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ON SOME EXOTIC SCHOTTKY GROUPS

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Abstract. We construct a Cartan-Hadamard manifold with pinched negative curvature whose group of isometries possesses divergent discrete free subgroups with parabolic elements that do not satisfy the so-called "parabolic gap condition" introduced in [4]. This construction relies on the comparaison between the Poincaré series of these free groups and the potential of some transfer operator which appears naturally in this context.

1. **Introduction.** Throughout this paper, X will denote a complete and simply connected Riemannian manifold of dimension $N \geq 2$ whose sectional curvature K is bounded between two negative constants $-B^2 \leq K \leq -A^2 < 0$. We denote by d the distance on X induced by the Riemannian metric and by ∂X the boundary at infinity; the isometries of X act as conformal transformations on ∂X when it is endowed by the so-called Gromov-Bourdon metric.

We denote by $\mathrm{Iso}(X)$ the group of orientation preserving isometries of X. There are three types of elements in $\mathrm{Iso}(X)$: the *elliptic* ones which fix a unique point in X, the *parabolic* ones which fix a unique point in ∂X and the *hyperbolic* ones which fix two points in ∂X. A subgroup of $\mathrm{Iso}(X)$ is said torsion free when it contains no elements of finite order; it is said non elementary when the set of accumulation points of its orbits is reduced to one or two points [1].

A *Kleinian group* of X is a non elementary torsion free and discrete subgroup Γ of orientation preserving isometries of X; this group Γ acts freely and properly discontinuously on X and the quotient manifold $M := X/\Gamma$ has a fundamental group which can be identified with Γ. One says that Γ is *a lattice* when the Riemannian volume of X/Γ is finite.

Our main result concerns a restrictive class of Kleinian groups, the so-called *Schottky groups*. Indeed, we will consider two isometries p and q such that there exist non-empty disjoint closed sets $\mathcal{U}_p$ and $\mathcal{U}_q$ in ∂X satisfying the following conditions:

$$\text{for all } k \in \mathbb{Z}^* \quad p^k\Big((X \cup \partial X) \setminus \mathcal{U}_p\Big) \subset \mathcal{U}_p \quad \text{and} \quad q^k\Big((X \cup \partial X) \setminus \mathcal{U}_q\Big) \subset \mathcal{U}_q.$$

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[1] in particular, the cyclic group generated by a parabolic or an hyperbolic isometry is elementary; note that, even in constant curvature -1, there may exist non abelian - and in particular non cyclic - elementary groups! (see for instance [11] in constant curvature -1, see also Section 2.1)

The group generated by p and q is a *Schottky group* ; it is discrete and free, its action on $X \cup \partial X$ may be thus encode which allows to use the thermodynamic formalism.

The elements of a Kleinian group Γ act as homeomorphism on the boundary ∂X, and, more precisely, on the subset of ∂X containing all the accumulation points of some (any) orbit $\Gamma \cdot \mathbf{x}$ of $\mathbf{x} \in X$, the so-called *limit set* Λ_Γ of Γ [(2)].

The set Λ_Γ carries almost all the information about the geodesic flow $(\phi_t)_t$ on the unit tangent bundle $T^1(X/\Gamma)$ of X/Γ since the elements of the *non-wandering set* Ω_Γ of $(\phi_t)_t$ are the projection on $T^1(X/\Gamma)$ of the unit tangent vectors of X whose points at infinity in both directions belong to Λ_Γ.

The convex-hull $C(\Lambda_\Gamma)$ of Λ_Γ is the trace on X of the smallest convex subset of $X \cup \partial X$ containing Λ_Γ; it is a Γ-invariant closed subset of X and the projection of Ω_Γ onto the manifold X/Γ is in fact equal to $C(\Lambda_\Gamma)/\Gamma$. The group Γ is said to be *convex co-compact* when it acts co-compactly on $C(\Lambda_\Gamma)$ and more generally *geometrically finite* when it acts like a lattice on some (any) ε-neighbourhood $C^\varepsilon(\Lambda_\Gamma)$ of $C(\Lambda_\Gamma)$ (in other words when $\mathrm{vol}(C^\varepsilon(\Lambda_\Gamma)/\Gamma) < +\infty$ for some (any) $\varepsilon > 0$). For instance, if Γ is a Schottky group generated by two hyperbolic isometries p and q, it is convex co-compact and its limit set has a fractal structure; this is false when p is parabolic, but in this case Γ remains geometrically finite.

It is shown in [9] that the existence and uniqueness of a measure of maximal entropy for the geodesic flow restricted to Ω_Γ is equivalent to the finiteness of a natural invariant Radon measure on $T^1(X/\Gamma)$ with support Ω_Γ, the so-called *Patterson-Sullivan measure* m_Γ. In this paper, we construct examples of isometry groups Γ for which the restriction of the geodesic flow $(\phi_t)_t$ to the set Ω_Γ exhibits particular properties with respect to ergodic theory. In particular, for those groups, the Patterson-Sullivan measure may be infinite and the associated dynamical system (ϕ_t, Ω_Γ) will thus have no measure of maximal entropy.

We now recall briefly the construction of the Patterson-Sullivan measure associated with a Kleinian group Γ. The critical exponent of Γ is the exponential growth of its orbital function defined by

$$\delta_\Gamma := \limsup_{r\to\infty} \frac{1}{r} \log \mathrm{card}\{\gamma \in \Gamma / d(\mathbf{x}, \gamma\cdot\mathbf{x}) \le r\}.$$

It does not depend on $\mathbf{x} \in X$ and coincides with the exponent of convergence of the Poincaré series of Γ defined by $\mathbf{P}_\Gamma(s, \mathbf{x}) := \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{o},\gamma\cdot\mathbf{o})}$; this series converges if $s > \delta_\Gamma$ and diverges when $s < \delta_\Gamma$. The group Γ is *divergent* when the Poincaré series diverges at the critical exponent; otherwise Γ is *convergent*.

A construction due to Patterson in constant curvature provides a family of δ_Γ-*conformal measures* $\sigma = (\sigma_\mathbf{x})_{\mathbf{x}\in X}$ supported on the limit set Λ_Γ. D. Sullivan showed also how to assign to σ an invariant measure m_Γ for the geodesic flow $(\phi_t)_t$ restricted to Ω_Γ. This construction has been extended by several people to the situation of a variable curvature space X and an arbitrary Kleinian group Γ acting on it [8], [16].

It is important to recall that the divergence/convergence of Γ has quite opposite effects on the dynamic of the group and the geodesic flow $(\phi_t)_t$ relatively to $\sigma\otimes\sigma$ and m_Γ respectively. For instance, the three following statements are equivalent: (i) Γ is divergent, (ii) the diagonal action of Γ on $(\partial X \times \partial X \setminus \mathit{diagonal}, \sigma\otimes\sigma)$ is completely conservative and ergodic, (iii) the geodesic flow on $(T^1(X/\Gamma), m_\Gamma)$ is completely conservative and ergodic (see [12] for a complete statement). By the Poincaré recurrence theorem, the finiteness of m_Γ implies the conservativity of $(\phi)_t$

[2] Λ_Γ is also the least non empty Γ-invariant closed subset of ∂X

relatively to m_Γ and consequently the divergence of Γ. There are thus two families of divergent groups: the groups Γ whose Patterson-Sullivan measure m_Γ is finite (in which case m_Γ is the unique measure of maximal entropy) and the groups Γ whose corresponding geodesic flow has no measure of maximal entropy but remains conservative and ergodic with respect to the (infinite) measure m_Γ.

It is well known since D. Sullivan's work [14] that co-compact, convex co-compact and more generally geometrically finite groups Γ of isometries of the N-dimensional hyperbolic space $\mathbb{H}^N$ are divergent and that their Patterson-Sullivan measure m_Γ is finite; this holds also for geometrically finite groups of isometries of rank one symmetric spaces [3].

For other explicit Kleinian groups, it is always subtle to decide whether there are convergent or divergent, even in the case of rank one symmetric spaces. For instance, as far as we know, the only ones explicit examples of convergent Kleinian groups are the normal subgroups Γ of co-compact groups Γ_0 for which the quotient $\Gamma_0/\Gamma \simeq \mathbb{Z}^k$ with $k \geq 3$: indeed, Γ_0 is divergent when $k = 1, 2$ and convergent when $k \geq 3$. The topological structure of the manifold X/Γ_0 is thus very influent for this question and the dichotomy divergence/convergence is related to the property of recurrence/transience of k-dimensional random Markov walks on the Euclidean space.

The situation is different in the general variable curvature case, even for geometrically finite groups - whose corresponding quotient manifolds have a quite simple topological structure though - because of the presence of parabolic subgroups. Indeed, the choice of the metric inside the horoballs fixed by these parabolic subgroups is much more flexible, and some curious phenomena may appear (see [5] for instance). There exist criteria which ensure that a geometrically finite group Γ is divergent. For instance, one may assume that its Poincaré exponent δ_Γ is strictly greater than the one of each of its parabolic subgroups [4, Théorème A]; this is the so-called *parabolic gap condition* (PGC), which is satisfied in particular when the parabolic subgroups of Γ are themselves divergent. Furthermore, the Patterson-Sullivan measure m_Γ of a divergent geometrically finite group Γ is finite if and only if, for any parabolic subgroup $\mathcal{P}$ of Γ, one has

$$\sum_{p \in \mathcal{P}} d(\mathbf{o}, p \cdot \mathbf{o}) e^{-\delta_{\mathcal{P}} d(\mathbf{o}, p \cdot \mathbf{o})} < +\infty, \tag{1}$$

where $\delta_{\mathcal{P}}$ denotes the critical exponent of $\mathcal{P}$ [4, Théorème B]; this holds of course in particular when the condition PGC is fullfilled.

In [4], one can find a construction of a Hadamard manifold (X, g) whose group of isometries admits non elementary geometrically finite groups Γ of convergent type; in this case, the condition PGC is not satisfied and the parabolic subgroups of Γ are convergent. Indeed, Γ is a Schottky group generated by an hyperbolic isometry and a parabolic one; in the dimension 2 case, the quotient manifold may for instance be a pair of pant with a cuspidal end and the curvature varies only inside the cusp. The fact that Γ may be convergent or divergent - with the importance consequences on the stochastic behavior of the corresponding geodesic flow that we describe above - is thus not only a consequence of the topology of the quotient manifold but depends also on the choice of the metric in a very subtle way.

In summary, for geometrically finite groups Γ, we have the two following classes of examples

1. Γ is convergent, or

2. Γ is divergent, but satisfies the condition PGC - this is for instance the case in constant negative curvature -

and, as far as we know, there are no known examples of geometrically groups Γ of divergent type which do not satisfy the critical gap property. We fill this gap, stating the

Main Theorem. *There exist Hadamard manifolds with pinched negative curvature whose group of isometries contains geometrically finite Schottky groups Γ of divergent type which do not satisfy the parabolic gap condition PGC. Furthermore, the Patterson-Sullivan measure m_Γ may be finite or infinite.*

which immediately leads to the

Corollary 1.1. *There exist negatively curved geometrically finite manifolds whose corresponding geodesic flow $(\phi_t)_t$ does not admit a (finite) measure of maximal entropy but is completely conservative and ergodic with respect to the associated Patterson-Sullivan measure.*

The paper is organized as follows: Section §2 deals with the construction of convergent parabolic groups; we recall in particular the results presented in [4]. Section §3 is devoted to the construction of Hadamard manifolds containing convergent parabolic elements and whose group of isometries is non elementary. In section §4 we construct Schottky groups with convergent parabolic factor and we explain how to choose the metric inside the corresponding cuspidal end to prove the main theorem.

