KPP equation, these variants being motivated by the fields of applications described above, in particular population dynamics, and to interpret, when possible, the results obtained with regard to the applications sought.

## 4 Organisation of the manuscript

Chapters 1 and 2 of this manuscript will deal with the Fisher-KPP equation with general heterogeneous coefficients : we only assume that these coefficients are uniformly continuous and bounded in  $(t, x)$ . The first chapter will be devoted to the expansion speeds for the solutions of the Cauchy problem, that is to say that we will construct as precise bounds as possible localizing the level lines of these solutions. These bounds will be optimal under different additional set of hypotheses on the coefficients : almost periodicity, unique ergodicity, homogeneous in each direction etc. The second chapter will be devoted to the existence of particular time-global solutions : *generalized transition waves*. These waves are global solutions spatially connecting two stationary states of the equation, with an interface depth uniformly bounded in time. The existence of such waves will be proved for the Fisher-KPP equations with coefficients depending only on  $t$ , or depending on  $t$  and being periodic in  $x$ , and for almost periodic coefficients in  $x$ . The alternative notion of *critical wave* will be discussed for one-dimensional equations. This notion is based on the counting of the number of zeros for the solutions of parabolic equations, a critical wave being a global solution in time steeper in  $x$  than all the other global solutions.

Chapter 3 investigates the existence and the asymptotic properties of traveling waves for Fisher-KPP equations with non-local saturation. Such equations no longer admit a comparison principle, thus most of the methods used to treat local equations are obsolete. In one of my PhD work [BNPR09], we have established the existence of traveling waves for an equation with symmetric non-locality, without however determining the asymptotic behavior of the traveling wave : it takes off from the unstable state

but it is not known if it converges to the homogeneous stationary state 1. In the first section, I present a new numerical approach allowing to construct these traveling waves, and supporting the conjecture of a convergence towards the homogeneous stationary state 1, even when this steady state is unstable. The second section discusses a “ toy-model ” with asymmetric kernel, for which we can construct a large variant of traveling waves, with various asymptotic behaviors. Finally, the last section investigates the Fisher-KPP delayed equation, corresponding to a non-local equation with asymmetric kernel, for which we show on the contrary that traveling waves have a well determined asymptotic behavior.

In the fourth chapter, I present recent results on a microscopic version of the Fisher-KPP equation : the density is no longer only parametrized in  $t$  and  $x$ , but also in speed  $v$ , the particles changing speed with a given probability density  $M(v)$  to reach the speed  $v$ . If  $M(v)$  is a sum of two Dirac masses around 0, then we get a Fisher-KPP equation with a hyperbolic term, for which we can construct smooth traveling waves, as for Fisher-KPP, or singular ones, rather close to hyperbolic shocks. This equation can be seen as a reaction-diffusion equation with a delay in the diffusion law. Such an equation has been used to understand the propagation of agriculture in Europe during the Neolithic, an application that I describe in the second section. For a kernel  $M(v)$  with compact support, we can still construct traveling waves and characterize their minimal speed, as described in the third section. The situation becomes more complicated when the kernel is not compactly supported : the traveling waves do not exist anymore and it is then a question of locating the level lines of the solutions of the problem of Cauchy, which grow superlinearly. We have considered the model case of a Gaussian kernel, for which we have managed to locate the level lines between two bounds proportional to  $t^{3/2}$ . In the fifth section, I present results on the linearization of the equation, for which one can make a passage to the limit with an appropriate scaling, before to deduce a conjecture on the localization of the level lines, which seem to behave exactly in  $t^{3/2}$ .

Finally, the fifth chapter investigates the dependence between the expansion speed and the coefficients in the case of a Fisher-KPP periodic space equation. Indeed, in this case, speed can be characterized using a family of eigenvalues, making this problem solvable. The first section mathematically formalizes the important problem in ecology of the influence of environmental fragmentation on invasion speed : it can be shown that in dimension 1, this question is mathematically addressed by comparing the speed related to a certain rate of growth and that associated with its rearrangement of Schwarz. We then show that fragmentation slows down the propagation, using a new formula for the eigenvalues of non-self-adjoint operators. This question is more difficult to formalize in dimension  $N$ , the rearrangement not being defined univocally. Section 2 investigates the simplest case : the optimization of a symmetric eigenvalue as a function of a potential taking only two values, the unknown being the zone where the potential is greatest (the habitat). And having a fixed area. Numerically, in dimension 2, the optimizer looks like a stripe, a ball or a ball complementary, but one can in fact analytically show that its boundary contains no sphere tip, which opens new questions. Finally, in the third section, I will briefly present other results of dependence on speed : the scaling limit for large periods, a series of counterexamples concerning, for example, the influence of diffusion and the case study of random stationary ergodic coefficients.

# Chapitre 1

## Expansion speeds in heterogeneous Fisher-KPP equations

This chapter and the following one are devoted to propagation phenomena for reaction-diffusion equations in heterogeneous media, that is, equations of the form :

$$\partial_t u - \sum_{i,j=1}^N a_{i,j}(t,x) \partial_{x_i x_j} u - \sum_{i=1}^N q_i(t,x) \partial_{x_i} u = r(t,x) u(1-u) \quad \text{in } I \times \mathbb{R}^N, \quad (1.1)$$

where  $I = (0, \infty)$  or  $I = \mathbb{R}$ ,  $A(t,x) = (a_{i,j}(t,x))_{i,j}$  is an elliptic matrix field ( $0 < \nu I_N \leq A(t,x) \leq \bar{\nu} I_N$  for all  $(t,x)$ ) and the reaction term  $r$  has positive infimum. Our goal is to understand the dynamics of this equation for coefficients depending in a general way on the time  $t$  and space  $x$  variables. We shall only assume that the coefficients are uniformly continuous and bounded in  $(t,x)$ .

Introducing heterogeneity into reaction-diffusion equations is relevant from a modeling point of view. If the models from genetics [6] or combustion [59] are naturally rather homogeneous, this can only be an approximation for population dynamics models. We refer to [32, 88, 96] for more detailed discussions on the origin of the heterogeneity and the related models.

In order to understand the effect of heterogeneity on reaction-diffusion equations, the community first considered in the 90s and 2000s particular heterogeneity classes, such as heterogeneity transverse to the propagation direction [21] or periodic (see [14, 95, 103] and numerous publications that followed). I contributed to this understanding during my PhD, where I studied the case of Fisher-KPP equations that are both periodic in space and time : existence, uniqueness and attractivity of a positive stationary state [N10], definitions and properties of the associated principal eigenvalues [N09-1], pulsating fronts [N09-2], relations between their speeds and coefficients [N11] and, in collaboration with H. Berestycki and F. Hamel, alternative proofs of the existence of an expansion speed [BHN08] to the one of H. Weinberger [101], based on PDE arguments.

The case of periodic equations having been widely studied in the 90s and 2000s, the actuality of the heterogeneous reaction-diffusion equations consists, among other things, today in the study of the equation (2.4) without making hypothesis on the coefficients other than their boundedness and their uniform smoothness. In other words, we consider heterogeneous media that do not fit in a particular class such as heterogeneity periodic or depending only on a transverse variable.

It would also be relevant to consider a homogeneous equation in a non-trivial domain  $\Omega$  instead of  $\mathbb{R}^N$ , in order to model for example the crossing of an obstacle [15] or a mountain pass [20] by a population, or the propagation of a cerebral vascular accident [10, 11, 33] (we also refer to [68] for the

study of the mean curvature equation, which is a limit case of the bistable reaction-diffusion equation, in a sawtooth cylinder). All the works that I will present below have been carried out in the case  $\Omega = \mathbb{R}^N$ , but it is obvious that a generalization of these results to more general domains is a natural outlook.

We will address in this chapter the following question :

**Question 1.** *For equations with coefficients depending on time and space, how to locate the level lines of the solutions of the Cauchy problem ?*

## 1.1 State of the art in the periodic framework

Let us first recall the results obtained in the 2000's in the case where the coefficients are periodic in  $t$  and in  $x$  : there exists  $(L_1, \dots, L_N, T)$  such that for every  $i = 1, \dots, N$  and  $(t, x) \in \mathbb{R} \times \mathbb{R}^N$  :

$$\begin{aligned} A(t, x + L_i e_i) &= A(t, x), & A(t + T, x) &= A(t, x), \\ q(t, x + L_i e_i) &= q(t, x), & q(t + T, x) &= q(t, x), \\ f(t, x + L_i e_i, u) &= f(t, x, u), & f(t + T, x, u) &= f(t, x, u), \end{aligned}$$

where  $(e_1, \dots, e_N)$  is the canonical basis of  $\mathbb{R}^N$ .

In this case, under certain reasonable hypotheses on the coefficients, equation (2.4) admits *pulsating wave solutions* [18, N09-2], that is, solutions  $u(t, x) = \phi(x \cdot e - ct, t, x)$ , where  $\phi = \phi(z, t, x)$  is periodic in  $t$  and  $x$ ,  $\phi > 0$ ,  $\phi(+\infty, t, x) = 0$  and  $\liminf_{z \rightarrow -\infty} \inf_{t, x} \phi(z, t, x) > 0$ ,  $c$  being the *speed* of the wave and  $e \in \mathbb{S}^{N-1}$  its *direction*. This speed can be characterized by a family of eigenvalues associated with parabolic operators

$$L_p \phi = -\partial_t \phi + \text{tr}(A \nabla^2 \phi) - 2pA \nabla \phi + q \cdot \nabla \phi + (pAp + r + q \cdot p)\phi. \quad (1.2)$$

Since the coefficients of these operators are periodic in  $(t, x)$ , the Krein-Rutman theory applies : for all  $p \in \mathbb{R}^N$ ,  $L_p$  admits a unique *periodic principal eigenvalue*  $k_p = k_p(A, q, c)$  associated with a periodic positive eigenfunction, called the *periodic principal eigenfunction*, that is to say a couple  $(k_p, \phi_p)$  solution of :

$$\begin{cases} L_p \phi_p = k_p \phi_p \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ \phi_p > 0, \\ \phi_p \text{ is periodic w.r.t. } (t, x). \end{cases} \quad (1.3)$$

The minimal speed of existence of pulsating waves is then characterized by :

$$c_e^* = \min_{\lambda > 0} \frac{k_{\lambda e}(A, q, c)}{\lambda}. \quad (1.4)$$

On the other hand, we can locate the level lines of  $u(t, \cdot)$ , where  $u$  is the solution of the Cauchy problem associated with the equation with a positive, non-zero, and compactly supported initial datum. These level sets no longer converge towards balls, as for the homogeneous equation, but approximate sets of the form  $t\mathcal{S}$ , in the following sense [43, BHN08, 101] :

$$\begin{cases} \text{for any compact } K \subset \text{int}\mathcal{S}, & \lim_{t \rightarrow +\infty} \left\{ \sup_{x \in tK} |u(t, x) - 1| \right\} = 0, \\ \text{for any closed set } F \subset \mathbb{R}^N \setminus \mathcal{S}, & \lim_{t \rightarrow +\infty} \left\{ \sup_{x \in tF} |u(t, x)| \right\} = 0. \end{cases} \quad (1.5)$$

The set  $\mathcal{S}$  is called the *expansion set*, or the *Wulff shape* by analogy with cristallography [102], and could be characterized with periodic eigenvalues :

$$\mathcal{S} = \{x, \forall p \in \mathbb{R}^N, k_p \geq p \cdot x\}. \quad (1.6)$$



We can reformulate this result by defining an expansion speed  $w_e^*$  in each direction  $e \in \mathbb{S}^{N-1}$  :

$$\begin{cases} \liminf_{t \rightarrow +\infty} u(t, x + wte) = 1 & \text{if } 0 \leq w < w^*(e), \\ \lim_{t \rightarrow +\infty} u(t, x + wte) = 0 & \text{if } w > w^*(e). \end{cases} \quad (1.7)$$

This speed is also related to the minimum speed of existence of waves via the following identity :

$$w_e^* = \min_{\xi \cdot e > 0} \frac{c_\xi^*}{e \cdot \xi} = \min_{p \cdot e > 0} \frac{k_p}{p \cdot e}. \quad (1.8)$$

## 1.2 The case of dimension 1

Let us now consider the Fisher-KPP heterogeneous equation in dimension 1 :

$$\begin{cases} \partial_t u - \partial_x(a(x)\partial_x u) = r(x)u(1 - u) & \text{dans } \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) = u_0(x) & \text{pour tout } x \in \mathbb{R}, \end{cases} \quad (1.9)$$

where  $u_0 \not\equiv 0$  is a continuous function with compact support, such that  $0 \leq u_0 \leq 1$  and  $a, c$  are uniformly bounded and continuous functions with a positive infimum. We studied the expansion speeds associated with equation (1.9) with H. Berestycki [BN12].

During my thesis, I obtained estimates with H. Berestycki and F. Hamel on the localization of the level lines of  $u(t, \cdot)$ , which showed in particular in the case of a compactly supported heterogeneity that the expansion speeds depend only on the heterogeneity of the coefficients at infinity. However, these estimates remained unsatisfactory since they were not optimal in the case of a periodic heterogeneity. In fact, what periodic coefficients teach us is that the heterogeneity of the coefficients must be taken into account through a notion of eigenvalues for the operators  $L_p$  defined by (1.2). However, for arbitrary heterogeneity, the operators  $L_p$  are no longer compact and the Krein-Rutman theorem no longer applies.

We have overcome this difficulty by using the notion of *generalized principal eigenvalues*, inspired by the notion introduced by H. Berestycki and his collaborators in [19, 22, 24]. Let us define two *generalized principal eigenvalues* associated with the  $L_p$  operators in  $(R, \infty)$  :

$$\begin{aligned} \underline{\lambda}_1(L_p, (R, \infty)) &:= \sup\{\lambda \mid \exists \phi \in \mathcal{A}_R, L_p \phi \geq \lambda \phi \text{ in } (R, \infty)\} \\ \overline{\lambda}_1(L_p, (R, \infty)) &:= \inf\{\lambda \mid \exists \phi \in \mathcal{A}_R, L_p \phi \leq \lambda \phi \text{ in } (R, \infty)\} \end{aligned} \quad (1.10)$$

the functions-tests being taken in

$$\mathcal{A}_R := \left\{ \phi \in \mathcal{C}^2(R, \infty), \phi'/\phi \in L^\infty, \phi > 0, \lim_{x \rightarrow +\infty} \frac{1}{x} \ln \phi(x) = 0 \right\}. \quad (1.11)$$

Note that the condition at infinity on the test functions  $\lim_{x \rightarrow +\infty} \frac{1}{x} \ln \phi(x) = 0$  is different (and more general) than the conditions in [19, 22], in order to obtain optimal results in the case of random stationary ergodic coefficients.

The following result allows us to verify that these notions are indeed generalizations of the classical notion of eigenvalue.

**Proposition 1.1.** *Assume that there exist  $\lambda \in \mathbb{R}$ ,  $R \in \{-\infty\} \cup \mathbb{R}$  and  $\phi \in \mathcal{A}_R$  such that  $\mathcal{L}\phi = \lambda\phi$  in  $(R, \infty)$ . Then,*

$$\lambda = \underline{\lambda}_1(\mathcal{L}, (R, \infty)) = \overline{\lambda}_1(\mathcal{L}, (R, \infty)).$$

In particular, if  $c$  is periodic, we can show that  $\underline{\lambda}_1(L_p, (R, \infty)) = \overline{\lambda}_1(L_p, (R, \infty)) = k_p$  is the periodic principal eigenvalue for any  $R$ .

We associate these two speeds with formulas analogous to the periodic case (1.8) :

$$\underline{w} = \min_{p>0} \sup_{R>0} \frac{\lambda_1(L_p, (R, \infty))}{p} \quad \text{and} \quad \overline{w} = \min_{p>0} \inf_{R>0} \frac{\overline{\lambda}_1(L_p, (R, \infty))}{p}. \quad (1.12)$$

**Theorem 1.2.** [BN12] Take  $u_0$  a measurable and compactly supported function such that  $0 \leq u_0 \leq 1$ ,  $u_0 \not\equiv 0$  and let  $u$  the solution of the associated Cauchy problem (1.9). One has

1. for all  $w > \overline{w}$ ,  $\lim_{t \rightarrow +\infty} \sup_{x \geq wt} |u(t, x)| = 0$ ,
2. for all  $w \in [0, \underline{w})$ ,  $\lim_{t \rightarrow +\infty} \sup_{0 \leq x \leq wt} |u(t, x) - 1| = 0$ .

The level lines of  $u(t, \cdot)$  are therefore well located between the lines  $\underline{w}t$  and  $\overline{w}t$ , up to a sub-linear correction.

In the case of periodic coefficients, we find the speed  $\underline{w} = \overline{w} = \min_{p>0} \frac{k_p}{p}$ . But we can prove the identity  $\underline{w} = \overline{w}$ , and thus locate the level lines of  $u(t, \cdot)$  accurately, for more general classes of heterogeneity, such as almost periodic, almost periodic at infinity, or random stationary ergodic coefficients.

### 1.3 Oscillations of the level lines and non-existence of an exact expansion speed

For general heterogeneities, it is not always true that there exists an exact expansion speed, such that  $\underline{w} = \overline{w}$ . I have constructed a counterexample in a work with J. Garnier and T. Giletti [GGN12], using a growth rate of the type

$$r(x) = r_0(x/L(x)) \quad \text{where } r_0 \text{ is periodic and } L(x) \text{ grows fast enough}$$

(such as  $x/(\ln x)^\alpha$ ,  $\alpha \in (0, 1)$  for example), then we can show that the level lines oscillate exactly between  $2\sqrt{\min r_0}t$  and  $2\sqrt{\max r_0}t$ . More precisely,

$$\forall w \in (2\sqrt{\min r_0}, 2\sqrt{\max r_0}), \text{ the } \omega\text{-limit set as } t \rightarrow +\infty \text{ of } t \mapsto u(t, wt) \text{ is equal to } [0, 1].$$

This is the first explicit counterexample to the existence of an exact expansion speed, although the general idea leading to such a phenomenon was more or less in all minds.

If on the contrary  $L(x)$  is slowly increasing (such as  $x/(\ln x)^\alpha$ ,  $\alpha > 1$  for example), then using Theorem 1.2 yields that  $\underline{w} = \overline{w} = w_\infty^*$ , where we find the speed  $w_\infty^*$  defined by (5.6) and obtained in [HNR11] as the limit of the expansion speeds associated with a growth rate  $c_0(x/L)$  when  $L \rightarrow +\infty$ .

Several other types of phenomena which can lead to oscillations of the level lines, and therefore to the non-existence of an exact expansion speed, have been identified these last years. Hamel and Sire [54] have found such phenomena for homogeneous reaction-diffusion equations of the ignition type, with initial data oscillating periodically between two values under the ignition temperature. We have also studied with F. Hamel [HN11] the case of a homogeneous Fisher-KPP equation with initial data decreasing exponentially between two decay rates. If the initial datum oscillates periodically between two exponentials, then it still exists an exact expansion speed, which can be explicitly characterized. But if the oscillations are more and more ample, then we can construct examples where there are always two expansion speeds  $\underline{w} < \overline{w}$ . The  $\omega$ -limit set of the function  $u(t, wt)$  when  $t \rightarrow +\infty$  is then exactly  $[0, 1]$  if  $w \in (\underline{w}, \overline{w})$ .

## 1.4 Random stationary ergodic media in dimension 1

Assume that the reaction term is a random variable  $r : (x, \omega, s) \in \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  defined on a probability set  $(\Omega, \mathbb{P}, \mathcal{F})$  and that  $a \equiv 1$  and  $q \equiv 0$  (in order to simplify the presentation). We suppose that  $r$  is stationary, in the sense that there exists a  $(\pi_x)_{x \in \mathbb{R}}$  transformation group preserving the probability measure such that  $r(x + y, \omega) = r(x, \pi_y \omega)$  for any  $(x, y, \omega) \in \mathbb{R} \times \mathbb{R} \times \Omega$ , and ergodic, in the sense that if  $\pi_x A = A$  for all  $x \in \mathbb{R}$  for some  $A \in \mathcal{F}$ , then  $\mathbb{P}(A) = 0$  or 1.

The stationarity hypothesis means that the statistical properties of the media do not depend on the place where it is observed. The ergodicity means that we can not separate  $\Omega$  into two non-neglectible subsets that are invariant by the group  $(\pi_x)_{x \in \mathbb{R}}$ .

Freidlin and Gartner [43] have shown that for almost every  $\omega \in \Omega$  there exists an expansion speed, which can be calculated using a family of Lyapunov exponents associated with the linearization of the equation around  $u \equiv 0$ . This result has been generalized by Nolen and Xin to different reaction-diffusion equations with random stationary ergodic coefficients in  $t$  and  $x$ , in dimension  $N$  [80, 81, 82].

We can find this result using Theorem 1.2, which gives us a new characterization of the speed from the generalized principal eigenvalues.

**Theorem 1.3.** *Under the hypotheses stated above, one has*

$$\overline{w}^\omega = \underline{w}^\omega = \min_{p>0} \frac{\overline{\lambda}_1(L_p, \mathbb{R})}{p} = \min_{p>0} \frac{\lambda_1(L_p, \mathbb{R})}{p} \quad (1.13)$$

for almost every  $\omega \in \Omega$ .

This result is indeed an immediate corollary of the following result and of Theorem 1.2.

**Theorem 1.4.** *For almost every  $\omega \in \Omega$ , one has  $\underline{\lambda}_1(L_p, \mathbb{R}) = \overline{\lambda}_1(L_p, \mathbb{R})$  for all  $p > 0$ .*

This result is obtained by explicitly constructing principal eigenfunctions and applying Proposition 1.1. Note that these are in the  $\mathcal{A}_{-\infty}$  class, that is, they are only sub-exponentials, but not necessarily bounded, which will be important for the extension of the results to  $\mathbb{R}^N$ .

