Stochastic dynamical systems in Ha Long Bay

from 10^{th} to 15^{th} July 2017

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These notes have been prepared and written with **Sara BROFFERIO**[(3)] during the first semester 2017.

The content has been taught by the two authors from 10th to 15th of July during a workshop on Stochastic Dynamical Systems in Vietnam (http://viasm.edu.vn/en/hdkh/sds2017)

Abstract. Consider a proper metric space $\mathbb{X}$ and a sequence $(F_n)_{n\geq 1}$ of i.i.d. random continuous mappings from $\mathbb{X}$ to $\mathbb{X}$. It induces the stochastic process $X_n^x = F_n \circ \cdots \circ F_1(x)$ starting at $x \in \mathbb{X}$. This process $(X_n^x)_{n\geq 0}$ is a **stochastic dynamical system (SDS)**. The courses focuses on the existence and uniqueness of invariant (finite or infinite) measures, as well as recurrence and ergodicity of this process.

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Throughout this course, we consider a proper metric space $(\mathbb{X}, d)$, a sequence $(F_n)_{n\geq 1}$ of independent and identically distributed (i.i.d.) random variables (r.v.) defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ and with values in the space $C(\mathbb{X})$ of continuous mappings from $\mathbb{X}$ to $\mathbb{X}$. We are interested in the asymptotic behavior of the sequences $L_n := F_n \circ \cdots \circ F_1$ ("left product") and $R_n := F_1 \circ \cdots \circ F_n$ ("right product") and their action on $\mathbb{X}$. We set $L_0 = R_0 = \mathrm{Id}$.

1 Iteration function systems

1.1 Definitions and notations

We consider the sequence $(X_n)_{n\geq 0}$ of $\mathbb{X}$-valued random variables defined by

$$\forall n \geq 0 \quad X_{n+1} = F_{n+1}(X_n)$$

where X_0 is a fixed random variable on $\mathbb{X}$. Then $X_n = L_n(X_0)$; when $X_0 = x$ with x fixed in $\mathbb{X}$, we denote $X_n = X_n^x$. The process $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{X}$ whose transition probability $P(x, B)$ is defined by: for any $x \in \mathbb{X}$ and any bounded Borel function $\phi : \mathbb{X} \to \mathbb{R}$,

$$P\phi(x) = \mathbb{E}(\ \phi(g_1 \cdot x)) = \int \ \phi(g \cdot x)\mu(\mathrm{d}g).$$

Such a sequence $(F_n)_{n\geq 1}$ is called an *iterated function system* (IFS) or *stochastic dynamical system* (SDS).

The behavior of the sequences $(L_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ strongly differ from one another: indeed,

1. the image of R_{n+1} is included in the one of R_n; thus, one may expect that, under reasonable conditions, for any $x \in \mathbb{X}$, the sequence $(R_n(x))_{n\geq 1}$ converges almost surely (a.s.) as $n \to +\infty$. This is formalized for instance by the existence of the so-called *Furstenberg's martingale* which converges $\mathbb{P}$-a.s.(see Theorem 3.2).

2. the image of L_n is included in the one of F_n, which changes at random at any step; that's why we study the recurrence/transience properties of the sequence $(L_n \cdot x)_{n\geq 0}$.

1.2 On the canonical probability space and invariant measures

From now on, we consider an IFS $(F_n)_{n\geq 1}$, we denote by $(X_n)_{n\geq 0}$ the Markov chain on $\mathbb{X}$ corresponding to the left products $L_n, n \geq 1$, and we introduce the "canonical probability space" associated with this chain:

$$\Big(\Omega = \mathbb{X}^{\otimes \mathbb{N}}, \mathcal{B}\Big(\mathbb{X}^{\otimes \mathbb{N}}\Big), (X_n)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0}\ \theta, \Big(\mathbb{P}_x\Big)_{x\in\mathbb{X}}\Big),$$

where

- $\mathcal{F}_n = \sigma(X_0, \ldots, X_n) = \sigma(X_0, F_1, \ldots, F_n)$ for any $n \geq 0$;
- θ is the shift operator on $\mathbb{X}^{\otimes \mathbb{N}}$ defined by $\theta((x_i)_{i\geq 0}) = (x_{i+1})_{i\geq 0}$ for any $(x_i)_{i\geq 0} \in \mathbb{X}^{\mathbb{N}}$;
- for any $x \in \mathbb{X}$, the probability $\mathbb{P}_x$ on Ω is rized by the property:

$$\mathbb{P}_x(A_0 \times \ldots \times B_k) = \mathbf{1}_{A_0}(x) \int_{A_1} P(x, \mathrm{d}x_1) \int_{A_2} P(x_1, \mathrm{d}x_2) \ldots, \int_{A_k} P(x_{k-1}, \mathrm{d}x_k)$$

for any $k \geq 0$ and any Borel sets $B_0, \ldots, B_k$ in $\mathbb{X}$.

For any Radon measure m on $\mathbb{X}$, let us denote $\mathbb{P}_m$ the Radon measure on $\mathbb{X}^{\otimes \mathbb{N}}$ defined by

$$\forall B \in \mathcal{B}\Big(\mathbb{X}^{\otimes \mathbb{N}}\Big) \quad \mathbb{P}_m(B) := \int_{\mathbb{X}} \mathbb{P}_x(B) m(\mathrm{d}x).$$

This measure is also the image of $m \otimes \mu^{\otimes \mathbb{N}}$ by the map $(x, g_1, g_2, \ldots) \mapsto (x, x_1 := g_1 \cdot x, x_2 := g_2 g_1 \cdot x, \ldots)$. When m is a probability measure, the same holds for $\mathbb{P}_m$; otherwise, $\mathbb{P}_m$ may be infinite.

For any Borel set $B \subset \mathbb{X}$ such that $m(B) > 0$, let us denote m_B be the probability measure on B induced by m and defined by $m_B(\cdot) = \dfrac{m(\cdot \cap B)}{m(B)}$.

The measure m is said to be *invariant* for the chain $(X_n)_{n\geq 0}$ if it satisfies the equality $mP = m$, i.e.[4] for any bounded Borel function $\phi : \mathbb{X} \to \mathbb{R}$

$$\int_{\mathbb{X}\times\mathbb{X}} \phi(y) m(\mathrm{d}x) P(x, \mathrm{d}y) = \int_{\mathbb{X}} \phi(y) m(\mathrm{d}y).$$

The measure m is P-invariant if and only if, for any bounded Borel function $\phi : \mathbb{X} \to \mathbb{R}$

$$m(\phi) = \int_{\mathbb{X}} \int_{C(\mathbb{X})} \phi(F \cdot x) \mu(\mathrm{d}F) m(\mathrm{d}x).$$

We also say that m is *μ-invariant* and write $m = \mu \star m$.

When m is P-invariant, the Radon measure $\mathbb{P}_m$ is θ-invariant on $\mathbb{X}^{\otimes \mathbb{N}}$. There arises an important question, that is to decide if $\mathbb{P}_m$ is ergodic, i.e. whether or not invariant Borel subsets A of $\mathbb{X}^{\otimes \mathbb{N}}$ satisfy $\mathbb{P}_m(A) = 0$ or $\mathbb{P}_m(\mathbb{X} \setminus A) = 0$. Let us recall that a Borel set $A \subset \mathbb{X}^{\otimes \mathbb{N}}$ is said to be *θ-invariant* (or simply *invariant*) if and only if $\theta(A) = A$; the collection of invariant Borel sets is a σ-algebra, called the *invariant σ-algebra* of $\mathbb{X}^{\otimes \mathbb{N}}$.

[4] i.e. means "that is to say"

1.3 Proximality and local proximality

Throughout this course, we assume that the functions which govern the transitions satisfy some "contraction" properties; we introduce the following

Definition 1.1. *A sequence $(f_n)_{n\geq 1}$ of continuous functions on $\mathbb{X}$ is said to be*

- **proximal** *on $\mathbb{X}$ if, for any $x, y \in \mathbb{X}$*

$$\lim_{n\to+\infty} d(f_n \cdot x, f_n \cdot y)) = 0.$$

- **locally proximal** *on $\mathbb{X}$ if, for any $x, y \in \mathbb{X}$ and any compact set $K \subset \mathbb{X}$,*

$$\lim_{n\to+\infty} d(f_n \cdot x, f_n \cdot y)) \ \mathbf{1}_K(f_n \cdot x) = 0.$$

Obviously,

(1) "local proximality" is a weaker property than "proximality";

(2) if $\lim_{n\to+\infty} f_n(x) = +\infty$, then the sequence $(f_n)_{n\geq 0}$ is locally proximal.

Examples. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is a strict contraction (i.e. $\|f(x) - f(y)\| \leq r\|x - y\|$ for some $0 \leq r < 1$ and any $x, y \in \mathbb{R}^d$), then the sequence $(f^n)_{n\geq 0}$ of iterates of f is proximal on $\mathbb{R}^d$.

1.4 On the attractor of a SDS. Conservativity and transience

Property 1.2. *If the sequence $(F_n)_{n\geq 1}$ is $\mathbb{P}$-almost surely locally proximal on $\mathbb{X}$ then*

(i) *either $\mathbb{P}(d(X_n^x, x) \to +\infty) = 0$ for any $x \in \mathbb{X}$,*

(ii) *or $\mathbb{P}(d(X_n^x, x) \to +\infty) = 1$ for any $x \in \mathbb{X}$.*

In the case (i)*, one says that $(X_n)_{n\geq 0}$ is* **conservative** *and in the case* (ii) *one says that $(X_n)_{n\geq 0}$ is* **transient***.*

Proof. We just present the principle and refer to [26] for the details. Let $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) > 0$ and such that the sequence $(X_n^x(\omega)_{n\geq 0}$ accumulates at a point $x_\omega \in \mathbb{X}$ for all $\omega \in \Omega_0$. For any $m \geq 1$ and $n \geq m + 1$, the equality $X_n^x(\omega) = F_n(\omega) \dots F_{m+1}(\omega)(X_m^x(\omega)$ implies that the sequence $(F_n(\omega) \cdots F_{m+1}(\omega)(X_m^x(\omega))_{n\geq m+1}$ accumulates also at x_ω. By local proximality, for $\mathbb{P}$- almost all $\omega \in \Omega_0$, so does the sequence $\Big(F_n(\omega) \cdots F_{m+1}(\omega)(y)\Big)_{n\geq m+1}$. Thus, the cluster point x_ω does not depend on $F_1, \cdots, F_m$. By Kolmogorov's 0-1 law, the set $\{\omega \mid (X_n^x(\omega))_{n\geq 0}$ accumulates in $\mathbb{X}\}$ has measure 0 or 1. □

Property 1.3. *If the sequence $(F_n)_{n\geq 1}$ is $\mathbb{P}$-almost surely locally proximal on $\mathbb{X}$ and conservative then there exists a set $L \subset \mathbb{X}$ such that*

$$\mathbb{P}\Big(\{\omega \in \Omega \mid L^x(\omega) = L \ for \ all \ x \in \mathbb{X}\}\Big) = 1$$

where $L^x(\omega)$ is the set of cluster points of the sequence $(X_n^x(\omega))_{n\geq 0}, \omega \in \Omega$. The set L is called the **attractor** *or the* **limite set** *of the IFS.*

Proof. We use the same argument as in Property 1.2. Let $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) > 0$ and $L^x(\omega) \neq \emptyset$ for any $\omega \in \Omega_0$. Fix $\omega \in \Omega_0$ and assume that sequence $(X_n^x(\omega)_{n\geq 0}$ accumulates at a point $x_\omega \in \mathbb{X}$. By the above, the value of x_ω does not depend on x neither on $F_1(\omega), \cdots, F_m(\omega)$ for any $m \geq 1$. Thus, the random set $L^x(\cdot)$ does not depend on x and is measurable with respect to the asymptotic σ-algebra associated with the sequence $(F_n)_{n\geq 1}$. It is thus constant $\mathbb{P}$-a.s. □

Consequently, the chain $(X_n^x)_{n\geq 0}$ is (topologically) recurrent on L i.e. every open set U such that $U \cap L \neq \emptyset$ is visited infinitely often with probability 1.

Property 1.4. *If the sequence $(F_n)_{n\geq 1}$ is $\mathbb{P}$-almost surely locally proximal on $\mathbb{X}$ and conservative then it possesses an invariant measure m and the support of m equals L.*

The existence of m follows from M. Lin's theorem (see section 9.3); the fact that it support equals L is straightforward.

2 Examples

2.1 The affine recursion

Let the state space $\mathbb{X}$ be $\mathbb{R}$ and the functions $F_n, n \geq 1$, be of the form $x \mapsto a_n x + b_n$, where a_n and b_n are random variables with values in $\mathbb{R}^*+$ and $\mathbb{R}$ respectively; the process $(L_n)_{n\geq 0}$ (resp. $(R_n)_{n\geq 0}$) is the "left" (resp. "right") *random walk* of the affine group of the real line.

For any $n \geq 0$, we set

$$X_{n+1} = a_{n+1} X_n + b_{n+1},$$

where X_0 is a fixed r.v. on $\mathbb{R}$. A direct computation yields for any $n \geq 1$,

$$R_n = F_1 \circ \cdots \circ F_n = (a_1, b_1) \ldots (a_n, b_n) = \Big(a_1 \ldots a_n, \sum_{k=1}^{n} a_1 \ldots a_{k-1} b_k \Big)$$

and

$$L_n = F_n \circ \cdots \circ F_1 = (a_n, b_n) \ldots (a_1, b_1) = \Big(a_n \ldots a_1, \sum_{k=1}^{n} b_k a_{k+1} \ldots a_n \Big).$$

2.2 Product of random matrices

We present briefly some elements on products of random matrices; we refer to [3], [7], [13], [17] and [23] for details and references therein. We consider the group $Gl(d,\mathbb{R})$ of invertible $d \times d$ matrices, $d \geq 1$, and denote $\mathbb{P}^1(\mathbb{R}^d)$ is the projective space corresponding to $\mathbb{R}^d$. The elements of $\mathbb{P}^1(\mathbb{R}^d)$ are denoted $\bar{x}$, where $x \in \mathbb{R}^d, x \neq 0$ and $\bar{x} = \{\lambda x \mid \lambda \neq 0\}$.

The group $Gl(d,\mathbb{R})$ acts on $\mathbb{X} = \mathbb{P}^1(\mathbb{R}^d)$ as follows; for any $g \in Gl(d,\mathbb{R})$ and $\bar{x} \in \mathbb{P}^1(\mathbb{R}^d)$

$$g \cdot \bar{x} = \overline{gx}.$$

Let us recall that any $g \in Gl(d,\mathbb{R})$ admit the following "polar decomposition" (or "Cartan decomposition") $g = kak'$ where k, k' are orthogonal matrices and a a diagonal one. We assume the diagonal of a is $(a(1), \ldots, a(d))$ with $a(1) \geq a(2) \geq \ldots \geq a(d)$; with this convention, the factor a is unique.

We consider a sequence $(g_n)_{n\geq 1}$ of i.i.d. random variables with values in $Gl(d,\mathbb{R})$ and defined on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$. Let μ be their common distribution; the closed semi-group generated by the support S_μ of μ is denoted T_μ.

We introduce two general conditions:

(I) "Irreducibility" : *there exists non finite union of proper subspaces of* $\mathbb{R}^d$ *which are invariant under the action of all elements of* T_μ

(P) "Proximality" *There exists a sequence* $(\xi_n)_{n\geq 1}$ *in* T_μ *and* $\bar{x}_0 \in \mathbb{P}^1(\mathbb{R}^d)$ *such that*

$$\forall \bar{x} \in \mathbb{P}^1(\mathbb{R}^d) \qquad \lim_{n\to+\infty} \xi_n \cdot \bar{x} = \bar{x}_0.$$ [5]

Under hypotheses (I) and (P), for any $\bar{x} \in \mathbb{P}^1(\mathbb{R}^d)$, the sequence $(R_n \cdot \bar{x})_{n\geq 0}$ converges $\mathbb{P}$-a.s. towards a random variable X_∞ with values in $\mathbb{P}^1(\mathbb{R}^d)$. Furthermore, the distribution of X_∞ is the unique probability measure on $\mathbb{P}^1(\mathbb{R}^d)$ invariant for the chain $(L_n \cdot X_0)_{n\geq 0}$ and we obtain

$$\lim_{n\to+\infty} \frac{1}{n} \mathbb{E}(\log |L_n|) = \int_{\mathbb{P}^1(\mathbb{R}^d)} \int_{Gl(d,\mathbb{R})} \log |gx| \mu(\mathrm{d}g) \nu(\mathrm{d}\bar{x}) := \gamma_\mu \quad \mathbb{P}-\text{a.s.}$$

The limit γ_μ is called the "Lyapunov exponent" of μ.

[5] in other words, if $\xi_n = k_n a_n k'_n$ is the Cartan decomposition of ξ_n, then $\lim_{n\to+\infty} \frac{a_n(2)}{a_n(1)} = 0$.

2.3 Transfer operators

Let us consider the following simple example. Let T be the map from $[0,1]$ to $[0,1]$ defined by

$$T(x) = 2x(\text{mod}1).$$

The Lebesgue measure $\mathrm{d}x$ on $[0,1]$ is T-invariant but is far to be the only one such probability measure: indeed, there exist many invariant measures, in particular the one supported by periodic orbits.

A probability measure m is said to *ergodic* if and only if the T-invariant Borel sets have 0 or 1 measure; equivalently, for any bounded Borel function $\varphi : \mathbb{X} \to \mathbb{R}$, it holds, $m(\mathrm{d}x)$-a.s.

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(T^k(x)) = \int_{\mathbb{X}} \varphi(y)m(\mathrm{d}y).$$

Ergodicity may be also stated as follows: for all $\varphi, \psi \in \mathbb{L}^2[0,1]$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \int_{\mathbb{X}} \varphi(T^k(x))\psi(x)m(\mathrm{d}x) = \int_0^1 \varphi(x)m(\mathrm{d}x) \times \int_0^1 \psi(x)m(\mathrm{d}x).$$

This last characterization of ergodicity yields naturally to the stronger notion of the mixing property: indeed, one says that T is *mixing* with respect to the measure m if

$$\lim_{n\to+\infty} \int_0^1 \varphi(T^n(x))\psi(x)m(\mathrm{d}x) = \int_0^1 \varphi(x)\mathrm{d}x \times \int_0^1 \psi(x)m(\mathrm{d}x).$$

In order to study the ergodic properties of the map T (ergodicity, mixing, ...), it is natural to consider the adjoint P of T with respect to the Lebesgue measure; the equality

$$\forall \varphi, \psi \in \mathbb{L}^2[0,1] \qquad \int_0^1 \varphi(T(x))\psi(x)\mathrm{d}x = \int_0^1 \varphi(x)P\psi(x)\mathrm{d}x \tag{2.1}$$

yields to the following definition of P:

$$\begin{aligned} P\psi(x) &:= \frac{1}{2}\psi\left(\frac{x}{2}\right) + \frac{1}{2}\psi\left(\frac{x+1}{2}\right) \\ &= \int \psi(f(x))\mu(\mathrm{d}f), \end{aligned}$$

where μ is the probability measure $\mu(\mathrm{d}f) = \frac{1}{2}(\delta_H + \delta_A)$ on $C[0,1]$ with $H(x) = \frac{x}{2}$ and $A(x) = \frac{x+1}{2}$.