We fix here once and for all some notation about asymptotic behavior of functions:

Notation 1.2. *Let f, g be two functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. We shall write $f \stackrel{c}{\asymp} g$ (or simply $f \asymp g$) when $\frac{1}{c} g(R) \leq f(R) \leq c g(R)$ for some constant $c \geq 1$ and R large enough.*

2. **On the existence of convergent parabolic groups.**

2.1. **The real hyperbolic space.**

We first consider the real hyperbolic space of dimension $N \geq 2$, identified to the upper half-space $\mathbb{H}^N := \mathbb{R}^{N-1} \times \mathbb{R}^{*+}$. In this model, the Riemannian hyperbolic metric is given by $\dfrac{dx^2 + dy^2}{y^2}$ where $dx^2 + dy^2$ is the classical euclidean metric on $\mathbb{R}^{N-1} \times \mathbb{R}^{*+}$. We denote by $\mathbf{i}$ the origin $(0, \cdots, 0, 1)$ of $\mathbb{H}^N$ and by $\|.\|$ the euclidean norm in $\mathbb{R}^N$.

Let p be a parabolic isometry of $\mathbb{H}^N$ fixing ∞; its induces on $\mathbb{R}^{N-1}$ an euclidean isometry which can be decomposed as the product $p = R_p \circ T_p = T_p \circ R_p$ of an affine rotation R_p and a translation T_p with vector of translation $\vec{s}_p$. By an elementary computation in hyperbolic geometry, one may check that the sequence $(d(\mathbf{i}, p^n \cdot \mathbf{i}) - 2 \ln n \|\vec{s}_p\|)_{n \geq 1}$ converges to 0. The Poincaré exponent of the group $\langle p \rangle$ is thus equal to $\frac{1}{2}$ and $\langle p \rangle$ is divergent.

More generally, for any parabolic subgroup $\mathcal{P}$ of the group of isometries of $\mathbb{H}^N$, the sequence $(d(\mathbf{i}, p \cdot \mathbf{i}) - 2 \ln \|\vec{s}_p\|)_p$ converges to 0 as $p \to \infty$ in $\mathcal{P}$. By one of Bieberbach's theorems, the group $\mathcal{P}$ contains a finite index abelian subgroup $\mathcal{Q}$ which acts by translations on a subspace $\mathbb{R}^k$ of $\mathbb{R}^{N-1}$; in other words, there exist k linearly independant vectors $\vec{s}_1, \cdots, \vec{s}_k$ and a finite set $F \subset \mathcal{P}$ such that any $p \in \mathcal{P}$

may be decomposed as $p = p_{\vec{s}_1}^{n_1} \cdots p_{\vec{s}_k}^{n_k} f$ with $n_1, \cdots, n_k \in \mathbb{Z}$ and $f \in F$ so that

$$\begin{aligned}\mathbf{P}_{\mathcal{P}}(s) = 1 + \sum_{p \in \mathcal{P}^*} e^{-sd(\mathbf{i}, p \cdot \mathbf{i})} &= 1 + \sum_{p \in \mathcal{P}^*} \frac{e^{so(p)}}{\|\vec{s}_p\|^{2s}} \\ &= 1 + \sum_{f \in F} \sum_{\bar{n}=(n_1,\cdots,n_k) \in (\mathbb{Z}^k)^*} \frac{e^{so(\overline{n})}}{\|n_1\vec{s}_1 + \cdots + n_k\vec{s}_k\|^{2s}}.\end{aligned} \tag{2}$$

The Poincaré exponent of $\mathcal{P}$ is thus equal to $\frac{k}{2}$ and the group is divergent.

All these calculations may be done in the following (less classical) model: using the natural diffeomorphism between $\mathbb{H}^N$ and $\mathbb{R}^N$ defined by $(x, y) \mapsto (x, t) := (x, \ln y)$ one may endow $\mathbb{R}^N$ with the hyperbolic metric $g_{hyp} := e^{-2t}dx^2 + dt^2$.

In this model, we fix the origin $\mathbf{o} = (0, \cdots, 0)$ and the vertical lines $\{(x, t)/t \in \mathbb{R}\}$ are clearly geodesics. For any $t \in \mathbb{R}$, we denote by $\mathcal{H}_t$ the hyperplane $\{(x, t) : x \in \mathbb{R}^{N-1}\}$; this corresponds to the horosphere centered at $+\infty$ and passing through $(0, \cdots, 0, t)$. For any $x, y \in \mathbb{R}^{N-1}$, the distance between $\mathbf{x}_t := (x, t)$ and $\mathbf{y}_t := (y, t)$ for the metric $e^{-2t}dx^2$ induced by g_{hyp} on $\mathcal{H}_t$ is equal to $e^{-t}\|x - y\|$; furthermore, if t is choosen in such a way that this distance is equal to 1 (namely $t = \ln \|x - y\|$), then the union of the 3 segments $[\mathbf{x}_0, \mathbf{x}_t], [\mathbf{x}_t, \mathbf{y}_t]$ and $[\mathbf{y}_t, \mathbf{y}_0]$ lies at a bounded distance of the hyperbolic geodesic joigning $\mathbf{x}_0$ and $\mathbf{y}_0$ which readily implies that $d(\mathbf{x}_0, \mathbf{y}_0) - 2 \ln \|x - y\|$ is bounded.

This crucial fact is the key to understand geometrically the estimations above; it first appeared in [4] and allowed the authors to construct negatively curved manifolds with convergent parabolic subgroups, we recall in the following subsection this construction.

2.2. **The metrics $T_{a,u}$ on $\mathbb{R}^N$.** We consider on $\mathbb{R}^{N-1} \times \mathbb{R}$ a Riemannian metric of the form $g = T^2(t)dx^2 + dt^2$, where dx^2 is a fixed euclidean metric on $\mathbb{R}^{N-1}$ and $T : \mathbb{R} \to \mathbb{R}^{*+}$ is a C^∞ non-increasing function. The group of isometries of g contains the isometries of $\mathbb{R}^{N-1} \times \mathbb{R}$ fixing the last coordinate. The sectional curvature at $(x, t) = (x_1, ..., x_{N-1}, t)$ does not depend on x : it is $K(t) = -\dfrac{T''(t)}{T(t)}$ on any plane $\langle \dfrac{\partial}{\partial X_i}, \dfrac{\partial}{\partial t} \rangle, 1 \le i \le N-1$, and $-K^2(t)$ on any plane $\langle \dfrac{\partial}{\partial X_i}, \dfrac{\partial}{\partial X_j} \rangle, 1 \le i < j \le N-1$ (when $N \ge 2$).

It is convenient to consider the non-decreasing function $u : \mathbb{R}^{*+} \to \mathbb{R}$ satisfying the following implicit equation

$$T(u(s)) = \frac{1}{s}. \tag{3}$$

Then, the value of the curvature of g is:

$$K(u(s)) := -\frac{T''(u(s))}{T(u(s))} = -\frac{2u'(s) + su''(s)}{s^2(u'(s))^3}. \tag{4}$$

Note that g has negative curvature if and only if T is convex. For instance, we have seen in the previous subsection that for $u(s) = \log s$ one gets $T(t) = e^{-t}$ and obtains a model of the hyperbolic space of constant curvature -1.

As it was seen in [4], the function u is of interest since it gives precise estimates (up to a bounded term) of the distance between points lying on the same horosphere $\mathcal{H}_t := \{(x, t) : x \in \mathbb{R}^{N-1}\}$ where $t \in \mathbb{R}$ is fixed. Namely, the distance between

$\mathbf{x}_t := (x,t)$ and $\mathbf{y}_t := (y,t)$ for the metric $T^2(t)dx^2$ induced by g on $\mathcal{H}_t$ is equal to $T(t)\|x-y\|$; for $t = u(\|x-y\|)$, this distance is thus equal to 1, and the union of the 3 segments $[\mathbf{x}_0, \mathbf{x}_t], [\mathbf{x}_t, \mathbf{y}_t]$ and $[\mathbf{y}_t, \mathbf{y}_0]$ lies at a bounded distance of the hyperbolic geodesic joigning $\mathbf{x}_0$ and $\mathbf{y}_0$ (see [4], lemme 4): this readily implies that $d(\mathbf{x}_0, \mathbf{y}_0) - 2u(\|x-y\|)$ is bounded.

In the sequel, we will assume that the function u coincides with the function $s \mapsto \ln s$ on $]0,1]$; in otherwords, the restriction to the set $]0,1]$ of the corresponding function $T_u(t)$ satisfying (3) is equal to $t \mapsto e^{-t}$. More generally, we will "enlarge " the region where $T_u(t)$ and e^{-t} coincides to the domain $\mathbb{R}^{N-1}\times]-\infty, a]$ with a arbitrary, introducing the following

Notation 2.1. *Let $a \in \mathbb{R}$ and $u : \mathbb{R}^{*+} \to \mathbb{R}$ be a C^2 non decreasing function such that*

- $u(s) = \ln s$ *for any* $s \in]0,1]$
- $K(u(s)) \in [-B^2, -A^2] \subset \mathbb{R}^{*-}$ *for any* $s > 0$.

We endow $\mathbb{R}^{N-1} \times \mathbb{R}$ with the metric $T^2_{a,u}(t)dx^2 + dt^2$, where $T_{a,u}$ is given by

$$\forall t \in \mathbb{R} \quad T_{a,u}(t) := \begin{cases} e^{-t} & if \quad t \leq a \\ \dfrac{e^{-a}}{u^{-1}(t-a)} & if \quad t \geq a \end{cases} \tag{5}$$

.

Note that this metric has constant curvature -1 on the domain $\mathbb{R}^{N-1}\times]-\infty, a]$.

2.3. **On the existence of metrics with convergent parabolic groups.** In this paragraph, we fix $a \in \mathbb{R}$ and endow $\mathbb{R}^{N-1}\times\mathbb{R}$ with the metric $T^2_{a,u}(t)dx^2+dt^2$ where $u(s) = \ln s + \alpha \ln\ln s$ for s large enough and some constant $\alpha > 0$; in this case, the curvature varies, nevertheless one has $\lim\limits_{s\to\infty} K(u(s)) = -1$ and all derivatives of $K(u(s))$ tend to 0 as $s \to +\infty$. We will first need the following

Lemma 2.2. *Fix $\kappa \in]0,1[$. For any $\alpha \geq 0$, there exists a constant $s_\alpha \geq 1$ and a non decreasing C^2 function $u_\alpha : \mathbb{R}^{*+} \to \mathbb{R}$ such that*

- $u_\alpha(s) = \ln s$ if $0 < s \leq 1$
- $u_\alpha(s) = \ln s + \alpha \ln\ln s$ if $s \geq s_\alpha$.
- $K(u_\alpha(s)) := -\dfrac{2u'_\alpha(s) + su''_\alpha(s)}{s^2(u'_\alpha(s))^3} \leq -\kappa^2$.