## 1.5 Statement of the result in dimension $N$

Let us now return to the multi-dimensional Fisher-KPP equation (2.4). The results we are going to state are true for an equation of the divergence form (in other words,  $q \equiv \nabla \cdot A$ ), as in dimension 1, or more generally for an advection term  $q$  not too different from the divergence of  $A$ , in the sense that :

$$\sup_{R>0} \inf_{t>R, |x|>R} \left( 4f'_u(t, x, 0) \min_{e \in \mathbb{S}^{N-1}} (eA(t, x)e) - |q(t, x) - \nabla \cdot A(t, x)|^2 \right) > 0. \quad (1.14)$$

We saw that in dimension 1, only the value of the coefficients in  $t > R$  and  $x > R$ , with  $R$  large, were relevant in the calculation of the expansion speeds. We must therefore first understand what domain in  $x$  we should restrict to in dimension  $N$ . Let  $C_{R,\alpha}(e)$  be the truncated truncated  $e$  and opening angle  $\alpha$  :

$$C_{R,\alpha}(e) := \left\{ (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad t > R, \quad |x| > R, \quad \left| \frac{x}{|x|} - e \right| < \alpha \right\}. \quad (1.15)$$

Let us immediately warn the reader that the expansion speeds in the  $e$  direction will not be computed solely by considering what happens in  $C_{R,\alpha}(e)$  : their characterization will be more complicated in general.

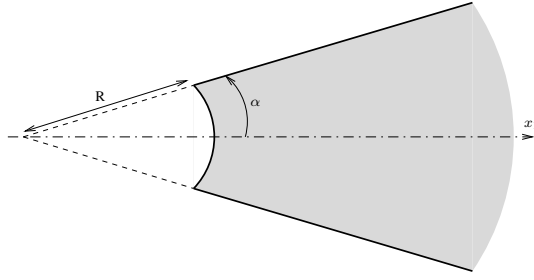


FIGURE 1.1 – Projection on the coordinates  $x$  of the truncated cone  $C_{R,\alpha}(e_1)$ .

Let us introduce the parabolic operators :

$$L_p \phi := e^{-p \cdot x} \mathcal{L}(e^{p \cdot x} \phi) = -\partial_t \phi + \text{tr}(A(t, x) \nabla^2 \phi) + (q(t, x) + 2A(t, x)p) \cdot \nabla \phi + (f'_u(t, x, 0) + p \cdot q(t, x) + pA(t, x)p) \phi. \quad (1.16)$$

and its generalized eigenvalues in any open set  $Q \subset \mathbb{R} \times \mathbb{R}^N$  :

$$\underline{\lambda}_1(\mathcal{L}, Q) := \sup\{\lambda \mid \exists \phi \in \mathcal{C}^{1,2}(Q) \cap W^{1,\infty}(Q), \inf_Q \phi > 0 \text{ and } \mathcal{L}\phi \geq \lambda \phi \text{ in } Q\}. \quad (1.17)$$

$$\overline{\lambda}_1(\mathcal{L}, Q) := \inf\{\lambda \mid \exists \phi \in \mathcal{C}^{1,2}(Q) \cap W^{1,\infty}(Q), \inf_Q \phi > 0 \text{ and } \mathcal{L}\phi \leq \lambda \phi \text{ in } Q\}. \quad (1.18)$$

This definition is less accurate than that used in dimension 1, since the test-functions are chosen from the bounded functions with positive infimum, whereas we considered the more general set defined in (1.11) of the sub-exponential functions in dimension 1. We have needed this boundedness hypothesis on the test-functions in order to obtain a comparison between  $\underline{\lambda}_1$  and  $\overline{\lambda}_1$  in the cone  $C_{R,\alpha}(e)$ . This gap will be particularly important in the case of random stationary ergodic coefficients.

We are now in a position to define two functions playing the role of Hamiltonians :

$$\overline{H}(e, p) := \inf_{R>0, \alpha \in (0,1)} \overline{\lambda}_1(L_p, C_{R,\alpha}(e)) \text{ and } \underline{H}(e, p) := \sup_{R>0, \alpha \in (0,1)} \underline{\lambda}_1(L_p, C_{R,\alpha}(e)) \quad (1.19)$$

for all  $p \in \mathbb{R}^N$  and  $e \in \mathbb{S}^{N-1}$ . These Hamiltonians satisfy the following properties.

- Proposition 1.5.** 1. The functions  $p \rightarrow \overline{H}(e, p)$  and  $p \rightarrow \underline{H}(e, p)$  are locally Lipschitz-continuous, uniformly with respect to  $e \in \mathbb{S}^{N-1}$ , and  $p \mapsto \overline{H}(e, p)$  is convex for all  $e \in \mathbb{S}^{N-1}$ .  
 2. For all  $p \in \mathbb{R}^N$ ,  $e \mapsto \underline{H}(e, p)$  is lower semicontinuous and  $e \mapsto \overline{H}(e, p)$  is upper semicontinuous.  
 3. There exist  $C \geq c > 0$  such that for all  $(e, p) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$  :

$$c(1 + |p|^2) \leq \underline{H}(e, p) \leq \overline{H}(e, p) \leq C(1 + |p|^2).$$

The last property allows to define the convex conjugates :

$$\underline{H}^*(e, q) := \sup_{p \in \mathbb{R}^N} (p \cdot q - \underline{H}(e, p)) \quad \text{and} \quad \overline{H}^*(e, q) := \sup_{p \in \mathbb{R}^N} (p \cdot q - \overline{H}(e, p)),$$

as well as the related "cost-functions" :

$$\begin{aligned} \underline{U}(x) &:= \inf \max_{t \in [0,1]} \left\{ \int_t^1 \underline{H}^*\left(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\right) ds, \quad \gamma \in H^1([0,1]), \gamma(0) = 0, \gamma(1) = x, \\ &\quad \forall s \in (0,1), \gamma(s) \neq 0 \right\}, \\ \overline{U}(x) &:= \inf \max_{t \in [0,1]} \left\{ \int_t^1 \overline{H}^*\left(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\right) ds, \quad \gamma \in H^1([0,1]), \gamma(0) = 0, \gamma(1) = x, \\ &\quad \forall s \in (0,1), \gamma(s) \neq 0 \right\}. \end{aligned} \quad (1.20)$$

Finally, we define the two expansion sets :

$$\underline{\mathcal{S}} := \text{cl}\{\underline{U} = 0\} \quad \text{and} \quad \overline{\mathcal{S}} := \{\overline{U} = 0\}. \quad (1.21)$$

It can be shown that the sets  $\underline{\mathcal{S}}$  and  $\overline{\mathcal{S}}$  are compact, star-shaped with respect to 0 and contain a ball centered at 0.

In the case of periodic coefficients, the Hamiltonians  $\underline{H}$  and  $\overline{H}$  are in fact independent of  $e$  and both are equal to the periodic eigenvalue  $k_p$  defined by (1.3), and we can compute explicitly  $\underline{U}(x) = \overline{U}(x) = \max\{0, H^*(-x)\}$ , where  $H(p) := k_p$ . We thus find the the Wulff shape  $\underline{\mathcal{S}} = \overline{\mathcal{S}} = \mathcal{S}$  defined by (1.6).

We are now in a position to state our main result.

**Theorem 1.6.** *Take  $u_0$  a measurable and compactly supported function such that  $0 \leq u_0 \leq 1$  and  $u_0 \not\equiv 0$  and let  $u$  the solution of (2.4) associated with this initial datum. One has*

$$\begin{cases} \text{for all compact set } K \subset \text{int}\underline{\mathcal{S}}, & \lim_{t \rightarrow +\infty} \left\{ \sup_{x \in tK} |u(t, x) - 1| \right\} = 0, \\ \text{for all closed set } F \subset \mathbb{R}^N \setminus \overline{\mathcal{S}}, & \lim_{t \rightarrow +\infty} \left\{ \sup_{x \in tF} |u(t, x)| \right\} = 0. \end{cases} \quad (1.22)$$

It is not possible, in this general framework, to simply express the expansion speeds in each direction as in the periodic case. At this stage, they can only be defined as  $\underline{w}_e := \sup\{w, w \in \underline{\mathcal{S}}\}$  and  $\overline{w}_e := \sup\{w, w \in \overline{\mathcal{S}}\}$ .

This result is not optimal because it does not cover the case of random stationary ergodic coefficients, for which the existence of exact expansion speeds is known [80], that is to say that we should expect  $\underline{\mathcal{S}} = \overline{\mathcal{S}}$ . We have not succeeded, unlike in the case of dimension 1, to construct sub-exponential eigenfunctions, nor to contradict such an existence. In homogenization papers, this difficulty is overcome by using a relaxed notion of corrector (see for example [67]), but this approach did not seem us adapted to deal with the general case.

However, this Theorem covers all other cases where the existence of an exact expansion rate has been known, and we also show that  $\underline{\mathcal{S}} = \overline{\mathcal{S}}$  in cases so far open : equations with coefficients that are uniquely ergodic or homogeneous in each direction.

## 1.6 Uniquely ergodic media

We will use in this manuscript the following definition of unique ergodicity.

**Definition 1.7.** *A uniformly continuous and bounded function  $f : \mathbb{R}^N \rightarrow \mathbb{R}^m$  is called uniquely ergodic if for any continuous function  $\Psi : \mathcal{H}_f \rightarrow \mathbb{R}$ , where  $\mathcal{H}_f := \text{cl}\{\tau_a f, a \in \mathbb{R}^N\}$ , where the closure is understood with respect to the locally uniform convergence, the limit*

$$\lim_{R \rightarrow +\infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \Psi(\tau_y f) dy$$

*exists and is uniform with respect to  $a \in \mathbb{R}^N$ .*

This limit defines a probability measure  $\mathbb{P}$  on  $\mathcal{H}_f$ , which is invariant with respect to translations  $g \mapsto g(\cdot + a)$  for any  $a \in \mathbb{R}^N$ . It can be shown that such a measure is unique, justifying the name of unique ergodicity. In the literature, it is rather this property that defines the unique ergodicity, the convergence of means on  $B_R(a)$  being seen as the consequence of the uniqueness of the measure  $\mathbb{P}$ .

It is easily verified that periodic, almost periodic and compactly supported functions are uniquely ergodic.

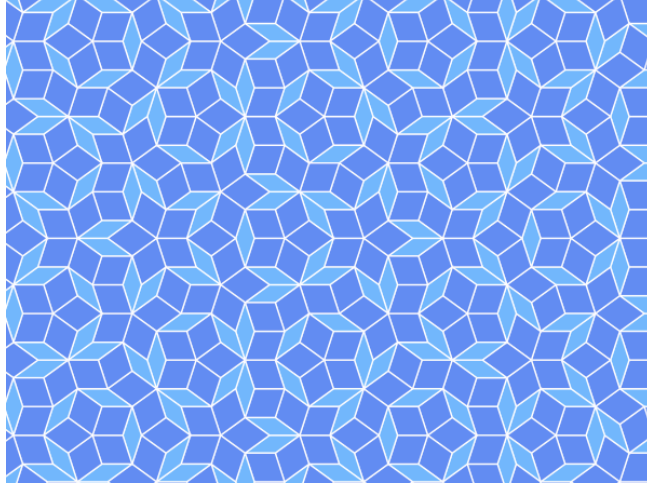


FIGURE 1.2 – A representation of the Penrose tiling

A classic example of uniquely ergodic function is constructed from the Penrose tiling. We refer to [85] for a definition of it. If one defines on each tile a compactly supported function, the function thus obtained on  $\mathbb{R}^N$  is uniquely ergodic [74, 85]. However, it is not almost periodic. The class of ergodic functions is therefore wider than that of almost periodic functions.

**Theorem 1.8.** *Assume that  $A$ ,  $q$  and  $c$  only depend on  $x$  and are uniquely ergodic, where  $c \in \mathcal{C}_{loc}^\delta(\mathbb{R}^N)$  is a given uniformly continuous and bounded function. Define the elliptic operator :  $\mathcal{L} = \text{tr}(A\nabla^2) + q \cdot \nabla + c$ . Then one has :*

$$\underline{\lambda}_1(\mathcal{L}, \mathbb{R}^N) = \overline{\lambda}_1(\mathcal{L}, \mathbb{R}^N).$$

We immediately derive the next result from Theorems 1.6 and 1.8.

**Corollary 1.9.** *Assume that  $A$ ,  $q$  and  $f'_u(\cdot, 0)$  only depend on  $x$  and are uniquely ergodic. Then  $\underline{\mathcal{S}} = \overline{\mathcal{S}}$  and*

$$\overline{w}(e) = \underline{w}(e) = \min_{p \cdot e > 0} \frac{\overline{\lambda}_1(L_{-p}, \mathbb{R} \times \mathbb{R}^N)}{p \cdot e} = \min_{p \cdot e > 0} \frac{\underline{\lambda}_1(L_{-p}, \mathbb{R} \times \mathbb{R}^N)}{p \cdot e}. \quad (1.23)$$

Theorem 1.8 is a powerful result from which other applications could be derived from the theory of generalized eigenvalues [24] : existence and uniqueness of nonlinear problem solutions, comparison principles for elliptic operators etc.

## 1.7 Directionally homogeneous media and convexity of the expansion sets

Consider now spatial heterogeneities converging when  $|x| \rightarrow +\infty$  in angular sectors of  $\mathbb{R}^2$ . Let us start with the case where the diffusion matrix  $A$  converges to two different values when  $x_1 \rightarrow \pm\infty$ .

### 1.7. Directionally homogeneous media and convexity of the expansion sets

**Proposition 1.10.** Assume that  $N = 2$ ,  $q \equiv 0$ ,  $f$  does not depend on  $(t, x)$  and  $A(x_1, x_2) = a(x_1)I_2$  is a smooth function such that  $\lim_{x_1 \rightarrow \pm\infty} a(x_1) = a_{\pm}$ , with  $a_+ > a_- > 0$ . Then  $\bar{\mathcal{S}} = \underline{\mathcal{S}}$  and this set is the convex envelope of

$$\{x \in \mathbb{R}^2, |x| \leq 2\sqrt{f'(0)a_+}, x_1 \geq 0\} \cup \{x \in \mathbb{R}^2, |x| \leq 2\sqrt{f'(0)a_-}, x_1 \leq 0\}.$$

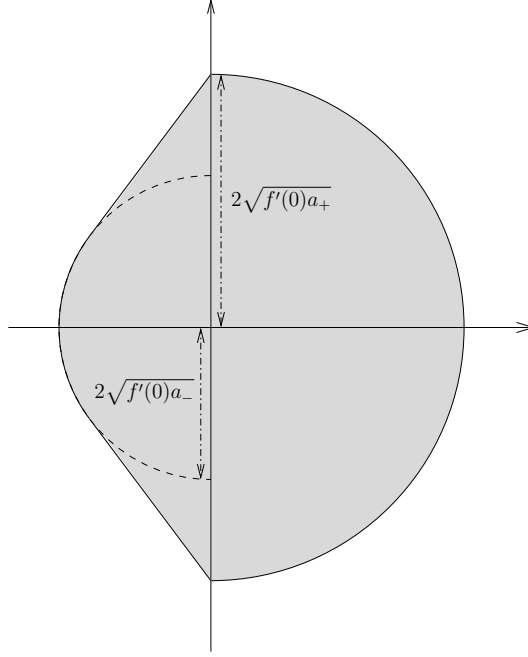


FIGURE 1.3 – The expansion set  $\bar{\mathcal{S}} = \underline{\mathcal{S}}$  given by Proposition 1.10 for  $N = 2$ .

We can show that for this type of coefficients, constant at infinity in angular sectors, there is always an exact expansion set, that is,  $\bar{\mathcal{S}} = \underline{\mathcal{S}}$ . We will pass on the statement of this quite technical type of result.

It is tempting to speculate that for this type of heterogeneity, it is always possible to characterize the expansion set as the convex envelope of the expansion sets associated with the values of the coefficients in each angular sector. This hypothesis is contradicted by the following result, which indeed contradicts more generally the convexity of the expansion sets.

**Proposition 1.11.** Assume that  $N = 2$ ,  $q \equiv 0$ ,  $f$  does not depend on  $(t, x)$  and  $A(x) = a(x)I_2$  is a smooth function such that

$$\lim_{x_1 \rightarrow +\infty} a(x_1, \alpha x_1) = \begin{cases} a_+ & \text{if } |\alpha| < r_0 \\ a_- & \text{if } |\alpha| > r_0 \end{cases}$$

where  $a_+ > a_- > 0$  and  $0 < r_0 < r := \sqrt{\frac{a_-}{a_+ - a_-}}$ . Then  $\bar{\mathcal{S}} = \underline{\mathcal{S}}$  and this set is :

$$\left\{ |x| < 2\sqrt{f'(0)a_+}, |x_2| \geq r_0 x_1 \right\} \cup \left\{ x_1 < \frac{1 - r_0 r}{r_0 + r} |x_2| + \frac{2\sqrt{f'(0)a_+(1 + r_0^2)}}{1 + r_0/r}, |x_2| \leq r_0 x_1 \right\}.$$

This expansion set is not convex if  $r_0 r < 1$ , as can be seen in the Figure illustrating Proposition 1.11.

The following result gives a condition guaranteeing the convexity of the expansion set for such heterogeneity.

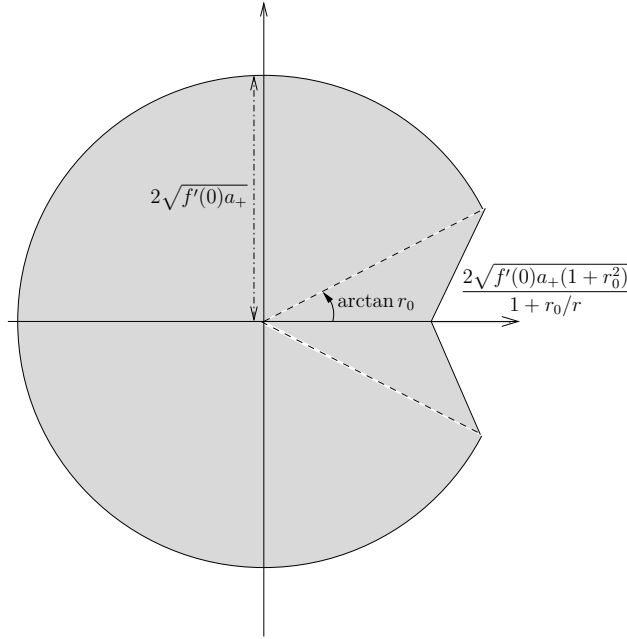


FIGURE 1.4 – The non-convex expansion set  $\bar{\mathcal{S}} = \underline{\mathcal{S}}$  given by Proposition 1.11.

**Proposition 1.12.** Assume that  $\underline{\mathcal{S}} = \bar{\mathcal{S}}$  and that the function  $x \in \mathbb{R}^N \setminus \{0\} \mapsto \underline{H}(x/|x|, p)$ , extended to 0 by  $\underline{H}(0, p) := \sup_{e \in \mathbb{S}^{N-1}} \underline{H}(e, p)$ , is quasiconcave over  $\mathbb{R}^N$  for all  $p \in \mathbb{R}^N$ , in the sense that

$$\text{the set } \{x \in \mathbb{R}^N, \underline{H}(x/|x|, p) \geq \alpha\} \text{ is convex for all } \alpha \in \mathbb{R}, p \in \mathbb{R}^N.$$

Then the set  $\bar{\mathcal{S}} = \underline{\mathcal{S}}$  is convex

A result by Alvarez, Lasry and Lions [2] ensures the convexity of the Hamilton-Jacobi equation solutions for concave Hamiltonians in  $x$ . In the examples presented in this section, the Hamiltonians are only quasi-concave and we have therefore to extend the result of [2] to this framework.

## 1.8 Scheme of proof

Let  $e \in \mathbb{S}^{N-1}$ , the idea of our proof is to compute  $\lim_{\varepsilon \rightarrow 0} u(t/\varepsilon, x/\varepsilon)$ . We introduce the Hopf-Cole transformation  $Z_\varepsilon$  defined by  $u(t/\varepsilon, x/\varepsilon) = e^{Z_\varepsilon(t, x)}$  and the semi-limits of this function

$$Z_*(t, x) := \liminf_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} Z_\varepsilon(s, y) \text{ and } Z^*(t, x) := \limsup_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} Z_\varepsilon(s, y).$$

It immediately follows that  $u(t, we) \rightarrow 0$  if  $Z^*(1, we) < 0$  and we can also show that  $u(t, we) \rightarrow 1$  if  $(1, we) \in \text{Int}\{Z_* = 0\}$ . Calculate the limit of  $u(t, we)$  when  $t \rightarrow +\infty$  therefore reduces to calculate the sign of  $Z^*(1, we)$  and  $Z_*(1, we)$ . We can estimate these semi-limits by verifying that they are solutions of certain inequations.

Our approach is classical : there are many such results in the literature (see, for example, [7, 8, 58, 67, 70]) for periodic, random stationary ergodic coefficients or to investigate small diffusion limits. The difficulty here was to consider coefficients depending in a general way on  $(t, x)$ . In particular, it was necessary to find good correctors to absorb the oscillations of the coefficients in  $(t/\varepsilon, x/\varepsilon)$ . These approximate correctors are precisely the generalized eigenvalues, restricted to the truncated cones  $C_{R, \alpha}(e)$ .



