Emergent supersymmetry

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Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Outline

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Introduction

- Supersymmetry, since its invention, has been considered as a "discretionary" property of physical systems.
- While it is assumed to be broken, how it can be, indeed, broken is, still, an open issue.
- Indeed, how it can be realized, in *physical* systems, is, also.
- *Why* should it be realized at all, is another question.
- "In order to address a "why" question, you have to be in some framework, where you allow something to be true." (R. Feynman, "Magnets", interview to the BBC, ca. 1983)

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Introduction

- Is it possible to find such a framework for supersymmetry ? Yes ! (Cf. Parisi & Sourlas (1982)) Supersymmetry isn't an optional–but an inevitable–property of *all* consistently closed physical systems.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Introduction

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Key steps in understanding the description of a physical system, in equilibrium with its fluctuations :

- The partition function of a physical system, describes a system, in equilibrium with its fluctuations.
- There exist, however, more than one partition functions–how are they related ?
- How can supersymmetry be, effectively, "hidden"–how can it, then, be revealed ? What are the appropriate probes ?

The system and the bath in equilibrium

One way to describe the dynamics of a physical system, in equilibrium with its fluctuations, is by writing down the map, between the "dynamical" degrees of freedom, ϕ and the fluctuations, η (the Langevin equation at equilibrium) :

$$\eta = -\frac{\partial U(\phi)}{\partial \phi}$$

This defines "additive" noise, that's assumed to be Gaussian :

$$\langle \eta \rangle = 0$$
$$\langle \eta\eta \rangle = \nu\delta$$

It's, also, possible to describe "multiplicative" noise, that's relevant for curved target spaces (relevant, in particular, for gauge theories).

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Many sorts of noise

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

$$\nu = \begin{cases} k_{\mathrm{B}} T \, \text{thermal} \\ \hbar \, \text{quantum} \\ \sigma^2 \, \text{annealed disorder} \end{cases}$$

The three partition functions–that are four

- *The partition function of the noise*

$$Z_\eta = \int [\mathscr{D}\eta]\, e^{-\frac{\eta^2}{2\nu}} = 1$$

- *The partition function from the Langevin map*

$$Z_\eta = Z_{\mathrm{L}} = \int [\mathscr{D}\phi]\ |U''(\phi)|\ e^{-\frac{U'(\phi)^2}{2\nu}} = 1$$

- *The canonical partition function*

$$Z = \int [\mathscr{D}\phi]\, e^{-\frac{U'(\phi)^2}{2\nu}} \stackrel{?}{=} 1$$

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

The fourth partition function

$Z_{\rm L}$ can be written as follows

$$Z_{\rm L} = \int [\mathscr{D}\phi] \; |U''(\phi)| \; e^{-\frac{U'(\phi)^2}{2\nu}} = \int [\mathscr{D}\phi] \, \mathrm{sign}(U''(\phi)) \, U''(\phi) \, e^{-\frac{U'(\phi)^2}{2\nu}} = \langle \mathrm{sign}(U''(\phi)) \rangle_{\rm SUSY} \, Z_{\rm SUSY}$$

where

$$Z_{\rm SUSY} \equiv \int [\mathscr{D}\phi] \, U''(\phi) \, e^{-\frac{U'(\phi)^2}{2\nu}} = \int [\mathscr{D}\phi][\mathscr{D}\psi][\mathscr{D}\chi] \, e^{-\frac{U'(\phi)^2}{2\nu} + \psi U''(\phi)\chi}$$

(Choice of units : $\nu \equiv 1$ henceforth).

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Emergent SUSY : the case of the one–dimenensional worldvolume

A concrete example : one–dimensional worldvolume, when

$$\eta(\tau) = \frac{d\phi}{d\tau} + \frac{\partial W}{\partial\phi} \equiv \frac{\partial U}{\partial\phi}$$

(This relation between η and ϕ is known as the "Nicolai map". It's not useful as a differential equation–rather, it describes a change of variables in the path integral !)