Proof. We first fix a C^2 non decreasing function $\phi : \mathbb{R} \to [0,\alpha]$, which vanishes on $\mathbb{R}^-$ and is equal to α on $[1,+\infty[$. For any $\varepsilon > 0$, we consider the function $v_\varepsilon : [e,+\infty[\to \mathbb{R}$ defined by

$$\forall s \geq 1 \quad v_\varepsilon(s) := \ln s + \phi_\varepsilon(s) \ln\ln s$$

where $\phi_\varepsilon(s) := \phi(\varepsilon \ln\ln s)$. A straightforward computation gives, for any $s \geq e$

$$\frac{2v'_\varepsilon(s) + sv''_\varepsilon(s)}{s^2(v'_\varepsilon(s))^3} = \frac{N_\varepsilon(s)}{D_\varepsilon(s)}$$

with

- $N_\varepsilon(s) := 1 + \dfrac{\phi_\varepsilon(s)}{\ln s} - \dfrac{\phi_\varepsilon(s)}{(\ln s)^2} + 2\phi'_\varepsilon(s)\Big(s\ln\ln s + \dfrac{s}{\ln s}\Big) + \phi''_\varepsilon(s)s^2\ln\ln s,$
- $D_\varepsilon(s) := \Big(1 + \dfrac{\phi_\varepsilon(s)}{\ln s} + \phi'_\varepsilon(s)s\ln\ln s\Big)^3,$
- $\phi'_\varepsilon(s) = \dfrac{\varepsilon}{s\ln s}\phi_\varepsilon(s)$ and $\phi''_\varepsilon(s) = \dfrac{\varepsilon^2 - \varepsilon(1+\ln s)}{(s\ln s)^2}\phi_\varepsilon(s).$

Consequently, one obtains, as $\varepsilon \to 0$ and uniformly on $[e, +\infty[$

$$\phi'_\varepsilon(s)\Big(s\ln\ln s + \frac{s}{\ln s}\Big) = \varepsilon\Big(\frac{\ln\ln s}{\ln s} + \frac{1}{(\ln s)^2}\Big)\phi_\varepsilon(s) \to 0$$

and

$$\phi''_\varepsilon(s)s^2\ln\ln s = \frac{\ln\ln s}{(\ln s)^2}\Big(\varepsilon^2 - \varepsilon(1+\ln s)\Big)\phi_\varepsilon(s) \to 0,$$

so that $\dfrac{2v'_\varepsilon(s) + sv''_\varepsilon(s)}{s^2(v'_\varepsilon(s))^3} \to 1$. One may thus choose $\varepsilon_0 > 0$ such that

$$\forall s \geq e \qquad -\frac{2v'_{\varepsilon_0}(s) + sv''_{\varepsilon_0}(s)}{s^2(v'_{\varepsilon_0}(s))^3} \leq -\kappa^2$$

and one sets

$$u_\alpha(s) := \begin{cases} \ln s & \text{if} \quad 0 < s \leq e \\ v_{\varepsilon_0}(s) & \text{if} \quad\ \ s \geq e, \end{cases} \tag{6}$$

with $s_\alpha := \exp(\exp(1/\varepsilon_0))$. $\square$

We thus fix $a, \alpha \geq 0$ and endow $\mathbb{R}^N = \mathbb{R}^{N-1}\times\mathbb{R}$ with the metric $T^2_{a,u_\alpha}(t)dx^2 + dt^2$ where u_α is given by Lemma 2.2. This metric has pinched negative curvature less than $-\kappa^2$ and constant negative curvature in the domain $\{(x,t) : t \leq\ a\}$.

Now, let $\mathcal{P}$ be a discrete group of isometries of $\mathbb{R}^{N-1}$ of rank $k \in \{1, \cdots, N-1\}$, i.e generated by k linearly independent translations $p_{\vec{\tau}_1}, \cdots, p_{\vec{\tau}_k}$ in $\mathbb{R}^{N-1}$. In order to simplify the notations, $\bar{n} = (n_1, \cdots, n_k) \in \mathbb{Z}^k$ will represent the translation of vector $n_1\vec{\tau}_1 + \cdots + n_k\vec{\tau}_k$ and $|\bar{n}|$ will denote its euclidean norm. These translations are also isometries of $\mathbb{R}^N$ endowed with the metric $T_{a,u_\alpha}(t)^2dx^2 + dt^2$ given above and the corresponding Poincaré series of $\mathcal{P}$ is given by

$$\begin{aligned}\mathbf{P}_\mathcal{P}(s) = 1 + \sum_{p\in\mathcal{P}^*} e^{-sd(\mathbf{o},p\cdot\mathbf{o})} &= 1 + \sum_{\bar{n}\in(\mathbb{Z}^k)^*} e^{-2su_{a,\alpha}(|\bar{n}|) - sO(\bar{n})} \\ &= 1 + \sum_{\bar{n}\in(\mathbb{Z}^k)^*} \frac{e^{-sO(\overline{n})}}{|\bar{n}|^{2s}\Big(\ln|\bar{n}|\Big)^{2s\alpha}}.\end{aligned} \tag{7}$$

We have thus proved the

Proposition 2.3. *Let $\mathbb{R}^N$ be endowed with the metric $T^2_{a,u_\alpha}(t)dx^2 + dt^2$ where u_α is given by Lemma 2.2. If $\mathcal{P}$ is a discrete group of isometries of $\mathbb{R}^{N-1}$ of rank k, its critical Poincaré exponent is equal to $k/2$; furthermore, the group $\mathcal{P}$ is convergent if and only if $\alpha > 1$.*

Remark 2.4. *One may also choose u is such a way that $u^{-1}(t) = e^{t/2-\sqrt{t}}$. If $r = 1$, the critical exponent of the associated Poincaré series is equal to $\frac{1}{2}$ and the group $\mathcal{P}$ is also convergent; this last example appears in [13], where some explicit results are given, in terms of the Poincaré series of the parabolic groups, which guarantee the equidistribution of the horocycles on geometrically finite negatively curved surfaces.*

3. **Weakly homogeneous Hadamard manifolds of type** (a, u_α)**.** In the previous section, we have endowed $\mathbb{R}^N$ with a metric $T_{a,u}(t)^2dx^2 + dt^2$; unfortunately, in this construction, except for some particular choice of u, all the isometries fix the same point at infinity and the group $Is(\mathbb{R}^N)$ is thus elementary. We need now to construct an Hadamard manifold with a metric of this inhomogeneous type in the neighbourhood of some points at infinity but whose group of isometries is non elementary.

3.1. **Metric of type (a, u) relative to some group Γ and some horoball $\mathcal{H}$.** Consider first a non uniform lattice Γ of isometries of $\mathbb{H}^N$. The manifold $M := \mathbb{H}^N/\Gamma$ has finite volume but is not compact; it thus possesses finitely many cusp $C_1, \cdots, C_l$, each cusp C_i being isometric to the quotient of some horoball $\mathcal{H}_i$ of $\mathbb{H}^N$ (centered at a point ξ_i) by a Bieberbach group $\mathcal{P}_i$ with rank $N-1$. Each group $\mathcal{P}_i$ also acts by isometries on $\mathbb{R}^{N-1} \times \mathbb{R}$ endowed with one of the metrics $T_{a,u}(t)^2dx^2 + dt^2$ given by Notation 2.1.

Now, we endow $\mathbb{R}^{N-1} \times \mathbb{R}$ with one of these metrics $T_{a,u}(t)^2dx^2 + dt^2$ and choose a in such a way we may paste the quotient $(\mathbb{R}^{N-1} \times [0, +\infty[)/\mathcal{P}_1$ with $M \setminus C_1$. The Riemannian manifold M remains negatively curved with finite volume. By construction, the group Γ acts isometrically on the universal covering $X \simeq \mathbb{R}^N$ of M endowed with the lifted metric $g_{a,u}$; note that $g_{a,u}$ coincides with the metric $T^2_{a,u}(t)dx^2 + dt^2$ on the preimage by Γ of the cuspidal end C_1 (3).

All this discussion gives sense to the following definition:

Definition 3.1. *Fix $a, \alpha \geq 0$, let u_α be the function given by Lemma 2.2 and (X, g) a negatively curved Hadamard manifold whose group of isometries contains a non uniform lattice Γ.*

Assume that X/Γ has one cusp C, let $\mathcal{P}$ be a maximal parabolic subgroup of Γ corresponding to this cusp, with fixed point ξ and let $\mathcal{H}$ be an horoball centered at ξ such that the $\gamma \cdot \mathcal{H}, \gamma \in \Gamma$, are disjoints or coincide.

One endows the manifold X with the metric $g_{a,u}$ defined by

1. $g_{a,u}$ has constant curvature -1 outside the set $\bigcup_{\gamma\in\Gamma} \gamma \cdot \mathcal{H}$

2. $g_{a,u}$ coincides with the metric $T_{a,u}(t)^2dx^2+dt^2$ inside each horoball $\gamma\cdot\mathcal{H}, \gamma \in \Gamma$.

One says that **the Riemannian manifold $(X, g_{a,u})$ has type (a, u) relative to the group Γ and the horoball $\mathcal{H}$.** *More generally, one says that* (X, g) **has type** u *when, for some $a \in \mathbb{R}$, some lattice Γ and some horoball $\mathcal{H}$, it has type (a, u) relative to Γ and $\mathcal{H}$.*

Remark 3.2. *If the metric g has type (a, u) relative to Γ and $\mathcal{H}$, the curvature remains equal to -1 in the stripe $\mathbb{R}^{N-1} \times [0, a] \subset \mathcal{H}$. In the limit case "$a = +\infty$", one recovers the hyperbolic metric of constant curvature -1.*

By construction, the elements of Γ are isometries of $(X, g_{a,u})$. It is a classical fact that the group of isometries of $\mathbb{H}^N$ is quite large since in particular it acts transitively on the hyperbolic space (and even on its unit tangent bundle). This property remains valid when X is symmetric, otherwise its isometry group is discrete ([6], Corollary 9.2.2); consequently, if g has type (a, u_α) with $\alpha > 0$, the group of isometries of (X, g) does not inherit this property of transitivity, it is is discrete and has Γ as finite index subgroup ([6], Corollary 1.9.34).

We set from now on the following

Notation 3.3. *The Hadamard manifold X has a fixed origin* **o** *and its group of isometries is supposed to contain a non uniform lattice Γ; we fix a maximal parabolic subgroup $\mathcal{P}$ of Γ with fixed point $\xi \in \partial X$ and an horoball $\mathcal{H}$ centered at ξ such that the horoballs $\gamma \cdot \mathcal{H}, \gamma \in \Gamma$, are disjoints or coincide.*

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[3] By the choice of C_1, the horoballs $\gamma\cdot\mathcal{H}_1, \gamma \in \Gamma$, are disjoint or coincide, they are also isometric to $\mathbb{R}^{N-1}\times\mathbb{R}^+$ endowed with the hyperbolic metric $e^{-2t}dx^2 + dt^2$; another way to endow $\mathbb{R}^N$ with the new metric $g_{a,u}$ is to replace inside each horoball $\gamma \cdot \mathcal{H}_1$ the hyperbolic metric with the restriction of $T^2_{a,u}(t)dx^2 + dt^2$ to the half space $\mathbb{R}^{N-1} \times \mathbb{R}^+$

We fix $\alpha \geq 0$ and we assume that, for any $a \geq 0$, the manifold X may be endowed with a metric $g_a := g_{a,u_\alpha}$ of type (a, u_α) relative to Γ and $\mathcal{H}$, *where u_α is given by Lemma 2.2.*

We denote by d_a the corresponding distance on X.