**Proposition 1.13.**

$$\begin{cases} \max\{\partial_t Z^* - \overline{H}(\frac{x}{|x|}, \nabla Z^*), Z^*\} \leq 0 & \text{in } (0, \infty) \times \mathbb{R}^N \setminus \{0\}, \\ \max\{\partial_t Z_* - \underline{H}(\frac{x}{|x|}, \nabla Z_*), Z_*\} \geq 0 & \text{in } (0, \infty) \times \mathbb{R}^N \setminus \{0\}, \\ \lim_{t \rightarrow 0^+} Z^*(t, x) = \lim_{t \rightarrow 0^+} Z_*(t, x) = \begin{cases} 0 & \text{if } x = 0, \\ -\infty & \text{if } x \neq 0, \end{cases} \\ Z^*(t, 0) = Z_*(t, 0) = 0 & \text{for all } t > 0, \end{cases}$$

$$\text{where } \overline{H}(e, p) := \lim_{R \rightarrow +\infty, \alpha \rightarrow 0} \overline{\lambda}_1(L_p, C_{R, \alpha}(e)), \quad \underline{H}(e, p) := \lim_{R \rightarrow +\infty, \alpha \rightarrow 0} \underline{\lambda}_1(L_p, C_{R, \alpha}(e)).$$

These equations are very singular : the Hamiltonians are not defined in  $x = 0$  and are semi-continuous in  $x$ , the initial datum is not bounded, nor continuous. In spite of everything, the following estimates can be obtained, from which Theorem 1.6 is derived.

**Proposition 1.14.**  $Z_*(t, x) \geq -t\underline{U}(x/t)$  and  $Z^*(t, x) \leq -t\overline{U}(x/t)$ , where  $\underline{U}$  and  $\overline{U}$  has been defined by (1.20).

The proof is based on the Evans and Souganidis [37] formulas for the solutions of first order Hamilton-Jacobi equations. These formulas are true for smooth Hamiltonians  $H$ , defined for all  $x \in \mathbb{R}^N$  and for bounded and smooth initial data, the difficulty consists in verifying that our singular equations can be approximated by smooth equations validating the hypotheses of [37], then pass to the limit .



# Chapitre 2

## Generalized transition waves for heterogeneous Fisher-KPP equations

This chapter is still devoted to reaction-diffusion equations in heterogeneous media :

$$\partial_t u - \sum_{i,j=1}^N a_{i,j}(t, x) \partial_{x_i x_j} u - \sum_{i=1}^N q_i(t, x) \partial_{x_i} u = r(t, x) u(1 - u) \quad \text{in } I \times \mathbb{R}^N, \quad (2.1)$$

where  $I = (0, \infty)$  or  $I = \mathbb{R}$ ,  $A(t, x) = (a_{i,j}(t, x))_{i,j}$  is an elliptic matrix field ( $0 < \underline{\nu} I_N \leq A(t, x) \leq \bar{\nu} I_N$  for all  $(t, x)$ ) and the reaction term  $r$  has positive infimum. The coefficients are uniformly continuous and bounded in  $(t, x)$ .

The notion of traveling waves is no longer adapted to fully heterogeneous equations, and we must therefore introduce an extension of this notion : generalized transition wave. We will then investigate existence results for such waves in various media.

The main question we investigate in this chapter is the following :

**Question 2.** *Are there generalized transition wave solutions of equations with coefficients depending on the time and space ?*

### 2.1 Definition, existence and non-existence

Let us begin by defining an extension of the notion of traveling waves. The first definition is due to Hamel and Berestycki [13] (see also [92]) : in dimension 1, a generalized transition wave is a time global positive solution  $u$  of (2.4), such that for any  $\varepsilon \in (0, 1)$ , the interfaces  $I_\varepsilon(t) := \{x \in \mathbb{R}, \varepsilon < u(t, x) < 1 - \varepsilon\}$  have lengths bounded uniformly in  $t$ . In other words, the transition between the states of equilibrium 0 and 1 is well localized in space, uniformly in  $t$ .

In dimension 1, the existence of such waves for a nonlinearity  $f$  depending only on  $x$  of "ignition type", that is,  $f(x, u) = 0$  if  $u < \theta$ , with  $\theta \in (0, 1)$ , has been proved in parallel by Nolen and Ryzhik [79] and Mellet, Roquejoffre and Sire [76]. These two teams then showed their uniqueness and stability together [75] still for this type of nonlinearities.

However, it has been shown by Nolen, Roquejoffre, Ryzhik and Zlatos [77] that generalized waves do not always exist for KPP nonlinearities. Consider a nonlinearity  $f(x, u) = r(x)u(1 - u)$ , with  $r(x) \equiv 1$  outside a compact, and  $r \geq 1$  on  $\mathbb{R}$ . Then if the compact heterogeneity is large enough, in the sense that  $\lambda_1(\mathcal{L}, \mathbb{R}) > 2$ , all global solutions in time are exponentially decreasing in  $|x| \rightarrow +\infty$ , thus contradicting the existence of generalized transitional waves.

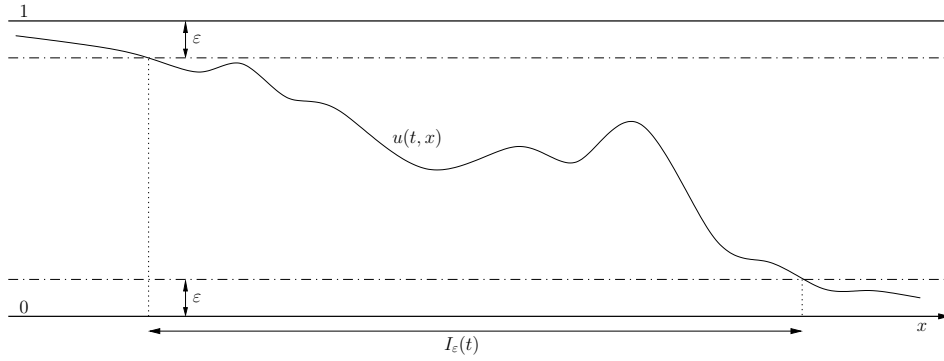


FIGURE 2.1 – The interface  $I_\varepsilon(t) = \{x, \varepsilon < u(t, x) < 1 - \varepsilon\}$  of a generalized transition wave  $u$ .

On the other hand, Zlatos [106] has given sufficient, but not necessary, conditions to the existence of generalized waves for KPP equations, always in dimension 1. These conditions are verified in particular when only the diffusion rate depends on  $x$ , but is no longer necessarily satisfied for periodic coefficients, whereas we know that some generalized transition waves (pulsating waves) exist in this case. On the other hand, he constructed [105] a counterexample to the existence of generalized waves for a nonlinearity of ignition type in dimension  $N \geq 2$ . Thus, the conditions guaranteeing the existence of generalized waves for heterogeneous coefficients in space are not entirely clear.

Finally, another definition of generalized waves has been introduced by Matano [73], based on a continuity of the front with respect to the coefficients. If the generalized wave is unique, as in the ignition case [75], then the two notions coincide. For KPP equations, there may be generalized families of waves (see [106] but also the papers [NR12, NR15, NR16] presented below). In this second case the existence of generalized fronts in the sense of Matano remains open.

## 2.2 An alternative notion in dimension $N = 1$ : the critical waves

Our goal is to introduce an alternative notion of waves for one-dimensional heterogeneous equations, which would exist under very general conditions. In this section, we will make the following hypotheses on the coefficients of equation (2.4) :

$$\begin{aligned} a, a_x, a_{xx}, 1/a, b, b_x &\in L^\infty(\mathbb{R}), f \in L^\infty(\mathbb{R} \times [0, 1]), \\ \exists C > 0 \text{ s.t. } |f(x, u) - f(x, v)| &\leq C|u - v| \text{ a.e. } (x, u, v) \in \mathbb{R} \times [0, 1] \times [0, 1], \\ f(x, 0) = f(x, 1) &= 0 \text{ a.e. } x \in \mathbb{R}. \end{aligned} \quad (2.2)$$

The definition is the following.

**Definition 2.1.** [N15-1] We say that a time-global solution  $u \in C^0(\mathbb{R} \times \mathbb{R})$  of (2.5), with  $0 < u < 1$ , is a critical traveling wave (to the right) if for all  $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$ , if  $v \in C^0(\mathbb{R} \times \mathbb{R})$  is a time-global solution of (2.5) such that  $v(t_0, x_0) = u(t_0, x_0)$  and  $0 < v < 1$ , then either  $u \equiv v$  or

$$u(t_0, x) > v(t_0, x) \quad \text{if } x < x_0 \quad \text{and} \quad u(t_0, x) < v(t_0, x) \quad \text{if } x > x_0. \quad (2.3)$$

This notion is based on the Angenent principle, which states that the number of zeros of the solution of a parabolic equation can only be nonincreasing in time. This principle has already been used to investigate the behavior of solutions in the case of random coefficients [92] or periodic ones [35].

**Theorem 2.2.** [N15-1]

## 2.2. An alternative notion in dimension $N = 1$ : the critical waves

1. (Existence and uniqueness) For all  $\theta \in (0, 1)$  and  $x_0 \in \mathbb{R}$ , equation (2.5) admits a unique critical transition wave  $u$  such that  $u(0, x_0) = \theta$ .
2. (Monotonicity in time)  $t \mapsto u(t, x)$  is either decreasing for all  $x \in \mathbb{R}$ , increasing for all  $x \in \mathbb{R}$  or constant for all  $x \in \mathbb{R}$ .
3. (Monotonicity in space) If  $f$  does not depend on  $x$ , then  $x \mapsto u(t, x)$  is nonincreasing for all  $t \in \mathbb{R}$ .

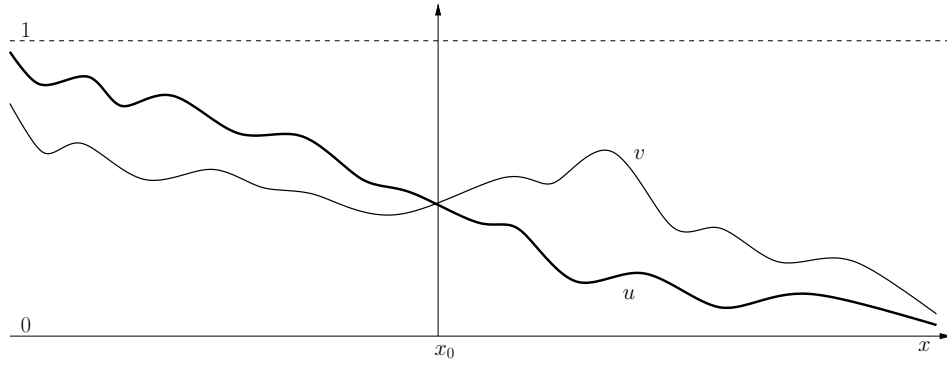


FIGURE 2.2 – The critical traveling wave  $u$  compared with a time-global solution  $v$  at  $t = t_0$ .

One could also prove that if  $f$  is monostable, in the sense that the equilibrium 1 is globally attractive and where  $s \mapsto f(t, x, s)/s$  is decreasing, then, in the case where there is a generalized transition wave, the critical wave is the wave with the smallest global speed (in the sense of [13]). But there are situations where the critical wave exists while there is no generalized wave [77], which gives all its relief to this notion.

In the case where the coefficients are random, stationary and ergodic in  $x$ , then the existence of generalized transitional wave is open in general. Numerical simulations, however, seem to contradict the existence of such fronts in general : the size of the interfaces  $I_\varepsilon(t)$  seems to be increasing and not to remain bounded when the wave alternately passes through zones with high and then low reaction rate. This case therefore potentially gives an additional example where the critical wave exists and presents some stationary properties, although the generalized waves probably do not exist.

Finally, this critical front is globally attractive for the Cauchy problem with an initial datum of the Heaviside type if and only if this front is continuous with respect to the media (in the sense of Matano [73]). This continuity, however, has since been contradicted in general in [52], but may remain true in some particular cases.

Thus, this approach was used in parallel by Ducrot, Giletti and Matano [35] to study the attractivity of the critical wave for general nonlinearities periodic in space and Heaviside initial data, and in the homogeneous case by Polacik, for general initial data [83]. In these two articles, the critical waves are rather called "terraces" or "minimal system of waves". The attractivity of the wave with minimal speed for periodic Fisher-KPP equations has also been proved by other methods, giving a better order of convergence, by Hamel, Nolen, Roquejoffre and Ryzhik [50].

## 2.3 Existence of generalized transition waves for time heterogeneous equations

Consider now Fisher-KPP equations depending only on  $t$ , for example :

$$\partial_t u - \partial_{xx} u = r(t)u(1 - u), \quad (2.4)$$

where  $r$  is a measurable, bounded function with positive infimum. For such equations, it is more appropriate to use an alternative and equivalent formulation of generalized fronts : they can be characterized as time-global positive solutions which could be written  $u(t, x) = U(t, x - \int_0^t \sigma(s)ds)$ , with  $U(t, -\infty) = 1$  and  $U(t, +\infty) = 0$  uniformly in  $t$ . The function  $\sigma(t)$  represents the local speed of the wave.

The existence of generalized waves for this type of equations has been demonstrated by Shen [93] in the case where  $r$  is uniquely ergodic, which implies in particular that the limit  $\langle r \rangle := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_t^{t+T} r(s)ds$  exists and is uniform in  $t \in \mathbb{R}$ . Such waves exist for a family of local speeds satisfying  $\langle \sigma \rangle \geq 2\sqrt{\langle r \rangle}$ . Our goal is to extend this result to general coefficients, no longer admitting a uniform mean, and to identify a quantity substituting for these means.

We proved with L. Rossi [NR12] that for every  $c > 2\sqrt{[r]}$ , there exists a generalized wave of speed  $\sigma$  such that  $[\sigma] = c$ , where one defines the least mean :

$$[r] := \sup_{T>0} \inf_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} r(s) ds.$$

This result easily extends to the case where  $c = 2\sqrt{[r]}$  using the notion of critical wave developed a posteriori in [N15-1]. Moreover, such fronts do not exist if  $c < 2\sqrt{[r]}$ . The quantity  $2\sqrt{[r]}$  thus plays the role of a critical speed on the least means.

Moreover, these fronts are decreasing in  $x$ , and in the case where the coefficients satisfy some properties (almost periodicity, ergodicity), then these properties are transmitted to the speed  $\sigma$  and to the profile of the front  $U$ . Indeed, the built  $\sigma$  is explicit :  $\sigma(t) = r(t)/\kappa + \kappa$ , with a  $\kappa$  well chosen to build sub and super solutions. A detailed study of the properties of the least means thus makes the condition  $\kappa \in (0, [r])$  emerge.

This result has then been partially extended by L. Rossi and L. Ryzhik [87] to media depending both time  $t$  in a general way and periodically on space  $x$ , but for a particular growth rate of the type  $r(t, x) = r_1(t) + r_2(x)$  and a diffusion matrix and an advection term depending only on  $x$ . Also, Shen [94] has proved the existence of generalized waves for periodic coefficients in  $x$  and uniquely ergodic in  $t$ . We have generalized this result to an equation of the type (2.4), whose coefficients are periodic in  $x$  and depend on  $t$  in a general way [NR15] : there exist  $c_* \leq c^*$  such that for all  $c > c^*$ , there exists a generalized wave of speed  $\sigma$  such that  $[\sigma] = c$ , whereas such a front does not exist if  $c < c_*$ . The speeds  $c_*$  and  $c^*$  are, of course, much more implicit than in the case where  $r$  depends only on  $t$ . Their characterization involves solutions of the problem linearized in 0. But we have not succeeded in demonstrating in general that  $c_* = c^*$ . This work allowed us to understand that without the hypothesis of a periodicity in  $x$ , the existence of generalized fronts can be obtained if we manage to prove a Harnack type inequality, global in  $x$ , for the linearization of the equation (2.4) in the neighborhood of  $u = 0$ .

## 2.4 Existence of generalized transition waves for Fisher-KPP equations with almost periodic coefficients

We have studied in this joint work with L. Rossi [NR16] The one-dimensional Fisher-KPP equations :

$$\partial_t u_t - \partial_x(a(x)\partial_x u) = r(x)u(1 - u), \quad t \in \mathbb{R}, x \in \mathbb{R}, \quad (2.5)$$

## 2.4. Existence of generalized transition waves for Fisher-KPP equations with almost periodic coefficients

with coefficients  $a \in \mathcal{C}^1(\mathbb{R})$ ,  $c \in \mathcal{C}(\mathbb{R})$  satisfying :

$$\inf_{\mathbb{R}} a > 0, \quad \inf_{\mathbb{R}} r > 0, \quad a, a', r \text{ are almost periodic.}$$

We will use here the Bochner definition of almost periodicity :

**Definition 2.3.** [27] A function  $a : \mathbb{R} \rightarrow \mathbb{R}$  is almost periodic (a.p. in the sequel) if from any sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}$  one can extract a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  such that  $a(x_{n_k} + x)$  converges uniformly in  $x \in \mathbb{R}$ .

These functions have many properties. They belong to the class of uniquely ergodic functions, that is, if  $r$  is almost periodic, then the limit  $\lim_{L \rightarrow +\infty} \frac{1}{L} \int_x^{x+L} r(y) dy$  exists and is uniform in  $x \in \mathbb{R}$ .

Let  $\mathcal{L}$  the operator associated with the linearized equation near 0 :

$$\mathcal{L}\phi := (a(x)\phi')' + r(x)\phi, \quad (2.6)$$

and  $\lambda_1(\mathcal{L}, \mathbb{R})$  its generalized principal eigenvalue, this time in the sense of Berestycki, Nirenberg and Varadhan [22] :

$$\lambda_1(\mathcal{L}, \mathbb{R}) := \inf\{\lambda \in \mathbb{R}, \exists \phi \in \mathcal{C}^2(\mathbb{R}), \phi > 0, \mathcal{L}\phi \leq \lambda\phi \text{ in } \mathbb{R}\}. \quad (2.7)$$

In dimension 1, the family  $(\phi_\gamma)_{\gamma > \lambda_1(\mathcal{L}, \mathbb{R})}$  can be defined, without having to make any almost periodicity hypothesis, by :

$$\mathcal{L}\phi_\gamma = \gamma\phi_\gamma \text{ in } \mathbb{R}, \quad \phi_\gamma(0) = 1, \quad \phi_\gamma > 0, \quad \lim_{x \rightarrow +\infty} \phi_\gamma(x) = 0. \quad (2.8)$$

If the coefficients are almost periodic, then  $\phi'_\gamma/\phi_\gamma$  is almost periodic and it is thus possible to define the limit

$$\mu(\gamma) := - \lim_{x \rightarrow \pm\infty} \frac{1}{x} \ln \phi_\gamma,$$

which is increasing and concave with respect to  $\gamma > \lambda_1(\mathcal{L}, \mathbb{R})$ .

**Theorem 2.4.** [NR16] Let

$$c^* := \inf_{\gamma > \lambda_1} \frac{\gamma}{\mu(\gamma)}, \quad \underline{c} := \frac{\lambda_1}{\underline{\mu}}, \quad \text{where } \underline{\mu} := \lim_{\gamma \searrow \lambda_1} \mu(\gamma).$$

The following properties hold :

1. If  $c^* < \underline{c}$  then for all  $c \in [c^*, \underline{c}]$ , there exists a time-increasing generalized transition front with average speed  $c$ ; for  $c > c^*$ , the front can be written as  $u(t, x) = U(\int_0^x \sigma - t, x)$ , where  $\sigma \in \mathcal{C}(\mathbb{R})$  is a.p. and has average  $1/c$  and  $U = U(z, x)$  is a.p. in  $x$  uniformly in  $z \in \mathbb{R}$ .
2. There are no generalized transition fronts with average speed  $c < c^*$ .

If  $\underline{\mu} = 0$ , then  $\underline{c} = +\infty$  and condition  $c^* < \underline{c}$  is always satisfied. An example of an almost periodic term of  $0 = r(x)$  for which  $\underline{\mu} > 0$  has been constructed by Sorets and Spencer [98] In the case of the discrete Laplacian and by Bjerklov [26] for the continuous Laplacian.

On the contrary, if  $\mathcal{L}$  admits a positive principal eigenfunction  $\varphi_1 \in \mathcal{C}^2(\mathbb{R})$ , that is, a solution of  $\mathcal{L}\varphi_1 = \lambda_1\varphi_1$ , which is almost periodic, then  $\underline{\mu} = 0$ . Such a function exists in particular

- if the coefficients are periodic and not just almost periodic,
- if  $r$  is constant,
- if  $a$  and  $r$  are quasiperiodic and if  $r$  has a sufficiently small amplitude.

We have also identified several conditions related to the criticality of the operator, that is, to the existence of sub and super eigenfunctions, guaranteeing the existence of an almost periodic eigenfunction.

Our result leaves open several interesting questions :

- What about the multi-dimensional case ? The construction of  $\phi_\gamma$  relies heavily on 1D arguments, are there similar solutions in larger dimensions ?
- In this case  $\underline{\mu} > 0$ , are there generalized waves of mean speed  $c > \underline{c}$  ? Is there any wave when  $\underline{c} = c^*$  ?
- What are the properties of the critical front of speed  $c = c^*$  ? Does it have an almost periodic profile ? Is it attractive, in a sense, for the Cauchy problem ?

## 2.5 Traveling waves for equations with prescribed speed

In this somewhat different work with J. Bouhours [BoN15], we aimed at establishing conditions guaranteeing the survival of a population in an environment evolving at constant speed in time because of climate change. H. Berestycki and several co-authors [12, 23, 89] have studied this question using a reaction-diffusion model (see also [84] as well as [64] for an integro-differential model ). The density of population  $u$  then satisfies an equation of type

$$u_t - u_{xx} = f(x - ct, u) \quad \text{dans } (0, \infty) \times \mathbb{R}, \quad (2.9)$$

where  $f(x, 0) = 0$ ,  $f(x, 1) \leq 0$ , and  $f(x, u) \leq -\delta u$  for all  $|x| \geq R$  and  $u \geq 0$ . In other words,  $u = 0$  is a state of equilibrium, when  $u = 1$  the media is saturated because of the finite quantity of resources and the growth rate  $f(x, \cdot)$  is very negative if  $x$  is too large, that is, the favorable zone (habitat) for the population is located in space. The quantity  $c$  represents then the rate at which the habitat evolves because of the global warming.