Notice that (2.1) readily implies: for any $n \geq 1$

$$\forall \varphi, \psi \in \mathbb{L}^2[0,1] \qquad \int_0^1 \varphi(T^n(x))\psi(x)\mathrm{d}x = \int_0^1 \varphi(x)P^n\psi(x)\mathrm{d}x. \tag{2.2}$$

The operator P is a Markov operator which acts on $C[0,1]$; the associated Markov chain $(X_n)_{n\geq 0}$ corresponds with an iterated function system with law μ.

The operator P has the Feller property on $[0,1]$, i.e. it acts on the space of continuous functions on $[0,1]$. We prove in the next sections that the Lebesgue measure on $[0,1]$ is the unique invariant probability measure for P and that, for any continuous function $\psi : [0,1] \to \mathbb{C}$ and $x \in [0,1]$, it holds

$$\lim_{n\to+\infty} P^n\psi(x) = \int_0^1 \psi(x)m(\mathrm{d}x).$$

The mixing property of T follows immediately from (2.2).

The operator P acts also on the space $\mathrm{Lip}[0,1]$ of Lipschitz functions on $[0,1]$ defined by

$$\mathrm{Lip}[0,1] := \{\psi : [0,1] \to [0,1] \mid \|\psi\| := |\psi|_\infty + m(f\psi) < +\infty\}$$

where $m(\psi) := \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|\psi(x)-\psi(y)|}{|x-y|}$; indeed, for any $\psi \in \mathrm{Lip}[0,1]$, it holds

$$\|P\psi\| \leq \frac{1}{2}\|\psi\| + |\psi|_\infty.$$

By [18], it implies that P satisfies the so-called "spectral gap property"; in particular, for any Lipschitz function ψ, the sequence $(P^n\psi(x))_{n\geq 0}$ converges exponentially fast towards $\int_0^1 \psi(x)\mathrm{d}x$. We refer to [18] and [25] for details and references therein.

2.4 The reflected random walk on $\mathbb{R}^+$ with absorption at 0

The reflected random walk on $\mathbb{R}^+$ with absorption at 0 is defined by : for any $n \geq 0$,

$$X_{n+1} = \max(X_n - Y_{n+1}, 0)$$

where X_0 is a fixed random variable on $\mathbb{R}$ and $(Y_n)_{n\geq 1}$ a sequence of i.i.d. real valued random variables whose distribution is denoted μ. When $X_0 = x$, we set $X_n = X_n^x$. The sequence $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{R}^+$, with probability transition $P(x,.)$ defined by: for any $x \in \mathbb{R}^+$ and any bounded Borel function $\phi : \mathbb{R}^+ \to \mathbb{R}$,

$$P\phi(x) = \mathbb{E}(\ \phi(\max(x - Y_1, 0)) = \int_{\mathbb{R}} \phi(\max(x-y,0))\mu(\mathrm{d}y).$$

We obtain

$$X_n^x = f_{Y_n} \circ f_{Y_{n-1}} \circ \cdots \circ f_{Y_1}(x),$$

where for any $a \in \mathbb{R}^+$, we denote by f_a the map on $\mathbb{R}$ defined by

$$\forall x \in \mathbb{R}^+ \quad f_a(x) := \max(x-a, 0). \tag{2.3}$$

Application. *We consider a queue and denote $\mathcal{A}_1, \mathcal{A}_2, \ldots$ the inter-arrival times between two successive customers; the arrival times of the customers are $0, \mathcal{A}_1, \mathcal{A}_1 + \mathcal{A}_2, \ldots$.*
We denote $\mathcal{S}_1, \mathcal{S}_2, \ldots$ the service time of the different customers.
We assume that $(\mathcal{A}_n)_{n\geq 0}$ and $(\mathcal{S}_n)_{n\geq 0}$ are two independent sequences of i.i.d random variables.
We set $W_0 = 0$, and, for any $n \geq 1$, we denote W_n the waiting time of the n^{th} customer in the queue.

If the n^{th} customer arrives at time t, he is served at time $t + W_n$ and leaves the queue at time $t + W_n + \mathcal{S}_n$; the customer $n+1$ arrives at time $t + \mathcal{A}_{n+1}$ and his waiting time in the queue W_{n+1} equals

- 0 *when* $\mathcal{A}_{n+1} \geq W_n + \mathcal{S}_n$
- $W_n + \mathcal{S}_n - \mathcal{A}_{n+1}$ *otherwise.*

In other words, setting $Y_{n+1} := \mathcal{A}_{n+1} - \mathcal{S}_n$, we may write

$$W_{n+1} := \max(W_n - Y_{n+1}, 0).$$

2.5 The reflected random walk on $\mathbb{R}^+$ with elastic collisions at 0

The reflected random walk $(X_n)_{n\geq 0}$ on $\mathbb{R}^+$ with elastic collisions at 0 is defined by: for any $n \geq 0$,

$$X_{n+1} = |X_n - Y_{n+1}|,$$

where X_0 is a fixed r.v. on $\mathbb{R}^+$ and $(Y_n)_{n\geq 1}$ a sequence of i.i.d. real valued random variables. When $X_0 = x$, we set $X_n = X_n^x$. The transition probability of $(X_n)_{n\geq 0}$ is given by: for any $x \in \mathbb{R}^+$ and any bounded Borel function $\phi : \mathbb{R}^+ \to \mathbb{R}$,

$$P\phi(x) = \mathbb{E}\Big(\ \phi(|x - Y_1|\Big) = \int_{\mathbb{R}} \phi(|x-y|)\mu(\mathrm{d}y).$$

Then $X_n^x = g_{Y_n} \circ g_{Y_{n-1}} \circ \cdots \circ g_{Y_1}(x)$, where, for any $a \in \mathbb{R}^+$, we denote by g_a the map on $\mathbb{R}$ defined by

$$g_a(x) := |x-a|. \tag{2.4}$$

2.6 The Diaconis-Friedman Markov chain

We describe here a Markov chain $(X_n)_{n\geq 0}$ on $[0,1]$ introduced by P. Diaconis and D. Friedman in [10]. If the chain is at $x \in [0,1]$ at time n, it selects at time $n+1$ one of the two intervals $[0,x]$ or $[x,1]$ with equal probability $\frac{1}{2}$ and moves to a uniformly chosen random point y in the selected interval. When $0<x<1$, the transition probability $P(x,\mathrm{d}y)$ of the chain $(X_n)_{n\geq 0}$ has a density $k(x,y)$ with respect to the Lebesgue measure on $]0,1[$ given by

$$k(x,y)=\frac{1}{2}\times\frac{1}{x}\mathbf{1}_{]0,x[}(y)+\frac{1}{2}\times\frac{1}{1-x}\mathbf{1}_{]x,1[}(y).$$

The probabilities $P(0,\mathrm{d}y)$ and $P(1,\mathrm{d}y)$ are respectively $\dfrac{1}{2}(\delta_0+\mathrm{d}x)$ and $\dfrac{1}{2}(\delta_1+\mathrm{d}x)$. It is shown in [10] that $(X_n)_{n\geq 0}$ possesses a unique invariant probability measure on $]0,1[$ whose density with respect to the Lebesgue measure is the famous arcsine density, defined by

$$\forall x\in]0,1[\quad f_{\frac{1}{2}}(x)=\frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{]0,1[}(x).$$

Similarly, the intervals $[0,x]$ and $[x,1]$ may be chosen with the respective probabilities $p\in]0,1[$ and $q=1-p$. In this case, the invariant probability measure is the Beta distribution $\mathcal{B}(q,p)$ of parameters q and p, given by its density

$$f_p(x)=\frac{1}{\Gamma(p)\Gamma(q)}x^{q-1}(1-x)^{p-1}\mathbf{1}_{]0,1[}(x).$$

The transition kernel P of $(X_n)_{n\geq 0}$ is defined by: for any Borel function $\phi:[0,1]\to\mathbb{C}$ and $x\in]0,1[$,

$$\begin{aligned} P\phi(x) &= \frac{p}{x}\int_0^x\phi(y)\mathrm{d}y+\frac{q}{1-x}\int_x^1\phi(y)\mathrm{d}y \\ &= p\int_0^1\phi(tx)\mathrm{d}t+q\int_0^1\phi(tx+1-t)\mathrm{d}t. \end{aligned} \tag{2.5}$$

This last expression is valid in fact for any $x\in[0,1]$.

Equality (2.5) is also of interest; it shows that the chain $(X_n)_{n\geq 0}$ fits into the framework of iterated random transformations. For any $0\leq t\leq 1$, we denote H_t the homothety $x\mapsto tx$ and A_t the affine transformation $x\mapsto tx+1-t$. We denote by μ the probability measure on $C[0,1]$ given by

$$\mu(\mathrm{d}T)=p\int_0^1\delta_{H_t}(\mathrm{d}T)\mathrm{d}t+q\int_0^1\delta_{A_t}(\mathrm{d}T)\mathrm{d}t,.$$

By (2.5), for any Borel function $\phi:[0,1]\to\mathbb{R}$,

$$P\phi(x)=\int_{C[0,1]}\phi(T\cdot x)\mu(\mathrm{d}T).$$

We refer to this chain as the Diaconis-Friedman's chain.

3 On the positive recurrence

3.1 The Furstenberg's principle

We claim that there is a strong interplay between the right products $(R_n)_{n\geq 0}$ and the left ones $(\L_n)_{n\geq 1}$. The following statement illustrates this fact.

Proposition 3.1. *Assume that there exists a random variable Z on $(\Omega,\mathcal{F},\mathbb{P})$ with values in $\mathbb{X}$ such that,* **for any** $x\in X$,

$$\lim_{n\to+\infty}R_n\cdot x=Z\qquad\mathbb{P}-a.s.$$

Then, the distribution ν of Z is the unique invariant probability measure for the process $(L_n \cdot x)_{n\geq 0}$ and, for any continuous function $\phi : X \to \mathbb{C}$ with compact support,

$$\lim_{n\to+\infty} \mathbb{E}(\phi(L_n \cdot x)) = \nu(\phi).$$

Proof. For any $x \in \mathbb{X}$, the sequence $(F_1 \circ \cdots \circ F_n \cdot x)_{n\geq 1}$ converges $\mathbb{P}$-a.s. to a random variable Z, with distribution ν. Let F_0 be a random map with the same distribution μ as F_1 and independent of the sequence $(F_n)_{n\geq 0}$. The function F_0 being continuous, the sequence $(F_0 \circ \cdots \circ F_n \cdot x)_{n\geq 0}$ converges $\mathbb{P}$-a.s. towards $Z_0 := F_0(Z)$. Since $(F_1 \circ \cdots \circ F_n \cdot x)_{n\geq 1}$ and $(F_0 \circ \cdots \circ F_n \cdot x)_{n\geq 0}$ have the same distribution, the same holds for the variables Z and Z_0. The variables F_0 and Z being independent, the distribution of $F_0(Z)$ equals $\mu \star \nu$. Eventually, $\mu \star \nu = \nu$, i.e. ν is μ-invariant.

If ν' is another μ-invariant probability measure, then for any continuous function ϕ with compact support and any $n \geq 1$,

$$\nu'(\phi) = \nu' \star \mu^{*n}(\phi) = \int_{\mathbb{X}} \mathbb{E}_x(\phi(L_n \cdot x))\nu'(\mathrm{d}x) = \int_{\mathbb{X}} \mathbb{E}_x(\phi(R_n \cdot x))\nu'(\mathrm{d}x).$$

Letting $n \to +\infty$, it yields $\nu'(\phi) = \int_{\mathbb{X}} \nu(\phi)\nu'(\mathrm{d}x)$. Then ν' is a finite measure and $\nu'(\phi) = \nu(\phi)$. Eventually, the measure ν is the unique μ-invariant probability measure on $\mathbb{X}$.

□

3.2 The Furstenberg's martingale

The following property, first stated by H. Furstenberg for the projective action of product of random matrices, is a very useful tool to study IFS and is a good illustration of the fact that the images of the R_n are nested within each other.

Theorem 3.2. *Assume that the Markov chain $(X_n)_{n\geq 0}$ admits an invariant probability measure ν on $\mathbb{X}$. Then for any bounded Borel function $\phi : \mathbb{X} \to \mathbb{R}$, the sequence $(R_n\nu(\phi))_{n\geq 0}$ is a bounded martingale on $\mathbb{R}$, thus it converges. Furthermore, when ϕ is uniformly continuous and bounded on $\mathbb{X}$,*

$$\lim_{n\to+\infty} R_n\nu(\phi) = \lim_{n\to+\infty} R_n\xi\nu(\phi). \tag{3.1}$$

for any ξ in the closed semi-group T_μ generated by the support of μ.

Proof. First, for any $n \geq 0$,

$$\begin{aligned}
\mathbb{E}(R_{n+1}\nu(\phi)/\mathcal{F}_n) &= \mathbb{E}(\nu(\phi \circ R_n \circ F_{n+1})/\mathcal{F}_n) \\
&= \int \nu(\phi \circ R_n \circ F)\mu(\mathrm{d}F) \\
&= \int \phi \circ R_n(F(x))\mu(\mathrm{d}F)\nu(\mathrm{d}x) \\
&= \int \phi \circ R_n(y)\mu \star \nu(\mathrm{d}y) \\
&= \int \phi \circ R_n(y)\nu(\mathrm{d}x) = R_n\nu(\phi)
\end{aligned}$$

since $\mu \star \nu = \nu$. This proves that $(R_n\nu(\phi))_{n\geq 0}$ is a bounded martingale on $\mathbb{R}$. Second, setting $M_n := R_n\nu(\phi)$, it holds for any $k \geq 0$,

$$\mathbb{E}(M_{n+k} - M_n)^2 = \mathbb{E}(M_{n+k}^2) - 2\mathbb{E}(M_{n+k}M_n) + \mathbb{E}(M_n^2) = \mathbb{E}(M_{n+k}^2) - \mathbb{E}(M_n^2)$$

which yields, for any $N \geq 1$,

$$\mathbb{E}\left(\sum_{n=0}^{N}(M_{n+k} - M_n)^2\right) = \sum_{l=0}^{k-1} \mathbb{E}(M_{N+l}^2) - \mathbb{E}(M_l^2) \leq 2k|\phi|_\infty.$$

Consequently

$$\mathbb{E}\left(\sum_{n=0}^{+\infty}(M_{n+k}-M_n)^2\right)=\mathbb{E}\left(\int_{\mathbb{X}}\int_{C(\mathbb{X})}\sum_{n=0}^{+\infty}(\phi(R_n\circ F(x))-\phi(R_n(x))^2\mu^{*k}(\mathrm{d}F)\nu(\mathrm{d}x)\right)<+\infty.$$

Hence, for ν-almost all $x\in\mathbb{X}$ and μ^{*k}-almost all $f\in C(\mathbb{X})$

$$\lim_{n\to+\infty}\phi(R_n\circ F(x))-\phi(R_n(x))=0\quad \mathbb{P}-\text{a.s.}$$

Hence, when ϕ is uniformly continuous on $\mathbb{X}$, there exists $\mathbb{X}_0\subset\mathbb{X}$, $\nu(\mathbb{X}_0)=1$, such that, for any $x\in\mathbb{X}_0$, any $k\geq 0$ and any F in the support of μ^{*k},

$$\lim_{n\to+\infty}\phi(R_n\circ F(x))-\phi(R_n(x))=0\quad \mathbb{P}-\text{a.s.}$$

□

This Theorem is used for instance for product of random matrices in order to apply Furstenberg' principle. Indeed, under condition (P), equality (3.1) yields, for any $\xi\in T_\mu$ and any contracting sequence $(\xi_k)_{k\geq 0}$ with limit point $\bar{x}_0$,

$$\lim_{n\to+\infty}R_n\nu(\phi)=\lim_{n\to+\infty}R_n\xi\circ\xi_k\nu(\phi)\overset{k\to+\infty}{\longrightarrow}=\lim_{n\to+\infty}\phi(R_n\circ\xi\cdot\bar{x}_0)\quad\mathbb{P}-a.s.,$$

which proves that $(R_n\circ\cdot x)_{n\geq 1}$ converges $\mathbb{P}$-a.s. towards some limit which does not depend on $x\in T_\mu\cdot x_0$. This allows to apply Furstenberg's principle to conclude.

□

This martingale property, and especially equality (3.1), is useful in many situations in order to apply Furstenberg's principle; it is the key argument for instance in the theory of product of random matrices ([3], [17], [7]).

3.3 A general criterion

We state and prove a general criterion which ensures that the Markov chain $(L_n\cdot x)_{n\geq 0}$ is positive recurrent.

Theorem 3.3. *Assume that*

1. *the sequence $(L_n)_{n\geq 0}$ is $\mathbb{P}$-a.s. locally proximal,*
2. *there exists on $\mathbb{X}$ a probability measure ν which is invariant for $(X_n)_{n\geq 0}$.*

Then, the measure ν is the unique invariant probability measure for $(X_n)_{n\geq 0}$ and this chain is ν- **topologically positive recurrent** *(or simpler positive recurrent), i.e for any open set $U\subset\mathbb{X}$ such that $\nu(U)>0$, the stopping time $\tau^U:=\inf\{n\geq 1\mid X_n\in U\}$ is finite $\mathbb{P}$-a.s. and*

$$\mathbb{E}_{\nu_U}(\tau^U)=\frac{1}{\nu(U)}.$$

Proof of Theorem 3.3. Assume that there exists another probability measure ν' invariant for the chain $(X_n)_{n\geq 0}$. We consider a continuous "test function" $\phi:\mathbb{X}\to\mathbb{C}$ with compact support K. For such a function ϕ, we may write, using both the facts that ν and ν' are invariant and probability measures

$$\begin{aligned}
|\nu(\phi)-\nu'(\phi)| &= \Big|\int_{\mathbb{X}}\mathbb{E}(\phi(L_n\cdot x))\nu(\mathrm{d}x)-\int_{\mathbb{X}}\mathbb{E}(\phi(L_n\cdot y))\nu'(\mathrm{d}y)\Big|\\
&\leq \int_{\mathbb{X}}\mathbb{E}\Big(\Big|\phi(L_n\cdot x)-\phi(L_n\cdot y)\Big|\Big)\nu(\mathrm{d}x)\nu'(\mathrm{d}y)\\
&\leq \int_{\mathbb{X}}\mathbb{E}\Big(\Big|\phi(L_n\cdot x)-\phi(L_n\cdot y)\Big|\Big(\mathbf{1}_K(L_n\cdot x)+\mathbf{1}_K(L_n\cdot y)\Big)\Big)\nu(\mathrm{d}x)\nu'(\mathrm{d}y)\quad\overset{n\to+\infty}{\longrightarrow}.
\end{aligned}$$

Since the term inside the integral converges $\nu \times \nu \times \mathbb{P}-$ a.s. to 0, this last term tends to 0 as $n \to +\infty$, by the dominated convergence theorem. It yields $\nu(\phi) = \nu'(\phi)$.
The second assertion of theorem is mainly based on Kac's formula ([28], see Theorem 9.1). By this formula, it suffices to check that $\mathbb{P}_\nu$ is ergodic for the shift operator θ; indeed, for any Borel set $B \subset \mathbb{X}$ such that $\nu(B) > 0$, it holds $\mathbb{P}_\nu(X_0 \in B) = \nu(B) > 0$ and τ_B is also the first return time to the "cylinder" $(X_0 \in B)$ of the map θ on $\mathbb{X}^{\mathbb{N}}$.