Note that, by construction, the sectional curvature of g_a is pinched between two non positive constants and is less than $-\kappa^2$ for somme constant $\kappa > 0$ which does not depend on a. Furthermore, using the fact that u_α is non negative and uniformly continuous on $[1, +\infty[$, one obtains the

Property 3.4. *For any a, a' and $\alpha \geq 0$, there exists a constant $K = K_{a,a',\alpha} \geq 1$ with $K_{a,a',\alpha} \to 1$ as $a' \to a$ such that*

$$\frac{1}{K}\, g_a \leq g_{a'} \leq K g_a,$$

so that

$$\frac{1}{K}\, d_a \leq d_{a'} \leq \ K d_a.$$

Remark 3.5. *Note that if $a' > a$, one has in fact $g_a \geq g_{a'}$ and so $d_a \geq d_{a'}$. It will be used in the last section.*

3.2. **On the metric structure of the boundary at infinity.** In this paragrah we describe the metric structure of the boundary at infinity of X; we need first to consider the Busemann function $\mathcal{B}^{(a)}_{\cdot}(\cdot,\cdot)$ defined by:

for any $x \in \partial X$ and any $\mathbf{p}, \mathbf{q}$ in X

$$\mathcal{B}^{(a)}_x(\mathbf{p},\mathbf{q}) = \lim_{\mathbf{x}\to x}\Big(d_a(\mathbf{p},\mathbf{x}) - d_a(\mathbf{q},\mathbf{x})\Big).$$

The Gromov product on ∂X, based at the origin $\mathbf{o}$, between the points x and y in ∂X is defined by

$$(x|y)^{(a)} := \frac{\mathcal{B}^{(a)}_x(\mathbf{o},\mathbf{z}) + \mathcal{B}^{(a)}_y(\mathbf{o},\mathbf{z})}{2}$$

where $\mathbf{z}$ is any point on the geodesic (x, y) (note that the value of $(x|y)^{(a)}$ does not depend on $\mathbf{z}$). By [2], the function

$$\begin{array}{rcl} D_a : \partial X \times \partial X & \to & \mathbb{R}^+ \\ (x,y) & \mapsto & D_a(x,y) := \begin{cases} \exp\Big(-\kappa (x|y)^{(a)}\Big) & if \quad x \neq y \\ 0 & if \quad x = y \end{cases}. \end{array} \tag{8}$$

is a distance on ∂X; furthermore, the cocycle property satisfied by the Busemann functions readily implies that for any $x, y \in \partial X$ and $\gamma \in \Gamma$

$$D_a(\gamma\cdot x, \gamma\cdot y) = \exp(-\frac{\kappa}{2}\mathcal{B}^{(a)}_x(\mathbf{o},\gamma^{-1}.\mathbf{o}))\exp(-\frac{\kappa}{2}\mathcal{B}^{(a)}_y(\mathbf{o},\gamma^{-1}.\mathbf{o}))D_a(x,y). \tag{9}$$

In other words, γ acts on $(\partial X, D_a)$ as a conformal transformation with coefficient of conformality

$$|\gamma'(x)|_a = \exp(-\kappa\mathcal{B}^{(a)}_x(\mathbf{o},\gamma^{-1}.\mathbf{o}))$$

at the point x, since equality (9) may be rewritten

$$D_a(\gamma\cdot x, \gamma\cdot y) = \sqrt{|\gamma'(x)|_a|\gamma'(y)|_a}\,D_a(x,y). \tag{10}$$

We will need to control the regularity with respect to a of the Busemann function $x \mapsto \mathcal{B}_x^{(a)}(\mathbf{o}, z)$. By Property 3.4, the spaces (X, d_0) and (X, d_a) are quasi-isometric and, for any $a_0 > 0$, there exist a constant $K_0 \geq 1$ such that

$$\forall a \in [0, a_0] \quad \frac{1}{K_0} d_0 \leq d_a \leq K_0 d_0. \tag{11}$$

Note that, by [7], one also gets

$$\frac{1}{K_{a,a',\alpha}}(y|z)_a \leq (y|z)_{a'} \leq K_{a,a',\alpha}(y|z)_a. \tag{12}$$

The corresponding distances D_a on ∂X are thus Hölder equivalent; more precisely, we have the

Property 3.6. *For any $a_0 > 0$, there exists a real $\omega_0 \in]0,1]$ such that, for all $a \in [0, a_0]$, one gets*

$$D_0^{1/\omega_0} \leq D_a \leq D_0^{\omega_0}.$$

The regularity of the Busemann function $x \mapsto \mathcal{B}_x^{(a)}(\mathbf{o}, \mathbf{p})$ where $\mathbf{p}$ is a fixed point in X is given by the following Fact, which refines a result due to M. Bourdon.

Fact 3.7. [1] *Let $E \subset \partial X$ and $F \subset X$ two sets whose closure $\overline{E}$ and $\overline{F}$ in $X \cup \partial X$ are disjoint. Then the family of functions $x \mapsto \mathcal{B}_x^{(a)}(\mathbf{o}, \mathbf{p})$, with $\mathbf{p} \in F$, is equi-Lipschitz continuous on E with respect to D_a.*

In particular, for $a_0 > 0$ fixed, there exist $\omega \in]0,1[$ and $C > 0$ such that, for all $a \in [0, a_0]$, one gets

$$\forall x, y \in E, \forall \mathbf{p} \in F \quad \left| \mathcal{B}_x^{(a)}(\mathbf{o}, \mathbf{p}) - \mathcal{B}_y^{(a)}(\mathbf{o}, \mathbf{p}) \right| \leq D_0(x, y)^{\omega}. \tag{13}$$

4. Divergent Schottky groups without PGC.

4.1. On the existence of convergent Schottky groups when $\alpha > 1$.

The fact that $\alpha > 1$ ensures that any subgroup of $\mathcal{P}$ is convergent. In [4], it is proved that Γ possesses also non elementary subgroups of convergent type; we first recall this construction and clarify the statement.

Proposition 4.1. *There exist Schottky subgroups G of Γ and $a_0 \geq 0$ such that*

- *G has Poincaré exponent $\frac{1}{2}$ and is convergent on (X, g_0)*
- *G has Poincaré exponent $> \frac{1}{2}$ and is divergent on (X, g_a) for $a \geq a_0$.*

Note that the group G necessarily contains a parabolic element, otherwise it would be convex co-compact and thus of divergent type, whatever metric g_a endows X.

Proof. We first work in constant negative curvature -1 and fix a parabolic isometry $p \in \mathcal{P}$. Since Γ is non elementary, there exists an hyperbolic isometry $q \in \Gamma$ whose fixed points are distinct from the one of p. If necessary, one may shrink the horoball $\mathcal{H}$ in such a way that the projection of the axis of q on the manifold $M = \mathbb{H}^N/\Gamma$ remains outside the cuspidal end $C \simeq \mathcal{H}/\mathcal{P}$; in others words, one may fix $\mathbf{o}$ on the axis of q and assume that for any $n \in \mathbb{Z}^*$ the geodesic segments $[\mathbf{o}, q^n \cdot \mathbf{o}]$ lie outside the set $\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{H}$ (and consequently in the region of constant curvature relative to the metric g_a).

By the dynamic of the elements of $\mathbb{H}^N$ there exist two compact sets $\mathcal{U}_p$ and $\mathcal{U}_q$ in $X \cup \partial X$ as follows:

1. $\mathcal{U}_p$ is a neigbourhood of the fixed point ξ_p of p;
2. $\mathcal{U}_q$ is a neigbourhood of the fixed points ξ_q^+ and ξ_q^- of q;
3. there exists $\theta > 0$ such that for any $\mathbf{x} \in \mathcal{U}_p$ and $\mathbf{y} \in \mathcal{U}_q$ the angle $\widehat{\mathbf{x}\,\mathbf{o}\,\mathbf{y}}$ is greater than θ;
4. for all $k \in \mathbb{Z}^*$ one has

$$p^k\Big((X \cup \partial X) \setminus \mathcal{U}_p\Big) \subset \mathcal{U}_p \quad \text{and} \quad q^k\Big((X \cup \partial X) \setminus \mathcal{U}_q\Big) \subset \mathcal{U}_q.$$

By the Klein's ping-pong lemma, the group $\langle p, q\rangle$ generated by p and q is free. Therefore each element $\gamma \in \langle p, q\rangle$, $\gamma \neq Id$, may be decomposed in a unique way as a product $\alpha_1^{n_1}\alpha_2^{n_2}\dots\alpha_k^{n_k}$ with $\alpha_i \in \{p, q\}, n_i \in \mathbb{Z}^*$ and $\alpha_i \neq \alpha_{i+1}$; the integer k is the **length** of γ and α_k is its **last letter**.

Let us now endow X with the metric $g_0 = g_{0,u_\alpha}$. If $\mathbf{x} \in \mathcal{U}_p$ and $\mathbf{y} \in \mathcal{U}_q$, the path which is the disjoint union of the geodesic ray $(\mathbf{x}, \mathbf{o}]$ and $[\mathbf{o}, \mathbf{y})$ is a quasi-geodesic in (X, g_0); therefore there exists a constant $C > 0$ which only depends on the sets $\mathcal{U}_p$ and $\mathcal{U}_q$ and on the bounds on the curvature - that is, on the choice of the function u_α - such that $d_0(\mathbf{x}, \mathbf{y}) \geq d_0(\mathbf{x}, \mathbf{o}) + d_0(\mathbf{o}, \mathbf{y}) - C$ The Poincaré series of this group equals

$$\mathbf{P}_{\langle p,q\rangle}(s) = 1 + \sum_{l\geq 1}\ \sum_{m_i,n_i\in\mathbb{Z}^*} e^{-sd_{\mathbf{o}}(\mathbf{o},p^{m_1}q^{n_1}\cdots p^{m_l}q^{n_l}\cdot\mathbf{o})}.$$

It follows that

$$\mathbf{P}_{\langle p,q\rangle}(s) \leq 1 + \sum_{l\geq 1}\Big(e^{2sC}\sum_{m\in\mathbb{Z}^*} e^{-sd_0(\mathbf{o},p^m\cdot\mathbf{o})}\sum_{n\in\mathbb{Z}^*} e^{-sd_0(\mathbf{o},q^n\cdot\mathbf{o})}\Big)^l.$$

Recall that $d_0(\mathbf{o}, p^m \cdot \mathbf{o}) = 2\ln m + 2\alpha \ln\ln |m| +$ *a bounded term*; since $\alpha > 1$, the series $\sum_{m\in\mathbb{Z}^*} e^{-sd_0(\mathbf{o},p^m\cdot\mathbf{o})}$ converges at its critical exponent $\delta_{\langle p\rangle} = \frac{1}{2}$.

We may now replace q by a sufficient large power q^k in order to get

$$e^C\sum_{m\in\mathbb{Z}^*} e^{-\frac{1}{2}d_0(\mathbf{o},p^m\cdot\mathbf{o})}\sum_{n\in\mathbb{Z}^*} e^{-\frac{1}{2}d_0(\mathbf{o},q^{kn}\cdot\mathbf{o})} < 1.$$

For such a k, it comes out that the critical exponent of the group G generated by p and $h := q^k$ is less than $\frac{1}{2}$ and that $\mathbf{P}_G(\frac{1}{2}) < +\infty$; since $p \in G$, one also gets $\delta_G \geq \delta_{\langle p\rangle} = \frac{1}{2}$. Finally $\delta_G = \frac{1}{2}$ and G is convergent, with respect to the metric d_0 on X.

Let us now prove that for a large enough the group G is divergent on (X, d_a). By the triangle inequality one first gets

$$\begin{aligned}\sum_{g\in G} e^{-sd_a(\mathbf{o},g\cdot\mathbf{o})} &\geq \sum_{l\geq 1}\ \sum_{n_i,m_i\in\mathbb{Z}^*} e^{-sd_a(o,p^{m_1}h^{n_1}\cdots p^{m_l}h^{n_l}\cdot\mathbf{o})}\\ &\geq \sum_{l\geq 1}\Big(\sum_{n\in\mathbb{Z}^*} e^{-sd_a(\mathbf{o},p^m\cdot\mathbf{o})}\sum_{m\in\mathbb{Z}^*} e^{-sd_a(\mathbf{o},h^n\cdot\mathbf{o})}\Big)^l.\end{aligned} \tag{14}$$

Recall first that, when the curvature is constant and equal to -1 (that is to say "$a = +\infty$" in the definition of g_a), the quantity $d_{\mathbb{H}^N}(\mathbf{o}, p^m\cdot\mathbf{o}) - 2\log|m|$ is bounded, so the parabolic group $\langle p\rangle$ is divergent with critical exponent $\frac{1}{2}$. There thus exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in]0, \varepsilon_0]$, one gets

$$\sum_{m\in\mathbb{Z}^*} e^{-(\frac{1}{2}+\varepsilon)d_{\mathbb{H}^N}(\mathbf{o},p^m\cdot\mathbf{o})}\sum_{n\in\mathbb{Z}^*} e^{-(\frac{1}{2}+\varepsilon)d_{\mathbb{H}^N}(\mathbf{o},h^n\cdot\mathbf{o})} > 1, \tag{15}$$

which proves that the critical exponent of G is strictly greater than $\frac{1}{2}$ for $a = +\infty$.