This equation has been studied exhaustively by Berestycki and Rossi [23] in the case where the nonlinearity is of KPP type, that is, the growth rate per individual  $f(x, u)/u$  is decreasing with respect to  $u$ . They showed that under these hypotheses there exists a critical speed  $c^*$  such that if  $c \in (0, c^*)$ , then  $u$  converges to a traveling wave solution when  $t \rightarrow +\infty$ , of the form  $u(t, x) = U(x - ct)$ , with  $U(\pm\infty) = 0$  and  $U > 0$ , whereas if  $c > c^*$ , then  $u$  tends to 0 locally. Thus, the critical speed  $c^*$  determines the persistence or extinction of the population : if the global warming is too fast, then the population goes extinct.

The KPP hypothesis on  $f$  is a relatively strong hypothesis. It is not fully realistic, biologists considering that it may be more natural to introduce an Allee effect, that is, a bistable nonlinearity : if the density  $u$  is too small, then the growth rate  $f(x, u)$  must be negative because the population is too dispersed to manage to reproduce. From the mathematical point of view, the hypothesis KPP allows to reduce to the study of the linearization of the equation around  $u = 0$ , which proves then to fully determine the dynamics of the equation. Our goal was to investigate the equation (2.9) under more general hypotheses on  $f$ .

Our first results were derived by noting that (2.9) has a variational structure. Let us introduce the energy :

$$E_c[u] = \int_{\mathbb{R}} e^{cz} \left\{ \frac{u_z^2}{2} - F(z, u) \right\} dz, \quad \forall u \in H_c^1(\mathbb{R}),$$

where  $H_c^1(\mathbb{R}) = H^1(\mathbb{R}, e^{cx} dx)$  and  $F(z, s) = \int_0^s f(z, t) dt$ . Then the traveling waves are indeed critical points of this energy functional. We could thus derive the following result :

**Theorem 2.5.** [BoN15] *If  $\inf_{u \in H^1(\mathbb{R})} E_0[u] < 0$ , then there exists  $\bar{c} \geq \underline{c} > 0$ , such that*  
*— for all  $c \in (0, \underline{c})$ , (2.9) admits a traveling wave solution in  $H_c^1(\mathbb{R})$ ,*



## 2.5. Traveling waves for equations with prescribed speed

— *for all  $c > \bar{c}$ , (2.9) does not admit any traveling wave solution, that is, 0 is the only stationary solution of (2.9).*

We have also shown that the solutions of the evolution problem (2.9) converge to stationary solutions in the moving frame of speed  $c$ , so to a traveling wave of speed  $c$  or to 0. Different conditions guarantee the convergence towards 0 or a traveling wave.

The existence of an exact critical speed  $\underline{c} = \bar{c} = c^*$ , analogous to the KPP case, is far from clear. Our numerical studies indicate that such speed should exist. Moreover, for nonlinearities  $f$  modeling an Allee effect, the steady state 0 becomes locally stable, unlike in the KPP case. The problem of minimization of the energy hidden in (2.9) is then non-trivial : several critical points can coexist, each corresponding to a traveling wave, but of variable stability. These phenomena are qualitatively very different from the KPP case.



# Chapitre 3

## The Fisher-KPP equation with nonlocal saturation

This chapter is devoted to the investigations of equation

$$u_t - u_{xx} = ru(1 - K_\sigma \star u) \quad (3.1)$$

where 0 and  $K$  is a probability density and

$$K_\sigma(x) := \frac{1}{\sigma} K\left(\frac{x}{\sigma}\right),$$

where  $\sigma$  is thus a parameter measuring the length of the non-locality, and  $\star$  is the standard convolution product. This equation was introduced by Genieys, Volpert and Auger [45] as a model Of the adaptive dynamics, the space parameter  $x$  represents here a trait evolving stochastically in time and the kernel  $K(x - y)$  measures the intensity of the competition between two line individuals  $x$  and  $y$ .

The main difficulty encountered in the study of this equation is, as for reaction-diffusion systems, the absence of a principle of comparison for the equation (3.1), because of the non-local term. Note that if the nonlinearity was  $K \star u(1 - u)$ , then the equation would still have a comparison principle, and the problem would then be of a totally different nature. This difficulty is therefore specific to the equation (3.1). Indeed, it is not a simple technical difficulty but a characteristic of the equation since it is this lack of comparison principle which allows the appearance of stationary non-trivial solutions, observed numerically in [45], via a Turing mechanism [100] : the homogeneous state 1 becoming unstable relative to well-chosen periodic perturbations, giving rise to branches of bifurcations.

We had demonstrated with H. Berestycki, B. Perthame and L. Ryzhik [BNPR09] that for every  $c \geq 2$ , (3.1) admits a traveling  $U(t, x) = U(x - ct)$ , with  $U > 0$ ,  $U(+\infty) = 0$  and  $\liminf_{z \rightarrow -\infty} U(z) > 0$ , this last condition being used to include any fronts connecting 0 to other state of equilibrium than 1. We also proved that if the Fourier transform  $\hat{\phi}$  was positive, then necessarily  $U(-\infty) = 1$ . Finally, we identified a parameter  $\sigma_M(c)$  such that for every  $\sigma > \sigma_M(c)$ , a traveling wave of speed  $c$  is necessarily non-monotonous. This result was innovative in this field because it was based on methods derived from local reaction-diffusion equations, giving global results rather than perturbative results as in most articles on non-local equations.

The existence of monotone waves, necessarily connecting 0 to 1, has subsequently been established for  $\sigma < \sigma_M(c)$  by Fang and Zhao [41], based on methods used In the framework of delayed equations [46, 66]. Alfaro and Coville [1] have also used energy methods to show that if the speed  $c$  is large enough (with  $\sigma$  fixed) then  $U(-\infty) = 1$ .

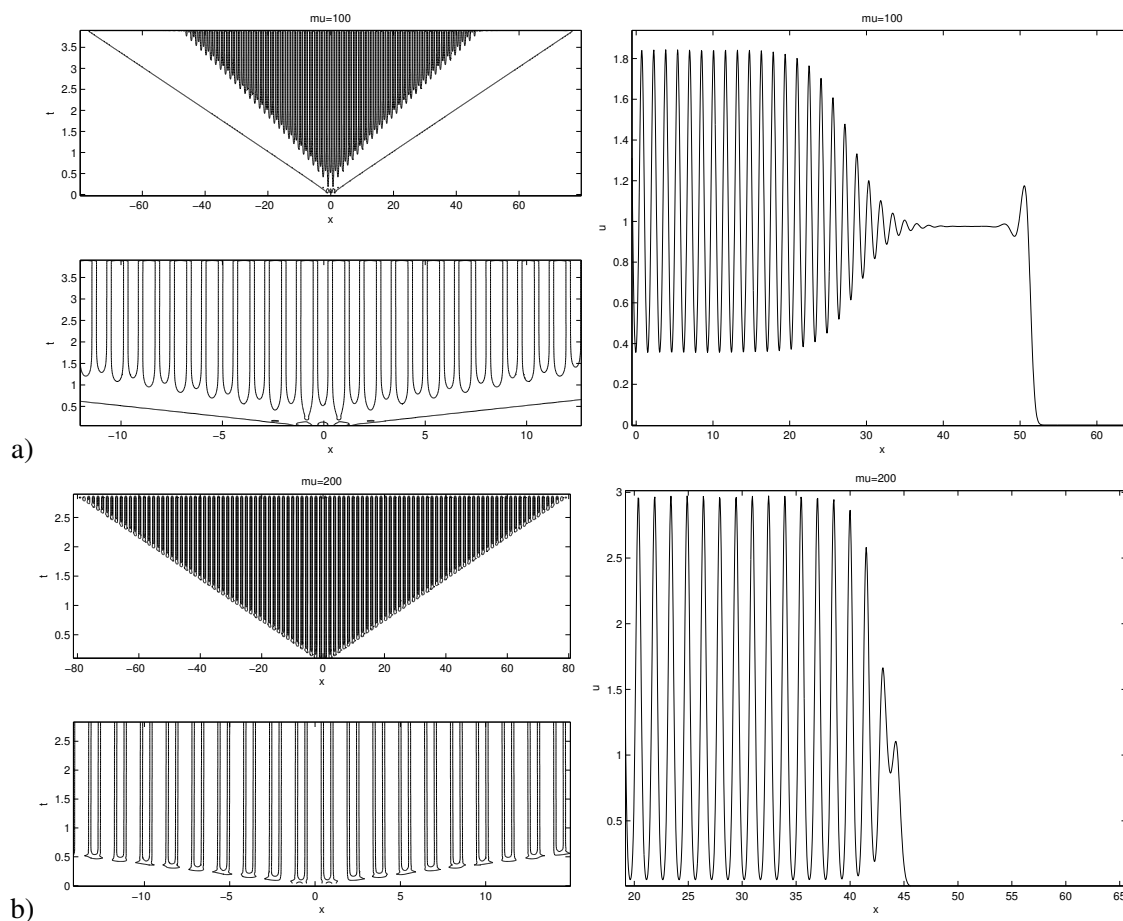


FIGURE 3.1 – Simulations of the Cauchy problem associated with (3.1) with the kernel  $K = 1_{(-1/2, 1/2)}$  on the domain  $[-80, 80]$ . Left : the level lines show that the solution does not converge to a traveling wave but has a more complex structure ; the figures at the bottom are zooms of the top figures. Right : the function connects the unstable state 0 to a periodic solution in  $x$ . Two values of  $r$  are used a)  $r = 100$  ; b)  $r = 200$ .

### 3.1. A new numerical approach to see emerging traveling waves

More recently, Faye and Holzer [38] constructed pulsating fronts connecting the stationary solution 1 with a very small speed to a periodic stationary solution in the neighborhood of the parameter  $\sigma$  for which the equilibrium 1 becomes unstable in Fourier variables, for the particular kernel

$$K(z) = \frac{3a}{2}e^{-a|z|} - e^{-|z|}, \quad a \in (2/3, 1).$$

This type of kernel makes it possible to transform the non-local equation into a system, for which the bifurcation methods apply.

On the other hand, the link between these traveling waves and the Cauchy problem associated with (3.1) is still poorly understood. Hamel and Ryzhik have constructed a uniform bound on the solutions of the Cauchy problem and the existence of an expansion speed [53], but there are no other results in this direction to our knowledge.

The existence of pulsating waves for other kernels, and further from the critical parameter, and / or connecting 0 to a periodic steady state remains open. In this chapter we present several approaches to partially answer this question.

**Question 3.** *What is the asymptotic behavior of traveling waves for the non-local Fisher-KPP equation (3.1) ?*

### 3.1 A new numerical approach to see emerging traveling waves

The numerical simulations presented in Figure 3 were limited to simulating the evolution equation associated with 3.1), thus observing the emergence of pulsating fronts connecting 0 to a periodic steady state for a non-locality  $\sigma$  large enough. But we can very well imagine that there are dynamically unstable traveling waves, therefore not observable by simulation of the evolution equation. On the basis of this hypothesis, we then directly simulated, with B. Perthame and M. Tang [NPT11], the construction method of the traveling waves used in [BNPR09].

Our algorithm approaches the problem in a bounded domain of length  $L = x_r - x_l$ ,  $x_l < 0 < x_r$  :

$$\begin{cases} \sigma^L \partial_x u^L + \partial_{xx} u^L + u^L(1 - K_\sigma \star u^L) = 0, & x_l < x < x_r, \\ u^L(x_l) = 1, \quad u^L(x_r) = 0, \quad u^L(0) = \varepsilon, \end{cases} \quad (3.2)$$

where the convolution is computed by extending  $u^L$  by 1 to the left and 0 to the right. We have analytically proved in [BNPR09] that if  $\varepsilon$  is small enough (under all possible oscillations of stationary periodic solutions), the sequence  $(u^L, \sigma^L)$  converges to a traveling wave when  $L \rightarrow +\infty$ .

In order to solve (3.2) we divide the domain into two parts :  $I_1 = [x_l, 0]$  and  $I_2 = [0, x_r]$ . For every  $\sigma$ , we then pass the elliptic equation with Dirichlet conditions at the boundary :

$$-\sigma \partial_{xx} u_i = \partial_{xx} u_i + u_i(1 - K_\sigma \star u), \quad u_i(0) = \varepsilon, \quad i = 1, 2, \quad u_1(x_l) = 1, \quad u_2(x_r) = 0. \quad (3.3)$$

The convolution term is computed using  $u_i$  on the interval  $I_i$ , 1 on  $(-\infty, x_l)$  and 0 on  $(x_r, +\infty)$ .

In order to compensate for the derivatives jump at  $x = 0$ , we would like  $\sigma^L$  to satisfy

$$\sigma^L = [\partial_x u_2(x_r) - \partial_x u_1(x_l)] + \int_{x_l}^{x_r} u(1 - K_\sigma \star u) dx. \quad (3.4)$$

We have thus reformulated the construction of the approximated solution  $u^L$  into a fixed point problem on  $(\sigma, u)$ , which can be solved by Newton Iterations. The method gives very different results than those obtained by simulating the evolution equation, shown in Figure 3.1.

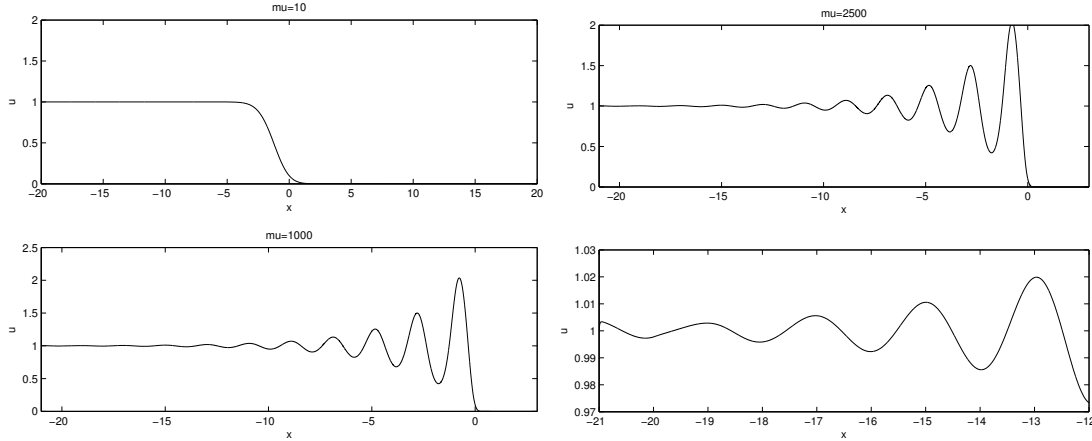


FIGURE 3.2 – traveling waves of equation (3.1) for the kernel  $K(z) := \frac{1}{2}bf1_{(-1,1)}$  for  $\mu = 10$  and  $\mu = 1000$  (left) and  $\mu = 2500$  (right, the upper figure shows the traveling wave while the bottom one is a zoom of its tail) obtained through the algorithm of [NPT11].

In particular, these simulations seem to show that traveling waves necessarily connect 0 to 1. This conjecture is natural since, for symmetric kernels  $K$ , it is necessary to add a periodic dependence in  $x$  so that a bifurcation to stationary periodic solutions can take place. It is therefore expected that these stationary solutions will be connected to 0 by pulsating fronts, as in [38] and not merely traveling waves.

This approach has then been generalized by M. Tang to several other types of reaction-diffusion equations, local or non-local, heterogeneous or not [99].

### 3.2 A “toy-model” exhibiting a wide variety of waves

Let us now consider the equation :

$$u_t - u_{xx} = ru(x)(1 - u(x - a)) \quad (3.5)$$

with  $a > 0$ . In other words, unlike the case discussed above, the kernel  $K = K(z)$  is here asymmetric : a Dirac mass at  $z = a$ . Let us simplify this equation by considering a piecewise linear nonlinearity in the neighborhoods of 0 and 1 :

$$\partial_t u(t, x) - \partial_{xx} u(t, x) = \begin{cases} ru(t, x) & \text{for } 0 \leq u(t, x) < \theta \\ 1 - u(t, x - a) & \text{for } u(t, x) \geq \theta, \end{cases} \quad (3.6)$$

with  $\theta \in (0, 1)$ ,  $a, r > 0$

We can then explicitly construct waves, that is, solutions of the form  $u(t, x) = U(x - ct)$ , by solving the two linear problems and by connecting the solutions thus obtained. We have shown, with B. Perthame, L. Rossi and L. Ryzhik [NRRP13], that there exists a large number of solutions according to the parameters :

- monotone traveling waves for any speed  $c \geq 2\sqrt{r}$  if  $r \geq (1 - \theta)/\theta$ ,
- non-monotonic traveling waves linking 0 to 1 with speed  $c$  for any  $c \geq 2\sqrt{r}$  and  $a$  large enough (depending on  $c$ ),
- $k+1$  wave trains if  $(2k+1)\pi \leq a < (2k+3)\pi$ , that is, solutions of the form  $u(t, x) = W(x - ct)$ , with  $W$  periodic,

### 3.3. Existence and asymptotic behavior of traveling waves for delayed equations

— a traveling wave connecting 0 to a wavetrain as soon as  $a \geq a_* := \frac{3}{2}\pi\sqrt{2r + \sqrt{4r^2 + 1}}$ .  
This result confirms the great richness of the equation (3.1).

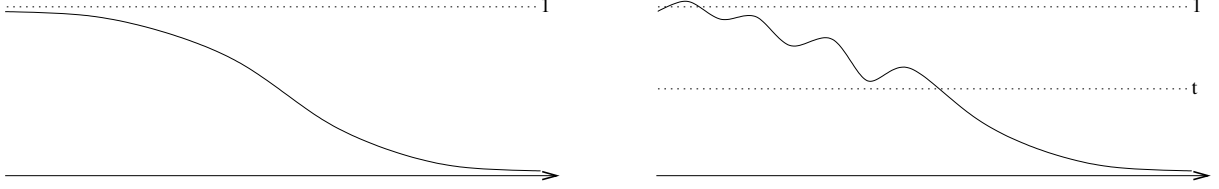


FIGURE 3.3 – Representations of the monotone traveling wave (left) and the non-monotone one (right) connecting 0 to 1.

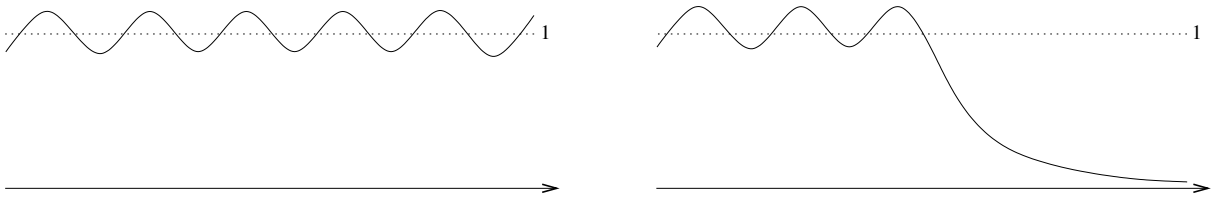


FIGURE 3.4 – Representations of the wavetrain (left) and its connexion to 0 (right).

### 3.3 Existence and asymptotic behavior of traveling waves for delayed equations

We have studied the Fisher-KPP equation with delay in time with A. Ducrot [DN14],

$$u_t - u_{xx} = u(t, x)(1 - u(t - \tau, x)),$$

with  $\tau > 0$  the delay. A traveling wave of this equation  $u(t, x) = U(x - ct)$  satisfies

$$-U'' - cU' = U(z)(1 - U(z + c\tau)) \quad (3.7)$$

with the additional conditions  $U > 0$ ,  $U(+\infty) = 0$  and  $\liminf_{z \rightarrow -\infty} U(z) > 0$ . The equation (3.7) is thus of the same form as that satisfied by the traveling waves of (3.5) studied in the previous section but with  $a = -c\tau < 0$ .

The existence of monotone traveling waves, converging to 1 at  $-\infty$ , has been proved in parallel by Kwong and Ou [66] and Gomez and Trofimchuk [46], for any  $\tau \leq \tau_{mon}(c)$  explicit. However, the existence of traveling waves and their convergence at  $-\infty$  for  $\tau \geq \tau_{mon}(c)$  remained open.

Equation (3.7) belongs to the theory of differential systems with positive loop delay (changing  $z$  by  $-z$ ) studied by Mallet-Paret and Sell in [71, 72]. Using this observation, we deduced [DN14], as well as Hasik and Trofimchuk [55] in parallel with a different method, that for any  $\tau > 0$  and  $c \geq 2$ , there are traveling waves, with the relaxed asymptotic condition  $\liminf_{z \rightarrow -\infty} U(z) > 0$ .

To determine the convergence at  $-\infty$  of the traveling waves is a difficult problem, but for which one can be more precise than for the kernel equation (3.1). Hence, Hasik and Trofimchuk [56] showed convergence to 1 for all  $\tau \leq 3/2$  and  $c \geq 2$ , as well as non-convergence to 1 when  $\tau \geq \pi/2$  and  $c \geq c^*(\tau)$ , with  $c^*(\tau)$  explicit.