For the ergodicity of $\mathbb{P}_\nu$, we must prove that for any $\phi \in \mathbb{L}^1(\mathbb{X}^{\otimes \mathbb{N}}, \mathbb{P}_\nu)$ and $\mathbb{P}_\nu$-almost all $\mathbf{x} \in \mathbb{X}^{\otimes \mathbb{N}}$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \phi(\theta^k(\mathbf{x})) = \int_{\mathbb{X}^{\otimes \mathbb{N}}} \phi(\mathbf{x})\mathbb{P}_\nu(d\mathbf{x}).$$

By an argument of density in $\mathbb{L}^1(\mathbb{X}^{\otimes \mathbb{N}}, \mathbb{P}_\nu)$, we may assume that ϕ depends only on finitely many coordinates, and more precisely, it is continuous with compact support in $\mathbb{X}^d$ for some $d \geq 1$. To simplify the notations, we assume that $d = 1$ [(6)].

Let $\mathcal{I}$ be the σ-algebra $\mathbb{X}^{\otimes \mathbb{N}}$ of θ-invariant sets. On the one hand, by Birkhoff's theorem, $\mathrm{d}\nu(x) \times \mathrm{d}\mu^{\otimes \mathbb{N}}(\omega)$-a.s., for any $l \geq 1$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(L_k(\omega)x) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=l+1}^{n-1} \phi(L_k(\omega)x) = \mathbb{E}_\nu[\phi(X_0)/\mathcal{I})(x,\omega).$$

On the other hand, by local proximality, for any $x, y \in \mathbb{X}$, it holds

$$d(L_k \cdot x, L_k \cdot y)\ \mathbf{1}_{(\phi(L_k\cdot x)>0)} \quad \overset{k\to+\infty}{\longrightarrow} \quad 0 \quad \mathbb{P}-\text{a.s.}$$

This implies that the former limit does not depend on x. Furthermore, writing $L_k = F_k \circ \cdots \circ F_{l+1} \circ L_l$ for any $k \geq l \geq 1$, it yields, $\mathbb{P}$-a.s.,

$$\mathbb{E}_\nu[\phi(X_0)/\mathcal{I})(\omega) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=l+1}^{n-1} \phi(F_k \dots F_{l+1}L_l(\omega)x) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=l+1}^{n-1} \phi(F_k \dots F_{l+1}(\omega)x).$$

Consequently, the random variable $\mathbb{E}_\nu(\phi(X_0)/\mathcal{I})(\cdot)$ does not depend on $F_1, \dots, F_l$, for any $l \geq 1$; it is thus measurable with respect to the "asymptotic" σ-algebra of $(F_n)_{n\geq 1}$. Kolmogorov' $0-1$ law readily implies that it is constant $\mathbb{P}$-a.s. This proves that, $\mathbb{P}_\nu$-a.s., the σ-algebra $\mathcal{I}$ equals $\{\emptyset, \Omega)$ and that $\mathbb{P}_\nu$ is ergodic.

□

4 On the null recurrence

Theorem 4.1. *Assume that*

1. *the sequence $(L_n)_{n\geq 1}$ is $\mathbb{P}$-a.s. locally proximal,*
2. *there exists a compact set $K_0 \subset \mathbb{X}$ such that*

$$\sum_{n=0}^{+\infty} \mathbb{P}(L_n \cdot x \in K_0) = +\infty. \tag{4.1}$$

*Then, there exists on $\mathbb{X}$ a unique invariant Radon measure m for $(X_n)_{n\geq 0}$; furthermore, when m is infinite, this chain is m-**topologically null recurrent** (or simpler m-null recurrent) that is to say, for any open set $U \subset \mathbb{X}$ such that $0 < m(U) < +\infty$, the stopping time $\tau^U := \inf\{n \geq 1 \mid X_n \in U\}$ is $\mathbb{P}$-a.s. finite and satisfies the equality*

$$\mathbb{E}_{m_U}(\tau^U) = +\infty.$$

[6]Indeed, when $d \geq 2$, we can lift each $F : \mathbb{X} \to \mathbb{X}$ to a continuous mapping $F^{(d)} : \mathbb{X}^d \to \mathbb{X}^d$ defined by

$$F^{(d)}(x_1, \dots, x_d) = (x_2, \dots, x_d, F(x_d)).$$

In this way, the random mappings F_n induce the SDS $F_n^{(d)} \circ \dots \circ F_1^{(d)}(x_1, \dots, , x_d)$ on $\mathbb{X}^d$; for $n \geq d-1$, this equals $(X_{n-d+1}^{x_d}, \dots, X_n^{x_d})$. We refer to [27] for the details.

Proof of Theorem 4.1. First, let us emphasize that (X_n) is a Feller chain, i.e. its transition probability acts on $C(\mathbb{X})$; the second condition of theorem implies that (X_n) is topologically conservative and the existence of on invariant Radon measure m is a direct consequence of Theorem 5.1 in [23] (see Appendix section 9.3). When m is finite, we apply Theorem 3.3 and conclude that m is unique, up to a multplicative constant, and $(X_n)_{n\geq 0}$ is positive recurrent.

Assume now that m is infinite. We can apply the ergodic theorem of Chacon Ornstein. Choosing an arbitrary positive function $g \in \mathbb{L}^1(\mathbb{X}^{\mathbb{N}}, \mathbb{P}_m)$ with

$$\mathbb{P}_m\left(\left\{\mathbf{x} \in \mathbb{X}^{\mathbb{N}} : \sum_{n=0}^{+\infty} g(\theta^n \mathbf{x}) < +\infty\right\}\right) = 0 \tag{4.2}$$

one has for any $f \in \mathbb{L}^1(\mathbb{X}^{\mathbb{N}}, \mathbb{P}_m)$

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} f(\theta^n \mathbf{x})}{\sum_{k=0}^{n-1} g(\theta^n \mathbf{x})} = \frac{\mathbb{E}_m(f/\mathcal{I})}{\mathbb{E}_m(g/\mathcal{I})} \quad \mathbb{P}_m - \text{a.s.} \tag{4.3}$$

In order to show ergodicity of θ, we need to show that the right hand side is just $\mathbb{E}_m(f)/\mathbb{E}_m(g)$. By an argument of density, it suffices to show this for non-negative functions that depend only on l coordinates for any $l \geq 1$; we detail only the case when $l = 1$.

From now on, we assume that f and g are non-negative, compactly supported, continuous functions on $\mathbb{X}$ that both are non zero; we assume that both functions f and g satisfy (4.2) (such functions do exist, by (4.1). We note $S_n f(x) := \sum_{k=0}^{n-1} f(L_k x)$ and $S_n g(x) := \sum_{k=0}^{n-1} g(L_k x)$. We must show that

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} f(\theta^n x)}{\sum_{k=0}^{n-1} g(\theta^n x)} = \frac{m(f)}{m(g)} \quad \mathbb{P}_m - \text{a.s.} \tag{4.4}$$

Step 1. First, let us check that the right term in (4.4) does not depend on x. Assume that the support of f and g is included in K and, for any $\delta >$, denote $K_\delta := \{y \in \mathbb{R} : d(y, K) \leq \delta\}$. Since f and g are uniformly continuous, we obtain by the local proximality: $\mathbb{P}$-a.s., for any $x, y \in \mathbb{X}$ and any $\epsilon > 0$, there exists a (random) number N such that, for $k \geq N$,

$$|f(L_k \cdot x) - f(L_k \cdot y)| \leq \epsilon \ \mathbf{1}_{K_\delta}(L_k \cdot y) \quad \text{and} \quad |g(L_k \cdot x) - g(L_k \cdot y)| \leq \epsilon \ \mathbf{1}_{K_\delta}(L_k \cdot y)$$

By (4.3), we may fix x in such a way $\left(\frac{S_n \ \mathbf{1}_{K_\delta}(x)}{S_n f(x)}\right)_{n\geq 0}$ converges $\mathbb{P}$-a.s. Hence

$$\limsup_{n\to+\infty} \Big| \frac{S_n f(x) - S_n f(y)}{S_n f(x)} \Big| \leq \epsilon \lim_{n\to+\infty} \frac{S_n \ \mathbf{1}_{K_\delta}(x)}{S_n f(x)} \quad \mathbb{P}_m - \text{a.s.}$$

Since this holds for any $\epsilon > 0$, it yields in fact $\mathbb{P}$-a.s., for any $x \in \mathbb{R}$, :

$$\lim_{n\to+\infty} \frac{S_n f(y)}{S_n f(x)} = 1 \quad \mathbb{P}_m - \text{a.s.}$$

The same applies to g in the place of f. Consequently,

$$\Big|\frac{S_n f(x)}{S_n g(x)} - \frac{S_n f(y)}{S_n g(y)}\Big| \leq \Big|\frac{S_n f(y)}{S_n g(y)}\Big|\Big|\frac{S_n f(x)}{S_n f(y)}\frac{S_n g(y)}{S_n g(x)} - 1\Big| \quad \overset{n\to+\infty}{\longrightarrow} \quad 0 \quad \mathbb{P}_m - \text{a.s.}$$

as $n \to +\infty$.

Eventually, there exists a random variable $Z_{f,g}$ and a set $\Omega_0 \subset \Omega$ of $\mathbb{P}_m$ measure 1 such that, for all $\omega \in \Omega_0$ and $x \in \mathbb{R}$,

$$\lim_{n\to+\infty} \frac{S_n f(x)}{S_n g(x)} = Z_{f,g}(\omega).$$

Step 2. We now prove that $Z_{f,g}$ is constant $\mathbb{P}$-a.s. By step 1, for any $\omega \in \Omega_0$, $\ell \geq 1$ and $x \in \mathbb{R}$,

$$\begin{aligned} Z_{f,g}(\omega) &= \lim_{n\to+\infty} \frac{S_n f(x)}{S_n g(x)} \\ &= \lim_{n\to+\infty} \frac{\sum_{k=\ell+1}^{n-1} f(L_k x)}{\sum_{k=\ell+1}^{n-1} g(L_k x)} \\ &= \lim_{n\to+\infty} \frac{\sum_{k=\ell+1}^{n-1} f(F_k \dots F_{\ell+1} L_\ell x)}{\sum_{k=\ell+1}^{n-1} g(F_k \dots F_{\ell+1} L_\ell x)} \\ &= \lim_{n\to+\infty} \frac{\sum_{k=\ell+1}^{n-1} f(F_k \dots F_{\ell+1} x)}{\sum_{k=\ell+1}^{n-1} g(F_k \dots F_{\ell+1} x)}. \end{aligned}$$

Thus, the random variable $Z_{f,g}$ is measurable with respect to the asymptotic σ-algebra associated with $(F_n)_{n\geq 1}$; it is thus $\mathbb{P}$-a.s. constant. It follows directly that $\mathbb{P}_m$ is ergodic for θ. The last statement of the theorem is a consequence of Kac's formula, applied to the dynamical system $(\mathbb{R}^{\otimes \mathbb{N}}, \mathcal{B}(\mathbb{R}^{\otimes \mathbb{N}}), \theta, \mathbb{P}_m)$.

□

5 The reflected random walk on $\mathbb{R}^+$ with absorption at 0

In this section, we study the Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{R}^+$ defined by: for any $n \geq 0$,

$$X_{n+1} = \max(X_n - Y_{n+1}, 0)$$

where X_0 is a fixed random variable on $\mathbb{R}$ and $(Y_n)_{n\geq 1}$ a sequence of i.i.d. real valued random variables with distribution μ. When $X_0 = x$, we set $X_n = X_n^x$. The sequence $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{R}^+$, with probability transition $P(x, .)$ defined by: for any $x \in \mathbb{R}^+$ and any bounded Borel function $\phi : \mathbb{R}^+ \to \mathbb{R}$,

$$P\phi(x) = \mathbb{E}(\ \phi(\max(x - Y_1, 0)) = \int_{\mathbb{R}} \phi(\max(x - y, 0))\mu(\mathrm{d}y).$$

We obtain

$$X_n^x = f_{Y_n} \circ f_{Y_{n-1}} \circ \cdots \circ f_{Y_1}(x)$$

where the maps $f_a, a \in \mathbb{R}^+$ are defined in (2.3).

As above, we set $L_n := f_{Y_n} \circ \ldots \circ f_{Y_1}$ and $R_n := f_{Y_1} \circ \cdots \circ f_{Y_n}$. A few simple remarks.

1. Each function f_a satisfies the following "weak" contraction property:

$$\forall x, y \in \mathbb{R} \quad |f_a(x) - f_a(y)| \leq |x - y|.$$

Consequently, the sequence $(|X_n^x - X_n^y|)_{n\geq 0}$ is non increasing.

2. Let us fix $x \leq y$ in $\mathbb{R}^+$. Let $(a_k)_{k\geq 1}$ be a sequence of real numbers such that there exists $a_1+\ldots+a_k \geq y$ for some $k \geq 1$; we denote by k_y the smallest integer satisfying this property. Thus, for any $k \geq k_y$

$$f_{a_k} \circ \cdots \circ f_{a_1}(x) = f_{a_k} \circ \cdots \circ f_{a_1}(y) = 0. \tag{5.1}$$

In other words, the trajectories X_k^x and X_k^y may coincide after a certain time; in particular the sequence $(L_n)_{n\geq 0}$ is proximal $\mathbb{P}$-a.s., as soon as $\mu(\mathbb{R}^{*+}) > 0$.

3. If the distribution μ of the Y_i is $\mathbb{N}$-valued and $x \in \mathbb{N}$, the chain $(X_n^x)_{n\geq 0}$ is also $\mathbb{N}$-valued. We do not focus on this case here and assume that μ is *adapted to* $\mathbb{R}$, i.e. that the support of the distribution μ generates a dense subgroup of $\mathbb{R}$.

Assume now that the Y_i are $\mathbb{R}$-valued with a strictly positive drift. Here arrives the following statement.

Theorem 5.1. *Set $Y_i^- := \max(-Y_i, 0)$ and $Y_i^+ := \max(Y_i, 0)$ and assume*

$$\mathbb{E}(Y_i^-) < +\infty \quad \text{and} \quad \mathbb{E}(Y_i^+) > \mathbb{E}(Y_i^-).$$

Then

1. *for any $x, y \in \mathbb{R}$, it holds $X_n^x - X_n^y = 0 \quad \mathbb{P}$-a.s. after a certain time (in particular the sequence $(L_n)_{n\geq 1}$ is $\mathbb{P}$-a.s. proximal on $\mathbb{R}$);*

2. *the chain $(X_n)_{n\geq 0}$ has a unique invariant probability measure ν on $\mathbb{R}^+$ and is ν-positive recurrent.*

Proof. By Theorem 3.3, it suffices to check that there exists on $\mathbb{R}^+$ an invariant probability measure for $(X_n)_{n\geq 0}$. Let $(T_n)_{n\geq 0}$ be the sequence of successive ascending strict ladder epoch of the random walk $(S_n := Y_1 + \ldots + Y_n)_{n\geq 1}$ (see Appendix, § 9.4). Under the hypotheses of Theorem 5.1, we obtain $\mathbb{P}(T_n < +\infty)$ (and $\mathbb{E}(T_n) < +\infty$) for any $n \geq 1$. Furthermore,

$$X_{T_n} = f_{S_{T_n} - S_{T_{n-1}}} \circ \cdots \circ f_{S_{T_2} - S_{T_1}} \circ f_{S_{T_1}}(X_0).$$

The random variables $(S_{T_n - T_{n-1}})_{n\geq 1}$ are i.i.d. and non negative; thus, $\mathbb{P}$-a.s. after a certain (random) time $n = n_\omega$, the sequence $X_{T_n(\omega)}(\omega)$ equals 0. Consequently

$$\sum_{k\geq 0} \mathbf{1}_{\{0\}}(X_k) = +\infty \quad \mathbb{P} - \text{a.s.}$$

and one concludes using Lin's theorem [23].

We may also use an explicit and classical construction of the invariant measure; let us present it in a few lines. The Dirac mass at 0 is invariant for the chain $(X_{T_n})_{n\geq 0}$. We thus consider the measure ν on $\mathcal{B}(\mathbb{R}^+)$ defined by: for any $B \in \mathcal{B}(\mathbb{R}^+)$,

$$\nu(B) := \mathbb{E}_0\Big(\sum_{k=0}^{T_1-1} \mathbf{1}_B(X_k)\Big).$$

By a classical trick called *balayage*, we may prove that ν is invariant for the chain $(X_n)_{n\geq 0}$. Indeed, if ρ is a probability measure on $\mathbb{R}^+$ which is invariant for $(X_{T_n})_{n\geq 1}$, the measure $\bar{\rho}$, defined by

$$\bar{\rho}(B) := \mathbb{E}_\rho\Big(\sum_{k=0}^{T_1-1} \mathbf{1}_B(X_k)\Big),$$

is invariant for $(X_n)_{n\geq 0}$; the argument works as follows

$$\begin{aligned}
\bar\rho P(B) &= \mathbb{E}_{\bar\rho}\Big(\mathbf{1}_B(X_1)\Big)\\
&= \int_{\mathbb{R}^+} \mathbb{E}_x(\mathbf{1}_B(X_1))\bar\rho(\mathrm{d}x)\\
&= \mathbb{E}_\rho\Big(\sum_{k=0}^{T_1-1} \mathbb{E}_{X_k}(\mathbf{1}_B(X_1))\Big)\\
&= \mathbb{E}_\rho\Big(\sum_{k=0}^{T_1-1} \mathbf{1}_B(X_{k+1})\Big)\\
&= \mathbb{E}_\rho\Big(\sum_{k=1}^{T_1} \mathbf{1}_B(X_k)\Big)\\
&= \nu(B) - \mathbb{E}_\rho(\mathbf{1}_B(X_0)) + \mathbb{E}_\rho(\mathbf{1}_B(X_{T_1}))\\
&= \nu(B)
\end{aligned}$$

where the last equality uses the fact that ρ is invariant for $(X_{T_n})_{n\geq 0}$ [7].