The same property holds in fact for finite but large enough values of a. Indeed, there exists $m_a \geq 1$, with $m_a \to +\infty$ as $a \to +\infty$, such that the geodesic segments $[\mathbf{o}, p^m \cdot \mathbf{o}]$ for $-m_a \leq m \leq m_a$ remain inside the stripe $\mathbb{R}^{N-1} \times [0, a] \subset \mathcal{H}$ corresponding to the cuspidal end of type (a, α) ; since g_a has curvature -1 in this stripe (see Remark 3.2), the quantities $d_a(\mathbf{o}, p^m \cdot \mathbf{o}) - 2\ln|m|$ remain also bounded for these values of m, uniformly in a ([4], lemme 4). Similarly, for any $a \geq 0$ the geodesic segments $[\mathbf{o}, h^n \cdot \mathbf{o}]$ lie in the region of constant curvature of the metric g_a so that $d_a(\mathbf{o}, h^n \cdot \mathbf{o}) = d_{\mathbb{H}^N}(\mathbf{o}, h^n \cdot \mathbf{o})$. So, by (15), for $\varepsilon \in]0, \varepsilon_0]$, one gets

$$\liminf_{a \to +\infty} \sum_{|m| \leq m_a} e^{-(\frac{1}{2}+\varepsilon) d_a(\mathbf{o}, p^m \cdot \mathbf{o})} \sum_{n \in \mathbb{Z}^*} e^{-(\frac{1}{2}+\varepsilon) d_a(\mathbf{o}, h^n \cdot \mathbf{o})} > 1.$$

There thus exists $a_0 > 0$ such that, for $a \geq a_0$ one gets $\sum_{g \in G} e^{-(\frac{1}{2}+\varepsilon) d_a(\mathbf{o}, g \cdot \mathbf{o})} = +\infty$. This last inequality implies that $\delta_G > \frac{1}{2}$ when $a \geq a_0$; by ([4], Proposition 1), the group G is thus divergent since $\delta_{\langle p \rangle} = \frac{1}{2}$.□

We now want to check that there exists some $a \in]0, a_0[$ such that the group G is divergent with $\delta_G = \frac{1}{2}$ when X is endowed with the metric g_a; to prove this, we need to compare the Poincaré series $\mathbf{P}_G(s)$ with the potential of some Ruelle operator $\mathcal{L}_{a,s}$ associated with G that we introduce in the following paragraph.

We fix from now on a Schottky group $G = \langle p, h \rangle$ satisfying the conclusions of Proposition 4.1 and subsets $\mathcal{U}_p$ and $\mathcal{U}_h$ in $X \cup \partial X$ satisfying conditions $(1), (2), (3)$ and (4) above.

4.2. **Spectral radius of the Ruelle operator and Poincaré exponent.** We introduce the family $(\mathcal{L}_{a,s})_{(a,s)}$ of *Ruelle operators* associated with $G = \langle p, h \rangle$ defined formally by: for any $a \in [0, a_0], s \geq 0, x \in \partial X$ and any bounded Borel function $\phi : \partial X \to \mathbb{R}$

$$\mathcal{L}_{a,s}\phi(x) = \sum_{\gamma \in \{p,h\}} \sum_{n \in \mathbb{Z}^*} 1_{x \notin U_\gamma} e^{-s\mathcal{B}_x^{(a)}(\gamma^{-n} \cdot \mathbf{o}, \mathbf{o})} \phi(\gamma^n \cdot x). \tag{16}$$

The sequence $(p^n \cdot \mathbf{o})_n$ accumulates at ξ_p. So, for any $x \in \mathcal{U}_h$ the angle at $\mathbf{o}$ of the triangle $\mathbf{x}\,\widehat{\mathbf{o}\; p^n} \cdot \mathbf{o}$ is greater than $\theta/2$ for n large enough and the sequence $(\mathcal{B}_x^{(a)}(p^{-n} \cdot \mathbf{o}, \mathbf{o}) - d_a(\mathbf{o}, p^n \cdot \mathbf{o}))_n$ is bounded uniformly in $x \in \mathcal{U}_h$ and $a \geq 0$. Similarly, the sequence $(\mathcal{B}_x^{(a)}(h^{-n} \cdot \mathbf{o}, \mathbf{o}) - d_a(\mathbf{o}, h^n \cdot \mathbf{o}))_n$ is bounded uniformly in $x \in \mathcal{U}_p$ and $a \geq 0$. It readily implies that $\mathcal{L}_{a,s}\phi(x)$ is finite when $s \geq \max(\delta_{\langle p \rangle}, \delta_{\langle h \rangle}) = \frac{1}{2}$ and that it acts on the space $\mathbb{L}^\infty(\partial X)$ of bounded Borel functions on ∂X.

By a similar argument, for any $k \geq 1$, the quantities

$$\Big(\mathcal{B}_y^{(a)}((p^{m_1}h^{n_1} \cdots p^{m_k}h^{n_k})^{-1} \cdot \mathbf{o}, \mathbf{o}) - d_a(\mathbf{o}, p^{m_1}h^{n_1} \cdots p^{m_k}h^{n_k} \cdot \mathbf{o})\Big)_n$$

and

$$\Big(\mathcal{B}_x^{(a)}((h^{n_1}p^{m_1} \cdots h^{n_k}p^{m_k})^{-1} \cdot \mathbf{o}, \mathbf{o}) - d_a(\mathbf{o}, h^{n_1}p^{m_1} \cdots h^{n_k}p^{m_k} \cdot \mathbf{o})\Big)_n$$

are bounded uniformly in $m_1, n_1, \cdots, m_k, n_k \in \mathbb{Z}^*$ and $x \in \mathcal{U}_p, y \in \mathcal{U}_h$. One thus gets

$$\sum_{m_1, \cdots n_k \in \mathbb{Z}^*} \exp(d_a(\mathbf{o}, p^{m_1}h^{n_1} \cdots p^{m_k}h^{n_k} \cdot \mathbf{o})) \asymp |\mathcal{L}_{a,s}^{2k} 1|_\infty$$

which states that the series $\mathbf{P}_G(s)$ diverges (resp., converges) if and only if $\sum_{k\geq 1} |\mathcal{L}_{a,s}^{2k} 1|_\infty$ diverges (resp., converges). Now, the fact that $\mathcal{L}_{a,s}$ is a non negative operator implies that the limit $\lim_{k\to+\infty} \Big(|\mathcal{L}_{a,s}^{2k} 1|_\infty\Big)^{\frac{1}{2k}}$ is equal to the spectral radius $\rho_\infty(\mathcal{L}_{a,s})$ of $\mathcal{L}_{a,s}$ on $\mathbb{L}^\infty(\partial X)$. We have thus established the

Fact 4.2. *The Poincaré series $\mathbf{P}_G(s)$ diverges (resp., converges) if and only the potential $\sum_{k\geq 1} |\mathcal{L}_{a,s}^{2k} 1|_\infty$ diverges (resp., converges). In particular, if δ_a denotes the Poincaré exponent of G for the metric g_a, one gets*

$$\delta_a = \sup\Big\{ s\geq 0 : \rho_\infty(\mathcal{L}_{a,s}) \geq 1\Big\} = \inf\Big\{ s\geq 0 : \rho_\infty(\mathcal{L}_{a,s}) \leq 1\Big\}.$$

Consequently, since G satisfies the conclusions of Proposition 4.1, one gets

- the series $\sum_{k\geq 1} |\mathcal{L}_{0,1/2}^{2k} 1|_\infty$ converges;
- for a_0 large enough, the series $\sum_{k\geq 1} |\mathcal{L}_{a_0,1/2}^{2k} 1|_\infty$ diverges

which implies in particular $\rho_\infty(\mathcal{L}_{0,1/2}) \leq 1$ and $\rho_\infty(\mathcal{L}_{a_0,1/2}) \geq 1$.

We will prove that for some value $a_* \in]0, a_0[$ one gets $\rho_\infty(\mathcal{L}_{a_*,1/2}) = 1$; the uniqueness of a_* will be proved in the last section.

We first need to control the regularity of the function $a \mapsto \rho_\infty(\mathcal{L}_{a_0,1/2})$. It will be quite simple to check that the function $a \mapsto \mathcal{L}_{0,1/2}$ is continuous from $\mathbb{R}^+$ to the space of bounded operators on $\mathbb{L}^\infty(\partial X)$; unfortunately, the function $\mathcal{L} \mapsto \rho_\infty(\mathcal{L})$ is in general only lower semi-continuous. In the case of the family of Ruelle operators we consider here, this function will be in fact continuous, because of the very special form of the spectrum in this situation.

4.3. **On the spectrum of the Ruelle operators.** Throughout this section, we will use the following

Notation 4.3. *For any $a \in [0, a_0], x \in \partial X$ and $\gamma \in G$, we will set*

- $\mathcal{L}_a = \mathcal{L}_{a,1/2}$ *and* $\rho_\infty(a) = \rho_\infty(\mathcal{L}_{a,1/2})$.
- $b_a(\gamma, x) = \mathcal{B}_x^{(a)}(\gamma^{-1}\mathbf{o}, \mathbf{o})$

Furthermore, δ_a will denote the Poincaré exponent of G with respect to the metric g_a and, for $\gamma \in \{p, h\}$ and $n \in \mathbb{Z}^*$, the "weight" function $w_a(\gamma^n, .)$ is defined by

$$\begin{aligned} w_a(\gamma^n, .) \quad : \quad & \partial X \to \mathbb{R}^+ \\ & x \mapsto 1_{x\notin U_\gamma} e^{-\delta_a b_a(\gamma^n, x)} \end{aligned} \tag{17}$$

With these notations, the Ruelle operator $\mathcal{L}_a$ introduced in the previous paragraph may be expressed as follows: for any $\phi \in \mathbb{L}^\infty(\partial X)$ and any $x \in \partial X$,

$$\mathcal{L}_a\phi(x) = \sum_{\gamma\in\{p,h\}} \sum_{n\in\mathbb{Z}^*} w_a(\gamma^n, x)\phi(\gamma^n \cdot x). \tag{18}$$

The iterates of $\mathcal{L}_a$ are given by

$$\mathcal{L}_a^k\phi(x) = \sum_{\gamma\in G(k)} w_a(\gamma, x)\phi(\gamma \cdot x) \tag{19}$$

where $G(k)$ is the set of $\gamma = \alpha_1^{n_1}\cdots\alpha_k^{n_k} \in G$ of length k (with $\alpha_{i+1} \neq \alpha_i^{\pm 1}$) and $w_a(\gamma, x) = 1_{x\notin U_{\alpha_k}} e^{-\delta_G b_a(\gamma,x)}$ when γ has last letter α_k; observe that we have the following "multiplicative cocycle property":

$$w_a(\gamma, x) = \prod_{i=1}^{k} w_a(\alpha_i^{n_i}, \alpha_{i+1}^{n_{i+1}}\cdots\alpha_k^{n_k}\cdot x). \tag{20}$$

We will see that the $\mathcal{L}_a, a \geq 0$, act on the space $C(\partial X)$ of real valued continuous functions on ∂X and that the map $a \mapsto \mathcal{L}_a$ is continuous. Nevertheless, the function $a \mapsto \rho_\infty(a)$ is only lower semi-continuous in general and may present discontinuities. The main idea to avoid this difficulty is to introduce a Banach space on which the $\mathcal{L}_a$ act quasi-compactly.

In the sequel, we will consider the restriction of the $\mathcal{L}_a$ to some subspace of $C(\partial X)$ of Hölder continuous functions.

Notation 4.4. *We denote $\mathbb{L}_{a,\omega}(\partial X)$ the space of Hölder continuous functions on ∂X defined by*

$$\mathbb{L}_{a,\omega}(\partial X) := \{\phi \in C(\partial X)/|\phi|_{a,\omega} = |\phi|_\infty + [\phi]_{a,\omega} < +\infty\}$$

where $[\phi]_{a,\omega} = \sup_{\gamma\in\{p,h\}} \sup_{\substack{x,y\in U_\gamma \\ x\neq y}} \dfrac{|\phi(x)-\phi(y)|}{D_a(x,y)^\omega}$ *denotes the ω-Hölder coefficient of ϕ with respect to the distance D_a.*

When $a = 0$ we will omit the index D_0 and set $\mathbb{L}_\omega(\partial X) := \mathbb{L}_{0,\omega}(\partial X)$.