We have characterized the convergence of traveling waves via a different approach. Indeed, by getting back into the proof of the results of [72], we can show that a traveling wave is completely determined by

the couple  $(U(z_0), U'(z_0))$  for any  $z_0 \in \mathbb{R}$ . In particular, the curve associated with  $U$  in the phase plane does not self-intersect, and therefore many methods from ODEs can be used.

**Proposition 3.1.** *Consider two solutions  $U, V \in \mathcal{C}^2(\mathbb{R})$  of (3.7). Assume that*

$$\liminf_{z \rightarrow +\infty} \frac{U(z)}{V(z)} \leq 1 \leq \limsup_{z \rightarrow +\infty} \frac{U(z)}{V(z)}$$

*and that there exists  $A < 0$  such that  $U(z) > V(z)$  for all  $z < A$ . Then there does not exist any  $X \in \mathbb{R}$  such that  $U(X) = V(X)$  and  $U'(X) = V'(X)$ .*

*In particular, if  $U \in \mathcal{C}^2(\mathbb{R})$  is a positive bounded solution of (3.7) such that  $U(+\infty) = 0$ , then the curve  $(U, U')$  does not self-intersect.*

This result is very surprising, since it is known that a delayed equation is well-posed only if it is associated initial datum defined over a time interval of length equal to the delay. The few additional properties included in the Proposition (positivity, definition on all  $\mathbb{R}$  and  $c$ .) are indeed sufficient to guarantee that the curve does not self-intersect in the phase plane.

It leads naturally to the following definition and convergence result.

**Definition 3.2.** *We say that a positive solution  $w \in \mathcal{C}^2(\mathbb{R})$  of (3.7) is a maximal wavetrain (of speed  $c$ ) if  $w$  is periodic with period  $L \geq c\tau$  and if the curve*

$$\mathcal{S} = \{(w(z), w'(z)), z \in \mathbb{R}\}$$

*is a Jordan curve and if for all wavetrain  $\tilde{w}$ , one has*

$$\{(\tilde{w}(z), \tilde{w}'(z)), z \in \mathbb{R}\} \subset \mathcal{S} \cup \mathcal{S}^{int}$$

*where  $\mathcal{S}^{int}$  is the interior of the Jordan curve  $\mathcal{S}$ .*

**Theorem 3.3** (Convergence to the maximal wavetrain). *Let  $c \geq 2$  be given. Then Equation (3.7) has a unique (up to translation) maximal wavetrain  $w \equiv w(z)$ . If  $U \equiv U(z)$  is a traveling wave solution of (3.7) then there exists  $\tau \in \mathbb{R}$  such that*

$$\lim_{z \rightarrow +\infty} (u(z) - w(z + \tau)) = 0.$$

*When nontrivial, this unique maximal wavetrain is slowly oscillating around the stationary state 1.*

It can be noticed that the maximum wavetrain is also the one with the largest oscillations, that is to say both the largest and the smallest minimum.

It is possible that the maximum wavetrain is actually the constant solution 1. Our result says that if there is a nontrivial wavetrain, then the traveling waves do not converge to 1 but to the maximum wavetrain, and vice versa : if the traveling wave converges to 1 then there is no wavetrain.

Although we have characterized the convergence of traveling waves as  $-\infty$ , we have not proved the uniqueness of these traveling waves. This problem is still open.

Let us conclude by noting that the results presented in this section are quite different from the results obtained in the previous section, although the two equations are very similar, up to the sign of  $a$  : while for  $a > 0$  a large variety of waves exists, for  $a < 0$  the limit at  $-\infty$  is well-defined, letting conjecture that the traveling wave is unique.

In conclusion, this work shows that the traveling waves constructed in [BNPR09] for a symmetric kernel  $K(z) = K(-z)$  probably always connect 0 to 1, even when 1 is unstable [NPT11], in which



### 3.3. Existence and asymptotic behavior of traveling waves for delayed equations

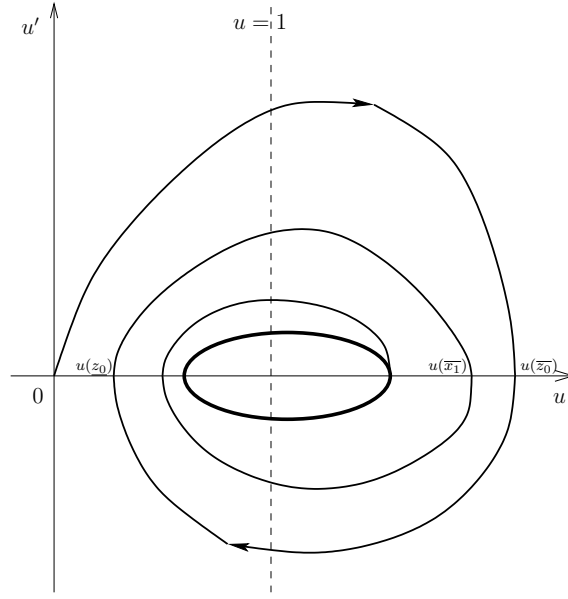


FIGURE 3.5 – The curve  $\mathcal{C}$  in the phase plane  $(U, U')$  and the curve associated with the maximal wavetrain  $(w, w')$ .

case a pulsating front would superimpose to connect 1 to a stable stationary state [38]. However, in the case of an asymmetric kernel, the traveling waves change in nature and their convergence depends on the position of the asymmetry : at the back of the front [NRRP13], it allows many wave-like solutions to coexist, whereas ahead of the front [DN14], the traveling waves connect to a well-defined object : the maximal wavetrain.



# Chapitre 4

## A microscopic point of view on the Fisher-KPP equation

This chapter is devoted to the study of the kinetic reaction-diffusion equation :

$$f_t + vf_x = M(v)\rho_f - f + r\rho_f(M(v) - f), \quad (4.1)$$

where  $f = f(t, x, v)$  represents a population density parameterized by time  $t$ , position  $x$  and local speed  $v \in V \subset \mathbb{R}$ ,  $\rho_f(t, x) := \int_V f(t, x, v)dv$  and  $M$  is a probability density that measures the probability that an individual has changed speed to acquire the speed  $v$  :

$$M \geq 0, \quad \int_V M(v)dv = 1, \quad \int_V vM(v)dv = 0, \quad \int_V v^2 M(v)dv = D < +\infty. \quad (4.2)$$

This equation has been introduced in a general form by Bisi and Desvillettes in 2006 [25] and in this form by Cuesta, Hittmeir and Schmeiser [34]. It is a kind of microscopic version of the equation of reaction-diffusion. Indeed, under the effect of the rescaling allowing more and more frequent changes of speeds, we arrive at the equation :

$$\varepsilon^2 f_t + \varepsilon v f_x = M(v)\rho_f - f + \varepsilon^2 r\rho_f(M(v) - f). \quad (4.3)$$

It can then be shown that the density  $\rho(t, x) = \int_V f(t, x, v)dv$  converges when  $\varepsilon \rightarrow 0$  to the solution of the reaction-diffusion equation

$$\partial_t \rho = D\partial_{xx}\rho + r\rho(1 - \rho),$$

where  $D := \int_V v^2 M(v)dv$ . It is by using such a rescaling and perturbative methods that Cuesta, Hittmeir and Schmeiser [34] proved the existence when  $\varepsilon$  is small enough of traveling waves for equation (4.3), when  $M$  is compactly supported in  $v$ . The steady state playing the role of 1 for the equation (4.1) is  $M(v)$ . So in this context a traveling wave is a solution of the form  $f(t, x, v) = U(x - ct, v)$ , with  $U > 0$ , and  $U(-\infty, v) = M(v)$ .

The existence of traveling waves for this equation was known only in the particular case of a uniform kernel :  $M(v) = \frac{1}{|V|}\mathbf{1}_V(v)$  [90, 91], for KPP, ignition or bistable nonlinearities.

Our aim in this chapter is to investigate the existence of traveling waves for (4.1) directly, that is to say, outside the perturbative onset and for general kernels.

**Question 4.** *Under which conditions on the kernel  $M(v)$  do there exist traveling waves for equation (4.1) ? If there do not exist any traveling wave, could we locate the level lines of the solution of the Cauchy problem ?*

## 4.1 Existence of traveling waves for a sum of two Dirac kernel

We have in a first time studied with E. Bouin and V. Calvez [BCN14] the case where  $M$  is replaced by two Dirac masses in  $v_m$  and  $-v_m$  and where the reaction term  $r\rho_f(M(v) - f)$  is replaced by the very similar term  $r\rho_f(M(v) - \rho_f)$ . In this case the equation simplifies to a scalar equation on the density  $\rho(t, x)$  :

$$\tau^2 \partial_{tt} \rho + (1 - \tau^2 + 2\tau^2 \rho) \partial_t \rho - \partial_{xx} \rho = r\rho(1 - \rho), \quad (4.4)$$

where  $\tau = 1/v_m$ . If  $\tau = 0$  we find the standard Fisher-KPP equation.

The equation has a dual nature, both hyperbolic (for  $\tau$  large) and parabolic (for  $\tau$  small). If  $\tau^2 r < 1$  (parabolic), there is a traveling wave of speed  $c$  if and only if  $c \geq c_\tau^*$ , where

$$c_\tau^* = \frac{2\sqrt{r}}{1 + \tau^2 r}. \quad (4.5)$$

This result has been proved by Hadeler [47] in 1988, but only for  $c \in [c_\tau^*, 1/\tau)$ , and with very different methods. We can immediately notice that the introduction of the term in  $\tau$  slows down the propagation in comparison with the standard Fisher-KPP equation, for which  $c^* = 2\sqrt{r}$ . These traveling waves change in nature if  $c \geq 1/\tau$  and can become singular. Finally, the traveling wave of speed  $c_\tau^*$  is stable in weighted  $L^2$  spaces.

If  $\tau^2 r \geq 1$  (hyperbolic regime) then there is a traveling wave of speed  $c$  if and only if  $c \geq 1/\tau$ , the traveling wave with speed  $1/\tau$  being discontinuous if  $\tau^2 r > 1$ . The proof of these results calls on both parabolic methods (sub and solutions method) and hyperbolic (shock constructions by hand).

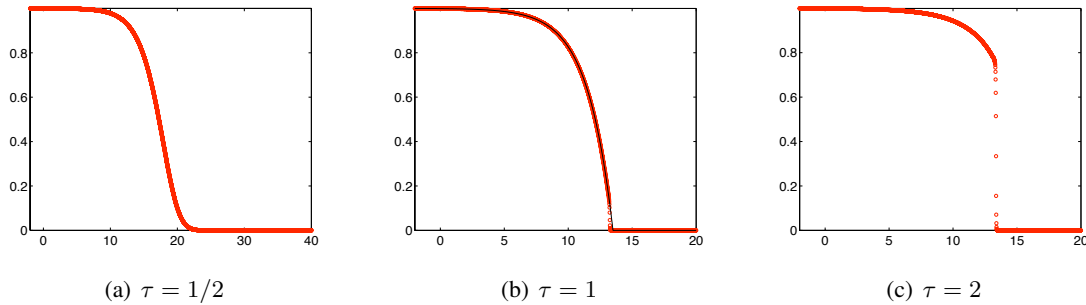


FIGURE 4.1 – Numerical simulations ( $r = 1$ ). If  $\tau < 1$ , the traveling wave is smooth and its speed is determined by the edge of the wave, where  $\rho \simeq 0$ . If  $\tau \geq 1$ , the traveling wave is no longer smooth : the population density jumps from  $\rho = 0$  to  $\rho = 3/4$  for  $\tau = 2$ .

## 4.2 An empirical validation of this model

Equation (4.4) was used in the 1990s by Fort and Mendez [40] to model the neolithic transition, that is, the spread of agriculture in Europe. This approach goes back to the work of the archaeologists Ammerman and Cavalli-Sforza [4], who modeled this transition using the Fisher-KPP equation (called in this community the "wave-of-advance model"). By estimating the rate of diffusion and reproduction rate of human populations in the Neolithic, using data from current agricultural populations, Ammerman and Cavalli-Sforza estimated an empirical speed propagation using archaeological data (distance and dating from excavation sites). The expansion speed thus obtained was clearly larger to the observed one, even at the first approximation (see Figure 4.1).

### 4.3. Existence of traveling waves for a compactly supported kernel

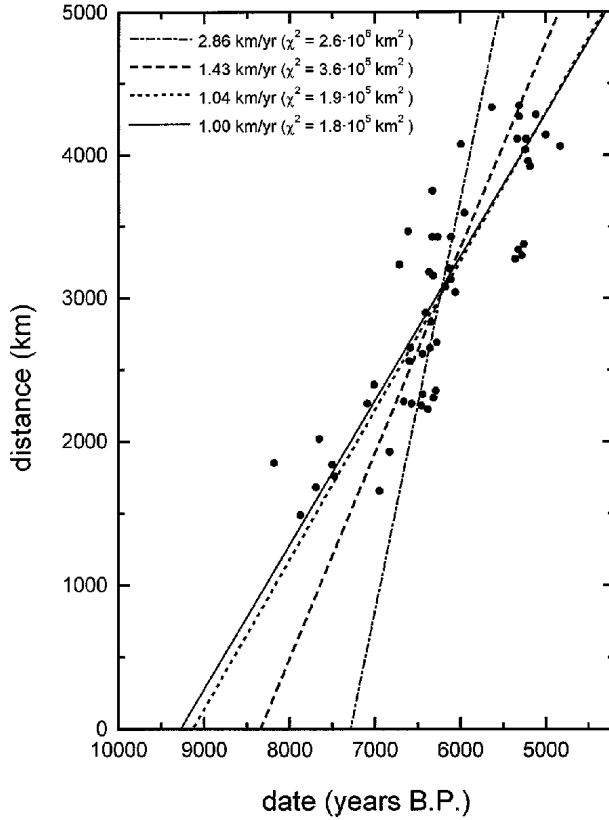


FIGURE 4.2 – Figure from [40] : comparison between the Fisher-KPP model and the delay equation model (4.4). Each point corresponds to a dated archaeological site where evidence of agricultural activity has been found, the distances being measured from Jericho (pre-empted place of agriculture). The solid line is the regression from the empirical data of [4].

The other three lines are computed from the Fisher-KPP model (dashed line and dashes) and the delay model (dashed line). We will not discuss here the dashed line, and refer to [40]. The delay model is clearly the most adjusted to the regression from the empirical data.

Fort and Mendez therefore assumed that Fick's law quantifying the dispersion was delayed by  $\tau = 25$  years, that is, it was necessary to wait for a generation until a migration takes place. This delay in the dispersion law transforms the Fisher-KPP equation into (4.4). The speed  $c_\tau^*$  thus obtained is much closer to the empirical speed, thus validating this model.

This model of Fisher-KPP equation with delay is finally very natural and has then been used by several physicists, for example to model forest fires or chemical reactions (see the references in [BCN14]).

I have presented the link between the different models and have mathematically formalized the heuristic results of [4, 40] in an interdisciplinary book chapter resulting from the conference "Interactions in complex systems" [Nchapitre].

### 4.3 Existence of traveling waves for a compactly supported kernel

Let us go back to the study of the equation (4.1). Based on the methods developed in [BCN14], we have solved [BCN15] the case where the density  $M(v)$  is compactly supported in  $v$ , for which exponential solutions of the linear equation can be found explicitly in the neighborhood of 0.

**Theorem 4.1.** *Suppose that  $V$  is compact and that  $M \in \mathcal{C}^0(V)$  satisfies (4.2). Let  $v_{\max} = \sup V$ . There is a speed  $c^* \in (0, v_{\max})$  such that (4.1) admits a traveling waves with speed  $c$  for all  $c \in [c^*, v_{\max})$ . These traveling waves are decreasing in  $z = x - ct$ .*

*Moreover, if  $\inf_V M > 0$  then there is no traveling wave of speed  $c$  in  $[0, c^*)$ .*

Naturally, the minimal speed  $c^*$  is computed by looking for solutions of the linearized equation near

0 of the form  $Q(v)e^{-\lambda(x-ct)}$ , which leads us to solve the dispersion relation :

$$(1+r) \int_V \frac{M(v)}{1+\lambda(c-v)} dv = 1. \quad (4.6)$$

We prove that for every  $\lambda > 0$  there exists a unique speed  $c(\lambda) \in (v_{\max} - \lambda^{-1}, v_{\max})$  satisfying this relationship, and we define

$$c^* = \inf_{\lambda > 0} c(\lambda).$$

We therefore fully generalized the anterior results of [34] out of the perturbative onset, with a characterization of the minimal speed. Our construction of traveling waves uses a classical method of sub and supersolutions constructed with the help of the linearized problem, with however a lack of compactness due to the transport term which complicates the passage to the limit. We can also show that the speed  $c^*$  is indeed an exact expansion speed for the Cauchy problem.

Note that we have not been able to prove the existence of “supersonic” traveling waves, that is, traveling waves of speed  $c \geq v_{\max}$ . By analogy with the case where  $M$  is a sum of two Dirac presented in Section 4.1, one can conjecture that such traveling waves exist, but are undoubtedly of a different nature, and in particular singular.

This dispersion relation is obviously very useful for investigating the dependence between the minimum speed  $c^* = c^*(M)$  and the dispersion kernel  $M$ . It can thus be shown that

$$c^*(M^*) \leq c^*(M) \leq c^*(M_\star),$$

where  $M^*$  is the Schwarz rearrangement of the  $M$  function (see Definition 5.1 below) and  $M_\star = -(-M)^*$  is its increasing rearrangement. Thus,  $M$  should rather concentrate its mass at the extremes of  $V$  in order to increase the expansion speed. By pushing the reasoning further, one can demonstrate that one maximizes the speed by putting two Dirac at the extremes of  $M$ , which brings us back to our previous article [BCN14]. More formally, we have the following result :

$$\begin{aligned} \frac{2\sqrt{rD}}{1+r} &\leq c^*(M) \leq \frac{2\sqrt{r}}{1+r} v_{\max} & \text{if } r < 1, \\ \sqrt{D} &\leq c^*(M) \leq v_{\max}, & \text{if } r \geq 1. \end{aligned}$$

In each of these inequalities, the right-hand side is the expansion speed computed in the previous Section 4.1.

## 4.4 Non-existence of traveling waves and superlinear propagation for a Gaussian kernel

The dynamics become more complex when  $M$  is no longer compactly supported. Thus, the traveling waves no longer exist, as the following result shows.

**Proposition 4.2.** *Suppose that  $M(v) > 0$  for all  $v \in \mathbb{R}$ . Then the equation (4.1) does not admit any traveling wave solution.*

One can, however, try to locate the transition between 0 and 1. Consider the case of a Gaussian kernel  $M(v) = e^{-v^2}/\sqrt{2\pi}$ . The numerical simulations presented in Figure 4.4 show that for Heaviside initial data, the solutions of the Cauchy problem propagate at a superlinear speed. In other words, if we define a certain  $X(t)$  locating a level line of  $\rho_f(t, \cdot)$ , for example  $\rho_f(t, X(t)) = 1/2$ , then  $X(t)/t^{3/2}$  converges

#### 4.4. Non-existence of traveling waves and superlinear propagation for a Gaussian kernel

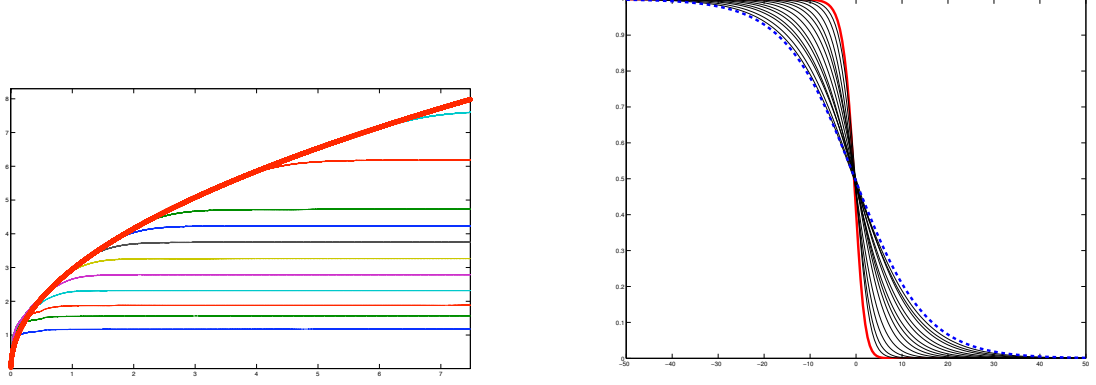


FIGURE 4.3 – Numerical simulations of the equation (4.1) with initial data  $f^0(x < 0, \cdot) = M(\cdot)$  and  $f^0(x > 0, \cdot) = 0$ . The distribution of the kernel  $M$  is Gaussian. Each plot corresponds to the evolution of the front speed for a truncation on  $V = [-A, A]$ , for (left)  $A = [(1 : 9), 15, 20]$ , and (right)  $A = (1 : 15)$ . The curves are ordered from bottom to top : the speed of the front is increasing in  $A$ . The  $t \mapsto t^{1/2}$  function is plotted in bold red : it seems to correspond to a very good approximation of the curve envelope. The front could thus propagate at the  $x \sim t^{3/2}$  scale. To the right are superimposed the macroscopic profiles  $\rho_f$  obtained at large time, for different truncations  $A = (1 : 15)$ . The profiles are translated in order to have  $\rho_f(t, 0) = \frac{1}{2}$ . We observe that the exponential decay rate is decreasing in  $A$ . Thus, the solution corresponding to  $V = \mathbb{R}$  should flatten when  $t \rightarrow +\infty$ .

when  $t \rightarrow +\infty$ . We can formally recover this growth rate of  $X(t)$  by solving  $M(X(t)/t) e^{rt} = 1$ , which gives  $X(t) \sim t^{3/2}$  in the case of a Gaussian  $M(v)$ .