To prove that ν is the unique invariant probability measure, we use Theorem 3.3 since the sequence $(f_{Y_n}\circ\cdots\circ f_{Y_1})_{n\geq 1}$ is $\mathbb{P}$-a.s. proximal; this is a direct consequence of the fact that $Y_1+\ldots+Y_n \to +\infty$ $\mathbb{P}$-a.s. as $n\to+\infty$ and of equality (5.1).

□

Let us now explore the "centered" case.

Theorem 5.2. *Assume that μ has moments of order 2 and $\mathbb{E}(Y_i)=0$. Then*

1. *the sequence $(L_n)_{n\geq 1}$ is $\mathbb{P}$-a.s. proximal on $\mathbb{R}$;*
2. *there exists on $\mathbb{R}^+$ a unique infinite Radon measure m invariant for $(X_n)_{n\geq 0}$ and the chain is m-null recurrent.*

Proof. We apply Theorem 4.1. Since $\mathbb{E}|Y_n|<+\infty$ and $\mathbb{E}(Y_n)=0$, we obtain

$$\limsup_{n\to+\infty} Y_1+\ldots+Y_n = +\infty \ \mathbb{P}\text{−a.s.}$$

and hence $\mathbb{P}(T_n<+\infty)=1$ for any $n\geq 1$. It yields

1. $\mathbb{P}(X_n^x=0 \text{ infinitely often})=1$, for any $x\in\mathbb{R}$.
2. for any $x,y\in\mathbb{R}^+$, the random variables X_n^x and X_n^y are equal $\mathbb{P}$-a.s. for n greater than some (random) integer. The sequence $(L_n)_{n\geq 0}$ is thus proximal $\mathbb{P}$-a.s.

We now prove that m is infinite. Assume that m is finite and let us apply Theorem 3.3; for any interval $B:=[0,b]$ such that $m(B)>0$ and m-almost all $x\in B$, it holds $\mathbb{E}_x(\tau^B)<+\infty$.

Fix $\delta>0$ such that $\mathbb{P}(Y_1\leq -\delta)>0$. Taking if necessary b smaller, we may assume that $m(]b-\delta,b])>0$ and choose $x\in]b-\delta,b]$; we obtain $\mathbb{E}_x(\mathbf{1}_{(Y_1\leq-\delta)}\times\tau^B)\leq \mathbb{E}_x(\tau^B)<+\infty$.

Notice that, for $n\geq 2$,

$$(Y_1\leq-\delta, Y_2\leq 0, Y_2+Y_3\leq 0,\ldots,Y_2+\ldots+Y_n\leq 0)\leq (X_1^x>b,\ldots,X_n^x>b)=(\tau^B\geq n). \tag{5.2}$$

Now, set $\tau^+:=\inf\{n\geq 1\mid S_n>0\}$; the r.v. τ^+ is a stopping time which is $\mathbb{P}$-a.s. finite and has infinite expectation, since $\mathbb{E}(Y_1)=0$. Inequality (5.2) may be rewritten as: for any $x\in B$,

$$\mathbb{P}(Y_1\leq-\delta)\times\mathbb{P}(\tau^+\geq n-1)\leq \mathbb{P}_x(\tau^B\geq n).$$

It yields $\mathbb{E}_x(\tau^B)\geq \mathbb{P}(Y_1\leq-\delta)\times\mathbb{E}(\tau^+)=+\infty$. Contradiction.

□

[7] **Warning!** In the "balayage" trick explained here, we do not know in general if the measure $\bar\rho$ is a Radon measure. In the present case, $\rho=\delta_0$ and $\bar\rho=\nu$ is finite since $\mathbb{E}(T_1)<+\infty$; when $\mathbb{E}(T_1)=+\infty$, it may be very difficult to decide whether or not $\bar\rho$ is a Radon measure.

6 The reflected random walk on $\mathbb{R}^+$ with elastic collisions at 0

In this section, we consider the reflected random walk $(X_n)_{n\geq 0}$ on $\mathbb{R}^+$ with elastic collisions at 0, defined by, for any $n \geq 0$

$$X_{n+1} = |X_n - Y_{n+1}|.$$

where X_0 is a fixed r.v. on $\mathbb{R}^+$ and $(Y_n)_{n\geq 1}$ a sequence of i.i.d. real valued random variables. When $X_0 = x$, we set $X_n = X_n^x$. The transition probability of the Markov chain $(X_n)_{n\geq 0}$ is given by: for any $x \in \mathbb{R}^+$ and any bounded Borel function $\phi : \mathbb{R}^+ \to \mathbb{R}$,

$$P\phi(x) \quad = \quad \mathbb{E}\Big(\phi(|x - Y_1|\Big) = \int_{\mathbb{R}} \phi(|x-y|)\mu(\mathrm{d}y).$$

It holds $\quad X_n^x = g_{Y_n} \circ g_{Y_{n-1}} \circ \cdots \circ g_{Y_1}(x) \quad$ where the maps $g_a, a \in \mathbb{R}^+$, are defined in (2.4).
As usual, we set $L_n := g_{Y_n} \circ \ldots \circ g_{Y_1}$. Some elementary remarks.

1. For any $x, y \in \mathbb{R}^+$ and any $a \in \mathbb{R}$ $\quad |g_a(x) - g_a(y)| \leq |x-y|$. Once again, the sequence $(|X_n^x - X_n^y|)_{n\geq 0}$ is decreasing.
2. If μ is $\mathbb{N}$-valued, we set $d := GCD\{n \geq 1/\mu(n) > 0\}$. When $x \in \mathbb{N}$, the chain $(X_n^x)_{n\geq 0}$ remains inside the countable set $S(x) := \{\pm x + d\mathbb{Z}\} \cap \mathbb{R}^+$. As in the previous section, we assume that μ is adapted to $\mathbb{R}$, i.e. the group generated by the support of μ is dense in $\mathbb{R}$.
3. If the support of μ is included in $[0, C]$, then after finitely many steps, the chain $(X_n^x)_{n\geq 0}$ remains inside $[0, C]$.

Lemma 6.1. *Assume that μ is adapted to $\mathbb{R}^+$. Then, the measure with density $\mu[x, +\infty[$ with respect to the Lebesgue measure on $\mathbb{R}^+$ is μ-invariant and this measure is finite if and only if $\mathbb{E}(Y_1) < +\infty$.*

Proof. Set $H(x) = \mu[x, +\infty[$. For any $x > 0$,

$$\begin{aligned}
\forall x > 0 \quad \nu P(x, +\infty[\quad &= \quad \mathbb{P}_\nu(|X_0 - Y_1| \geq x)\\
&= \quad \mathbb{P}_\nu(X_0 \geq x + Y_1) + \mathbb{P}_\nu(X_0 < Y_1 - x)\\
&= \quad \int_x^{+\infty} \mathbb{P}(Y_1 \leq t - x)H(t)\mathrm{d}t + \int_0^{+\infty} \mathbb{P}(Y_1 > t + x)H(t)\mathrm{d}t\\
&= \quad \int_x^{+\infty} (1 - H(t-x)H(t)\mathrm{d}t + \int_0^{+\infty} H(t+x)H(t)\mathrm{d}t\\
&= \quad \int_x^{+\infty} H(t)\mathrm{d}t - \int_x^{+\infty} H(t-x)H(t)\mathrm{d}t + \int_0^{+\infty} H(t+x)H(t)\mathrm{d}t\\
&= \quad \nu[x, +\infty[.
\end{aligned}$$

□

The contraction properties of the sequences $(L_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ are not easy to obtain. Locally, the maps F_i are Lipschitz functions with Lipschitz coefficient 1 (such maps are called "contractions"). Nevertheless, "sometimes", the two paths $(R_n \cdot x)_{n\geq 0}$ and $(R_n \cdot y)_{n\geq 0}$ become strictly closer to each other; it happens at time n when there is a reflexion for one and only one of the sequences $(R_n \cdot x)_{n\geq 0}$ and $(R_n \cdot y)_{n\geq 0}$. In other words, when the measure μ is adapted to $\mathbb{R}$, the closed semi-group generated by the maps $F_a, a \in \mathrm{S}_\mu$, contains a constant function [20]. By [26], Theorem 4.2., we obtain the following lemma.

Lemma 6.2. *If μ is adapted to $\mathbb{R}$, then for any $x, y \in \mathbb{R}^+$,*

$$\lim_{n\to+\infty} \Big(L_n \cdot x - L_n \cdot y \Big) = \lim_{n\to+\infty} \Big(R_n \cdot x - R_n \cdot y \Big) = 0 \quad \mathbb{P} - \text{a.s.}$$

Element of proof. We explain here how the adaptation condition on μ implies that the closed semi-group T_μ generated by the maps $g_a, a \in S_\mu$ contains the constant function 0, i.e. that there exists a sequence of functions $(\xi_k)_{k\geq 1}$ in T_μ such that, for any $x \in \mathbb{R}^+$,

$$\lim_{k\to+\infty} \xi_k \cdot x = 0.$$

To simplify the argument, we assume that the support of μ contains at least two real numbers $\beta > \alpha > 0$ such that $\beta/\alpha \notin \mathbb{Q}$; this is the first case of adaptation considered in [20], and the others cases follow by a variation of the argument presented below.

First, recall that any $x \in \mathbb{R}$ may be decomposed in a unique way as

$$x = [x] + \{x\}$$

where $[x] \in \mathbb{Z}$ and $0 \le \{x\} < 1$. The value $[x]$ is called the *integer part* of x and $\{x\}$ its *fractional part*. Let T be the map from $]0,1]$ to $]0,1]$ defined by $T(x) = \left\{\frac{1}{x}\right\}$ for any $x \in]0,1]$. This map is associated to the decomposition of reals numbers in continuous fraction; indeed, if $x \in]0,1]$ decomposes as

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \cdots}}},$$

with $n_1, n_2 \ldots \ge 1$, then $T(x)$ decomposes as $\dfrac{1}{n_2 + \frac{1}{n_3+\cdots}}$. Furthermore, this decomposition is infinite if and only if $x \notin \mathbb{Q}$.

A straightforward computation yields: for any $0 < \alpha_1 < \alpha_0$ and any $x \in [0, \alpha_1]$,

$$g_{\alpha_1}^{[\alpha_0/\alpha_1]} \circ g_{\alpha_0}(x) = g_{\alpha_1\{\alpha_0/\alpha_1\}}(x). \tag{6.1}$$

In particular, the function $\phi_{\alpha_1,\alpha_0} := g_{\alpha_1}^{[\alpha_0/\alpha_1]} \circ g_{\alpha_0}$ maps $[0, \alpha_1]$ into $[0, \alpha_1\{\alpha_0/\alpha_1\}]$.

Let us set $\gamma := \beta/\alpha, \alpha_0 := \beta, \alpha_1 := \alpha$ and $\alpha_n := \alpha T(\gamma)T^2(\gamma)\ldots T^{n-1}(\gamma)$ for any $n \ge 1$, and let us consider the sequence $(\Phi_n)_{n\ge 0}$ of continuous functions defined by

$$\forall n \ge 0 \qquad \Phi_: = \phi_{\alpha_n,\alpha_{n-1}} \circ \cdots \circ \phi_{\alpha_2,\alpha_1} \circ \phi_{\alpha_1,\alpha_0}.$$

Each function $\Phi_n, n \ge 1$, belongs to the closed semi-groupe generated by the support of μ and maps the interval $[0, \alpha_1]$ into $[0, \alpha_{n+1}]$.

If $\alpha_n \to 0$ as $n \to +\infty$, then the sequence $(\Phi_n \circ g_{\alpha_1}^n)_{n\ge 0}$ converges to the constant function which equals 0. The sequence $(\alpha_n)_{n\ge 0}$ is strictly decreasing and it suffices to show that

$$S_n(\gamma) := -\log\gamma - \log T(\gamma) - \ldots - \log T^n(\gamma) \to +\infty \quad \text{as} \quad n \to +\infty.$$

Let us decompose γ as $\gamma = \dfrac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3+\cdots}}}$ with $k_1, k_2, \ldots \ge 1$. There are two cases to consider:

- if $k_\ell \ge 2$ infinitely often (i.o.) , then $-\log T^\ell(\gamma) \ge \log 2 > 0$ i.o.
- if $k_\ell = 1$ for $\ell \ge \ell_0$, then $T^\ell(\gamma) = \frac{1}{1+T^\ell(\gamma)}$, i.e. $T^\ell(\gamma) = \frac{\sqrt{5}-1}{2}$ for $\ell \ge \ell_0$.

In both cases, $S_n(\gamma) \to +\infty$ as $n \to +\infty$.

When μ is adapted but α_0, α_1 as above do not exist in its support, we adapt the same strategy, with minor modifications [20].

□

Using these results, we may apply Theorem 3.3 to obtain the following statement.

Theorem 6.3. *Assume that μ is adapted to $\mathbb{R}$ with support in $\mathbb{R}^+$ and $0 < m := \mathbb{E}(Y_i) < +\infty$. Then*

1. *the sequence $(L_n)_{n\ge1}$ is $\mathbb{P}$-a.s. proximal on $\mathbb{R}$;*
2. *the probability measure $\nu(\mathrm{d}x) = \dfrac{1}{m}\mu[x, +\infty[\mathrm{d}x$ is the unique P-invariant probability measure and $(X_n)_{n\ge0}$ is ν-positive recurrent on $[0, C]$, where $C := \sup C_\mu$.*

When the Y_i are $\mathbb{R}$-valued, with a strictly positive drift, we use "balayage trick" to construct an invariant measure for $(X_n)_{n\ge0}$.

Theorem 6.4. *Assume that μ is adapted to $\mathbb{R}$, $\mathbb{E}|Y_i| < +\infty$ and $0 < m := \mathbb{E}(Y_i) < +\infty$. Then*

1. *the sequence* $(L_n)_{n\geq 1}$ *is* $\mathbb{P}$*-a.s. proximal on* $\mathbb{R}$*;*

2. *there exists on* $\mathbb{R}^+$ *a unique* P*-invariant probability measure* ν *(with support* $\mathbb{R}^+$*) an the chain* $(X_n)_{n\geq 0}$ *is* ν*-positive recurrent on* $\mathbb{R}^+$*.*

Remark. When $\mu(\mathbb{R}^{*-}) > 0$, the support of ν equals $\mathbb{R}^+$, even if the Y_i are bounded from above; it is a direct consequence of the construction of ν, based on the balayage trick.

Proof. Let $(T_n^+)_{n\geq 0}$ be the sequence of strict ladder epochs of the random walk $(S_n)_{n\geq 0}$, where $S_0 = 0$ and $S_n := Y_1 + \ldots + Y_n, n \geq 1$.

By Theorem 3.2, chap. XII in [12], it holds $\mathbb{P}(T_1^+ < +\infty) = 1$ and $\mathbb{P}(S_{T_1^+} < +\infty) = 1$; furthermore, the (non negative) random variables $T_1^+$and $S_{T_1^+}$ have finite expectation (more precisely, Wald's formula states $\mathbb{E}(S_{T_1^+}) = \mathbb{E}(T_1^+) \times \mathbb{E}(Y_1)$.)

The key point is that, for any $n \geq 0$,

$$X_{T_n^+} = g_{S_{T_n^+} - S_{T_{n-1}^+}} \circ \cdots \circ g_{S_{T_1^+}}(X_0).$$

By Corollary 6.3, there exists on $\mathbb{R}^+$ a (unique) probability measure ν' which is invariant for the chain $(X_{T_n^+})_{n\geq 0}$ [8]. By the balayage trick presented above, the measure ν defined by

$$\nu(B) := \mathbb{E}_{\nu'}\Big(\sum_{k=0}^{T_1-1} \mathbf{1}_B(X_k)\Big)$$

for any Borel set $B \subset \mathbb{R}^+$ is invariant for $(X_n)_{n\geq 0}$. Furthermore, $\nu(\mathbb{R}^+) = \nu'(\mathbb{R}^+) \times \mathbb{E}(T_1) = \mathbb{E}(T_1) < +\infty$, which readily implies that ν is finite; we normalize it in such a way it is a probability measure.

To prove that ν is unique, we use Theorem 3.3 and thus check that $(f_{Y_n} \circ \cdots \circ f_{Y_1})_{n\geq 1}$ is $\mathbb{P}$-a.s. proximal; it follows by Lemma 6.2.

□

Let us now deal with the centered case.

Corollary 6.5. *Assume that* μ *has moments of order* 2 *and* $\mathbb{E}(Y_i) = 0$*. Then*

1. *the sequence* $(L_n)_{n\geq 1}$ *is* $\mathbb{P}$*-a.s. proximal on* $\mathbb{R}$*.*

2. *the chain* $(X_n)_{n\geq 0}$ *has a unique invariant Radon measure* m *and is* m*-null recurrent.*

Proof. We check that hypotheses of Theorem 4.1 hold. The sequence$(L_n)_{n\geq 1}$ is $\mathbb{P}$-a.s. proximal on $\mathbb{R}^+$ (and a fortiori $\mathbb{P}$-a.s. locally proximal) by Lemma 6.2. Thus, condition (1) of Theorem 4.1 holds.

Now, let $(T_n)_{n\geq 0}$ be the sequence of strictly ascending ladder epochs of the random walk $(S_n)_{n\geq 0}$. By Theorem 3.2, chap. XII de [12], it holds $\mathbb{P}(T_1 < +\infty) = 1$ and $\mathbb{E}(T_1) = +\infty$ and $\mathbb{E}(S_{T_1}) < +\infty$. Thus, by Theorem 4.1, the reflected random walk on $\mathbb{R}^+$ associated with the sequence $\Big(S_{T_n} - S_{T_{n-1}}\Big)_{n\geq 0}$ is positive recurrent. Since $X_{T_n} = f_{S_{T_n} - S_{T_{n-1}}} \circ \cdots \circ f_{S_{T_1}}(X_0)$, the chain $(X_n)_{n\geq 0}$ is also recurrent on $\mathbb{R}^+$ and possesses an invariant Radon measure, by M. Lin's theorem. This measure is infinite; otherwise we could apply Theorem 3.3 which yields to a contradiction with the equality $\mathbb{E}(T_1) = +\infty$.

□

There exist in fact other situations of null recurrence, described in [26]. For instance, when the r.v. Y_i are positive, but without moment of oder one, the associated reflected random walk $(X_n)_{n\geq 0}$ may be either null-recurrent or transient; as a consequence, the null recurrence may also happen in the centered case, without moment of oder two. We do not get into the details here but state the main results from [26]. They relies on the fact that, under these assumptions, the sub-process $(X_{\mathrm{r}_n})_{n\geq 0}$ of $(X_n)_{n\geq 0}$ at the successive reflexion times admits an invariant probability measure, thus it is positive recurrent, which yields to the recurrence of $(X_n)_{n\geq 0}$.

Theorem 6.6. *Assume that* μ *is adapted to* $\mathbb{R}$*. Then,*

[8] The unicity of ν' is a consequence of the fact that the distribution of the random variable $S_{T_1^+}$ is adapted on $\mathbb{R}$, which can be proved in a elementary way. We do not use this property in the sequel.