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Emergent SUSY : the case of the one–dimenensional worldvolume

Then

$$S[\phi,\psi,\chi] = \int d\tau \left\{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}W'(\phi)^2 - \frac{1}{2}\left(\psi\dot{\chi} - \dot{\psi}\chi\right) - W''(\phi)\frac{1}{2}\left(\psi\chi - \chi\psi\right)\right\} =$$

$$\int d\tau \left\{\frac{1}{2}\dot{\phi}^2 - \frac{F^2}{2} + FW'(\phi) - \frac{1}{2}\left(\psi\dot{\chi} - \dot{\psi}\chi\right) - \frac{1}{2}W''(\phi)\left[\psi,\chi\right]\right\}$$

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Emergent SUSY

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

The action is invariant (up to total derivatives) under the transformations

$$\begin{aligned} Q_1\phi &= -\chi & Q_2\phi &= \psi \\ Q_1\chi &= 0 & Q_2\chi &= \dot{\phi} - F \\ Q_1\psi &= -\dot{\phi} - F & Q_2\psi &= 0 \\ Q_1 F &= \dot{\chi} & Q_2 F &= \dot{\psi} \end{aligned}$$

that close on the translations :

$$\{Q_1, Q_2\} = -2\frac{d}{d\tau} \Leftrightarrow [\zeta_1 Q_1, \zeta_2 Q_2] = 2\zeta_1\zeta_2\frac{d}{d\tau} = -\zeta_\alpha \varepsilon^{\alpha\beta}\zeta_\beta\frac{d}{d\tau}$$

Emergent SUSY

This example can be generalized to the case

$$\frac{\partial U}{\partial \phi} \equiv \mathsf{s}\frac{\partial \phi}{\partial u} + \frac{\partial W}{\partial \phi}$$

then the anticommutator of two such transformations does close on the generators of the translations (provided the s either commute, or generate a Clifford algebra). The former case describes worldvolume supersymmetry ; the latter allows for target space supersymmetry. Cf. arXiv :1712.07045 for the $\mathcal{N} = 2, D = 2$ WZ model.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Some subtleties

We remark that

$$Z_{\mathrm{L}} = \int [\mathscr{D}\phi] \, |U''(\phi)| \, e^{-\frac{U'(\phi)^2}{2}}$$

is a manifestly non–negative quantity and, insofar as it can be well–defined, is equal to 1.
If $U''(\phi)$ is of constant sign, then $\langle \mathrm{sign}(U''(\phi)) \rangle_{\mathrm{SUSY}} = 1$ and $Z_{\mathrm{SUSY}} = 1$.
If $U''(\phi)$ isn't of constant sign, then Z_{SUSY} isn't well–defined, so neither is $\langle \mathrm{sign}(U''(\phi)) \rangle_{\mathrm{SUSY}}$.
However, the product of $\langle \mathrm{sign}(U''(\phi)) \rangle_{\mathrm{SUSY}} Z_{\mathrm{SUSY}} = Z_{\mathrm{L}}$, which can provide a consistent definition for it.
Can we check this, while avoiding having to deal with each factor separately ? And what about the canonical partition function ? ? ?

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

A new look at the canonical partition function

The canonical partition function

$$Z = \int [\mathscr{D}\phi]\, e^{-\frac{U'(\phi)^2}{2}}$$

describes a physical system, in equilibrium with its fluctuations. Therefore, one would expect that it be equal to the other three partition functions, namely equal to 1 ; and that it be invariant under supersymmetry transformations.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

A new look at the canonical partition function

A "slight" problem : it only takes into account the ϕ and doesn't seem to know anything about the ψ and the χ. Nevertheless, since $(\delta_\zeta\phi)^n = 0$, for all $n > 1$ and $\delta_\zeta\phi$ is a constant shift in the ϕ, it's straightforward to show that, even though $U'(\phi)^2/2 = S[\phi]$, isn't invariant under $\phi \rightarrow \phi + \delta_\zeta\phi$, Z is–provided that $U''(\phi)$ is of constant sign.

This seems too good to be true ! What's the catch ?

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

The catch–and when it's not

The catch is that, since

$$\eta = U'(\phi)$$

$U'(\phi)$ is a free field.
Therefore, if $Z_{\mathrm{L}}, Z_{\mathrm{SUSY}}$ and Z describe the same physics, $U'(\phi)$ must have the same correlation functions, whatever the partition function used. The easiest partition function to sample is Z.
What this means is that it is useful to check, whether fluctuations are able to "reproduce" the presence of $|U''(\phi)|$ in the path integral for Z. This isn't at all trivial to show-even if the sign of $U''(\phi)$ is fixed, in the infinite–dimensional case, relevant for quantum mechanics and quantum field theory. It is possible to provide numerical evidence for subsets of the identities that $U'(\phi)$ should satisfy, if it's a Gaussian field.