We have also analytically shown [Nchapitre] the existence of lower and upper bounds propagating in  $t^{3/2}$  on the localization of the level lines for the Gaussian kernel.

**Theorem 4.3.** *Let  $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$  for all  $v \in \mathbb{R}$ . Let  $f^0 \in L^\infty(\mathbb{R} \times V)$  such that  $0 \leq f^0(x, v) \leq M(v)$  for all  $(x, v) \in \mathbb{R} \times V$ . Let  $f$  the solution of the Cauchy problem (4.1) with initial datum  $f^0$ .*

— Assume that there exist  $a \geq b \geq 1$  such that

$$\forall (x, v) \in \mathbb{R} \times V, \quad f^0(x, v) \leq \frac{1}{b} M\left(\frac{x}{b}\right) M(v) e^{ra}.$$

Then, for all  $\varepsilon > 0$ , one has

$$\lim_{t \rightarrow +\infty} \sup_{|x| \geq (1+\varepsilon)\sigma\sqrt{2rt^{3/2}}} \rho_f(t, x) = 0.$$

— If there exist  $\gamma \in (0, 1)$  and  $x_L \in \mathbb{R}$  such that

$$\forall (x, v) \in \mathbb{R} \times V, \quad f^0(x, v) \geq \gamma M(v) 1_{x < x_L},$$

then, for all  $\varepsilon > 0$ , one has

$$\lim_{t \rightarrow +\infty} \sup_{x \leq (1-\varepsilon)\sigma\left(\frac{rt}{r+2}\right)^{3/2}} \rho_f(t, x) \geq 1 - \gamma.$$

In other words, for Heaviside initial data (in  $x$ ), one could estimate the location the level lines by :

$$\sigma\left(\frac{r}{r+2}\right)^{3/2} \leq \liminf_{t \rightarrow +\infty} \frac{X(t)}{t^{3/2}} \quad (4.7)$$

whereas for initial data with compact support in  $x$ , we have :

$$\limsup_{t \rightarrow +\infty} \frac{X(t)}{t^{3/2}} \leq \sigma\sqrt{2r}. \quad (4.8)$$

This result necessitated a fine understanding of the contributions of each term (growth, transport and redistribution of speeds), by successive isolations of each phenomenon.

This type of behavior is similar to other linear propagations obtained in several recent articles for the Fisher-KPP equation with fractional diffusion [30], for non-exponentially bounded nonlocal dispersions [44] or for singular diffusions [28]. Note that in the case of a fractional diffusion, the propagation rate is not the same for Heaviside initials with compact support [30], contrary to the case with local diffusion. We can therefore expect similar phenomena in our case.

## 4.5 A large deviations theorem for Gaussian kernels

In order to better understand the localization of the level lines of the solutions of the nonlinear equation (4.1) with Gaussian kernel (with variance  $\sigma = 1$  in this section to simplify the exposition), we have returned, together with E. Bouin, V. Calvez and E. Grenier [BCGN], to the study of its linearization near  $f \simeq 0$  :

$$\partial_t f + v \cdot \nabla_x f = (1+r)M(v)\rho - f. \quad (4.9)$$

The change of variables  $f^\varepsilon(t, x, v) := e^{-rt/(1+r)\varepsilon} f(t/(1+r)\varepsilon, x/(1+r)\varepsilon^{3/2}, v/\sqrt{\varepsilon})$  eliminates the dependency in  $r$  and corresponds to the expected scaling in  $(t, x, v)$ , thus leading to the equation :

$$\partial_t f^\varepsilon(t, x, v) + v \cdot \nabla_x f^\varepsilon(t, x, v) = \frac{1}{\varepsilon} (M_\varepsilon(v)\rho^\varepsilon(t, x) - f^\varepsilon(t, x, v)), \quad t > 0, x \in \mathbb{R}^n, v \in \mathbb{R}^n \quad (4.10)$$

where  $M_\varepsilon(v) := M(v/\varepsilon^{1/2})/\varepsilon^{n/2}$  and  $\rho^\varepsilon(t, x) := \int_{\mathbb{R}^n} f^\varepsilon(t, x, v) dv$ .

We deal with this problem using the Hopf-Cole transformation, which was already used in Section 1.8 :

$$u^\varepsilon(t, x, v) := -\varepsilon \log f^\varepsilon(t, x, v),$$

which verifies the equation :

$$\partial_t u^\varepsilon(t, x, v) + v \cdot \nabla_x u^\varepsilon(t, x, v) = 1 - \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{u^\varepsilon(t, x, v) - u^\varepsilon(t, x, v') - |v|^2/2}{\varepsilon}\right) dv'. \quad (4.11)$$

**Theorem 4.4** (Convergence). *Assume the initial datum  $u_0$  satisfies :*

$$[A] \quad u_0 - \frac{|v|^2}{2} \in W^{1,\infty}(\mathbb{R}^{2n}), \quad (4.12)$$

$$[B] \quad -\det(\text{Hess}_v(u_0(x, v))) \neq 0, \quad D_v^{(3)} Z_0 \in L_{loc}^\infty. \quad (4.13)$$



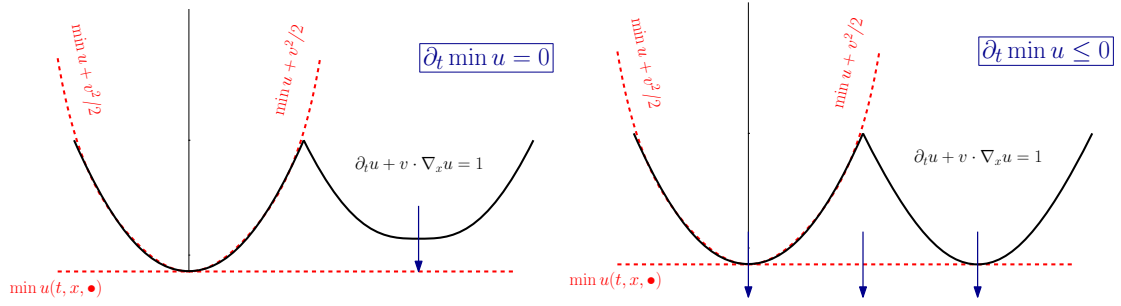


FIGURE 4.4 – Dynamics of the equation (4.14).

Let  $u^\varepsilon$  the solution of (4.11) with initial datum  $u^\varepsilon(0, \cdot) = u_0$ . Then,  $u^\varepsilon$  converges locally uniformly as  $\varepsilon \rightarrow 0$  to  $u$ , the unique viscosity solution of

$$\begin{cases} \max \left( \partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v) - 1, \quad u(t, x, v) - \min_{w \in \mathbb{R}^n} u(t, x, w) - \frac{|v|^2}{2} \right) = 0, \\ \partial_t \left( \min_{w \in \mathbb{R}^n} u(t, x, w) \right) \leq 0, \\ \partial_t \left( \min_{w \in \mathbb{R}^n} u(t, x, w) \right) = 0, \quad \text{if } \mathcal{S}(u)(t, x) = \{0\}, \\ u(0, x, v) = u_0(x, v). \end{cases} \quad (4.14)$$

where  $\mathcal{S}(u)(t, x) := \{v \in \mathbb{R}^n \mid u(t, x, v) = \min_{w \in \mathbb{R}^n} u(t, x, w)\}$ .

The notion of viscosity solution for this equation is not natural. Similarly, the fact that this equation satisfies a principle of comparison (which implies the uniqueness of the solution) necessitates long technical developments, that one can overcome by the two conditions [A, B], which are not optimal. We refer to [BCGN] on these points.

In order to convince the reader that equation (4.14) is well-posed, we will briefly and heuristically describe its dynamics. The first condition of (4.14) imposes the parabolic constraint :

$$u(t, x, v) \leq \min_{w \in \mathbb{R}^n} u(t, x, w) + \frac{|v|^2}{2}. \quad (4.15)$$

Consequently, the solution reaches its global minimum with respect to  $v$  at  $v = 0$ . Moreover, we have the following dichotomy :

1. either the constraint is saturated :  $u = \min_v u + |v|^2/2$ ,
2. or the solution evolves by free transport and creation of matter :  $\partial_t u + v \cdot \nabla_x u = 1$ .

Two other cases are then distinguished. If  $v = 0$  is the only global minimizer in  $v$  (ie  $\mathcal{S}(u)(t, x) = \{0\}$ ) then  $\min_v u$  does not decrease, and the parabolic constraint (4.15) is unchanged. However, the solution goes on evolving by free transport and growth in the unsaturated zone. If the global minimum in speed is reached for another speed than  $v = 0$  (ie  $\mathcal{S}(u)(t, x) \neq \{0\}$ ), this minimum can decrease, which modifies the parabolic constraint (4.15). It is therefore the decay in the unsaturated zone that determines the decay in the saturated zone.

#### Chapitre 4. A microscopic point of view on the Fisher-KPP equation

Let us now define the function from which we can build a representation formula for the solutions of (4.14) :

$$\mu(t, x; w) := \begin{cases} \frac{x}{w}, & \text{si } 0 \leq \frac{x}{w} \leq t \\ \frac{3}{2}|x|^{2/3}, & \text{si } |x| \leq t^{3/2} \text{ and } \frac{x}{w} > t \\ \frac{|x|^2}{2t^2} + t, & \text{si } |x| \geq t^{3/2} \text{ and } \frac{x}{w} > t. \end{cases} \quad (4.16)$$

**Theorem 4.5.** Assume that  $N = 1$ . Let  $u_0 : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  a bounded function. Then, the viscosity solution of (4.14) is given by :

$$u(t, x, v) = \inf_{(y, w)} (\phi(t, x - y, v; w) + u_0(y, w)), \quad (4.17)$$

where

$$\phi(t, x, v; w) = \frac{|v|^2}{2} + \min(\mu(t, x; v), \mu(t, x; w)),$$

except in the particular case  $v = w = x/t$ , for which :

$$\phi(t, x, v; w) = \frac{x}{w}.$$

From this representation formula and the characterization of  $\mu$ , we can compute a localization of the level lines for the initial linear equation (4.9) :

$$X(t) \simeq \left( \frac{((2/3)r)^{3/2}}{1+r} \right) t^{3/2}.$$

To conclude that this rate of growth is the one associated with the nonlinear equation (4.1), there are still several obstacles to be lifted.

- First, even for the linear equation, it is not easy to determine from the representation formula (4.17) which conditions to impose on the initial datum in order to reach this growth rate, computed from the function  $\mu$ . In addition, we have applied the rescaling  $(t/\varepsilon, x/\varepsilon^{3/2}, v/\sqrt{\varepsilon})$  to the equation but not to the initial data. We also need to make sure that it does not change the spread rate.
- Then we do not have a representation formula for the linear equation (4.1). One can always limit the propagation rate from above with solutions of the linear problem, but it is always more difficult to construct sub-solutions.

# Chapitre 5

## Shape and eigenvalue optimization problems in Fisher-KPP equations

We consider in this chapter the Fisher-KPP reaction-diffusion equation :

$$u_t - \nabla \cdot (A(t, x) \nabla u) + q(t, x) \cdot \nabla u = r(t, x)u - u^2 \quad (5.1)$$

with diffusion coefficients  $A$ , advection  $q$  and growth rate  $r$  that are in  $x \in \mathbb{R}^N$ . Let us recall here the results of writings in the section 1.1 : one can characterize the minimum speed of existence of the fronts by

$$c_e^* = \min_{\lambda > 0} \frac{k_{\lambda e}(A, q, c)}{\lambda}, \quad (5.2)$$

and this speed is connected to the exact expansion speed  $w_e^*$  in the direction  $e$  for the Cauchy problem with compactly supported initial data through the identity  $w_e^* = \min_{\xi \cdot e > 0} c_{\xi}^* / e \cdot \xi$  [BHN08]. This characterization makes it possible to address the problem of optimization of the speeds with respect to the coefficients.

**Question 5.** *What is the influence of environmental heterogeneity on the population invasion speed ?*

### 5.1 Influence de la fragmentation de l'habitat sur la speed

In ecology, habitat fragmentation is defined as the emergence of discontinuities within the ecosystem following loss of surface area, habitat being defined as the part of the environment most favorable to the organism considered. The causes of this fragmentation can be both natural (geological processes, climate change) or human (agriculture, extension of urban areas).

Habitat fragmentation is one of the main causes of species extinction. Biologists have identified several mechanisms in this direction : increased competition in the remaining habitats, threshold effects on the size of habitat fragments, and the impossibility for the species to migrate to other habitats. However, very few mathematical models have been used to investigate this question.

But how to define mathematically the fragmentation of a habitat ? A first approach, derived from Shigesada, Kawasaki and Teramoto [95], consists in comparing the environment associated with the coefficients  $A_L(x) := A(x/L)$ ,  $q_L(x) = q(x/L)$  and  $c_L(x) = r(x/L)$ , with  $L > 1$ , to the one, intuitively more fragmented, associated with  $A(x)$ ,  $q(x)$  and  $r(x)$ .

I thus confirmed analytically [N09-3] numerical simulations [96] showing that under the hypothesis  $\nabla \cdot q = 0$ , the function  $L \mapsto k_p(A_L, q_L, r_L)$  is increasing for all  $p$  in  $\mathbb{R}^N$  and thus the speed

$w_e^*(A_L, q_L, r_L)$  is increasing in  $L$ , which means that the fragmentation of the environment slows down the biological invasion.

Another approach, due to Berestycki, Hamel and Roques [17] (see also [31]), is to optimize the speed as a function of the growth rate  $r$  among the class of functions valued  $r_+$  on a set  $E$  of given measure  $|E| = m$  and  $r_-$  on its complementary, with  $A \equiv I_N$  and  $q \equiv 0$ . In dimension  $N = 1$ , the least fragmented environment is the one associated with the centered interval<sup>1</sup> of measure  $m$ , which will be denoted by  $E^*$ .

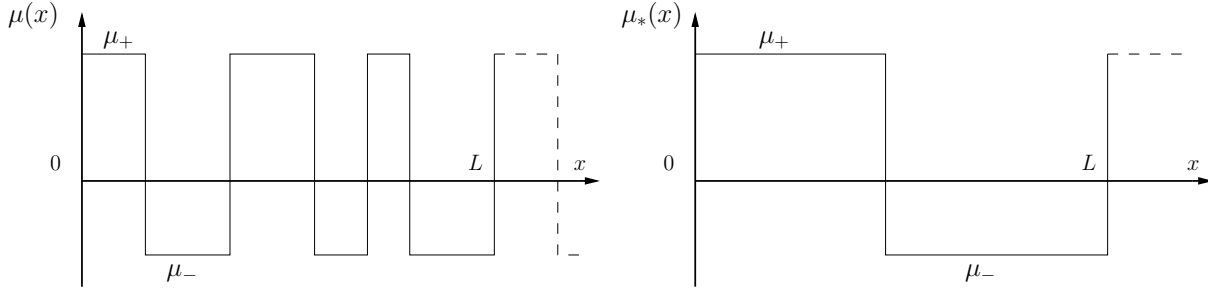


FIGURE 5.1 – Left : a growth rate  $\mu$  corresponding to the less "fragmented" habitat. Right : its Schwarz rearrangement  $\mu_*$ . \*\*\*r\*\*\*efaire les figures

In fact, it is possible to generalize this construction to each level set of a simple function and then to any bounded function  $r$ . We call the *Schwarz rearrangement*  $r^*$  of this function.

**Definition 5.1.** *The Schwarz rearrangement of a periodic measurable and bounded function  $r : \mathbb{R} \rightarrow \mathbb{R}$  is the unique periodic measurable function  $r^*$*

- *with the same distribution function,*
- *symmetric with respect to  $x = 0$ ,*
- *nonincreasing on  $(0, L/2)$ .*

It can therefore be considered that the Schwarz rearrangement  $r^*$  of the growth rate corresponds to the least fragmented habitat among habitat with a given distribution function.

This rearrangement satisfies several powerful inequalities such as those of Polya-Szego or Hardy-Littlewood (see [61]). Using the Rayleigh quotient characterization of the symmetric eigenvalue  $k_0(1, 0, r)$  :

$$k_0(1, 0, r) = - \min_{\alpha \in H_{per}^1(0, L)} \frac{\int_0^L |\alpha'|^2 - \int_0^L r(x) \alpha^2}{\int_0^L \alpha^2}, \quad (5.3)$$

Berestycki, Hamel and Roques have deduced [17] a Faber-Krahn-like inequality on the eigenvalue associated with  $p = 0$  :

$$k_0(1, 0, r^*) \geq k_0(1, 0, r).$$

The sign of this eigenvalue determining the large time behavior of the solution of the Cauchy problem associated with

$$\partial_t u = \partial_{xx} u + r(x)u - u^2,$$

1. We consider in this section periodic coefficients in  $x$ . Consequently, one can easily verify that the speed does not vary when the coefficients are translated. The position of the interval within the periodicity cell  $(0, L)$  is therefore irrelevant. This is no longer true if one considers Dirichlet, Neumann or Robin, boundary conditions (see Section 5.2).

## 5.2. Optimisation of self-adjoint eigenvalues in dimension $N$

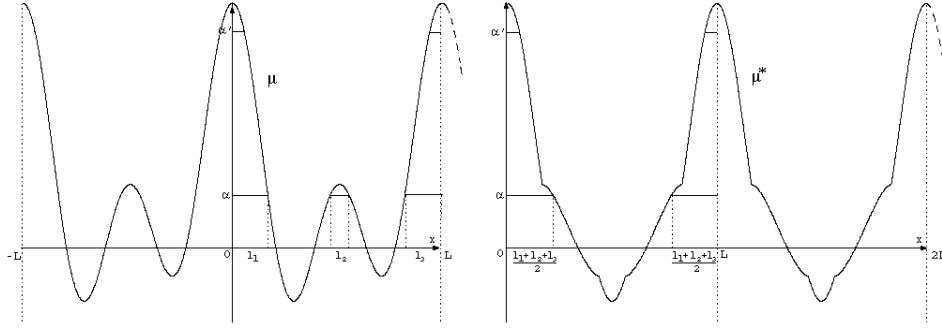


FIGURE 5.2 – A function  $\mu$  and its Schwarz rearrangement  $\mu^*$  [Berestycki-Hamel-Roques 05]

we can thus observe situations for which the solution of the equation with a rearranged growth rate  $r^*$  tends to a positive solution of the stationary equation when  $t \rightarrow +\infty$ , we can then speak of persistence of the population, while that associated with the growth rate  $r$  tends to 0, we speak then of extinction, whereas the inverse situation is always impossible. Thus, the least fragmented habitat is the most favorable to the survival of the population. But when the species associated with  $r$  and  $r^*$  persist, how to compare their behaviors ?

In dimension 1, one can show [N09-3] that

$$\forall p \in \mathbb{R}, \quad k_p(1, 0, r^*) \geq k_p(1, 0, r), \quad \text{and thus} \quad c^*(1, 0, r^*) \geq c^*(1, 0, r),$$

that is to say that the least fragmented habitat gives the expansion speed the biggest.

The main difficulty in obtaining these results was the non-symmetric character of the operators  $L_p$  when  $p \neq 0$ . Due to the periodic boundary conditions, we cannot get rid of the 1st order term by a change of variable. This non-symmetry prevents the eigenvalue  $k_p$  from being written as a Rayleigh quotient, that is, the maximum of a quotient of integrals. Since the properties of the rearrangement are integral rather than pointwise, there are very few results on the effect of the rearrangement on non-symmetric eigenvalues and the existing results are all based on a symmetry of the first-order term [3, 48].

I overcame this difficulty by finding a new integer characterization of this eigenvalue :

$$k_p(1, 0, r) = - \min_{\alpha \in H_{per}^1(0, L)} \left( \int_0^L \alpha'^2 - \int_0^L r(x) \alpha^2 - \frac{Lp^2}{\frac{1}{L} \int_0^L \frac{1}{\alpha^2}} \right) / \int_0^L \alpha^2. \quad (5.4)$$

Let us note at the outset that this characterization is not the minimum of a quadratic form, which confirms that it was not possible to rewrite this eigenvalue as a Rayleigh quotient. The inequality  $k_p(1, 0, r^*) \geq k_p(1, 0, r)$  then follows immediately from this formula and of the classical inequalities verified by the rearrangement.

This characterization comes from a more general formula allowing to characterize any eigenvalue of a non-symmetric operator as minimum of a family of eigenvalues of self-adjoint operators :

$$k_p(A, q, c) = \min_{\beta \in C_{per}^1} k_0 \left( A, 0, (\nabla \beta + p)A(\nabla \beta + p) + q \cdot (\nabla \beta + p) + c - \nabla \cdot q/2 \right). \quad (5.5)$$

## 5.2 Optimisation of self-adjoint eigenvalues in dimension $N$

The optimization of the speed  $c_e^*(I_N, 0, r)$  as a function of the growth rate  $r$  is much more difficult in dimension  $N$ .