1. *if* $\mathbb{P}(Y_i \geq 0) = 1, \mathbb{E}(Y_i) = +\infty$ *and* $\mathbb{E}(\sqrt{Y_i}) < +\infty$*, then the measure* $\nu(\mathrm{d}x) = \mu[x, +\infty[\mathrm{d}x$ *is the unique* P*-invariant probability measure and* $(X_n)_{n\geq 0}$ *is* ν*-null recurrent on* $\mathbb{R}^+$*.*

2. *if* $\mathbb{E}(Y_i^{3/2}) < +\infty$ *and* $\mathbb{E}(Y_i) = 0$*, there exists on* $\mathbb{R}^+$ *a unique (infinite) Radon measure* m *invariant for the chain* $(X_n)_{n\geq 0}$ *and the chain is* m*-null recurrent.*

7 The affine recursion

We consider here the affine recursion $(X_n)_{n\geq 0}$ on $\mathbb{R}$ defined by: for any $n \geq 0$,

$$X_{n+1} = a_{n+1} X_n + b_{n+1},$$

where X_0 is a fixed r.v. on $\mathbb{R}$ and the $(a_n, b_n), n \geq 1$, are $\mathrm{Aff}(\mathbb{R})$-valued i.i.d. random variables with distribution μ. The Markov chain $(X_n)_{n\geq 0}$ is generated by the affine IFS $(g_n)_{n\geq 1}$ with $g_n(x) := a_n x + b_n$ for any $x \in \mathbb{R}$ and $n \geq 1$.

7.1 Background on the affine group of the real line

Let $\mathrm{Aff}(\mathbb{R})$ be the groupe of affine maps of the real line $\mathbb{R}$, that is the groups of maps $g : \mathbb{R} \to \mathbb{R}$ of the form

$$\forall x \in \mathbb{R}, \quad g(x) := ax + b,$$

where $a = a(g) \in \mathbb{R}^{*+}$ and $b = b(g) \in \mathbb{R}$. We identify g and $(a, b) = (a(g), b(g)) \in \mathbb{R}^{*+} \times \mathbb{R}$; if $g_1 = (a_1, b_1)$ and $g_2 = (a_2, b_2)$ belong to $\mathrm{Aff}(\mathbb{R})$, their product $g_1 g_2$ is the map defined by

$$\forall x \in \mathbb{R} \quad g_1 g_2(x) = g_1 \circ g_2(x) = a_1 a_2 x + a_1 b_2 + b_1,$$

i.e.

$$g_1 g_2 = (a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1).$$

The group $\mathrm{Aff}(\mathbb{R})$ is non abelian and equals the semi-direct product $\mathbb{R}^{*+} \rtimes \mathbb{R}$ of $\mathbb{R}^{*+}$ and $\mathbb{R}$.

The group G may be identified with the upper hyperbolic plane $\mathbb{H} := \mathbb{R}^{*+} \times \mathbb{R}$. We endow $\mathbb{H}$ with the hyperbolic metric $\dfrac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y}$; its boundary at infinity $\partial\mathbb{H}$ equals $\mathbb{R} \bigcup \{\infty\}$, it is metrizable in such a way $\mathbb{H} \bigcup \partial\mathbb{H}$ is compact. The set $\mathbb{H} \bigcup \partial\mathbb{H}$ is called the compactification of $\mathbb{H}$. The affine maps $(x, y) \mapsto (ax + b, y)$ are isometries of $\mathbb{H}^2$ which fixes $+\infty$.

We now recall the definition of Haar measures on a topological group G. The right (resp. left) Haar measure on a topological group G is a σ-finite measure on this group, which is invariant under the right (resp. left) action by translation; it is unique up to a multiplicative constant.

The right and left Haar measure coincide when G is abelian. For instance, the Lebesgue measure on $\mathbb{R}$ is the Haar measure of the group $(\mathbb{R}, +)$; similarly, $\frac{\mathrm{d}a}{a}$ on $\mathbb{R}^{*+}$ is the Haar measure of the group $(\mathbb{R}^{*+}, \times)$.

When G is not abelian, these Haar measures may differ or coincide. For instance, the right Haar measure of $G = \mathrm{Aff}(\mathbb{R})$ is $\frac{\mathrm{d}a\ \mathrm{d}b}{a}$ and the left Haar measure equals $\frac{\mathrm{d}a\ \mathrm{d}b}{a^2}$; one says that $\mathrm{Aff}(\mathbb{R})$ is a *non unimodular* group. Let us notice that there exist unimodular and non abelian groups; it is for instance the case of the group $O(3)$ of 3×3 orthogonal matrices.

Now, consider a sequence $(g_n)_{n\geq 1}$ of i.i.d. G-valued random variables and the associated right and left random walks $R_n = g_1 \ldots g_n$ and $L_n = g_n \ldots g_1$. The unimodularity of G has an important influence on the behavior of these random walks:

Theorem 7.1. *([15] Theorem 51) If the random walks* $(R_n)_{n\geq 0}$ *(or* $(L_n)_{n\geq 0}$*) is recurrent on* G*, then* G *is unimodular.*

In particular, the right and left random walks on $\mathrm{Aff}(\mathbb{R})$ are all transient.

In the sequel, we assume that the law μ of the random variables g_n satisfies the following condition of "non degeneration":

Hypothesis H1 $\quad \forall x \in \mathbb{R} \quad \mu\{g \in G : g \cdot x = x\} < 1 \quad \text{and} \quad \mu\{g \in G : a(g) = 1\} < 1.$

Let us achieve this subsection with the following important remark which emphasizes the connections between the group $\mathrm{Aff}(\mathbb{R})$ and some subgroups of $Gl(2, \mathbb{R})$. Let us associate to each (random) affine map $g_n = (a_n, b_n)$ the (random) matrice $G_n := \begin{pmatrix} a_n & b_n \\ 0 & 1 \end{pmatrix}$. With this identification, the functions R_n and L_n correspond respectively to $G_1 \dots G_n$ and $G_n \dots G_1$; furthermore, the action of the affine maps R_n (resp. L_n) on $\mathbb{R}$ corresponds to the projective action of the corresponding matrices since, for any $x \in \mathbb{R}$,

$$\begin{pmatrix} R_n(x) \\ 1 \end{pmatrix} = G_1 \dots G_n \begin{pmatrix} x \\ 1 \end{pmatrix} \qquad \left(\text{resp. } \begin{pmatrix} L_n(x) \\ 1 \end{pmatrix} = G_n \dots G_1 \begin{pmatrix} x \\ 1 \end{pmatrix}\right)$$

Let us notice that the matrices $\begin{pmatrix} a_n & b_n \\ 0 & 1 \end{pmatrix}$ all fixes the line $\mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus, the condition of irreducibility (I) introduced for product of random matrices does not holds here; thus, the present study of affine IFS concerns also product of random matrices in the case when condition (I) fails.

7.2 Applications: perpetuities and ARCH (or GARCH) families

There exist many domains where the stochastic affine recursion may apply; we refer to [6] for a long list of examples and present here only two of them.

7.2.1 Perpetuities

The notion of perpetuities originates from life insurance and financial contracts. At the beginning of the n-th period, a payment b_n is made in your life insurance portfolio; during the previous $n-1$ periods, the amount X_{n-1} has been accumulated. In each period, the previous payments are subject to interest, given by the random variables $a_n = 1 + \delta_n$. Thus, the values of a perpetuity in the first n periods is given by the equation

$$X_n := a_n X_{n-1} + b_n = (1 + \delta_n) X_{n-1} + b_n.$$

• The point of view of the customer

Assume that the initial deposit X_0 equals 0 and that X_n equals the values of the portfolio at the beginning of the period n; it holds

$$X_n = F_n \circ \dots \circ F_1(0) = L_n(0)$$

where $F_n(x) := a_n x + b_n = (1 + \delta_n)x + b_n$ and $L_n = F_n \circ \dots \circ F_1$.

• The point of view of the "bank"

The contract states that the initial deposit equals 0 and that the deposits at the end of the period $n \geq 1$ will be b_n and the (random) expected interest rate during this period equals δ_n. Thus, the value Y_n of the portfolio at time n equals

$$Y_0 = 0, Y_1 = +\frac{b_1}{1+\delta_1}, Y_3 = b_0 + \frac{b_1}{1+\delta_0} + \frac{b_2}{(1+\delta_0)(1+\delta_1)} \dots$$

so that $Y_n = G_1 \circ \dots \circ G_n(b_0) = R'_n(b_0)$ where $G_n(x) = \frac{x}{1+\delta_n} + b_n$ and $R'_n = G_0 \circ \dots \circ G_{n-1}$.

7.2.2 The ARCH's family

In 1982, Engle introduced a model for the "log-return" $\mathcal{L}_n := \log P_{n+1} - \log P_n$ of speculative prices $P_0, P_1, P_2, \dots$ In its simplest form, it is assumed that

$$\mathcal{L}_n = \sigma_n Z_n \quad \text{with} \quad \sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \mathcal{L}_{n-i}^2,$$

- $p \geq 1$;
- $\alpha_0, \dots, \alpha_p \geq 0$ and $\alpha_0, \alpha_p \neq 0$;

- $(Z_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with law $\mathcal{N}(0,1)$.

The quantity σ_n is the "volatility" of $\mathcal{L}_n$ and $(\mathcal{L}_n)_{n\geq 0}$ is an **auto**r**e**gressive **c**onditionally **h**eteroscedastic process of oder p (ARCH(p) for short).

It is easy to see that the squared ARCH(1) process $(X_n)_{n\geq 0}$, with $X_n = \mathcal{L}_n^2$, satisfies the recursive equation:

$$X_n = a_n X_{n-1} + b_n$$

with $a_n := \alpha_1 Z_n^2$ and $b_n := \alpha_0 Z_n^2$.

7.3 The affine recursion in the "contracting" case

The transition probability of the Markov chain $(X_n)_{n\geq 0}$ is given by: for any $x \in \mathbb{R}^+$ and any Borel set $B \subset \mathbb{R}^+$,

$$P(x,B) = \mathbb{E}(\ \mathbf{1}_B(g_1\ x)) = \int_{\mathrm{Aff}(\mathbb{R})} \mathbf{1}_B(gx)\mu(\mathrm{d}g) = \int_{\mathbb{R}^{*+}\times\mathbb{R}} \mathbf{1}_B(ax+b)\mu(\mathrm{d}a\mathrm{d}b).$$

As usual, we consider the right (resp. left) random walks on $\mathrm{Aff}(\mathbb{R})$ with distribution μ defined by $R_0 = L_0 = (1,0)$, and, for any $n \geq 1$,

$$R_n = g_1 \circ \cdots \circ g_n = (a_1,b_1)\dots(a_n,b_n) = \Big(a_1\dots a_n, \sum_{k=1}^n a_1\dots a_{k-1}b_k\Big)$$

and

$$L_n = g_n \circ \cdots \circ g_1 = (a_n,b_n)\dots(a_1,b_1) = \Big(a_n\dots a_1, \sum_{k=1}^n b_k a_{k+1}\dots a_n\Big).$$

Theorem 7.2. *Assume that* $\mathbb{E}\Big(\log^+ a_n + \log^+ |b_n|\Big) < +\infty$ *and* $\mathbb{E}(\log a_n) < 0$. *Then,*

1. *for any $x \in \mathbb{R}$, the sequence $(R_n(x))_{n\geq 0}$ converges $\mathbb{P}$-a.s. towards a finite random variable Z,*
2. *the sequences $(R_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ are $\mathbb{P}$-a.s. proximal on $\mathbb{R}$,*
3. *the distribution ν of Z is the unique invariant probability measure on $\mathbb{R}$ for the affine recursion $(X_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$ is ν-positive recurrent.*

Proof. For any $x \in \mathbb{R}$ it holds $R_n(x) = a_1\dots a_n x + \sum_{k=1}^n a_1\dots a_{k-1}b_k$. On the one hand, the strong law of large numbers on $\mathbb{R}$ yields, for $\mathbb{P}$- almost all $\omega \in \Omega$

$$\frac{1}{n}\Big(\log a_1(\omega) + \ldots + \log a_n(\omega)\Big) \overset{n\to+\infty}{\longrightarrow} \mathbb{E}(\log a_1) < 0,$$

and hence $a_1(\omega)\dots a_n(\omega) \overset{n\to+\infty}{\longrightarrow} 0$. On the other hand $\mathbb{E}(\log^+|b_n|) < +\infty$, it holds for any $\epsilon > 0$,

$$\sum_{k=1}^{+\infty} \mathbb{P}(\log^+|b_k| \geq k\epsilon) = \mathbb{E}\left(\sum_{k=1}^{+\infty} \mathbf{1}_{(\log^+|b_k|\geq k\epsilon}\right) \leq \frac{1}{\epsilon}\mathbb{E}\left(\log^+|b_1|\right) < +\infty.$$

Thus $\mathbb{P}(\limsup_{k\to+\infty}(\log|b_k| \geq k\epsilon)) = 0$; it yields $\limsup_n \frac{1}{k}\log|b_k| \leq 0$ $\mathbb{P}$-a.s. so that

$$\begin{aligned}\limsup_{n\to+\infty}(a_1\dots a_{k-1}b_k)^{\frac{1}{k}} &= \limsup_{n\to+\infty}\exp\Big(\frac{\log a_1 + \ldots + \log a_{k-1} + \log|b_k|}{k}\Big)\\ &\leq \exp(\mathbb{E}(\log a_1)) < 1.\end{aligned}$$

By Cauchy criterion, the series $\sum_{k=1}^{+\infty} a_1\dots a_{k-1}b_k$ converges $\mathbb{P}$ a.s. towards a random variable

$$Z := \sum_{k=1}^{+\infty} a_1\dots a_{k-1}b_k.$$

Thus, Furstenberg's principle implies that the distribution ν of Z is the unique invariant probability measure for the Markov chain $(L_n \cdot x)_{n\geq 1}$ on $\mathbb{R}$. The unicity of this measure is also a consequence of Theorem 3.3, since for any $x, y \in \mathbb{R}$, it holds

$$|L_n \cdot x - L_n \cdot y| = a_1 \dots a_n |x - y| \quad \stackrel{n\to+\infty}{\longrightarrow} \quad 0 \quad \mathbb{P}-\text{a.s.}$$

□

7.4 The affine recursion in the centered case

We introduce the following conditions:

Hypothesis H2 $\quad \displaystyle\int \Big(|\log a(g)|^2 + (\log^+ |b(g)|)^{2+\eta} \Big) \mu(\mathrm{d}g) < +\infty$, for some $\eta > 0$

Hypothesis H3 $\quad \displaystyle\int \log a(g) \mu(\mathrm{d}g) = 0.$

The random walks $(L_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ with distribution μ on $\mathrm{Aff}(\mathbb{R})$ are transient since $\mathrm{Aff}(\mathbb{R})$ is non unimodular (see Theorem 7.1); thus, for any continuous function with compact support $\phi : \mathrm{Aff}(\mathbb{R}) \to \mathbb{R}^+$ and any $g \in \mathbb{R}$

$$U\phi(g) := \mathbb{E}\Big(\sum_{n=0}^{+\infty} \phi(R_k g) \Big) = \mathbb{E}\Big(\sum_{n=0}^{+\infty} \phi(L_k g) \Big) < +\infty.$$

Theorem 7.3. *Under hypotheses H1, H2 and H3, the sequence of random functions $(L_n)_{n\geq 0}$ is $\mathbb{P}$-a.s. locally proximal on $\mathbb{R}$. Furthermore, there exists on $\mathbb{R}$ a unique invariant (infinite) Radon measure m for $(X_n)_{n\geq 1}$ and this chain is m-null recurrent.*

Proof. Let $(S_n)_{n\geq 0}$ be the random walk on $\mathbb{R}$ defined by $S_0 = 0$ and $S_n := \log a(R_n)$. We consider the sequence $(T_n)_{n\geq 0}$ of strictly ascending ladder epochs of this walk, defined by $T_0 = 0$, and for $n \geq 1$,

$$T_n := \inf\{k > T_{n-1} : S_k < S_{T_{n-1}}\}.$$

The sequence $(g_{T_n} \circ \dots \circ g_1)_{n\geq 0}$ is a random walk on $\mathrm{Aff}(\mathbb{R})$ whose increments have the same distribution as the random variable $g_{T_1} \circ \dots \circ g_1$. On the one hand, by Proposition 9.5, under hypotheses H2 and H3, we obtain $\mathbb{E}(\log a(g_{T_1} \circ \dots \circ g_1)) \in]-\infty, 0[$; on the other hand, it holds $\mathbb{E}(\log^+ b(g_{T_1} \circ \dots \circ g_1)) < +\infty$ [23]. By Theorem 7.2, for any $x \in \mathbb{R}$, the chain $(g_{T_n} \circ \dots \circ g_1 \cdot x)_{n\geq 0}$ is positive recurrent on open sets of $\mathbb{R}$, so is the chain $(L_n \cdot x)_{n\geq 0}$ and the existence of m follows from M. Lin's theorem. Thus, hypotheses of Theorem 4.1 hold and it remains to check the local proximality.

We first establish the following proposition which describes how $(R_n)_{n\geq 0}$ tends to $+\infty$ in $\mathrm{Aff}(\mathbb{R})$.

Theorem 7.4. *Under hypotheses H1, H2 and H3, for almost all $g \in \mathrm{Aff}(\mathbb{R})$ with respect to the Haar measure,*

$$\lim_{n\to+\infty} \max\{a(gR_n), |b(gR_n)|\} = \infty.$$

Corollary 7.5. *Under hypotheses H1, H2 and H3, the sequence $(L_n)_{n\geq 0}$ is $\mathbb{P}$-a.s. locally proximal on $\mathbb{R}$.*

Proof of corollary 7.5. We fix a compact set $K \subset \mathrm{Aff}(\mathbb{R})$ and $k > 0$ such that $K \subset [-k, k]$.

Notice that $L_n = g_n \dots g_1 = (g_1^{-1} \dots g_n^{-1})^{-1} = \check{R}_n^{-1}$ where $(\check{R}_n)_{n\geq 0}$ denotes the right random walk with law $\check{\mu}$. Since $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$, the term $b(L_n)$ equals

$$b(L_n) = -\frac{b(\check{R}_n)}{a(\check{R}_n)} = -a(L_n) b(\check{R}_n).$$

The equality $L_n \cdot y = a(L_n)y + b(L_n) \in K$ yields $|b(L_n)| \leq k + a(L_n)|y|$, so that $|b(\check{R}_n)| \leq \dfrac{k}{a(L_n)} + |y|$. Finally

$$L_n \cdot y \in K \quad \Rightarrow \quad \max(a(\check{R}_n), b(\check{R}_n)) \leq (k \vee 1) \frac{1}{a(L_n)} + |y|.$$

Now, the right random walk $(\check{R}_n)_{n\geq 0}$ satisfies hypotheses of Theorem 7.4; consequently $\max(a(\check{R}_n), b(\check{R}_n)) \to +\infty$ and $a(L_n)\ \mathbf{1}_{L_n\cdot y\in K} \to 0$ $\mathbb{P}$-a.s.