The spaces $(\mathbb{L}_{a,\omega}(\partial X), |.|)$ are $\mathbb{C}$-Banach space, furthermore for any $a \geq 0$, the identity map from $(\mathbb{L}_{a,\omega}(\partial X), |.|_{a,\omega})$ to $(C(\partial X), |.|_\infty)$ is compact.

We now want to prove that each operator $\mathcal{L}_a$ acts on $\mathbb{L}_{a,\omega}(\partial X)$; in fact, we need a stronger result, i.e that each $\mathcal{L}_a$, for $0 \leq a \leq a_0$, acts on $\mathbb{L}_\omega(\partial X)$. It will be a direct consequence of the following:

Lemma 4.5. *There exists $\omega_0 \in]0,1[$ such that for all $\omega \in]0,\omega_0]$, $\gamma \in \{p,h\}$, $a \in [1,a_0]$ and $n \in \mathbb{Z}^*$, the function $w_a(\gamma^n, .)$ belongs to $\mathbb{L}_\omega(\partial X)$; furthermore, the sequence $\left(e^{\delta_G d_a(\mathbf{o},\gamma^n\cdot\mathbf{o})}|w_a(\gamma^n,.)|_\omega\right)_n$ is bounded.*

Proof. The cluster points of the sequence $(\gamma^n\cdot\mathbf{o})_n$ belong to U_γ. Since the curvature is pinched, the quantity $b_a(\gamma^n, x)) - d(\mathbf{o}, \gamma^n\cdot\mathbf{o})$ is bounded uniformly in $n \in \mathbb{Z}^*, x \in \partial X \setminus U_\gamma$ and $a \in [0, a_0]$; so is the sequence $\left(e^{\delta_G d_a(\mathbf{o},\gamma^n\cdot\mathbf{o})}|w_a(\gamma^n,.)|_\infty\right)_n$. In order to control the ω-Hölder coefficient of $w_a(\gamma^n, .)$, we use Fact 3.7: the functions $x \mapsto \mathcal{B}_x^{(a)}(\mathbf{o}, \gamma^{-n}\cdot\mathbf{o})$ are equi-Lipschitz continuous on $\partial X \setminus U_\alpha$ with respect to D_a, since once again the cluster points of the sequence $(\gamma^n\cdot\mathbf{o})_n$ belong to U_γ. More precisely, the sequence $\left(e^{\delta_G d_a(\mathbf{o},\gamma^n\cdot\mathbf{o})}|w_a(\gamma^n,.)|_\omega\right)_n$ is bounded for any $a \in [0,a_0]$ and for ω given by inequality (13) .□

From now on and for once, we fix $\omega_0 \in]0,1[$ satisfying the conclusion of the above Lemma. We know that, for $a \in [0,a_0]$, the operator $\mathcal{L}_a$ acts on $\mathbb{L}_\omega(\partial X)$ whenever $\omega \in]0,\omega_0]$; let $\rho_\omega(a)$ denote the spectral radius of $\mathcal{L}_a$ on $\mathbb{L}_\omega(\partial X)$. We have the

Proposition 4.6. *For any $\omega \in]0,\omega_0]$ and $a \in [1,a_0]$, one gets*

- $\rho_\omega(a) = \rho_\infty(a)$
- *$\rho_\omega(a)$ is a simple eigenvalue of the operator $\mathcal{L}_a$ acting on $\mathbb{L}_\omega(\partial X)$ and the associated eigenfunction is non negative on ∂X.*

Furthermore, the operator $\mathcal{L}_a$ is quasi-compact on $\mathbb{L}_\omega(\partial X)$: there exists $r < 1$ such that the essential spectral radius of $\mathcal{L}_a$ on $\mathbb{L}_\omega(\partial X)$ is less than $r\rho_\omega(a)$.

In particular the eigenvalue $\rho_\omega(a)$ is isolated in the spectrum of $\mathcal{L}_a$, it is simple and the corresponding eigenfunction is non-negative.

Proof. Fix $x, y \in \partial X \cap U_p$; one gets

$$|\mathcal{L}_a^k\phi(x) - \mathcal{L}_a^k\phi(y)| \leq \sum_{\gamma\in G(k)} w_a(\gamma, x)|\phi(\gamma\cdot x) - \phi(\gamma\cdot y)| + \sum_{\gamma\in G(k)} |w_a(\gamma, x) - w_a(\gamma, y)| \times |\phi|_\infty. \tag{21}$$

Note that in these sums, it is sufficient to consider the $\gamma \in G(k)$ with last letter $\alpha_k \neq p$. For such γ the quantity $b_a(\gamma, x)$ is greater than $d_a(\mathbf{o}, \gamma\cdot\mathbf{o}) - C$ for some constant C which depends only on the angle θ_0 and the bounds on the curvature; in particular $b_a(\gamma, x) \geq 1$ for all but finitely many $\gamma \in G$ with last letter $\neq p$. It readily follows that $\liminf_{\substack{\gamma\in G(k)\\ k\to+\infty}} \frac{b_a(\gamma, x)}{k} > 0$, uniformly in $x \in U_p$. In other words, there exists $0 < r < 1$ and $C > 0$ such that

$$|\gamma'(x)|_a \leq Cr^k$$

for any $k \geq 1, x \in U_p$ and $\gamma \in G(k)$ with last letter $\neq p$. The same argument works when $x, y \in \partial X \cap U_h$.

We thus obtain the inequality

$$[\mathcal{L}_a^k\phi]_\omega \leq r_k[\phi]_\omega + R_k|\phi|_\infty$$

with $r_k = CK_\omega^2 r^k |\mathcal{L}_a^k|_\infty$ and $R_k = \sum_{\gamma\in G(k)}[w_a(\gamma, .)]_\omega$.

Note that $\mathcal{L}_a$ is a non-negative operator, so that the quantity $\limsup_k |\mathcal{L}_a^k 1|_\infty^{1/k}$ is equal to the spectral radius $\rho_\infty(a)$ of $\mathcal{L}_a$ on $C(\Lambda)$. Using a version due to H. Hennion of the Ionescu-Tulcea-Marinescu's theorem concerning quasi-compact operators, one may conclude that the essential spectral radius of $\mathcal{L}_a$ on $\mathbb{L}_\omega(\partial X)$ is less than $r\rho_\infty(a)$; in other words, the spectral values of $\mathcal{L}_a$ with modulus $\geq r\rho_\infty(a)$ are isolated eigenvalues with finite multiplicity in the spectrum of $\mathcal{L}_a$. This implies in particular that $\rho_\omega(a) = \rho_\infty(a)$. The inequality $\rho_\omega(a) \geq \rho_\infty(a)$ is obvious since the function 1 belongs to $\mathbb{L}_\omega(\partial X)$. Furthermore, the strict inequality would imply the existence of a function $\phi \in \mathcal{L}_a$ such that $\mathcal{L}_a\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$ of modulus $> \rho_\infty(a)$; this would give $|\lambda||\phi| \leq \mathcal{L}_a|\phi|$ so that $|\lambda| \leq \rho_\infty(a)$, which is a contradiction.

It remains to control the value $\rho_\omega(a)$ in the spectrum of $\mathcal{L}_a$. The operator $\mathcal{L}_a$ being non negative and compact on $\mathbb{L}_\omega(\partial X)$, its spectral radius $\rho_\omega(a)$ is an eigenvalue with associated eigenfunction $\phi_a \geq 0$.

Assume that ϕ_a vanishes at $x_0 \in \partial X$ and let $g \in \{p, h\}$ such that $x_0 \in U_g$; the equality $\mathcal{L}_a\phi_a(x_0) = \rho_\omega(a)\phi_a(x_0)$ implies that $\phi_a(\gamma\cdot x_0) = 0$ for any $\gamma \in G$ with last letter $\neq g$. By minimality of the action of G on ∂X one thus has $\phi_a = 0$ on ∂X. Consequently, the function ϕ_a is non negative.

Let us now check that $\rho_\omega(a)$ is a simple eigenvalue of $\mathcal{L}_a$. Consider the operator P defined formaly by $P(f) = \frac{1}{\rho_\omega(a)\phi_a}\mathcal{L}_a(f\phi_a)$; this operator is well defined on $\mathbb{L}_\omega(\partial X)$ since ϕ_a does not vanish, it is non negative, quasi-compact with spectral radius 1 and Markovian (that is to say $P1 = 1$). If $f \in \mathbb{L}_\omega(\partial X)$ satisfies the equality $Pf = f$ one considers a point $x_0 \in \partial X$ such that $|f(x_0)| = |f|_\infty$ and $g \in \{p, h\}$ such that $x_0 \in U_g$. An argument of convexity applied to the inequality $P|f| \leq |f|$ readily implies $|f(x_0)| = |f(\gamma \cdot x_0)|$ for any $\gamma \in G$ with last letter $\neq g$; by minimality of the action of G on ∂X it follows that the modulus of f is constant on ∂X. Applying again an argument of convexity and the minimality of the action of G on its limit set, one proves that f is in fact constant on ∂X; it follows that $\mathbb{C}\phi_a$ is the eigenspace associated with $\rho_\omega(a)$ on $\mathbb{L}_\omega(\partial X)$.□

4.4. **Regularity of the function $a \mapsto \mathcal{L}_a$.** In this section we will establish the following

Proposition 4.7. *For any $0 < \omega < \omega_0$, the function $a \mapsto \mathcal{L}_a$ is continuous from $[1, a_0]$ to the space of continuous linear operators on $(\mathbb{L}_\omega(\partial X), |.|_\omega)$.*

Proof. It suffices to check that, for $\gamma \in \{h, p\}, a, a' \in [1, a_0]$ and $0 < \omega < \omega_0$ one has

$$\lim_{a' \to a} \sup_{n \in \mathbb{Z}} e^{\delta_G d_0(\mathbf{o}, \gamma^n \cdot \mathbf{o})} |w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)|_\omega = 0.$$

First one gets

$$\begin{aligned} |w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)| &= e^{-\delta_G b_a(\gamma^n, .)} |e^{-\delta_G (b_{a'}(\gamma^n, .) - b_a(\gamma^n, .))} - 1| \\ &\leq C e^{-\delta_G d_a(\gamma^n \mathbf{o}, \mathbf{o})} |e^{-\delta_G (b_{a'}(\gamma^n, .) - b_a(\gamma^n, .))} - 1| \end{aligned} \tag{22}$$

where the constant C depends only one the bounds on the curvature.