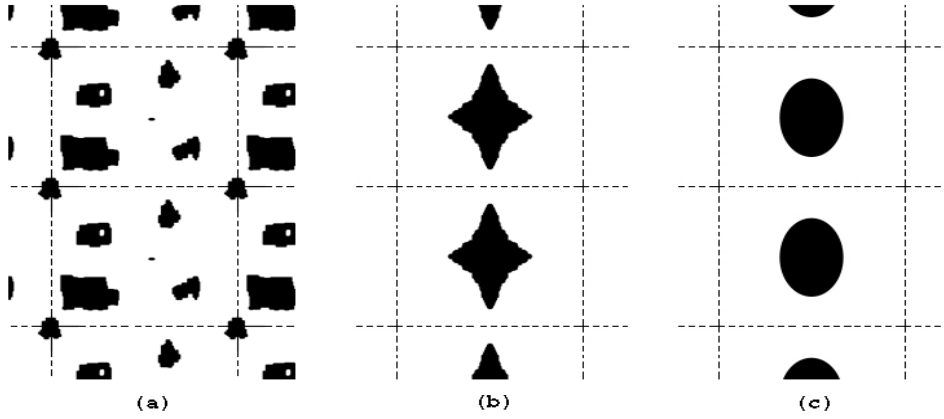


FIGURE 5.3 – (a) A set  $E$ , (b) its Steiner symmetrization  $E^\#$ , (c) another Steiner symmetric set having the same area as  $E$ , but not obtained from  $E$  by Steiner symmetrization [Berestycki-Hamel-Roques 05]

A useful notion in dimension  $N$  is the *Steiner symmetrization*. Consider a periodic function  $r$  of  $N$  variables  $(x_1, \dots, x_N)$ . We can construct a function  $r^{*1}$  by freezing  $(x_2, \dots, x_N)$  and applying the Schwarz rearrangement to the function  $x_1 \mapsto r(x_1, \dots, x_N)$ . We then freeze  $(x_1, x_3, \dots, x_N)$  and rearrange  $x_2 \mapsto r^{*1}(x_1, x_2, \dots, x_N)$ . The function obtained after having been rearranged in each variable is the Steiner symmetrization  $r^\#$  symmetry of  $r$ . It has the same distribution function as  $r$ , and it is symmetric and decreasing with respect to each hyperplane  $\{x_k = 0\}$ . However, these properties no longer suffice to characterize such a function, which is therefore no longer unique (see Figure 5.2). For example, we could get another Steiner symmetric invariant function by first rearranging in  $x_N$ , then in  $x_{N-1}$  etc.

This symmetry always verifies the classical inequalities of Hardy-Littlewood and Polya-Szego, which makes it possible to prove [17] that

$$k_0(I_N, 0, r^\#) \geq k_0(I_N, 0, r).$$

However, unlike dimension 1, we can construct counter-examples [N09-3] for which the Steiner symmetry of  $r$  decreases the expansion speed instead of increasing it.

It turns out that even the simplest problem, that of the minimization of the principal eigenvalue

$$F(E) := k_0(I_N, 0, B1_E)$$

of the Schrodinger operator  $-\Delta - B1_E$  with periodic boundary conditions as a function of the domain  $E \subset C$  with given area, is poorly understood.

Let us recall the interpretation of this problem in population dynamics :  $E$  being seen as the favorable zone or the habitat of the species under consideration, we want to understand where to place this habitat in order to maximize the chances of survival of the species [17, 31].

We can show [86] that there exists a maximizer  $\hat{E}$  of function  $F$ , and that this maximizer is necessarily Steiner symmetric,  $\hat{E}^\# = \hat{E}$ . In dimension 1,  $\hat{E}$  is therefore an interval, but can one better characterize  $\hat{E}$  in dimension  $N$ ?

Numerical simulations show that the optimal set  $\hat{E}$  looks like a ball, the complementary set of a ball, or a stripe, depending on the amplitude  $B$  (see [60, 86] and Figure 5.2). L. Roques and F. Hamel have shown [86] analytically that the ball could not be optimal for any amplitude  $B$  and Kao, Lou and Yanagida [60] have proved that the stripe ceased to be a local minimizer for some parameters.

We have demonstrated with J. Lamboley, A. Laurain and Y. Privat [LLNP], that the boundary  $\partial\hat{E}$  of a maximizing set cannot contain a piece of sphere, contradicting the conjectures based on numerical simulations.

**Theorem 5.2.** [LLNP] *If  $\partial\hat{E} \cap \Omega$  is analytic, then  $\partial\hat{E} \cap \Omega$  does not contain any piece of sphere.*

We have also considered the case of Neumann or Robin boundary conditions in a domain  $\Omega$  which is not necessarily a hypercube  $C$  associated with periodic boundary conditions. For these conditions, we proved that if the set  $\hat{E}$  (resp.  $\Omega \setminus \hat{E}$ ) is radially symmetric with respect to a point of  $\Omega$  and sufficiently regular, then  $\Omega$  is necessarily a sphere, as well as  $\hat{E}$  (or  $\Omega \setminus \hat{E}$ ). On the other hand, we have also considered the open problem of dimension 1 ( $\Omega = (0, 1)$ ) with Robin boundary conditions and showed that  $\hat{E}$  is necessarily a central interval or an interval touching the boundary, depending on the position of the Robin parameter  $\beta$  with respect to a critical parameter computed in [57].

Finally, we studied the case where  $\Omega = B(0, 1)$  is a ball with Neumann boundary conditions. For  $N = 2, 3, 4$ , the set  $\hat{E}$  maximizing the principal eigenvalue of  $-\Delta - B1_E$  in  $\Omega$  with Neumann boundary conditions can never be a ball. The numerical simulations presented in Figure 5.2 argue for a  $\hat{E}$  set that resembles a part of disk, but we have neither confirmed nor negate analytically these observations.

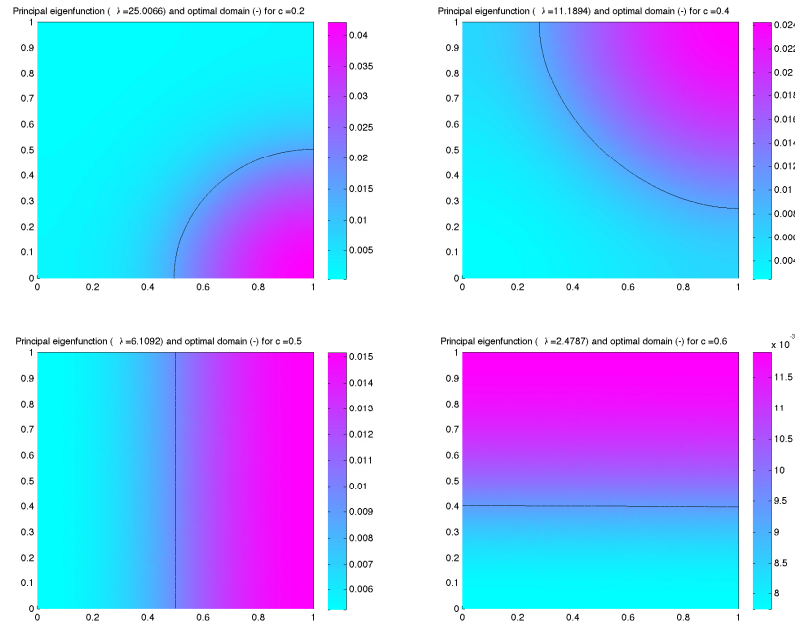


FIGURE 5.4 – Optimal domains for various parameters in the case of Neumann boundary conditions in  $\Omega = (0, 1)^2$ .

## 5.3 Other dependence results

### 5.3.1 Computation of the speed limit for large periods

Formula (5.5) is extremely useful for investigating the optimization of the eigenvalues  $k_p(A, q, r)$ . We used it with F. Hamel and L. Roques [HNR11] to calculate the limit of the speed  $w_e^*(A_L, q_L, r_L)$  when  $L \rightarrow +\infty$ . We were able to characterize this limit by methods of viscosity solutions, using correctors

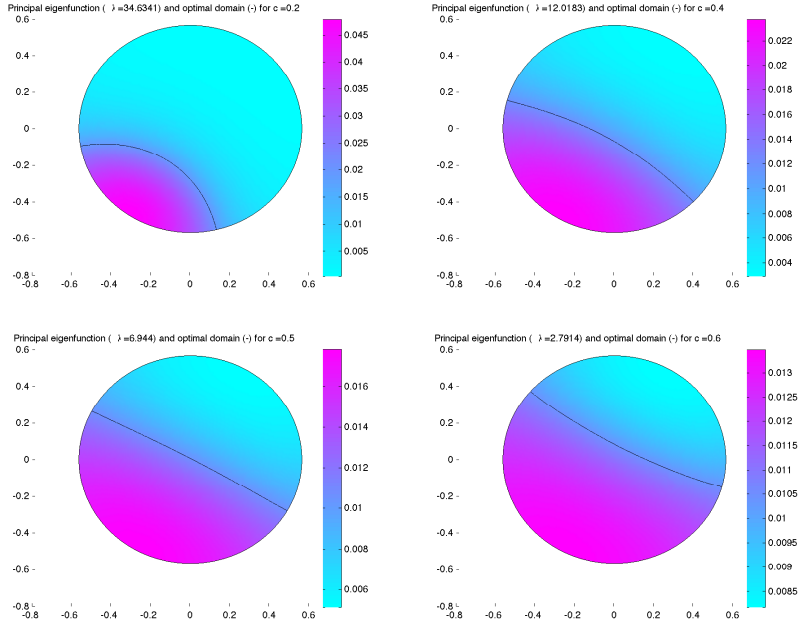


FIGURE 5.5 – Optimal domains for various parameters in the case of Neumann boundary conditions in  $\Omega = B(0, 1/\sqrt{\pi})$ .

associated with a quadratic Hamilton-Jacobi equation of order 1. In the case where  $N = 1$ ,  $A = 1$  and  $q \equiv 0$ , we can then characterize the limit by

$$w_{\infty}^* = \min_{k \geq \|r\|_{\infty}} k/j(k), \quad \text{with} \quad j(k) := \int_0^1 \sqrt{k - r(x)} dx, \quad (5.6)$$

This characterization extends a preliminary work of F. Hamel and L. Roques with J. Fayard [51] in the case of a term  $r$  taking only two values.

### 5.3.2 Small dependence results

I have also used the formula (5.5) in an article [N11] solving several open questions on the optimization of expansion speed. I showed the following results :

- for  $N = 1$ , the speed  $w^*(d, 0, r)$  is not necessarily increasing with respect to the diffusion coefficient  $d$  if  $r$  is not constant in  $x$ ,
- if  $q = \nabla Q$  is a gradient vector field, then the speed is smaller than in the absence of an advection term :  $w_e^*(I_N, \nabla Q, r_0) \leq w_e^*(I_N, 0, r_0) = 2\sqrt{r_0}$ ,
- taking the mean of  $r$  in  $t$  or in  $x$  decreases the speed  $w_e^*(I_N, 0, r)$ .

Berestycki, Hamel and Nadirashvili [16] have shown the monotonicity of  $\kappa \mapsto w_e^*(\kappa A, 0, r_0)$  in arbitrary dimension, for a diffusion matrix  $A$  depending on  $x$  and a constant growth rate  $r_0$ . El Smailly [36] has exhibited two matrices  $A$  and  $B$  such that  $A \geq B$  in the sense of positive symmetric matrices, but  $w_e^*(B, 0, r_0) > w_e^*(A, 0, r_0)$ , showing that the inequality obtained in [16] could only be valid for proportional diffusion matrices. The first result stated above shows that such a result of monotonicity with respect to diffusion cannot be true if the growth rate  $r$  depends on  $x$ .

It is known that an advection term with null divergence accelerates the propagation [16] (see also [62] and the many references that followed about the quantification of this acceleration). We show here



### 5.3. Other dependence results

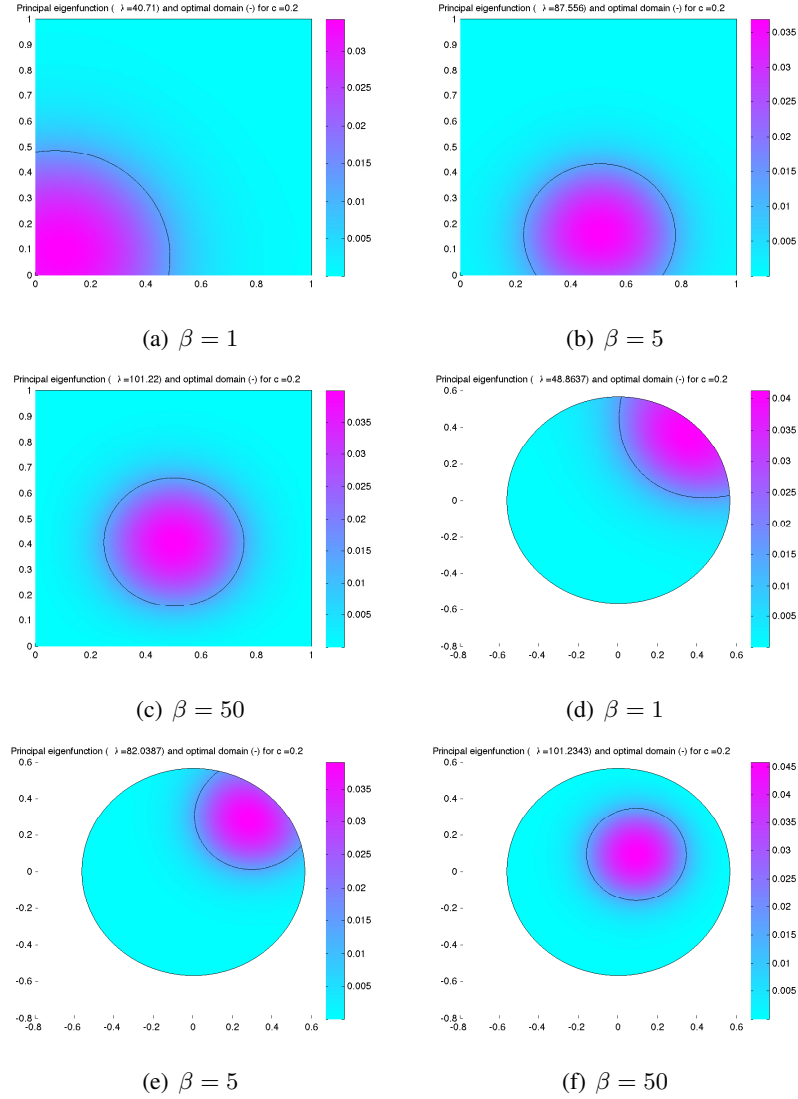


FIGURE 5.6 – Optimal domains for various parameters in the case of Robin boundary conditions in  $\Omega = B(0, 1)$  and  $\Omega = (0, 1)^2$ . the results of [LLNP] show that the domains in (c) and (f) are not balls.

the inverse effect for advection gradient terms, which are in a sense the supplementary set of functions with null divergence.

It is easy to show that the mean of  $r$  in  $t$  and in  $x$  decreases the speed. The last result is more precise : we can separate the effect of temporal and spatial means.

On the same subject, formula (5.5) was also used by Liang, Lin and Matano [69] to study the maximization of the speed as a function of the direction of propagation for a growth rate depending only on  $x_1$  in dimension  $N$ .

#### 5.3.3 Random stationary ergodic dependence

The characterization using generalized eigenvalues of the expansion speed for the Cauchy problem associated with the unidimensional Fisher-KPP equation with random stationary ergodic coefficients described in Section 1.4 makes it possible to use the methods developed above to investigate the dependence

of this speed on the coefficients of the equation. It is thus possible to extend several results from the periodic case to the random case. Thus, I showed [N15-2] that the increase of the amplitude of the reaction term  $r$  or the rescaling  $x \rightarrow x/L$ , with  $L > 1$ , of the coefficients, accelerate the spread of the population. It is now necessary to obtain similar results for problems with no equivalence in the case of periodic : for example, numerical simulations [96] seem to show that to increase the variance of the heterogeneity (ie the degree of stochasticity) will accelerate the propagation.

# Publications list

- [BCGN] E. Bouin, V. Calvez, E. Grenier, G. Nadin. Large deviations for velocity- jump processes and non-local Hamilton-Jacobi equations *Soumis*.
- [BNsoumis] H. Berestycki, G. Nadin. Asymptotic spreading for general heterogeneous Fisher-KPP type equations. *Soumis*.
- [NR16] G. Nadin, L. Rossi. Generalized transition fronts for one-dimensional almost periodic Fisher-KPP equations. , *Arch. Rat. Mec. Anal.*, 223(3) :1239–1267, 2017.
- [LLNP] A. Laurain, J. Lamboley, G. Nadin, et Y. Privat. Properties of minimizers of the principal eigenvalue with indefinite weight and robin conditions, *Calc. Var and PDE*, 55(6), 2016.
- [Nchapitre] G. Nadin. Hyperbolic traveling waves in neolithic populations modelling. *Accepted for publication in the book “Interactions in Complex Systems”*.
- [BoN15] J. Bouhours, G. Nadin. A variational approach to reaction-diffusion equations with forced speed in dimension 1. *Discrete Contin. Dyn. Syst.* 35(5) : 1843–1872, 2015.
- [NR15] G. Nadin, L. Rossi. Transition waves for Fisher-KPP equations with general time-heterogeneous et space-periodic coefficients. *Analysis and PDE*, 8(6) :1351 ?1377, 2015.
- [GN15] L. Girardin, G. Nadin. traveling waves for diffusive et strongly competitive systems : relative motility et invasion speed. *Eur. J. Appl. Math.*, 26(4) :521–534, 2015.
- [BCN15] E. Bouin, V. Calvez, et G. Nadin. Propagation in a kinetic reaction-transport equation : traveling waves et accelerating fronts. *Arch. Ration. Mech. Anal.*, 217(2) : 571–617, 2015.
- [N15-1] G. Nadin. Critical traveling waves for general heterogeneous one-dimensional reaction-diffusion equations. *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 32(4) :841–873, 2015.
- [N15-2] G. Nadin. How does the spreading speed associated with the Fisher-KPP equation depend on random stationary diffusion et reaction terms ? *Discrete Contin. Dyn. Syst. Ser. B*, 20(6) :1785–1803, 2015.
- [BCN14] E. Bouin, V. Calvez, et G. Nadin. Hyperbolic traveling waves driven by growth. *Math. Models Methods Appl. Sci.* 24(6) :1165–1195, 2014.
- [DN14] A. Ducrot, G. Nadin, Asymptotic behaviour of traveling waves for the delayed Fisher-KPP equation. *Journal of differential equations*, 256(9) :3115–3140, 2014.
- [NRRP13] G. Nadin, L. Rossi, L. Ryzhik et B. Perthame. Wave-like solutions for non-local reaction-diffusion equations : a toy model. *Math. Model. Nat. Phenom.*, 8(2) :33–41, 2013.
- [BN12] H. Berestycki et G. Nadin. Spreading speeds for one-dimensional monostable reaction-diffusion equations. *J. Math. Phys.*, 53(11), 2012.
- [NR12] G. Nadin et L. Rossi. Propagation phenomena for time heterogeneous KPP reaction diffusion equations. *J. Math. Pures Appl.*, 98(9) :633–653, 2012.

## Publications list

- [GGN12] J. Garnier, T. Giletti et G. Nadin. Maximal et minimal spreading speeds for reaction diffusion equations in nonperiodic slowly varying media. *J. Dynam. Differential Equations*, 24(3) :521–538, 2012.
- [N11] G. Nadin. Some dependence results between the spreading speed et the coefficients of the space-time periodic Fisher-KPP equation. *Eur. J. Appl. Math.*, 22 :169–185, 2011.
- [NPT11] G. Nadin, B. Perthame et M. Tang. Can a traveling wave connect two unstable states ? The case of the non-local Fisher equation. *C. R. Math. Acad. Sci. Paris*, 349(9-10) :553–557, 2011.
- [HN11] F. Hamel et G. Nadin. Spreading properties et complex dynamics for monostable reaction-diffusion equations. *Comm. Part. Diff. Equations*, 37 :511–537, 2012.
- [HNR11] F. Hamel, G. Nadin, et L. Roques. A viscosity solution method for the spreading speed formula in slowly varying media. *Indiana Univ. Math. J.*, 60 :1229–1248, 2011.

## Other articles, not displayed in this manuscript

- [NSVsoumis] G. Nadin, M. Strugarek, N. Vauchelet. Hindrances to bistable front propagation : application to Wolbachia invasion, *soumis*.
- [GNsoumis] L. Girardin, G. Nadin. Competition in periodic media : II – Segregative limit of pulsating fronts and "Unity is not Strength"-type result, *soumis*.
- [NPsoumis] G. Nadin, Y. Privat. An extremal eigenvalue problem arising in heat conduction. *J. Math. Pures et Appl.*, 105(6) :845–872, 2016.
- [CKNR14] M. Cristofol, I. Kaddouri , G. Nadin et L. Roques. Coefficient determination via asymptotic spreading speeds. *Inverse problems*, 30, 2014.
- [MNP14] G. Marck, G. Nadin, et Y. Privat. What is the optimal shape of a fin for one-dimensional heat conduction ? *SIAM J. Appl. Math.* 74 (2014), no. 4, 1194–1218.
- [Het13] C. Hauzy, G. Nadin, E. Canard, I. Gounet, N. Mouquand et B. Ebenman. Confronting the Paradox of Enrichment to the Metacommunity Perspective *PLoS ONE*, 8(12), 2013.

## PhD articles

- [N10] G. Nadin. Existence et uniqueness of the solutions of a space-time periodic reaction-diffusion equation, *J. Diff. Eq.*, 249(6) :1288–1304, 2010.
- [N09-1] G. Nadin. The principal eigenvalue of a space-time periodic parabolic operator. *Ann. Mat. Pura Appl.*, 4 :269–295, 2009.
- [N09-2] G. Nadin. Traveling fronts in space-time periodic media. *J. Math. Pures Appl.*, 92 :232–262, 2009.
- [N09-3] G. Nadin. The effect of the Schwarz rearrangement on the periodic principal eigenvalue of a nonsymmetric operator. *SIAM J. Math. Anal.*, 4 :2388–2406, 2009.
- [BNPR09] H. Berestycki, G. Nadin, B. Perthame et L. Ryzhik, The non-local Fisher-KPP equation : traveling waves et steady states. *Nonlinearity*, 22 :2813–2844, 2009.
- [BHN08] H. Berestycki, F. Hamel, et G. Nadin. Asymptotic spreading in heterogeneous diffusive excitable media. *J. Func. Anal.*, 255(9) :2146–2189, 2008.
- [NPR08] G. Nadin, B. Perthame, et L. Ryzhik. Traveling waves for the Keller-Segel system with Fisher birth term. *Interface free bound.*, 10 :517–538, 2008.