□

7.4.1 Local proximality: proof of Theorem 7.4

Proof of Theorem 7.4. First we state the following general result.

Lemma 7.6. *Let H be a second-countable locally compact group with denumerable basis. Denote dh its right Haar measure and $U(h,\cdot) := \sum_{n=0}^{+\infty} \delta_h * \mu^{*n}, h \in H$, the potential of a transient random walk $(R_n)_{n\geq 0}$ on H with law μ. For any $\phi \in \mathbb{L}^1(H)$, the map $h \mapsto U\phi(h)$ is dh-a.s. finite.*

Proof. Let us denote $\check{\mu}$ the image of the measure μ by the transformation $g \mapsto g^{-1}$ and $\check{U}$ the potential of the random walk $(\check{R}_n)_{n\geq 0}$ on H with law $\check{\mu}$. To prove Lemma 7.6, it suffices to check that the map $h \mapsto U\phi(h)$ is locally integrable on H with respect to dh. Let K be a compact set; using the fact that dh is right-invariant, we may write

$$\begin{aligned}\int_H U\phi(g)\ \mathbf{1}_K(g)\mathrm{d}g &= \int_H\int_H \phi(gh)\ \mathbf{1}_K(g)U(\mathrm{d}h)\mathrm{d}g\\ &= \int_H\int_H \phi(k)\ \mathbf{1}_K(kh^{-1})U(\mathrm{d}h)\mathrm{d}k\\ &= \int_H \phi(k)\check{U}\ \mathbf{1}_K(k)\mathrm{d}k.\end{aligned}$$

The random walk $(L_n)_{n\geq 0}$ is transient on H, the same holds for the random walk $(\check{L}_n)_{n\geq 0}$ with law $\check{\mu}$; thus, the potential $\check{U}\ \mathbf{1}_K(h)$ is finite, and even uniformly bounded by the maximum principle; it yields

$$\int_H U\phi(F_1)\ \mathbf{1}_K(F_1)\mathrm{d}F_1 \leq \sup_{h\in H} \check{U}\ \mathbf{1}_K(h)\ \int_H \phi(h)\mathrm{d}g < +\infty.$$

□

Now, we consider the following subset C of $\mathbb{H} = \mathbb{R}\times\mathbb{R}^{*+}$.

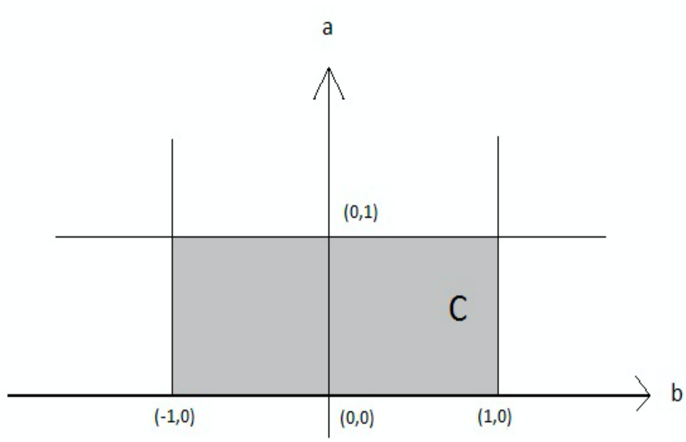


Figure 1: The set $C = \Big\{g = (a, b) \mid a \in]0, 1[\text{ and } b \in] -1, 1[\Big\}$

The following lemmas states that, $\mathbb{P}$-a.s., the right random walk $(gR_n)_{n\geq 0}$ does not cross infinitely often ∂C.

Lemma 7.7. *Denote and assume H1 and H2. For* dadb*-almost all* $g = (a, b) \in \text{Aff}(\mathbb{R})$*,*

$$\mathbb{P}\Big(gR_{n+1} \in C, gR_n \notin C \text{ infinitely often}\Big) = 0.$$

By Borel-Cantelli's lemma, it suffices to prove that $\sum_{n=0}^{+\infty} \mathbb{P}(gR_{n+1} \in C, gR_n \notin C) < +\infty$.

Yet, $\mathbb{P}(gR_{n+1} \in C, gR_n \notin C) = \mathbb{E}\Big(\mathbb{P}(gR_n g_{n+1} \in C/R_n)\ \mathbf{1}_{C^c}(gR_n)\Big) = \mathbb{E}(\phi(gR_n))$, where ϕ is the positive Borel function from $\text{Aff}(\mathbb{R})$ to $\mathbb{R}$ defined by $\phi(g) := \mathbb{P}(gg_1 \in C)\ \mathbf{1}_{C^c}(g)$. In other words,

$$\sum_{n=0}^{+\infty} \mathbb{P}(gR_{n+1} \in C, gR_n \notin C) = U\phi(g).$$

Thus, it suffices to check that ϕ is integrable with respect to the right Haar measure $\mathrm{d}g = \frac{\mathrm{d}a\ \mathrm{d}b}{a}$. Indeed, the random walk $(L_n)_{n\geq 0}$ is transient since $\text{Aff}(\mathbb{R})$ is non unimodular, thus we may apply Lemma 7.6. On the one hand $\int\int \phi(a,b)\ \mathbf{1}_{(a\geq 1)}\phi(a,b)\frac{\mathrm{d}a\ \mathrm{d}b}{a} < +\infty$; indeed,

$$\begin{aligned}
\int\int \phi(a,b)\ \mathbf{1}_{(a\geq 1)}\frac{\mathrm{d}a\ \mathrm{d}b}{a} &= \mathbb{E}\Big(\int_1^{+\infty}\int_{\mathbb{R}} \mathbf{1}_C((a,b)(a_1,b_1))\frac{\mathrm{d}a\ \mathrm{d}b}{a}\Big)\\
&= \mathbb{E}\Big(\int_1^{+\infty}\int_{\mathbb{R}} \mathbf{1}_{(aa_1<1)}\ \mathbf{1}_{(|ab_1+b|<1)}\frac{\mathrm{d}a\ \mathrm{d}b}{a}\Big)\\
&= \mathbb{E}\Big(\int_1^{+\infty} \mathbf{1}_{(aa_1<1)}\Big(\int_{-1-ab_1}^{1-ab_1}\mathrm{d}b\Big)\frac{\mathrm{d}a}{a}\Big)\\
&= 2\mathbb{E}\Big(\log\Big(\frac{1}{a_1}\vee 1\Big)\Big) = 2\mathbb{E}(\log^-(a_1)).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int\int \phi(a,b)\ \mathbf{1}_{(a<1,|b|\geq 1)}\frac{\mathrm{d}a\ \mathrm{d}b}{a} &\leq \mathbb{E}\Big(\int_0^1\Big(\int_{\mathbb{R}} \mathbf{1}_{(|b|\geq 1)}\ \mathbf{1}_{(|ab_1+b|<1)}\mathrm{d}b\Big)\frac{\mathrm{d}a}{a}\Big)\\
&\leq \mathbb{E}\Big(\int_0^1 (|ab_1|\wedge 2)\frac{\mathrm{d}a}{a}\Big)\\
&\qquad \text{because} \quad \int_{\mathbb{R}} \mathbf{1}_{(|b|\geq 1)}\ \mathbf{1}_{(|b-x|<1)}\mathrm{d}b \leq |x|\wedge 2\\
&= \mathbb{E}\Big(\ \mathbf{1}_{(0<|b_1|\leq 2)}\int_0^1 (|ab_1|\wedge 2)\frac{\mathrm{d}a}{a}\\
&\qquad + \mathbf{1}_{(|b_1|\geq 2)}\int_0^1 (|ab_1|\wedge 2)\frac{\mathrm{d}a}{a}\Big)\\
&= \mathbb{E}\Big(|b_1|\ \mathbf{1}_{(0<|b_1|\leq 2)}) + (2 + 2\log(|b_1|/2))\ \mathbf{1}_{(|b_1|\geq 2)}\Big)\\
&\leq 2 + 2\mathbb{E}(\log|b_1|) < +\infty.
\end{aligned}$$

The proof is complete.

□

Theorem 7.4. By Lemma 7.7, for almost all $g \in \text{Aff}(\mathbb{R})$, $\mathbb{P}$-a.s., after a certain (random) time, the sequence $(gR_n)_{n\geq 0}$ belongs either to C or to C^c. The condition $\mathbb{E}(\log a_1) = 0$ implies that $(\log(a_n \dots a_1))_{n\geq 0}$ is recurrent on $\mathbb{R}$, thus gR_n belongs to C^c when n is great enough. Eventually, for almost all $g \in \text{Aff}(\mathbb{R})$, the sequence $(R_n)_{n\geq 0}$ stays inside gC^c after a certain (random) time. Observe that

$$gC^c = \Big\{(a,b)|a > a(g) \text{ or } b > b(g) + a(g) \text{ or } b < b(g) - a(g)\Big\}$$

and that, for all $k \geq 1$, the set $\Big\{g \in \text{Aff}(\mathbb{R}) \mid gC^c \subset \{(a,b)| \max\{a,|b|\} > k\Big\}$ has positive Haar measure. Thus, we may choose a sequence $(g_k)_{k\geq 0}$ in Aff($\mathbb{R}$) such that $R_n \in \{(a,b)| \max\{a,|b|\} > k\}$ for n great enough (i.e. n greater than some m depending on ω and k). In particular

$$\lim_{n\to+\infty} \max\{a(R_n), |b(R_n)|\} = \infty.$$

For a generic starting point g_0, observe that

$$\lim_{n\to+\infty} \max\{a(g_0R_n), |b(g_0R_n)|\} = \lim_{n\to+\infty} \max\{a(g_0)a(R_n), |a(g_0)b(R_n) + b(g_0)|\} = \infty.$$

□

7.4.2 Local proximality: an alternative proof

We propose here and alternative proof of Corollary 7.5, studying directly the left products L_n on Aff($\mathbb{R}$). First, we state the analogous of Lemma 7.5 in this context.

Lemma 7.8. *Let H be a second-countable locally compact group with denumerable basis. Denote dh its left Haar measure and $U(h,\cdot) := \sum_{n=0}^{+\infty} \mu^{*n} * \delta_h, h \in H$, the potential of a transient random walk $(L_n)_{n\geq 0}$ on H with law μ. For any $\phi \in \mathbb{L}^1(H)$, the map*

$$h \mapsto U\phi(h) := \sum_{n=0}^{\infty} \mathbb{E}(\phi(L_n h))$$

is dh-a.s. finite.

Proof. Let us denote $\check{\mu}$ the image of the measure μ by the transformation $g \mapsto g^{-1}$ and $\check{U}$ the potential of the random walk $(\check{L}_n)_{n\geq 0}$ on H with law $\check{\mu}$. To prove Lemma 7.8, it suffices to check that the map $h \mapsto U\phi(h)$ is locally integrable on H with respect to dh. Let K be a compact set; using the fact that dh is left-invariant, we may write

$$\begin{aligned}
\int_H U\phi(g)\ \mathbf{1}_K(g)\mathrm{d}g &= \int_H \int_H \phi(hg)\ \mathbf{1}_K(g) U(\mathrm{d}h)\mathrm{d}g \\
&= \int_H \int_H \phi(k)\ \mathbf{1}_K(h^{-1}k) U(\mathrm{d}h)\mathrm{d}k \\
&= \int_H \phi(k)\check{U}\ \mathbf{1}_K(k)\mathrm{d}k.
\end{aligned}$$

The random walk $(L_n)_{n\geq 0}$ is transient on H, the same holds for the random walk $(\check{L}_n)_{n\geq 0}$ with law $\check{\mu}$; thus, the potential $\check{U}\ \mathbf{1}_K(h)$ is finite, and even uniformly bounded by the maximum principle; it yields

$$\int_H U\phi(F_1)\ \mathbf{1}_K(F_1)\mathrm{d}F_1 \leq \sup_{h\in H} \check{U}\ \mathbf{1}_K(h) \int_H \phi(h)\mathrm{d}g < +\infty.$$

□

Now, we may prove directly that the sequence $(L_n)_{n\geq 0}$ is $\mathbb{P}$-a.s. locally proximal on $\mathbb{R}$. Let

$$D = \{(a,b)|a \geq 1 \text{ and } |b| \leq a\} = \{(a,b)|a \geq \max\{a,|b|\}.$$

First, we prove that, $\mathbb{P}$-a.s., the left random walk $(L_n g)_{n\geq 0}$ does not cross infinitely often ∂D, i.e. that for dadb-almost all $g = (a,b) \in \text{Aff}(\mathbb{R})$,

$$\mathbb{P}\Big(L_{n+1}g \in D, L_n g \notin D \text{ infinitely often}\Big) = 0. \tag{7.1}$$

By Borel-Cantelli's lemma, it suffices to prove that $\sum_{n=0}^{+\infty} \mathbb{P}(L_{n+1}g \in D, L_n g \notin D) < +\infty$. First, we write the quantity $\mathbb{P}(L_{n+1}g \in D, L_n g \notin D)$ as

$$\mathbb{P}(L_{n+1}g \in D, L_n g \notin D) = \mathbb{E}\Big(\mathbb{P}(g_{n+1}L_n g \in D/L_n)\ \mathbf{1}_{D^c}(L_n g)\Big) = \mathbb{E}(\phi(L_n g)),$$

where ϕ is the positive Borel function from $\mathrm{Aff}(\mathbb{R})$ to $\mathbb{R}$ defined by

$$\phi(g) := \mathbb{P}(g_1 g \in D)\ \mathbf{1}_{D^c}(g) = \mathbb{P}((a_1 a, a_1 b + b_1) \in D)\ \mathbf{1}_{D^c(a,b)}.$$

It yields

$$\sum_{n=0}^{+\infty} \mathbb{P}(L_{n+1}g \in D, L_n g \notin D) = U\phi(g).$$

Since $\mathrm{Aff}(\mathbb{R})$ is non unimodular, the random walk $(L_n)_{n\geq 0}$ is transient; by Lemma 7.8, it suffices to check that ϕ is integrable with respect to the left Haar measure $\mathrm{d}g = \frac{\mathrm{d}a\ \mathrm{d}b}{a^2}$. Noticing that

$$D^c = \{(a,b) | a \geq 1 \text{ and } b > |a|\} \cup \{(a,b) | a < 1\},$$

we split the integral in two parts. On the one hand,

$$\begin{aligned}
\int \mathbb{P}((a_1 a, a_1 b + b_1) \in D)\ \mathbf{1}_{(a<1)} \frac{\mathrm{d}a\mathrm{d}b}{a^2} &= \mathbb{E}\left(\int \mathbf{1}_{(a,b)\in D}\ \mathbf{1}_{(a<a_1)} \frac{\mathrm{d}a\mathrm{d}b}{a^2}\right) \\
&= \mathbb{E}\left(\int\int \mathbf{1}_{(|b|\leq a)} \mathrm{d}b\ \mathbf{1}_{(1\leq a<a_1)} \frac{\mathrm{d}a}{a^2}\right) \\
&= \mathbb{E}\left(\int 2a\ \mathbf{1}_{(1\leq a<a_1)} \frac{\mathrm{d}a}{a^2}\right) \\
&= \mathbb{E}(2\log^+ a_1);
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\int \mathbb{P}((a_1 a, a_1 b + b_1) \in D)\ \mathbf{1}_{(a\geq 1, |b|>a)} \frac{\mathrm{d}a\mathrm{d}b}{a^2} &= \mathbb{E}\left(\int \mathbf{1}_{(a_1 a \geq 1, |a_1 b + b_1| \leq a_1 a)}\ \mathbf{1}_{(a\geq 1, |b|>a)} \frac{\mathrm{d}a\mathrm{d}b}{a^2}\right) \\
&= \mathbb{E}\left(\int\int \mathbf{1}_{(|a_1 b + b_1| \leq a_1 a, |b|>a)} \mathrm{d}b\ \mathbf{1}_{(a \geq \max\{a_1, 1)\}} \frac{\mathrm{d}a}{a^2}\right) \\
&\leq \mathbb{E}\left(\int\int \mathbf{1}_{(|b+\frac{b_1}{a_1}|\leq a, |b|>a)}\ \mathrm{d}b\ \mathbf{1}_{(a\geq 1)} \frac{\mathrm{d}a}{a^2}\right) \\
&\leq \mathbb{E}\left(\int_1^\infty \mathbf{1}_{(|\frac{b_1}{a_1}|\leq a)} 2|\frac{b_1}{a_1}| + \mathbf{1}_{(|\frac{b_1}{a_1}|>a)} 2a \frac{\mathrm{d}a}{a^2}\right) \\
&\leq \mathbb{E}\left(\int_{\max\{|\frac{b_1}{a_1}|, 1\}}^\infty 2|\frac{b_1}{a_1}| \frac{\mathrm{d}a}{a^2}\right) + 2\mathbb{E}(\log^+ |\frac{b_1}{a_1}|) \\
&= 2\mathbb{E}\left(\frac{|\frac{b_1}{a_1}|}{\max\{|\frac{b_1}{a_1}|, 1\}}\right) + 2\mathbb{E}(\log^+ |\frac{b_1}{a_1}|) \\
&\leq 2 + 2\mathbb{E}(\log^+ |\frac{b_1}{a_1}|).
\end{aligned}$$

The proof of (7.1) is complete; let us now conclude. Since $\liminf_{n\to+\infty} a(L_n) = 0$ $\mathbb{P}$-a.s., for every $g \in \mathrm{Aff}(\mathbb{R})$, the random walk $L_n g$ visits D^c infinitely often. Then, $\mathbb{P}(d\omega)$-a.s and $\mathrm{d}g$-a.s., there exists an integer $N = N(\omega, g)$ such that, for any $n \geq N$,

$$a(L_n g) \leq \max\{1, |b(L_n(\omega)g)|\}.$$

Thus, for all $x \in \mathbb{R}$, it holds, $\mathbb{P}$-a.s, $\mathrm{d}a\mathrm{d}b$-almost surely and for $n \geq N = N(\omega, (a,b))$,

$$a(L_n(\omega))a \leq \max\{1, |a(L_n(\omega))(x+b) + b(L_n)|\} \leq \max\{1, |a(L_n(\omega))x + b(L_n(\omega))|\} + a(L_n(\omega))|b|. \quad (7.2)$$

In particular, if $|b| < a$,

$$a(L_n(\omega)) \leq \frac{1}{a - |b|} \max\{1, |a(L_n(\omega))x + b(L_n(\omega))|\}, \text{ for all } n > N.$$

For any $k \in \mathbb{N}$, we can chose an element (a_k, b_k) in the set of positive measure $\{a - |b| > k\}$ such that inequality (7.2) holds for any ω in a set of full measure, denoted by Ω_k. In particular, this inequality holds for all $k \geq 0$ and all $\omega \in \bigcap_{k \in \mathbb{N}} \Omega_k$ (which is of full $\mathbb{P}$-measure). Eventually, $\mathbb{P}$-a.s., it holds, for all $k \in \mathbb{N}$ and all n large enough,

$$a(L_n) \leq \frac{1}{k} \max\{1, |a(L_n)x + b(L_n)|\}.$$

Consequently, for all $x \in \mathbb{R}$, all $y \in \mathbb{R}$ and all $C > 0$,

$$\lim_{n\to\infty} |L_n(x) - L_n(y)| \, \mathbf{1}_{(|L_n(x)| \leq C)} = \lim_{n\to\infty} a(L_n)|x-y| \, \mathbf{1}_{(|a(L_n)x + b(L_n)| \leq C)} = 0 \qquad \mathbb{P}\text{-a.s.}$$

8 Asymptotically affine SDS: tail of the invariant measure

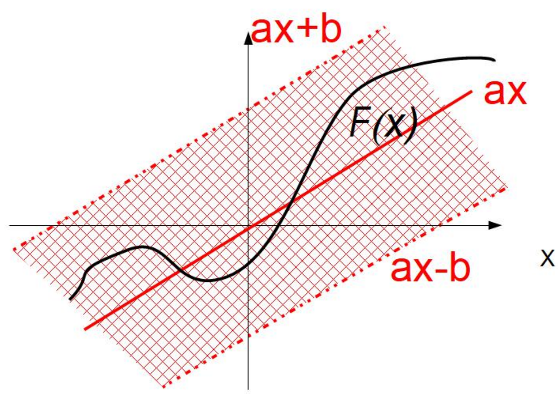
In this section, we study a large class of positive recurrent SDSs $(F_n)_{n\geq 1}$ on $\mathbb{R}$ and study the tail of their invariant probability ν. In other words, we are interested in the behavior as $t \to +\infty$ of the function $t \mapsto \nu[t, +\infty[$. This yields information on the existence of moments for μ and limit theorems for some functional of the Markov chain $(X_n)_{n\geq 0}$.