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Beyond the classical partition function

We wish to check that the correlation functions of $\eta = U'(\phi)$, sampled according to Z, describe a Gaussian field, with ultralocal 2–point function. This isn't trivial, since Z doesn't include $|\det U''|$, so what's at stake is, whether the fluctuations, described by Z, can generate $|\det U''|$–or not.
However, Z, that contains only the action of the commuting field(s), ϕ, can be sampled perfectly well.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

The lattice approach

$$S = \int d\tau \left\{ \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\partial W}{\partial \phi}\right)^2 \right\} \rightarrow$$
$$\sum_{n=0}^{N-1} \left\{ -\phi_n \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{2a} + \frac{a}{2}\left(\frac{\partial W}{\partial \phi_n}\right)^2 \right\}$$

Example : The polynomial superpotential

$$W = c\phi + \frac{m^2\phi^2}{2} + \frac{\lambda}{K!}\phi^K \Leftarrow \frac{\partial W}{\partial \phi} = c + m^2\phi + \frac{\lambda}{(K-1)!}\phi^{K-1}$$

Define the lattice parameters

$$\varphi_n \equiv a^{-1/2}\phi_n, \quad m^2_{\mathrm{latt}} \equiv m^2 a, \quad \lambda_{\mathrm{latt}} \equiv \lambda a^{\frac{K}{2}}, \quad c_{\mathrm{latt}} \equiv c a^{1/2}$$

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

Noise on the lattice

It is possible to write the action, on the lattice, as a sum of perfect squares, that define the lattice version of the noise field :

$$-\varphi_n\varphi_{n+1}+\varphi_n^2+\frac{1}{2}\left(c_{\mathrm{latt}}+m_{\mathrm{latt}}^2\varphi_n+\frac{\lambda_{\mathrm{latt}}}{(K-1)!}\varphi_n^{K-1}\right)^2=$$
$$\frac{1}{2}\left((\varphi_{n+1}-\varphi_n)^2+\left(c_{\mathrm{latt}}+m_{\mathrm{latt}}^2\varphi_n+\frac{\lambda_{\mathrm{latt}}}{(K-1)!}\varphi_n^{K-1}\right)^2\right)=$$
$$\frac{1}{2}\left(\varphi_{n+1}-\varphi_n+c_{\mathrm{latt}}+m_{\mathrm{latt}}^2\varphi_n+\frac{\lambda_{\mathrm{latt}}}{(K-1)!}\varphi_n^{K-1}\right)^2$$
$$-(\varphi_{n+1}-\varphi_n)\,W'(\varphi_n;c_{\mathrm{latt}},m_{\mathrm{latt}}^2,\lambda_{\mathrm{latt}})$$

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Some more tricks

$$g \equiv \frac{\lambda}{|m|^K} = \frac{\lambda_{\text{latt}}}{|m_{\text{latt}}|^K}$$
$$\frac{c_{\text{latt}}}{|m_{\text{latt}}|} = \frac{c}{|m|} \equiv C$$
$$s \equiv \frac{|m^2|}{\lambda} = a^{\frac{2-K}{2}} \frac{|m_{\text{latt}}|^2}{\lambda_{\text{latt}}} \varphi_n \equiv \left(\frac{|m_{\text{latt}}|^2}{\lambda_{\text{latt}}} \right)^{\alpha} \boldsymbol{\varphi}_n$$

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook

The lattice action

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Thus

$$S_{\rm latt} = \frac{1}{g^{2/(K-2)}|m_{\rm latt}|^2}\sum_{n=0}^{N-1}\left[-\varphi_n\varphi_{n+1} + \varphi_n^2 + \frac{|m_{\rm latt}|^4}{2}\left(c_{\rm latt}g^{\frac{1}{K-2}} + {\rm sign}(m_{\rm latt}^2)\varphi_n + \frac{\varphi_n^{K-1}}{(K-1)!}\right)^2\right]$$

This is the lattice action, that expresses the physics most clearly.

Target space supersymmetry in two dimensions

$$S = \int d^2x \left\{ \frac{1}{2}\left(\partial_A \phi_J\right)^2 - \frac{F_I^2}{2} + F_I \frac{\partial W}{\partial \phi_I} + \sigma_A^{IJ}\partial_A\phi_J \frac{\partial W}{\partial \phi_I} - \psi\sigma_A\partial_A\chi - \psi W''(\phi)\chi \right\}$$

For the case of the cubic superpotential,

$$\frac{\partial W}{\partial \phi_1} = g\left(\phi_1^2 - \phi_2^2\right)$$
$$\frac{\partial W}{\partial \phi_2} = 2g\phi_1\phi_2$$

the crossterm is a total derivative. Using the tricks presented above, we can deduce the lattice action and the noise field on the lattice.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Numerical results : One–dimensional worlvolume