Since the axis of h lies in the region of X where the curvature is -1, the quantity $d_a(\mathbf{o}, h^n \cdot \mathbf{o}) - |n| l_h$, where l_h denotes the hyperbolic length of the closed geodesic associated with h, is bounded uniformly in $a \in [0, a_0]$ and $n \in \mathbb{Z}^*$; the same holds for the quantity $d_a(p^n \cdot \mathbf{o}, \mathbf{o}) - d_0(p^n \cdot \mathbf{o}, \mathbf{o})$. Consequently

$$|w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)| \leq C' e^{-\delta_G d_0(\gamma^n \mathbf{o}, \mathbf{o})} |e^{-\delta_G (b_{a'}(\gamma^n, .) - b_a(\gamma^n, .))} - 1|$$

and we have to study the regularity of the function $a \mapsto b_a(\gamma^n, x)$, for any point $x \notin U_\gamma$. By inequalities (12), one gets

$$(y|z)_{a'} \to (y|z)_a \quad \text{as} \quad a' \to a,$$

when $(y|z)_a$ remains bounded. There are thus two cases to consider:

- We first consider the case $\gamma = p$. For any $n \in \mathbb{Z}^*$ let y_n be the point in ∂X such that $\mathbf{o}$ belongs to the geodesic ray $[p^n \cdot \mathbf{o}, y_n)$ (for the metric g_a); this ray is in fact a quasi-geodesic for any $a' \in [0, a_0]$, so the point $\mathbf{o}$ belongs to some bounded neigbourhood of the geodesic ray (for $g_{a'}$) from $p^n \cdot \mathbf{o}$ to x_n (note that $\inf_{n \in \mathbb{Z}^*} D_0(y_n, \xi_p) > 0$ by convexity of the horospheres). For any $n \in \mathbb{Z}^*$ and $a' \in [0, a_0]$ one gets

$$b_{a'}(p^n, x) = (p^n \cdot x | y_n)_{a'} - (x | p^{-n} \cdot y_n)_{a'} - b_{a'}(p^n, p^{-n} \cdot y_n).$$

 Since $p^n \cdot x \to \xi_p$ as $|n| \to +\infty$ and $\inf_{n \in \mathbb{Z}^*} D_0(y_n, \xi_p) > 0$, one gets

$$(p^n \cdot x | y_n)_{a'} \to (p^n \cdot x | y_n)_a$$

 as $a' \to a$, uniformly in $n \in \mathbb{Z}^*$ and $x \notin \mathcal{U}_p$. Similarly, since $\inf_{n \in \mathbb{Z}^*} D_0(y_n, \xi_p) > 0$, the sequence $(p^{-n} \cdot y_n)_n$ converges to ξ_p as $|n| \to +\infty$ so that $(x | p^{-n} \cdot y_n)_{a'} \to (x | p^{-n} \cdot y_n)_a$ uniformly in $n \in \mathbb{Z}^*$ and $x \notin \mathcal{U}_p$. Finally, one has $b_{a'}(p^n, p^{-n} \cdot x_n) = \mathcal{B}^{(a')}_{x_n}(\mathbf{o}, p^n \cdot \mathbf{o}) = d_{a'}(\mathbf{o}, p^n \cdot \mathbf{o})$; the geodesic segment $[\mathbf{o}, p^n \cdot \mathbf{o}]$ is included in the horosphere $\mathcal{H}$, so that

$$b_{a'}(p^n, p^{-n} \cdot x_n) \to b_a(p^n, p^{-n} \cdot x_n)$$

as $a' \to a$, uniformly in $n \in \mathbb{N}^*$.

- Consider now the case when $\gamma = h$; for any $n \geq 1$, one gets

$$b_a(h^n, x) = (h^n \cdot x | h^n \cdot \xi_h^+)_a - (x|\xi_h^+)_a - b_a(h^n, \xi_h^+)$$

with $b_a(h^n, \xi_h^+) = nl_h$. The facts that $x \notin U_h$ and $\xi_h^+ \in U_h$ readily implies $(x|\xi_h^+)_{a'} \to (x|\xi_h^+)_a$ as $a' \to a$. On the other hand $h^n \cdot x \to x_+$ as $n \to +\infty$ so that $(h^n \cdot x | h^n \cdot \xi_h^+)_a \to (x_+|\xi_h^+)_a$; since $\xi_h^+ \neq x_+$, the Gromov product $(x_+|\xi_h^+)_a$ is equal to $-\log d_a(\mathbf{o}, (x_+ \xi_h^+))$ up to a bounded term and the sequence $((h^n \cdot x | h^n \cdot \xi_h^+)_a)_{n \geq 1}$ is bounded uniformly in $a \in [0, a_0], x \notin U_h$ and $n \in \mathbb{N}$. It readily follows that $(h^n \cdot x | h^n \cdot \xi_h^+)_{a'} \to (h^n \cdot x | h^n \cdot \xi_h^+)_a$ as $a' \to a$, for any $n \geq 1$. A similar argument holds for $n \leq -1$. Finally, $b_{a'}(h^n, x) - b_a(h^n, x) \to 0$ uniformly in $n \geq 0$ and $x \notin U_h$ and the lemma is proved for $\gamma = h$.

Finally one has proved that for $\gamma \in \{h, p\}$ and $a, a' \in [0, a_0]$ one has

$$\lim_{a' \to a} \sup_{n \in \mathbb{Z}} e^{\delta_G d_0(\mathbf{o}, \gamma^n \cdot \mathbf{o})} |w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)|_\infty = 0.$$

To conclude the proof of the proposition, we use the classical fact that if a bounded sequence $(f_n)_n$ in $\mathbb{L}_{\omega_0}(\partial X)$ converges uniformly to some (continuous) function f, then the convergences remains valid in $\mathbb{L}_\omega(\partial X)$ for any $0 < \omega < \omega_0$: namely, we may fix $\varepsilon > 0$ and note that, for $0 < \omega \leq \omega_0$, the following inequality holds

$$[w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)]_\omega \leq \frac{2|w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)|_\infty}{\varepsilon^\omega} + [w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)]_{\omega_0} \varepsilon^{\omega_0 - \omega}$$

which immediately gives

$$|w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)|_\omega \leq \big(\frac{2}{\varepsilon^\omega} + 1\big) |w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)|_\infty + |w_{a'}(\gamma^n, .) - w_a(\gamma^n, .)|_{\omega_0} \varepsilon^{\omega_0 - \omega}.$$

One concludes letting $a' \to a$ and $\varepsilon \to 0$.□

4.5. **Proof of the main theorem.** We can now conclude the proof of the main theorem. We fix $\omega \in]0, \omega_0[$.

Since the spectral radius $\rho_\omega(a)$ of the operator $\mathcal{L}_a$ acting on $\mathbb{L}_\omega$ is an eigenvalue and is isolated in the spectrum of $\mathcal{L}_a$, the function $a \mapsto \rho_\omega(a)$ has the same regularity than $a \mapsto \mathcal{L}_a$; it is thus continuous on $[1, a_0]$. Furthermore, for any $a \in [1, a_0]$, the eigenfunction ϕ_a associated with $\rho_\omega(a)$ is non negative on ∂X. So one has $\phi_a \asymp 1$, which readily implies that $|\mathcal{L}_a^{2k} \phi_a|_\infty \asymp |\mathcal{L}_a^{2k} 1|_\infty$ uniformly in $k \geq 1$. By the equality $\mathcal{L}_a \phi_a = \rho_\omega(a) \phi_a$, it follows that $\rho_\omega(a) = \rho_\infty(a)$.

By the choice of the metrics g_a, we have $\rho_\infty(0) \leq 1$ and $\rho_\infty(a_0) \geq 1$; so there exists $a_* \in]0, a_0[$ such that $\rho_\omega(a_*) = \rho_\infty(a_*) = 1$.

On the other hand, the function $s \mapsto \rho_\omega(\mathcal{L}_{a_*, s})$ is strictly decreasing on $\mathbb{R}^+$. Fix $s > \delta_{\langle p \rangle}$; one has $\rho_\omega(a_*) < 1$ and the series $P_G(s)$ thus converges when X is endowed with the metric g_{a^*}. This proves that for the value a_* of the parameter a the critical exponent of G is less than $\delta_{\langle p \rangle}$; since $p \in G$, one has in fact $\delta_G = \delta_{\langle p \rangle}$.

Finally, since $\phi_{a_*} \asymp 1$, one has $\sum_{k \geq 1} |\mathcal{L}_{a_*, \delta_{\langle p \rangle}}^{2k} 1|_\infty \asymp \sum_{k \geq 1} |\mathcal{L}_{a_*, \delta_{\langle p \rangle}}^{2k} \phi_{a_*}|_\infty$; these two series diverge in fact because of the equality $\mathcal{L}_{a_*, \delta_{\langle p \rangle}} \phi_{a_*} = \phi_{a_*}$. By the Fact 4.2, it follows that for the value a_* of the parameter a, the series $P_G(\delta_G)$ diverges.

By criteria (1), one easily sees that m_Γ is finite when $\alpha > 2$ and infinite when $\alpha \in]1, 2]$.

This achieves the proof of the main theorem. □

4.6. **Complement.** A natural question concerns the uniqueness of the value a_* of the parameter a such that the spectral radius $\rho_\infty(a)$ of $\mathcal{L}_a$ is equal to 1; this uniqueness is not necessary to prove the main Theorem but nevertheless it is of interest to describe for instance the behavior of the orbital function of G when a varies, which is the subject of a forthcoming work.

By the continuity of the function $a \mapsto \rho_\infty(a)$, the uniqueness of a_* is a direct consequence of the strict monotonicity of this function. We thus have to prove that $\rho(\mathcal{L}_a) < \rho(\mathcal{L}_{a'})$ for any a, a' in $[0, a_0]$ such that $a < a'$. Note first that, for any fixed $\mathbf{x} \in X$ one gets

$$\rho(\mathcal{L}_a) = \rho_\infty(\mathcal{L}_a) = \lim_{k\to+\infty}\Big(\Big\| \sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}} e^{-\frac{1}{2}\mathcal{B}^{(a)}_{\cdot}(\gamma^{-1}\cdot\mathbf{x},\mathbf{x})}\Big\|_\infty\Big)^{\frac{1}{2k}} = \lim_{k\to+\infty}\Big(\sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}} e^{-\frac{1}{2}d_a(\mathbf{x},\gamma\cdot\mathbf{x})}\Big)^{\frac{1}{2k}}$$

and we have thus to check that there exists $C > 0$ and $\rho := \rho(a, a') < 1$ such that, for any $n \geq 1$, one gets

$$\sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}} e^{-\frac{1}{2}d_a(\mathbf{x},\gamma\cdot\mathbf{x})} \leq C\rho^k \sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}} e^{-\frac{1}{2}d_{a'}(\mathbf{x},\gamma\cdot\mathbf{x})}. \tag{23}$$

For any $x \in \partial X$ and $\mathbf{y}$, we will denote by $\mathcal{H}^{(a)}_x(\mathbf{y})$ the horoball (with respect to the metric g_a) centered at x and passing through $\mathbf{y}$; furthermore, for any $\mathbf{x} \in X$ we denote by $\psi_{x,\mathbf{y}}(\mathbf{x})$ its projection (with respect to g_a) on the horosphere $\partial\mathcal{H}^{(a)}_x(\mathbf{y})$.

In order to simplify the argument, one first assume that the two following conditions hold

- $(\mathbf{C}_1)$ for any $x \in \mathcal{U}_p \cap \partial X$ the points $h^n \cdot \mathbf{o}, n \in \mathbb{Z}^*$, lie outside the horoball $\mathcal{H}^{(a)}_x(\mathbf{o})$.
- $(\mathbf{C}_2)$ for any $x \in \mathcal{U}_h \cap \partial X$ the points $p^m \cdot \mathbf{o}, m \in \mathbb{Z}^*$, lie outside the horoball $\mathcal{H}^{(a)}_x(\mathbf{o})$.

Fix $k \geq 1$ and $\gamma \in \Gamma_{2k}$ with last letter in h. Let us decompose γ into $a_{2k}a_{2k-1}\cdots a_1$ with $a_{2i} = p^{m_i}$ and $a_{2i-1} = h^{n_i}$ for $1 \leq i \leq k$; set $\gamma_0 := Id$ and $\gamma_j := a_j \cdots a_1$ for $1 \leq j \leq 2k$. We fix $x \in \mathcal{U}_p \cap \partial X$; by the ping-pong dynamic, there exists $c > 0$ independent of γ such that the distances $d_a(\mathbf{o}, \psi_{x,\mathbf{o}}(\gamma^{-1}\cdot\mathbf{o}))$ and $d_{a'}(\mathbf{o}, \psi_{x,\mathbf{o}}(\gamma^{-1}\cdot\mathbf{o}))$ are both $\leq c$.