8.1 Asymptotically linear SDS

We enlarge here the class of affine recursions on $\mathbb{R}$ to SDSs whose corresponding random maps F_n are "asymptotically" linear maps.

From now on, we consider a SDS $(X_n)_{n\geq 0}$ generated by a sequence of i.i.d. Lipschitz functions $(F_n)_{n\geq 1}$ with law μ acting on an unbounded closed subset $\mathbb{X}$ of the real line. In the most interesting examples $\mathbb{X}$ equals $\mathbb{R}$, $[0, +\infty)$ or the set of natural numbers $\mathbb{N}$. We say that this SDS in **asymptotically linear**, with bounded error, if for μ-almost all F, there exist two real numbers $a = a(F) \geq 0$ and $b = b(F)$ such that

$$\forall x \in \mathbb{X} \qquad |F(x) - ax| \leq b. \quad (8.1)$$



Example of an asymptotic linear function.

We set $a_n = a(F_n)$ and $b_n = b(F_n)$. For any $x \in \mathbb{R}$, the Markov chain $(X_n^x)_{n\geq 0}$ is bounded from below and above by two affine recursions $(\underline{X}_n^x)_{n\geq 0}$ and $(\overline{X}_n^x)_{n\geq 0}$. More precisely, for any $n \geq 1$

$$\underline{X}_n^x \;\leq\; X_n^x \;\leq\; \overline{X}_n^x \quad (8.2)$$

where $\underline{X}_n^x = a_n \underline{X}_{n-1}^x - b_n$ and $\overline{X}_n^x = a_n \overline{X}_{n-1}^x + b_n$ for any $n \geq 1$.

Lemma 8.1. *Assume that $\int \log^+ a(F)\mu(dF) < \infty$, $\int \log^+ b(F)\mu(dF) < \infty$ and $\int \log a(F)\mu(dF) < 0$. Then, the asymptotically linear SDS $(X_n^x)_{n\geq 0}$ is topologically positive recurrent and possesses (at least) one invariant probability measure ν.*

Let us emphasize that measure ν is not necessarily unique since the random maps F_n are not necessarily locally proximal. Indeed, the random maps may fix two proper closed sets, $\mathbb{X}_1$ and $\mathbb{X}_2$, of $\mathbb{X}$ (i.e. $F_n(\mathbb{X}_1) \subseteq \mathbb{X}_1$ and $F_n(\mathbb{X}_2) \subseteq \mathbb{X}_2$ $\mathbb{P}$-a.s.), in which case each subset $\mathbb{X}_1$ and $\mathbb{X}_2$ may support an invariant measure. Let us for instance consider the SDS defined by

$$F_n(x) = a_n x + a'_n \frac{x}{1+|x|},$$

where

(i) a_n and a'_n are bounded random variables with values in $\mathbb{R}^+$,

(ii) $\mathbb{E}(\log a_n) < 0$,

(iii) $\mathbb{E}(\log(a_n + a'_n)) > 0$.

This SDS admits one invariant measure supported by $[0, +\infty)$ and an another one supported by $(-\infty, 0]$.

Proof of Lemma 8.2. Let $(\underline{R_n})_{n\geq 0}$ (resp. $(\overline{R_n})_{n\geq 0}$)be the right random walks on $\mathrm{Aff}(\mathbb{R})$ corresponding to the maps $x \mapsto a_k x - b_k$ and $x \mapsto a_k x + b_k$. Under the above assumptions, for any $x \in \mathbb{R}$, the sequence $(\underline{R_n} \cdot x, \overline{R_n} \cdot x)_{n\geq 0}$ converges $\mathbb{P}$-a.s. to a $\mathbb{R}^2$-valued random variable Z_∞; thus, $(\underline{X_n}^x, \overline{X_n}^x)_{n\geq 0}$ converges in distribution to Z_∞ and is recurrent in $\mathbb{R}^2$.

The existence of the invariant probability follows from M. Lin's theorem.

□

The affine recursion is the key example of "asymptotically linear SDS", but this class is much wider and flexible. Many SDSs are asymptotically linear after a change of the metric of $\mathbb{R}$; in other words, they are asymptotically linear after conjugacy. More precisely, let $r : \mathbb{R} \to \mathbb{R}$ be a continuous increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ (or a subset of $\mathbb{R}$) and set $\widetilde{F}_n := r \circ F_n \circ r^{-1}$ for any $n \geq 1$. The SDS $(\widetilde{X}_n)_{n\geq 0}$ associated to the sequence $(\widetilde{F}_n)_{n\geq 1}$ satisfies the following equality:

$$\forall x \in \mathbb{R}, \qquad \widetilde{X}_n^x = \widetilde{F}_n \cdots \widetilde{F}_1(x) = r \circ F_n \cdots F_1 \circ r^{-1}(x) = r(X_n^{r^{-1}(x)}).$$

This expression yields to the followings elementary remarks.

- $(X_n)_{n\geq 0}$ is recurrent if and only if $(\widetilde{X}_n)_{n\geq 0}$ is recurrent.
- ν is F-invariant measure if and only if $r * \nu$ is $\widetilde{F}$-invariant.

Example. The reflected random walk on $\mathbb{R}^+$ with elastic collisions at 0 is conjugated to an asymptotic linear SDS. Indeed, Notice that $F_n(x) = |x - Y_n| = x - Y_n$ for $x > Y_n$; thus, we may set $r(x) = e^x$ for any $x \in \mathbb{R}$ so that $r \circ F_n \circ r^{-1}(y) = e^{\log y - Y_n} = e^{-Y_n} y$ for $y > 0$ large enough.

8.2 Renewal in direction opposite to the drift

In the contracting case, the (unique) invariant probability measure of the affine recursion is given by the law of the random variable $Z_\infty = \sum_{n=0}^{+\infty} a_1 \cdots a_n b_{n+1}$; the general term of this series converges to zero exponentially fast, thus the quantity Z_∞ seems to be of the same order as the random variable $\max\{a_1 \cdots a_n \mid n \geq 0\}$, which is $\mathbb{P}$-a.s. finite since $a_1 \cdots a_n \to 0$ $\mathbb{P}$-a.s. as $n \to +\infty$.

From now on, for any $n \geq 1$, we set $u_n := \log a_n$ and $S_n := u_1 + \cdots + u_n$ so that $\log a_1 \ldots a_n = S_n$. By convention $S_0 = 0$. The random walk $(S_n)_{n\geq 0}$ goes $\mathbb{P}$-a.s. to $-\infty$; to understand how far it can go in the opposite direction to its drift, we now control the tail of the ($\mathbb{P}$-a.s. finite) random variable $M := \max\{S_n \mid n \geq 0\}$ when $E(u_1) < 0$.

The random walk $(S_n)_{n\geq 0}$ being transient, its Green function

$$U\ \mathbf{1}_{[a,b]}(s) := \sum_{n=0}^{\infty} \mathbb{E}\left(\ \mathbf{1}_{[a,b]}(-s+S_n)\right)$$

is finite for any $s \in \mathbb{R}$ and any compact interval $[a, b]$; the quantity $U\ \mathbf{1}_{[a,b]}(s)$ also equals the mean number of visits of the interval $[s+a, s+b]$ by the random walk $(S_n)_{n\geq 0}$, its behavior as $s \to -\infty$ yields some information about the way $(S_n)_{n\geq 0}$ tends to $-\infty$. It is the focus of the "renewal theory": the description of the asymptotic behavior of the Green functions $Ug(s) := \sum_{n=0}^{\infty} \mathbb{E}\left(g(-s+S_n)\right)$ when $s \to +\infty$ and $\mathbb{E}(u_1) < 0$, for a class of test functions g to be precised.

The test functions g we consider are continuous on $\mathbb{R}$ and satisfies the condition

$$\sum_{n\in\mathbb{Z}} \sup_{x\in[n,n+1]} |g(x)| < \infty. \tag{8.3}$$

Such a function g can be uniformly approximated by step functions; in other words, it is a *directly Riemann integrable* functions [9].

Now, we may state the main result of this section.

Theorem 8.2. *Let $(u_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with values in $[-\infty, +\infty)$ such that*

(i) the law of the u_n is adapted to $\mathbb{R}$,

(ii) the r.v. u_1^+ is integrable and $\mathbb{E}(u_1) < 0$,

(iii) there exists $\alpha > 0$ such

$$\mathbb{E}(e^{\alpha u_1}) = 1 \text{ and } \mathbb{E}(|u_1|e^{\alpha u_1}) < +\infty. \tag{8.4}$$

(with the convention $e^{-\infty} = 0$).

Then, for any continuous function g from $\mathbb{R}$ to $\mathbb{R}$ such that the function $x \mapsto e^{-\alpha x}g(x)$ satisfies condition (8.3), it holds

$$\lim_{t\to+\infty} e^{\alpha t} \sum_{n=0}^{\infty} \mathbb{E}\left(g(S_n - t)\right) = \frac{1}{\mathbb{E}_\delta(u_1 e^{\alpha u_1})} \int_{\mathbb{R}} g(y) e^{-\alpha x} \mathrm{d}x \tag{8.5}$$

and there exists a constant $C > 0$ such that

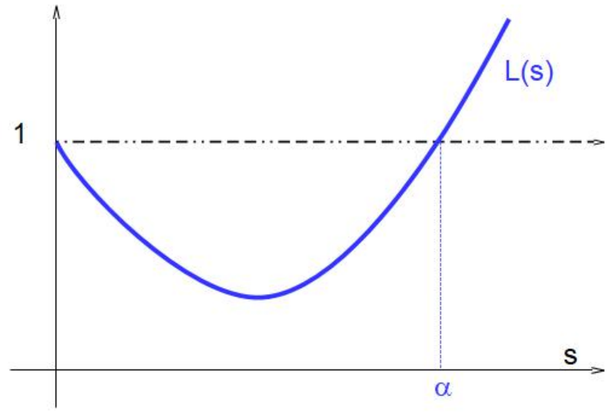
$$\lim_{t\to+\infty} e^{\alpha t} \mathbb{P}(\max_{n\geq 0} S_n > t) = C. \tag{8.6}$$

Proof. Following Kesten's approach, we construct a "perturbation" $(\widetilde{S}_n)_{n\geq 0}$ of $(S_n)_{n\geq 0}$ with a strictly positive drift, and we apply the classical renewal theorem to $(\widetilde{S}_n)_{n\geq 0}$. Thus, the proof is decomposed into two steps.

Step 1: change of the distribution of the increments

The Laplace transform L_μ of the distribution μ is defined formally by $L_\mu(s) := \mathbb{E}(e^{su_1})$. It is defined on $[0, a]$ as soon as $\mathbb{E}(e^{au_1}) < +\infty$ and it is convex and continuous on $[0, a]$; furthermore, $L(0) = \mathbb{P}(u_1 > -\infty) \leq 1$ and $\lim_{s\to+\infty} L_\mu(s) = +\infty$ if $\mathbb{P}(u_1 > 0) > 0$.

[9] A non-negative function f, defined on the real line or on a half-line, is said to be directly Riemann integrable (dRi) if the upper and lower Riemann sums of f over the whole (unbounded) domain converge to the same finite limit, as the mesh of the partition vanishes.
Heuristically, dRi functions are not too "wide", they can be uniformly approximated by step functions.



The Laplace transform $L_\mu(s)$ on $\mathbb{R}^+$

From now on, we assume that there exists a real number $\alpha > 0$ such that $\mathbb{E}(e^{\alpha u_1}) = 1$. The measure $\tilde{\mu}(\mathrm{d}x) = e^{\alpha x}\mu(\mathrm{d}x)$ is a probability measure on $\mathbb{R}$ and, for any $n \geq 0$,

$$\tilde{\mu}^{\star n}(\mathrm{d}x) = e^{\alpha x}\mu^{\star n}(\mathrm{d}x). \tag{8.7}$$

If μ is adapted on $\mathbb{R}$, the same holds for $\widetilde{\mu}$. Let $(\widetilde{u}_n)_{n\geq 1}$ be a sequence of i.i.d. real random variables with law $\widetilde{\mu}$. Equality (8.7) may be extended as follows: for any $n \geq 1$ and bounded Borel function $\phi : \mathbb{R}^n \to \mathbb{R}$, it holds

$$\mathbb{E}(\phi(u_1, \cdots, u_n)) = \mathbb{E}\Big(e^{-\alpha \widetilde{S}_n}\phi(\widetilde{u}_1, \cdots, \widetilde{u}_n)\Big) \tag{8.8}$$

The Green functions of the r.w. $(S_n)_{n\geq 0}$ and $(\widetilde{S}_n)_{n\geq 0}$ are closely related; indeed, for any dRI continuous function $g : \mathbb{R} \to \mathbb{R}$, it holds

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}\left(g(S_n - t)\right) &= \sum_{n=0}^{\infty} \mathbb{E}\left(e^{-\alpha \widetilde{S}_n} g(\widetilde{S}_n - t)\right) \\ &= e^{-\alpha t}\sum_{n=0}^{\infty} \mathbb{E}\left(e^{-\alpha(\widetilde{S}_n - t)} g(\widetilde{S}_n - t)\right)\end{aligned}$$

Step 2: the renewal theorem for random walks on $\mathbb{R}$ with positive drift

The random walk $(\widetilde{S}_n)_{n\geq 0}$ has positive drift $\mathbb{E}\widetilde{u}_n > 0$ and is adapted to $\mathbb{R}$; it thus satisfies the following statement.

Proposition 8.3 (Renewal Theorem)**.** *For any $t > 0$, set $\widetilde{T}_t := \inf\{n \geq 0, \widetilde{S}_n > t\}$ and let $\widetilde{R}_t$ be the "residual time" $\widetilde{R}_t = \widetilde{S}_{\widetilde{T}_t} - t$.*

If the continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies (8.3), then

$$\mathbb{E}\Big[\sum_{n\geq 0} g(-t + \widetilde{S}_n)\Big] \to \frac{1}{\mathbb{E}(\widetilde{u}_1)}\int_{\mathbb{R}} g(x)\mathrm{d}x, \qquad \text{as } t \to +\infty. \tag{8.9}$$

Furthermore, if $\displaystyle\int_0^{+\infty} g(x)\mathrm{d}x > 0$, there exists $C(g) > 0$ such that

$$\mathbb{E}\left(g(\widetilde{R}_t)\ \mathbf{1}_{(\widetilde{T}_t<\infty)}\right) \to C(g) > 0, \quad \text{as} \quad t \to +\infty. \tag{8.10}$$

Notice that $\mathbb{P}(\widetilde{T}_t < +\infty) = 1$ since $\widetilde{S}_n \to +\infty$ $\mathbb{P}$-a.s. as $n \to +\infty$. The convergence (8.9) is the classical renewal theorem on $\mathbb{R}$; we refer to [4] for the proof. Now, we explain how (8.10) follows, in the case when the $\widetilde{u}_n$ are $\mathbb{R}^+$-valued; it holds

$$\begin{aligned}
\mathbb{E}\Big(g(-t+\widetilde{S}_n)\ \mathbf{1}_{(\widetilde{T}_t<\infty)}\Big) &= \sum_{n=1}^{+\infty}\mathbb{E}(g(\widetilde{R}_t);\widetilde{T}_t=n)\\
&= \sum_{n=1}^{+\infty}\mathbb{E}\left(g(-t+\widetilde{S}_{n-1}+\widetilde{u}_n);t-\widetilde{u}_n<\widetilde{S}_{n-1}\le t\right)\\
&\quad \text{(this is here that we use the fact that the } \widetilde{u}_n \text{ are } \mathbb{R}^+-\text{valued)}\\
&= \int_0^\infty\sum_{n=1}^{+\infty}\mathbb{E}(g(-t+\widetilde{S}_{n-1}+y)\ \mathbf{1}_{]-y,0]}(-t+\widetilde{S}_{n-1})\widetilde{\mu}(\mathrm{d}y)\\
&\overset{t\to+\infty}{\longrightarrow} \frac{1}{\mathbb{E}(\widetilde{u}_1)}\int_0^\infty g(x)\widetilde{\mu}(x,+\infty)\mathrm{d}x>0.
\end{aligned}$$

Now, let us achieve the proof of Theorem 8.2. Convergence (8.5) follows applying Proposition 8.3. To prove (8.6) observe that (8.8) yields

$$\begin{aligned}
e^{\alpha t}\mathbb{P}\left(\sup_{n\ge 0}S_n>t\right) &= e^{\alpha t}\mathbb{P}(\widetilde{T}_t<\infty)\\
&= e^{\alpha t}\sum_{n\ge 1}\mathbb{E}\left(\ \mathbf{1}_{(\widetilde{T}_t=n)}\right)\\
&= \sum_{n\ge 1}\mathbb{E}\left(\ \mathbf{1}_{(\widetilde{T}_t=n)}e^{-\alpha(\widetilde{S}_n-t)}\right)\\
&= \mathbb{E}\left(e^{-\alpha\widetilde{R}_t}\ \mathbf{1}_{(\widetilde{T}_t<\infty)}\right).
\end{aligned}$$

We conclude applying (8.10).