# Bibliography

- [1] M. Alfaro, J. Coville. Rapid traveling waves in the non-local Fisher equation connect two unstable states. *Appl. Math. Lett.*, 25 :2095–2099, 2012.
- [2] O. Alvarez, J.-M. Lasry, P.-L. Lions. Convex viscosity solutions and state constraints. *J. Math. Pures Appl.*, 76(3) :265–288, 1997.
- [3] A. Alvino, G. Trombetti, P.-L. Lions. Comparison results for elliptic and parabolic equations via Schwarz symmetrization *Ann I.H.P. Anal non lin.*, 7(2) :37–65, 1990.
- [4] A.J. Ammerman, L.L. Cavalli-Sforza. The Neolithic Transition et the Genetics of Populations in Europe. *Princeton University Press*, 1984.
- [5] S. Angenent. The zero set of a solution of a parabolic equation. *J. Reine Angew. Math.*, 390 :79–96, 1988.
- [6] D.G. Aronson, H.F. Weinberger. Multidimensional nonlinear diffusions arising in population genetics. *Adv. Math.*, 30 :33–76, 1978.
- [7] G. Barles, and B. Perthame. Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.*, 26(5) :1133–1148, 1988.
- [8] G. Barles, and B. Perthame. Discontinuous solutions of deterministic optimal stopping time problems. *RAIRO Modél. Math. Anal. Numér.*, 21(4) :557–579, 1987.
- [9] N. Barton, M. Turelli. Spatial waves of advances with bistable dynamics : cytoplasmic and genetic analogues of Allee effects. *The Am. Nat.*, 78(3), 2011.
- [10] H. Berestycki, J. Bouhours et G. Chapuisat Front propagation and blocking phenomena in cylinder with varying cross section. *Calc. Var. Partial Differential Equations*, 55(3), 2016.
- [11] H. Berestycki et G. Chapuisat. Traveling fronts guided by the environment for reaction-diffusion equations. *Netw. Heterog. Media* 8(1) :79–114, 2013.
- [12] H. Berestycki, O. Diekmann, K. Nagelkerke, P. Zegeling Can a species face a climate change ?. *Bull. Mathematical Biology*, 71(2), 2008.
- [13] H. Berestycki, et F. Hamel. Generalized traveling waves for reaction-diffusion equations. *Perspectives in Nonlinear Partial Differential Equations. In honor of H. Brezis, Contemp. Math.*, 446 :101–123, 2007.
- [14] H. Berestycki, et F. Hamel. Front propagation in periodic excitable media. *Comm. Pure Appl. Math.*, 55 :949–1032, 2002.
- [15] H. Berestycki, F. Hamel, et H. Matano. Bistable traveling waves around an obstacle. *Comm. Pure Appl. Math.*, 62(6) :729–788, 2009.
- [16] H. Berestycki, F. Hamel, et N. Nadirashvili. The speed of propagation for kpp type problems. I - periodic framework. *J. Europ. Math. Soc.*, 7 :173–213, 2005.

## Bibliography

- [17] H. Berestycki, F. Hamel, et L. Roques. Analysis of the periodically fragmented environment model. I. Species persistence. *J. Math. Biol.*, 51 :75–113, 2005.
- [18] H. Berestycki, F. Hamel, et L. Roques. Analysis of the periodically fragmented environment model : II. Biological invasions et pulsating traveling fronts, *J. Math. Pures Appl.*, 84 :1101–1146, 2005.
- [19] H. Berestycki, F. Hamel et L. Rossi. Liouville-type results for semilinear elliptic equations in unbounded domains. *Ann. Mat. Pura Appl.*, 186(4) :469–507, 2007.
- [20] H. Berestycki, A. Kretzschmar, L. Roques, et A. Roques. A population facing climate change : joint influences of Allee effects et environmental boundary geometry. *Population Ecology*, 50(2) :215–225, 2008.
- [21] H. Berestycki, L. Nirenberg. traveling fronts in cylinders. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9(5) :497–572, 1992.
- [22] H. Berestycki, L. Nirenberg, et S. R. S. Varadhan. The principal eigenvalue et maximum principle for second order elliptic operators in general domains. *Comm. Pure Appl. Math.*, 47 :47–92, 1994.
- [23] H. Berestycki, L. Rossi. Reaction-diffusion equations for population dynamics with forced speed, I - The case of the whole space. *Discrete et Continuous Dynamical System Series B*, 21(1) :41–67, 2008.
- [24] H. Berestycki, et L. Rossi. Generalizations et properties of the principal eigenvalue of elliptic operators in unbounded domains *Comm. Pure Appl. Math.*, 68(6) :1014–1065, 2015.
- [25] M. Bisi et L. Desvillettes. From reactive Boltzmann equations to reaction-diffusion systems. *J. Stat. Phys.*, 125(1) :249–280, 2006.
- [26] K. Bjerklov. Positive Lyapunov exponents for continuous quasi-periodic Schrödinger equations. *J. Math. Phys.*, 47(2), 2006.
- [27] S. Bochner. Beitrage zur theorie der fastperiodischen funktionen. *Math. Ann.*, 96 :119–147, 1926.
- [28] E. Bouin, V. Calvez, N. Meunier, S. Mirrahimi, B. Perthame, G. Raoul, R. Voituriez, Invasion fronts with variable motility : phenotype selection, spatial sorting et wave acceleration. *C. R. Math. Acad. Sci. Paris*, 350(15-16) : 761–766, 2012.
- [29] M.D. Bramson, Convergence of solutions of the Kolmogorov equation to traveling waves. *Mem. Amer. Math. Soc.*, 44 (1983).
- [30] X. Cabré et J.-M. Roquejoffre. Front propagation in Fisher-KPP equations with fractional diffusion. *C. R. Math. Acad. Sci. Paris*, 347 :1361–1366, 2009.
- [31] R.S. Cantrell et C. Cosner, Diffusive logistic equations with indefinite weights : population models in disrupted environments. *Proc. Roy. Soc. Edinburgh*, 112 :293–318, 1989.
- [32] R. S. Cantrell, C. Cosner. Spatial ecology via reaction-diffusion equations. John Wiley et Sons Ltd., Chichester, UK, 2003, 428 pp.
- [33] G. Chapuisat, E. Grenier. Existence et nonexistence of traveling wave solutions for a bistable reaction-diffusion equation in an infinite cylinder whose diameter is suddenly increased. *Commun. Partial Differ. Equations*, 30(10-12) :1805–1816, 2005.
- [34] C. Cuesta, S. Hittmeir et C. Schmeiser. Traveling waves of a kinetic transport model for the KPP-Fisher equation. *SIAM J. Math. Anal.*, 44, 4128–4146.
- [35] A. Ducrot, T. Giletti, et H. Matano. Existence et convergence to a propagating terrace in one-dimensional reaction-diffusion equations. *Trans. Amer. Math. Soc.*, 366 :5541–5566, 2014.
- [36] M. ElSmaily. PhD thesis, Section 2.7, “Équations de réaction-diffusion dans des milieux hétérogènes non-bornés”, 2008.

- [37] L. C. Evans, et P. E. Souganidis. A PDE approach to geometric optics for certain semilinear parabolic equations. *Indiana Univ. Math.*, 38(1) :141–172, 1989.
- [38] G. Faye, et M. Holzer. Modulated traveling fronts for a non-local Fisher-KPP equation : a dynamical system approach. *J. of Differential Equations*, 258(7) :2257–2289, 2015.
- [39] R. A. Fisher. The advance of advantageous genes. *Ann. Eugenics*, 7 :335–369, 1937.
- [40] J. Fort, et V. Méndez. Time-Delayed Theory of the Neolithic Transition in Europe. *Phys. Rev. Let.*, 82(4), 1999.
- [41] J. Fang et X.-Q. Zhao, Monotone wavefronts of the non-local Fisher-KPP equation. *2011 Nonlinearity*, 24(11) :3043–3054, 2011.
- [42] M. Freidlin. Functional Integration et Partial Differential Equations. *Ann. Math. Stud.*, 109, Princeton University Press, Princeton, NJ, 1985.
- [43] M. Freidlin et J. Gärtner. On the propagation of concentration waves in periodic et random media. *Sov. Math. Dokl.*, 20 :1282–1286, 1979.
- [44] J. Garnier, Accelerating solutions in integro-differential equations *SIAM J. Math. Anal.*, 43(4)1955–1974, 2011.
- [45] S. Genieys, V. Volpert et P. Auger. Pattern et Waves for a Model in Population Dynamics with non-local Consumption of Resources. *Mathematical Modelling of Natural Phenomena*, 1(1) :65–82, 2006.
- [46] A. Gomez, S. Trofimchuk. Monotone traveling wavefronts of the KPP-Fisher delayed equation. *J. Diff. Eqs.*, 250 : 1767–1787, 2011.
- [47] K.P. Hadeler. Hyperbolic traveling fronts. *Proc. Edinburgh Math. Soc.*, 31 :89–97, 1988.
- [48] F. Hamel, N. Nadirashvili, and E. Russ. Rearrangement inequalities and applications to isoperimetric problems for eigenvalues. *Annals of Math.*, 174 :647–755, 2011.
- [49] F. Hamel, J. Nolen, J.-M. Roquejoffre et L. Ryzhik. A short proof of the logarithmic Bramson correction in Fisher-KPP equations. *Netw. Heterog. Media*, 8(1) :275–289, 2013.
- [50] F. Hamel, J. Nolen, J.-M. Roquejoffre et L. Ryzhik. The logarithmic time delay of KPP fronts in a periodic medium. *J. European Math. Society*, to appear.
- [51] F. Hamel et L. Roques et J. Fayard. Spreading speeds in slowly oscillating environments. *Bull. Math. Biol.*, 72(5) :1166–1191, 2010.
- [52] F. Hamel et L. Rossi. Admissible speeds of transition fronts for non-autonomous monostable equations. *SIAM J. Math. Anal.*, 47(5) :3342–3392, 2015.
- [53] F. Hamel and L. Ryzhik. On the non-local fisher-KPP equation : steady states, spreading speeds, and global bounds. *Nonlinearity*, 27, 2014.
- [54] F. Hamel et Y. Sire. Spreading speeds for some reaction-diffusion equations with general initial conditions. *SIAM J. Math. Anal.*, 42 :2872–2911, 2010.
- [55] K. Hasik et S. Trofimchuk. Slowly oscillating wavefronts of the KPP-Fisher delayed equation. *Discrete Contin. Dyn. Syst.*, 34(9) :3511–3533, 2014.
- [56] K. Hasik ; S. Trofimchuk. An extension of the Wright’s  $3/2$ -theorem for the KPP-Fisher delayed equation. *Proc. AMS*, 143(7) :,3019–3027, 2015.
- [57] M. Hintermüller, C.-Y. Kao, et A. Laurain. Principal eigenvalue minimization for an elliptic problem with indefinite weight et Robin boundary conditions. *Appl. Math. Optim.*, 65(1) :111–146, 2012.
- [58] H. Ishii. A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 16(1) :105–135, 1989.

## Bibliography

- [59] Ja. I. Kanel. Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory. *Mat. Sb. (N.S.)*, 59(101) : 245–288, 1962.
- [60] C.-Y. Kao, Y. Lou, et E. Yanagida. Principal eigenvalue for an elliptic problem with indefinite weight on cylindrical domains. *Math Biosci Eng*, 5(2) :315–35, 2008.
- [61] B. Kawohl. On the isoperimetric nature of a rearrangement inequality et its consequences for some variational problems. *Arch. Ration. Mech. Anal.*, 94 :227–243, 1986.
- [62] A. Kiselev et L. Ryzhik. Enhancement of the traveling front speeds in reaction-diffusion equations with advection. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18 :309–358, 2001.
- [63] A.N. Kolmogorov, I.G. Petrovsky, et N.S. Piskunov. Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bulletin Université d'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.)*, 1–26, 1937.
- [64] Y. Zhou, et M. Kot, Discrete-time growth-dispersal models with shifting species ranges. *Theor Ecol*, 4 :13–25, 2011.
- [65] S. M. Kozlov. Ground states of quasiperiodic operators. *Dokl. Akad. Nauk SSSR*, 271(3) :532–536, 1983.
- [66] M. K. Kwong, et C. Ou. Existence et nonexistence of monotone traveling waves for the delayed Fisher equation. *J. Diff. Eq.*, 249(3) :728–745, 2010.
- [67] P.-L. Lions, et P. E. Souganidis. Homogenization of “viscous” Hamilton-Jacobi equations in stationary ergodic media. *Comm. Partial Differential Equations*, 30(1-3) :335–375, 2005.
- [68] B. Lou, H. Matano, et K.-I. Nakamura. Recurrent traveling waves in a two-dimensional saw-toothed cylinder et their average speed. *J. Differential Equations*, 255(10) :3357–3411, 2013.
- [69] X. Liang, X. Lin, et H. Matano. Maximizing the spreading speed of KPP fronts in two-dimensional stratified media. *Proc. London Math. Soc.* 109(5) :1137–1174, 2014.
- [70] A. J. Majda, and P. E. Souganidis. Large-scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales. *Nonlinearity*, 7(1) :1–30, 1994.
- [71] J. Mallet-Paret et G. Sell. Systems of differential delay equations : Floquet multipliers et discrete Lyapunov functions. *J. Diff. Eq.*, 125 :385–440, 1996.
- [72] J. Mallet-Paret et G. Sell. The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay *J. Diff. Eq.*, 125 (1996), pp 441–489.
- [73] H. Matano. Traveling waves in spatially random media. *RIMS Kokyuroku*, 1337 :1–9, 2003.
- [74] H. Matano, et M. Nara. Large time behavior of disturbed planar fronts in the Allen-Cahn equation. *J. Differential Equations*, 251(12) : 3522–3557, 2011.
- [75] A. Mellet, J. Nolen, J.-M. Roquejoffre, L. Ryzhik, Stability of Generalized Transition Fronts. *Comm. Partial Diff. Eq.*, 34 :521–552, 2009.
- [76] A. Mellet, J.-M. Roquejoffre, et Y. Sire. Generalized fronts for one-dimensionnal reaction-diffusion equations. *Discrete Contin. Dyn. Syst.*, 26(1) :303–312, 2009.
- [77] J. Nolen, J.-M. Roquejoffre, L., Ryzhik, A. Zlatos. Existence et Non-existence of Fisher-KPP Transition Fronts. *Arch. Rat. Mech. Anal.*, 203 :217–246, 2012.
- [78] J. Nolen, M. Rudd, et J. Xin. Existence of KPP fronts in spatially-temporally periodic advection et variational principle for propagation speeds. *Dynamics of PDE*, 2(1) :1–24, 2005.
- [79] Nolen J., Ryzhik L.. Traveling waves in a one-dimensional random medium. *Ann. de l'Inst. Henri Poincare, C. Analyse non lineaire*, 26 :1021–1047, 2009.



- [80] J. Nolen, et J. Xin. Asymptotic Spreading of KPP Reactive Fronts in Incompressible Space-Time random Flows. *Ann. de l'Inst. Henri Poincaré – Analyse Non Lineaire*, 26(3) :815–839, 2008.
- [81] J. Nolen, and J. Xin. KPP Fronts in 1D Random Drift. *Discrete and Continuous Dynamical Systems B*, 11(2), 2009
- [82] J. Nolen, and J. Xin. Variational Principle of KPP Front Speeds in Temporally Random Shear Flows with Applications. *Communications in Mathematical Physics*, 269 :493–532, 2007.
- [83] P. Polacik. Propagating terraces and the dynamics of front-like solutions of reaction-diffusion equations on  $\mathbb{R}$  *Mem. Amer. Math. Soc.*, to appear.
- [84] A. B. Popatov, et M. Lewis, Climate et competition : the effect of moving range boundaries on habitat invasibility. *Bull. Math. Biol.*, 6(5) :975–1008, 2004.
- [85] E. A. Robinson. The dynamical properties of Penrose tilings. *Trans AMS*, 348 :4447–4464, 1996.
- [86] L. Roques et F. Hamel. Mathematical analysis of the optimal habitat configurations for species persistence. *Math. Biosciences*, 210 :34–59, 2007.
- [87] L. Rossi, et L. Ryzhik. Transition waves for a class of space-time dependent monostable equations. *Commun. Math. Sci.*, 12(5) :879–900, 2014.
- [88] L. Roques. Modèles de réaction-diffusion pour l'écologie spatiale. Editions Quae, 2013.
- [89] L. Roques, A. Roques, H. Berestycki et A. Kretschmar. A population facing climate change : joint influences of Allee effects et environmental boundary geometry. *Population Ecology*, 50(2) :215–225, 2008.
- [90] H.R. Schwetlick. traveling fronts for multidimensional nonlinear transport equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17 :523–550, 2000.
- [91] H.R. Schwetlick. Limit sets for multidimensional nonlinear transport equations. *J. Differential Equations*, 179 :356–368, 2002.
- [92] W. Shen. Traveling waves in diffusive random media. *J. Dynam. Differential Equations*, 16(4) :1011–1060, 2004.
- [93] W. Shen. Existence, uniqueness, et stability of generalized traveling waves in time dependent monostable equations. *J. Dynam. Differential Equations*, 23(1) :1–44, 2011.
- [94] W. Shen. Existence of Generalized Traveling Waves in Time Recurrent et Space Periodic Monostable Equations. *Journal of Applied Mathematics et Computation*, 1 :69–94, 2011.
- [95] N. Shigesada, K. Kawasaki, et E. Teramoto. Traveling periodic waves in heterogeneous environments. *Theoretical Population Biology*, 30(1) :143–160, 1986.
- [96] N. Shigesada et K. Kawasaki. Biological invasions : theory et practice. *Oxford Series in Ecology et Evolution*, Oxford : Oxford University Press, 1997.
- [97] J. G. Skellam. Random Dispersal in Theoretical Populations. *Biometrika*, 38(1-2) :196–218, 1951.
- [98] E. Sorets, et T. Spencer. Positive Lyapunov Exponents for Schrodinger Operators with Quasi-Periodic Potentials *Commun. Math. Phys.*, 142 :543–566, 1991.
- [99] M. Tang. A relaxation method for the pulsating front simulation of the periodic advection diffusion reaction equation. *Commun. Math. Sciences*, 11(3) :651–678, 2013.
- [100] A. M. Turing. The Chemical Basis of Morphogenesis. *Phil. Trans. R. Soc. Lond. B*, 237(641) (1952), pp. 37-72.
- [101] H. Weinberger. On spreading speed et traveling waves for growth et migration models in a periodic habitat. *J. Math. Biol.*, 45 :511–548, 2002.

## Bibliography

- [102] G. Wulff. Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflagen. *Z. Kryst. Mineral*, 34 :449, 1901.
- [103] X. Xin. Existence et stability of traveling waves in periodic media governed by a bistable nonlinearity. *J. Dyn. et Diff. Eq.*, 3(4) :541–573, 1991.
- [104] Y.B. Zeldovich, G.I. Barenblatt, V.B. Libovich, G.M. Mackviladze. The mathematical theory of combustion and explosions. *Cons. Bureau*, New York, 1985.
- [105] A. Zlatoš. Generalized traveling waves in disordered media : existence, uniqueness, et stability. *Arch. Ration. Mech. Anal.*, 208(2) :447–480, 2013.
- [106] A. Zlatoš. Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations. *J. Math. Pures Appl.*, 98 :89–102, 2012.



# Abstract

## Propagation phenomena in various reaction-diffusion models

Reaction-diffusion equations generate propagation phenomena, which can be quantified mathematically through two notions : by locating the level lines of the solutions (expansion speeds) or using traveling waves. The purpose of this manuscript is to discuss these two concepts for different variants of the Fisher-KPP equation.

In the first and the second chapters, we will study these phenomena for heterogeneous Fisher-KPP equations depending in a general way on  $t$  and  $x$ . Bounds on the expansion speeds in each direction can be obtained with the aid of generalized eigenvalues, giving exact expansion speeds in particular for coefficients that are almost periodic, uniquely ergodic or constant at infinity in angular sectors. We will then discuss the notion of generalized transition waves for this equation and show their existence when the coefficients depend only on  $t$  or else are almost periodic in  $x$ .

The third chapter deals with non-local and delayed Fisher-KPP equations. Since these equations no longer admit a comparison principle, the asymptotic behavior of traveling waves is still poorly understood. A new numerical method supports the conjecture of the convergence of one homogeneous state towards the other for symmetric kernels. For asymmetric kernels, the convergence of the traveling waves depends on the position of the asymmetry with respect to the direction of propagation.

A kinetic reaction-diffusion equation, with a probability kernel measuring the velocity changes, will be studied in the fourth chapter. If this kernel is compactly supported, the existence of traveling waves can be proved and their velocities can be characterized. In the case where the kernel is a sum of two Dirac masses, we find a Fisher-KPP equation with a delay in the diffusion law, of which an application in archaeology will be discussed. In the case of a Gaussian kernel, the level lines propagate in  $t^{3/2}$ , and we will seek to determine this propagation rate in two different ways.

Finally, in the fifth part, we present several results on the dependence between the coefficients of the equation and the speed of propagation in the case of a periodic equation in space. In particular, one can show that the Schwarz rearrangement of the growth term accelerates the propagation in dimension 1, but that the situation is much less understood in higher dimensions.

**Keywords:** reaction-diffusion equations, traveling waves, expansion speeds, Hamilton-Jacobi equations, nonlocal and delayed equations, principal eigenvalues, Schwarz rearrangement.