The cocycle property of the Busemann function thus leads to the following

$$\begin{aligned} d_a(\mathbf{o}, \gamma\cdot\mathbf{o}) &\geq d_a(\mathbf{o}, \psi_{x,\mathbf{o}}(\gamma^{-1}\cdot\mathbf{o})) - c \\ &= \mathcal{B}^{(a)}_x(\gamma^{-1}\cdot\mathbf{o}, \mathbf{o}) - c \\ &= \sum_{j=0}^{2k-1} \mathcal{B}^{(a)}_x(\gamma_{j+1}^{-1}\cdot\mathbf{o}, \gamma_j^{-1}\cdot\mathbf{o}) - c \\ &= \sum_{j=0}^{2k-1} \mathcal{B}^{(a)}_{\gamma_j\cdot x}(a_{j+1}^{-1}\cdot\mathbf{o}, \mathbf{o}) - c, \end{aligned} \tag{24}$$

and one may thus write, as in (19)

$$\sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}} e^{-\frac{1}{2}d_a(\mathbf{x},\gamma\cdot\mathbf{x})} \leq e^{\frac{c}{2}}\mathcal{L}_a^{2k}1(x). \tag{25}$$

By the previous assumption, all the quantities $\mathcal{B}_x^{(a)}(\gamma_{j+1}^{-1}\cdot\mathbf{o},\gamma_j^{-1}\cdot\mathbf{o})$ above are non negative and we want to compare them with a similar quantity involving $g_{a'}$. For any $x\in\partial X$ and $\mathbf{x},\mathbf{y}\in X$, the quantity $\mathcal{B}_x(\mathbf{x},\mathbf{y})$ is equal to the "signed" length (for g_a) of $[\mathbf{x},\psi_{x,\mathbf{y}}(\mathbf{x})]_a$, the geodesic segment (for g_a) joining $\mathbf{x}$ and $\psi_{x,\mathbf{y}}(\mathbf{x})$; in other words, with obvious notations, one gets

$$\mathcal{B}_x^{(a)}(\mathbf{x},\mathbf{y})=\int_{[\mathbf{x},\psi_{x,\mathbf{y}}(\mathbf{x})]_a}dg_a$$

where the integral is non negative when $\mathbf{x}$ is outside $\mathcal{H}_x^{(a)}(\mathbf{y})$ and negative when it lies inside. Similarly, we introduce the quantity $\beta_x(\mathbf{x},\mathbf{y})$ defined by

$$\beta_x(\mathbf{x},\mathbf{y})=\beta_x^{(a,a')}(\mathbf{x},\mathbf{y}):=\int_{[\mathbf{x},\psi_{x,\mathbf{y}}(\mathbf{x})]_a}dg'_a.$$

Note that for any $\mathbf{x},\mathbf{y},\mathbf{z}$ in X and $\gamma\in\Gamma$ one gets $\beta_x(\mathbf{x},\mathbf{y})+\beta_x(\mathbf{y},\mathbf{z})=\beta_x(\mathbf{x},\mathbf{z})$ and $\beta_x(\mathbf{x},\mathbf{y})=\beta_{\gamma\cdot x}(\gamma\cdot\mathbf{x},\gamma\cdot\mathbf{y})$.

Since $d_{a'}(\mathbf{o},\psi_{x,\mathbf{o}}(\gamma^{-1}\cdot\mathbf{o}))$ is $\le c$, we may write, as above

$$\begin{aligned}
d_{a'}(\mathbf{o},\gamma\cdot\mathbf{o}) &\le d_{a'}(\mathbf{o},\psi_{x,\mathbf{o}}(\gamma^{-1}\cdot\mathbf{o}))+c\\
&\le \beta_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})+c\\
&= \sum_{j=0}^{2k-1}\beta_x(\gamma_{j+1}^{-1}\cdot\mathbf{o},\gamma_j^{-1}\cdot\mathbf{o})+c \qquad (26)\\
&= \sum_{j=0}^{2k-1}\beta_{\gamma_j\cdot x}(a_{j+1}^{-1}\cdot\mathbf{o},\mathbf{o})+c.
\end{aligned}$$

which leads to the following inequality

$$\sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}}e^{-\frac{1}{2}d_{a'}(\mathbf{x},\gamma\cdot\mathbf{x})}\ge e^{-\frac{c}{2}}\mathcal{K}^{2k}1(x), \qquad (27)$$

where $\mathcal{K}\phi(y):=\sum_{\gamma\in\{p,h\}}\sum_{n\in\mathbb{Z}^*}1_{x\notin U_\gamma}e^{-\frac{1}{2}\beta_y(\gamma^{-n}\cdot\mathbf{o},\mathbf{o})}\phi(\gamma^n\cdot y)$ for any function $\phi\in\mathbb{L}^\infty(\partial X)$ and any $y\in\partial X$. To prove (23) it is thus sufficient to compare the spectral radius of $\mathcal{L}_a$ and $\mathcal{K}$; we will use the following

Fact 4.8. *For any $y\in\partial X$ and $\mathbf{x},\mathbf{y}\in X$ one gets*

$$\Big|\beta_y(\mathbf{x},\mathbf{y})\Big|\le\Big|\mathcal{B}_y^{(a)}(\mathbf{y},\mathbf{y})\Big|.$$

Furthermore, for any $n\in\mathbb{Z}^*$, there exists $\eta(n)\ge 0$, with $\eta(n)>0$ when $|n|$ is large enough, such that

$$\forall y\in\mathcal{U}_h\quad 0\le\beta_y(p^n\cdot\mathbf{o},\mathbf{o})\le\mathcal{B}_y^{(a)}(p^n\cdot\mathbf{o},\mathbf{o})-\eta(n).$$

Proof. The first inequality is a direct consequence of the Remark 3.5, namely $g_{a'}\le g_a$. To prove the second one, we note that for any $y\in\mathcal{U}_h$ and any $n\in\mathbb{Z}$ with $|n|$ large enough, the geodesic segment $[p^n\cdot\mathbf{o},\psi_{x,\mathbf{o}}(p^n\cdot\mathbf{o})]_a$ inters sufficiently into the horoball $\mathcal{H}$ centered at ξ_p and in particular in the region where g_a and $g_{a'}$ differ (ie $g_{a'}>g_a$); consequently $\beta_y(p^n\cdot\mathbf{o},\mathbf{o})-\mathcal{B}_y^{(a)}(p^n\cdot\mathbf{o},\mathbf{o})>0$. the existence of $\eta(n)>0$ follows by an argument of continuity with respect to y.□

By this Fact, if $y \in \mathcal{U}_p$, one gets

$$\mathcal{L}_a 1(y) = \sum_{n\in\mathbb{Z}^*} e^{-\frac{1}{2}\mathcal{B}^{(a)}_y(h^{-n}\cdot\mathbf{o},\mathbf{o})} \leq \sum_{n\in\mathbb{Z}^*} e^{-\frac{1}{2}\beta_y(h^{-n}\cdot\mathbf{o},\mathbf{o})} = \mathcal{K}1(y).$$

Assume now $y \in \mathcal{U}_h$ and fix $n_0 \geq 1$ such that $\eta(n_0) > 0$. By Property 3.4, one gets $0 \leq \beta_y(p^{-n_0}\cdot\mathbf{o},\mathbf{o}) \leq K_0 d_0(p^{-n_0}\cdot\mathbf{o},\mathbf{o})$ where K_0 is the constant which appears in (11); consequently

$$e^{-\frac{1}{2}\beta_y(p^{-n_0}\cdot\mathbf{o},\mathbf{o})} \geq \delta_0 := e^{-\frac{K_0}{2}d_0(p^{-n_0}\cdot\mathbf{o},\mathbf{o})}.$$

On the other hand, by the above

$$\sum_{n\in\mathbb{Z}^*} e^{-\frac{1}{2}\beta_y(p^{-n}\cdot\mathbf{o},\mathbf{o})} \leq \sum_{n\in\mathbb{Z}^*} e^{-\frac{1}{2}(d_{a'}(p^{-n}\cdot\mathbf{o},\mathbf{o})-c)} \leq \Delta_0 := \sum_{n\in\mathbb{Z}^*} e^{-\frac{1}{2K_0}(d_0(p^{-n}\cdot\mathbf{o},\mathbf{o})-c)}.$$

It follows

$$\begin{aligned}\mathcal{L}_a 1(y) &= e^{-\frac{1}{2}\mathcal{B}^{(a)}_y(p^{-n_0}\cdot\mathbf{o},\mathbf{o})} + \sum_{\substack{n\in\mathbb{Z}^*\\ n\neq n_0}} e^{-\frac{1}{2}\mathcal{B}^{(a)}_y(p^{-n}\cdot\mathbf{o},\mathbf{o})}\\ &\leq e^{-\frac{\eta(n_0)}{2}} \times e^{-\frac{1}{2}\beta_y(p^{-n_0}\cdot\mathbf{o},\mathbf{o})} + \sum_{\substack{n\in\mathbb{Z}^*\\ n\neq n_0}} e^{-\frac{1}{2}\beta_y(p^{-n}\cdot\mathbf{o},\mathbf{o})}\\ &\leq \rho \sum_{n\in\mathbb{Z}^*} e^{-\frac{1}{2}\beta_y(p^{-n}\cdot\mathbf{o},\mathbf{o})} = \rho\mathcal{K}1(y),\end{aligned} \tag{28}$$

with $\rho := 1 - \left(1 - e^{-\frac{\eta(n_0)}{2}}\right)\frac{\delta_0}{\Delta_0} \in]0,1[$.

Combining the two inequalities $\mathcal{L}_a 1(y) \leq \mathcal{K}1(y)$ for $y \in \mathcal{U}_p$ and $\mathcal{L}_a 1(y) \leq \rho\mathcal{K}1(y)$ for $y \in \mathcal{U}_h$, one obtains by iteration

$$\forall k \geq 1 \quad \mathcal{L}_a^{2k}1(.) \leq \rho^k \mathcal{K}^{2k}1(.)$$

We put together this inequality with (25) and (27) and obtain finally

$$\sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}} e^{-\frac{1}{2}d_a(\mathbf{x},\gamma\cdot\mathbf{x})} \leq e^c\rho^k \sum_{\substack{\gamma\in\Gamma_{2k}\\ l(\gamma)=h}} e^{-\frac{1}{2}d'_a(\mathbf{x},\gamma\cdot\mathbf{x})}.$$

This gives the expected inequality (23), in the case when conditions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$ hold.

When one or both of these conditions do not hold, one replaces the family $\{h^n : n \in \mathbb{Z}^*\}$ (resp. $\{p^n : n \in \mathbb{Z}^*\}$) by the countable set $H := \{g \in \Gamma^{2N+1}/l(g) = h\}$ (resp. $P := \{g \in \Gamma^{2N+1}/l(g) = p\}$), where N is choosen large enough such that

- for any $x \in \mathcal{U}_p \cap \partial X$, the points $g\cdot\mathbf{o}, g \in H$, lie outside the horoball $\mathcal{H}^{(a)}_x(\mathbf{o})$.
- for any $x \in \mathcal{U}_h \cap \partial X$, the points $g\cdot\mathbf{o}, g \in P$, lie outside the horoball $\mathcal{H}^{(a)}_x(\mathbf{o})$.

Any γ in $\Gamma_{2k(2N+1)}$ with last letter h may be decomposed into $\gamma = a_{2k}\cdots a_1$ with $a_{2i} \in P$ and $a_{2i-1} \in H$ for $1 \leq i \leq k$; the same argument as above, with obvious modifications, leads to the inequality

$$\sum_{\substack{\gamma\in\Gamma_{2k(2N+1)}\\ l(\gamma)=h}} e^{-\frac{1}{2}d_a(\mathbf{x},\gamma\cdot\mathbf{x})} \leq e^c\rho^k \sum_{\substack{\gamma\in\Gamma_{2k(2N+1)}\\ l(\gamma)=h}} e^{-\frac{1}{2}d'_a(\mathbf{x},\gamma\cdot\mathbf{x})},$$

and (23) follows again.

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