For the case of a one–dimensional worldvolume (i.e. Euclidian quantum mechanics, if $\nu = \hbar$) it's possible to find

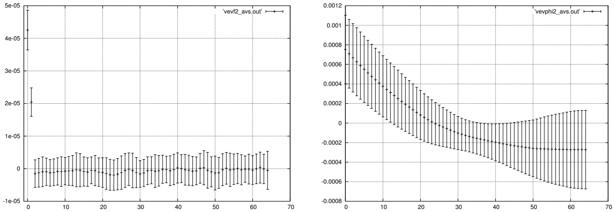

Figure – The connected 2–point functions for the noise field, $\langle \eta_{|n-n'|}\eta_0 \rangle - \langle \eta \rangle^2$ and of the scalar field, $\langle \Phi_{|n-n'|}\Phi_0 \rangle - \langle \Phi \rangle^2$, for $N = 128$, $g = 0.9$.The value of $m^2_{\rm latt} = 0.0001$.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Numerical results : Two–dimensional worlvolume ; Target space SUSY

For the case of a two–dimensional worldvolume (i.e. Euclidian WZ model, if $\nu = \hbar$) it's possible to find

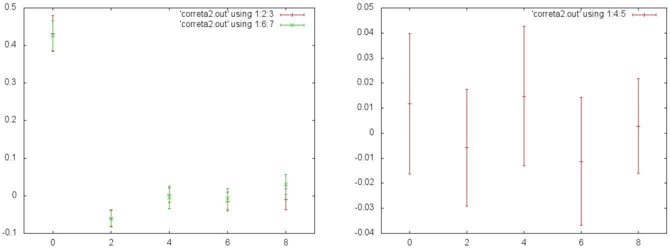

Figure – Typical results for the cubic superpotential : $\langle \eta_n^I \eta_{n+d}^J \rangle$ for $I = J$ (left panel) and $I \neq J$ (right panel) and $d = 0, 2, 4, 6, 8$, on the 17×17 square lattice. $g^2_{\mathrm{latt}} = 0.7$. The diagonal noise term is a $\delta-$function, while the off-diagonal noise term vanishes, to numerical precision.

Conclusions and outlook

- "Supersymmetric quantum mechanics" is a repetition : Quantum mechanics, just like any field theory, of one–dimensional worldvolume, in equilibrium with its fluctuations, is, anyway, supersymmetric.
- Quantum field theories, with two–dimensional worldvolume, in equilibrium with their fluctuations, also, are supersymmetric, since continuous symmetries can't be broken in (less than) two dimensions.
- In both cases, the canonical partition function can generate $|\det U''(\phi)|$; therefore, it does describe all possible fluctuations.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Conclusions and outlook

- What is remarkable is that, even when supersymmetry is "spontaneously broken" in these cases, only the 1–point function of $U'(\phi)$ is modified ; $U'(\phi)$ remains a Gaussian field, with ultra–local 2–point function.
- There do exist cases, when the canonical partition function does *not* describe all possible fluctuations, in the sense that it can't generate $|\det U''(\phi)|$. That's how it's possible to "hide" supersymmetry. This can be shown in zero–dimensional models, where the "anomalies" of the stochastic identities can be explicitly computed and the distribution of the various fields can be determined. It may, also, occur, in theories with worldvolume that's three–dimensional, or higher (or whose target space is three–dimensional or higher).

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Conclusions and outlook

- In these cases new, boundary, degrees of freedom become relevant for consistently closing the system. Identifying them in three dimensions or higher is non–trivial ; in particulat since a term that isn't, manifestly, a boundary term, has to be checked that it doesn't, in fact, contribute. But now one can start imagining how to frame these questions–so it's possible to set up the calculations, that will provide the answers.
- Gauge theories describe, in fact, target spaces that aren't flat ; therefore the noise, that's relevant, isn't additive, but multiplicative. Attempts to describe the degrees of freedom, that "close" the system in such cases, can be tested in magnetic systems, where the Landau–Lifshitz–Gilbert equation is relevant.

Introduction
Fluctuations and supersymmetry
Many partition functions : One system
A new property emerges
Numerical results
Conclusions and outlook

Conclusions and outlook

"Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning." (W. Churchill)

Introduction

Fluctuations and supersymmetry

Many partition functions : One system

A new property emerges

Numerical results

Conclusions and outlook