□

8.3 Tail of the invariant measure

Theorem 8.4. *Let $(X_n)_{n\ge 0}$ be an asymptotically linear SDS associated to the sequence of i.i.d. functions $(F_n)_{n\ge 1}$ and let $a_n=a(F_n)$ and $b_n=b(F_n)$ for any $n\ge 1$.*

Suppose that a_1 is not supported on a discrete subgroup of $\mathbb{R}_+^*$ and that there exists $\alpha>0$ such that

- $\mathbb{E}(a_1^\alpha)=1$ *and* $\mathbb{E}(a_1^\alpha|\log a_1|)<\infty$,
- $\mathbb{E}(b_1^\alpha)<\infty$.

Then, it holds $\mathbb{E}(\log a_1)<0$ and there exists on $\mathbb{R}$ at least one invariant probability measure for ν for $(X_n)_{n\ge 0}$. For any bounded Lipschitz function ϕ on $(0,+\infty)$ whose support is bounded from above, let $g_\phi:(0,+\infty)\to\mathbb{R}$ be defined by

$$\forall s\in\mathbb{R},\quad g_\phi(s):=\int_{\mathbb{R}}\mathbb{E}(\phi(e^s x)-\phi(e^s a_1 x))\nu(\mathrm{d}x)$$

.

Then, the function $s\mapsto e^{-\alpha s}g_\phi(s)$ is dRI (i.e. satisfies condition (8.3)) and

$$\lim_{z\to+\infty}z^\alpha\int_{\mathbb{R}}\phi(z^{-1}x)\nu(\mathrm{d}x)=\frac{1}{\mathbb{E}(a_1^\alpha\log a_1)}\int_{\mathbb{R}}e^{-\alpha s}g_\phi(s)\mathrm{d}s\in\mathbb{R}. \tag{8.11}$$

In particular, there exists $C\ge 0$ such that $\lim_{z\to+\infty}z^\alpha\nu(z,\infty)=C$, with $C>0$ if the support of the measure ν is not bounded from above.

Proof. Since $\mathbb{E}(\log a_1) < 1$, there exists on $\mathbb{R}$ (at least) one invariant probability measure ν; let R be a random variable with law ν, independent on $(F_n)_{n\geq 1}$.

We want to study the behavior as $t \to +\infty$ of the quantity $\mathbb{E}(\phi(e^{-t}R)) = \int_{\mathbb{R}} \phi(e^{-t}x)\nu(\mathrm{d}x)$.

Step 1: From the tail to the potential.

The inequality $\mathbb{E}(\log a_1) < 1$ implies that, for all bounded measurable function ϕ whose support does not contain 0, it holds

$$\mathbb{E}(\phi(e^{-t}R)) = \sum_{n=0}^{\infty} \mathbb{E}\left(g_\phi(S_n - t)\right). \tag{8.12}$$

Indeed, let a be a copy of a_1 independent on the sequence $(a_n)_{n\geq 1}$; it holds

$$\begin{aligned}
\mathbb{E}(\phi(e^{-t}R)) &= \sum_{j=0}^{n-1} \mathbb{E}\left(\phi(e^{-t}a_1\cdots a_jR) - \phi(e^{-t}a_1\cdots a_ja_{j+1}R)\right) \\
&\qquad\qquad + \mathbb{E}\left(\phi(e^{-t}a_1\cdots a_nR)\right) \\
&= \sum_{j=0}^{n-1} \mathbb{E}\left(\phi(e^{S_j-t}R) - \phi(e^{S_j-t}aR)\right) + \mathbb{E}\left(\phi(e^{-t}a_1\cdots a_nR)\right) \\
&= \sum_{j=0}^{n-1} \mathbb{E}\left(g_\phi(S_j - t)\right) + \mathbb{E}\left(\phi(e^{-t}a_1\cdots a_nR)\right)
\end{aligned}$$

Since the support of ϕ does not contains 0 and $a_1\cdots a_n \to 0$ $\mathbb{P}$-a.s., the sequence $(\phi(e^{-t}a_1\cdots a_nR))_{n\geq 0}$ converges $\mathbb{P}$-a.s. to 0; hence, $\mathbb{E}\left(\phi(e^{-t}a_1\cdots a_nR)\right) \to 0$, the function ϕ being bounded.

If we can ensure that the function $s \mapsto g_\phi(s)e^{-\alpha s}$ is directly Riemann integrable, then convergence (8.11) is a direct consequence of the renewal Theorem 8.2. This is the aim of the following step.

Step 2. The function $s \mapsto e^{-\alpha s}g_\phi(s)$ is directly Riemann integrable. Observe that the function g_ϕ is continuous on $\mathbb{R}$ and that, using the fact that ν is invariant, it holds

$$g_\phi(s) = \mathbb{E}(\phi(e^sR) - \phi(e^sa_1R)) = \mathbb{E}(\phi(e^sF_1(R)) - \phi(e^sa_1R)).$$

We have to check that condition (8.3) holds, i.e. to establish the convergence of the series

$$\sum_{n=-\infty}^{+\infty} \sup_{n\leq s<(n+1)} e^{-\alpha s}|g_\phi(s)|.$$

First, we may write

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \sup_{n\leq s<(n+1)} e^{-\alpha s}|g_\phi(s)| &= \sum_{n=-\infty}^{+\infty} \sup_{n\leq s<n+1} e^{-\alpha s}\left|\mathbb{E}\left(\phi(e^sF_1(R)) - \phi(e^sa_1R)\right)\right| \\
&\leq \sum_{n=-\infty}^{+\infty} \sup_{n\leq s<n+1} e^{-\alpha s}\mathbb{E}\left(|\phi(e^sF_1(R)) - \phi(e^sa_1R)|\right) \\
&\leq \mathbb{E}\left(\underbrace{\sum_{n=-\infty}^{+\infty} \sup_{n\leq s<n+1} e^{-\alpha s}|\phi(e^sF_1(R)) - \phi(e^sa_1R)|}_{\Sigma(\phi)}\right)
\end{aligned} \tag{8.13}$$

We distinguish three cases according that α is smaller, greater or equal to 1.

Case $0 < \alpha < 1$.

We split the series $\Sigma(\phi)$, according to the value of n, smaller or bigger then $-\log b_1$. On the one hand,

$$\begin{aligned}\Sigma_1(\phi) &:= \sum_{n=-\log b_1}^{+\infty} \sup_{n\le s<n+1} e^{-\alpha s}|\phi(e^s F_1(R)) - \phi(e^s a_1 R)| \\ &\le |2\phi|_\infty \sum_{n=-\log b_1}^{+\infty} e^{-\alpha n} = \frac{2|\phi|_\infty}{1-e^{-\alpha}} b_1^\alpha,\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\Sigma_2(\phi) &:= \sum_{n=-\infty}^{-\log b_1} \sup_{n\le s<n+1} e^{-\alpha s}|\phi(e^s F_1(R)) - \phi(e^s a_1 R)| \\ &\le m(\phi) \sum_{n=-\infty}^{-\log b_1} \sup_{n\le s<n+1} e^{-(\alpha-1)s} b_1 \\ &= \frac{m(\phi)}{1-e^{1-\alpha}} b_1^\alpha,\end{aligned}$$

where $m(\phi)$ is the Lipschitz coefficient of ϕ. Thus for some constant $C = C_{\alpha,\phi} < \infty$

$$\sum_{n=-\infty}^{+\infty} \sup_{n\le s<(n+1)} e^{-\alpha s}|g_\phi(s)| \le \mathbb{E}\left(\Sigma(\phi)\right) \le C_{\phi,\alpha}\mathbb{E}(b_1^\alpha) < \infty.$$

Case $\alpha > 1$. The argument is more subtle. First recall that ϕ is Lipschitz continuous, with support included in $[\delta, +\infty)$ for some $\delta > 0$. Thus, for any $x, y \in \mathbb{R}$

$$|\phi(x) - \phi(y)| \le m(\phi)|x-y| \ \mathbf{1}_{(\max\{|x|,|y|\}>\delta)}.$$

Hence, by (8.1)

$$\begin{aligned}|\phi(e^s F_1(R)) - \phi(e^s a_1 R)| &\le m(\phi) e^s |F_1(R) - a_1 R| \ \mathbf{1}_{(\max\{e^s|F_1(R)|, e^s a_1|R|\}>\delta)} \\ &= m(\phi) e^s b_1 \ \mathbf{1}_{(e^s M>\delta)}\end{aligned}$$

where M is the random variable (depending on F_1 and R) defined by

$$M := \max\{|F_1(R)|; a_1|R|\}.$$

Then

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \sup_{n\le s<n+1} e^{-\alpha s}|\phi(e^s F(R)) - \phi(e^s a_1 R)| &\le m(\phi) \sum_{n=-\infty}^{+\infty} \sup_{n\le s<n+1} e^{-(\alpha-1)s} b_1 \ \mathbf{1}_{(e^s M>\delta)} \\ &\le m(\phi) \sum_{n=\log(\delta/M)-1}^{+\infty} \sup_{n\le s<n+1} e^{-(\alpha-1)s} b_1 \\ &\le m(\phi) \sum_{n=\log(\delta/M)-1}^{+\infty} e^{-(\alpha-1)n} b_1 \\ &\le m(\phi) \left(\frac{M}{\delta}\right)^{\alpha-1} \frac{e^{\alpha-1}}{1-e^{1-\alpha}} b_1.\end{aligned}$$

Now, it remains to check that $M^{\alpha-1}b_1 = \max\{|F_1(R)|, a_1|R|\}^{\alpha-1} b_1$ is integrable. The inequality $|F_1(R)| \le |a_1 R| + b_1 \le a_1|R| + b_1$ yields, for some constant $C = C(\alpha,\phi) > 0$,

$$\begin{aligned}\mathbb{E}\left(M^{\alpha-1}b_1\right) &\le C\mathbb{E}\left(a_1^{\alpha-1} b_1 |R|^{\alpha-1} + b_1^\alpha\right) \\ &= C\mathbb{E}\left(a_1^{\alpha-1} b_1\right)\mathbb{E}\left(|R|^{\alpha-1}\right) + C\mathbb{E}\left(b_1^\alpha\right) \\ &\le C\mathbb{E}\left(a_1^\alpha\right)^{\frac{\alpha-1}{\alpha}} \mathbb{E}\left(b_1^\alpha\right)^{\frac{1}{\alpha}} \mathbb{E}\left(|R|^{\alpha-1}\right) + C\mathbb{E}\left(b_1^\alpha\right)\end{aligned}$$

It is sufficient to prove that $\mathbb{E}\left(|R|^{\alpha-1}\right) < \infty$.

For this purpose, let us consider the process $Z_n = g_1 \cdots g_n(0), n \geq 0$, associated to the random maps $g_n : x \mapsto a_n x + b_n$, and its limit Z_∞ (in the sense of $\mathbb{P}$-a.s. convergence). One gets, for any $n \geq 1$,

$$Z_n := \sum_{k=0}^{n-1} a_1 \cdots a_k b_{k+1} \quad \text{and} \quad Z_\infty := \sum_{k=0}^{\infty} a_1 \cdots a_k b_{k+1}.$$

The fact that $a_1 \cdots a_k \to 0$ $\mathbb{P}$-a.s. as $k \to \infty$ readily implies

$$\mathbb{P}(|R| > t) = \mathbb{P}(|g_1 \cdots g_n(R)| > t) \leq \mathbb{P}(Z_n + a_1 \cdots a_k R > t) \quad \stackrel{k \to +\infty}{\longrightarrow} \quad \mathbb{P}(Z_\infty > t).$$

Thus, the random variable $|R|$ is stochastically smaller than Z_∞ and Minkowski inequality implies

$$\begin{aligned}
\left(\mathbb{E}|R|^{\alpha-1}\right)^{1/(\alpha-1)} &\leq \left(\mathbb{E}|Z_\infty|^{\alpha-1}\right)^{1/(\alpha-1)} \\
&= \left(\mathbb{E}\left[\left(\sum_{k=0}^{\infty} a_1 \cdots a_k b_{k+1}\right)^{\alpha-1}\right]\right)^{1/(\alpha-1)} \\
&\leq \sum_{k=0}^{\infty} \left(\mathbb{E}\left[(a_1 \cdots a_k)^{\alpha-1}\right]\right)^{1/(\alpha-1)} \mathbb{E}(b_1^{\alpha-1})^{1/(\alpha-1)} \\
&= \mathbb{E}(b_1^{\alpha-1})^{1/(\alpha-1)} \sum_{k=0}^{\infty} \mathbb{E}(a_1^{\alpha-1})^{\frac{n}{\alpha-1}} < \infty.
\end{aligned}$$

since $\mathbb{E}(a_1^{\alpha-1}) = L(\alpha-1) < 1$.

Case $\alpha = 1$. To prove the convergence of the series (8.13) in this case, we mix the previous arguments. For n greater than $-\ln^+ b_1$,

$$\begin{aligned}
\Sigma_1(\phi) \quad &:= \quad \sum_{n=-\ln^+ b_1}^{+\infty} \sup_{n \leq s < n+1} e^{-s} |\phi(e^s F_1(R)) - \phi(e^s a_1 R)| \\
&\leq \quad |2\phi|_\infty \sum_{n=-\ln^+ b_1}^{+\infty} e^{-n} = \frac{2|\phi|_\infty}{1-e^{-1}} \max\{b_1, 1\},
\end{aligned}$$

Similarly, when $n \leq -\ln^+ b_1$,

$$\begin{aligned}
\Sigma_2(\phi) : \quad &= \quad \sum_{n=-\infty}^{-\ln^+ b_1} \sup_{n \leq s < n+1} e^{-s} |\phi(e^s F(R)) - \phi(e^s a_1 R)| \\
&\leq \quad m(\phi) \sum_{n=-\infty}^{-\ln^+ b_1} \sup_{n \leq s < n+1} e^{-s+s} b_1 \ \mathbf{1}_{(e^s M > \delta)} \\
&\leq \quad m(\phi) \sum_{n=\log(\delta/M)-1}^{-\ln^+ b_1} b_1 \ \mathbf{1}_{(\log(\delta/M)-1 < -\ln^+ b_1)} \\
&\leq \quad m(\phi) b_1 (-\ln^+ b_1 - \log(\delta/M) + 1) \ \mathbf{1}_{(\max\{b_1,1\} < \frac{eM}{\delta})} \\
&\leq \quad m(\phi) \left(\ln^+ \left(\frac{M}{\delta \max\{b_1, 1\}}\right) b_1 + b_1\right).
\end{aligned}$$

It remains to prove that the random variable $\ln^+\left(\frac{M}{\delta\max\{b_1,1\}}\right) b_1$ is integrable. Indeed,

$$\begin{aligned}
\mathbb{E}\left(\ln^+\left(\frac{M}{\delta\max\{b_1,1\}}\right) b_1\right) \quad &\leq \quad \mathbb{E}\left(\sqrt{\frac{M}{\delta\max\{b_1,1\}}}\ b_1\right) \\
&\leq \mathbb{E}\left(\sqrt{\frac{a_1|R_1|+b_1}{\delta\max\{b_1,1\}}}\ b_1\right) \\
&\leq \mathbb{E}\left(\sqrt{\frac{a_1|R_1|}{\delta\max\{b_1,1\}}}\ b_1\right) + \mathbb{E}\left(\sqrt{\frac{b_1}{\delta\max\{b_1,1\}}}\ b_1\right) \\
&\leq \frac{1}{\sqrt{\delta}}\mathbb{E}\left(\sqrt{a_1\max\{b_1,1\}}\right)\mathbb{E}\left(\sqrt{|R_1|}\right) + \frac{1}{\sqrt{\delta}}\mathbb{E}\left(b_1\right) \\
&\leq \frac{1}{\sqrt{\delta}}\mathbb{E}(\sqrt{a_1})\mathbb{E}(\sqrt{\max\{b_1,1\}})\mathbb{E}\left(\sqrt{|R_1|}\right) + \frac{1}{\sqrt{\delta}}\mathbb{E}\left(b_1\right)
\end{aligned}$$

with $\mathbb{E}\left(|R_1|^{1/2}\right) < +\infty$ since $1/2 < 1 = \alpha$.
Step 3. Lower bound. To achieve the proof, we still need to show that if the support of ν is unbounded from above, then

$$\liminf_{z\to+\infty} z^\alpha \nu(z,\infty) > 0.$$

Fix $t > 0$ and let us consider the "Furstenberg's martingale" $(M_n)_{n\geq 0}$ given by $M_0 := \nu(t,+\infty)$ and, for any $n \geq 1$,

$$M_n = \mathbb{P}(F_1\cdots F_n(R) > t | F_1,\ldots,F_n) = \int \mathbf{1}_{(t,+\infty)}(F_1\cdots F_n(x))\nu(\mathrm{d}x).$$

We also introduce the stopping times $T_t := \inf\{n \geq 1 : a_1\cdots a_n > t\}$. It holds

$$\begin{aligned}
\nu(t,+\infty) = \mathbb{E}(M_0) \quad &\geq \quad \mathbb{E}\left(M_T\ \mathbf{1}_{(T<\infty)}\right) \\
&= \quad \int \mathbb{P}\left(F_1\cdots F_T(x) > t, T<\infty\right)\right)\nu(\mathrm{d}x).
\end{aligned}$$

The inequality $F_1\cdots F_T(x) \geq a_1\cdots a_T x - Z_T \geq tx - Z_\infty$ readily implies

$$(F_1\cdots F_T(x) > t) \supseteq (tx - Z_\infty > t) = (tx - t > Z_\infty). \tag{8.14}$$

For any $K > 0$, we get

$$\begin{aligned}
\nu(t,+\infty) \quad &\geq \quad \int_K^{+\infty} \mathbb{P}(T<\infty, (K-1)t > Z_\infty)\nu(\mathrm{d}x) \\
&\geq \quad \Big(\mathbb{P}\left(T<\infty\right) - \mathbb{P}\left((K-1)t \leq Z_\infty\right)\Big) \times \nu(K,\infty)
\end{aligned}$$

Theorem 8.2 implies $\mathbb{P}\left(T < \infty\right) = \mathbb{P}\left(\max_n a_1\cdots a_n > t\right) > C_1 e^{-\alpha\log t}$ and convergence (8.11) yields $\mathbb{P}(Z_\infty > t) \leq C_2 t^\alpha$, for some constants $C_1, C_2 > 0$. Consequently,

$$\nu(t,+\infty) \geq (C_1 - C_2(K-1)^{-\alpha})\nu(K,\infty)t^{-\alpha} = C_K t^{-\alpha}$$

with $C_K > 0$ for K great enough, since the support of ν is unbounded.

□

9 Appendix

9.1 Markov chains on denumerable state spaces

We recall here briefly the notions of irreducibility and positive/null recurrence. for denumerable Markov chains.

We assume that $(X_n)_{n\geq 0}$ is a Markov chain on a denumerable state space $\mathbb{X}$; the transition matrix $(p_{,x,y})_{x,y\in X}$ may be infinite and controls the transitions of the Markov chain.

$$\forall n\geq 1, \forall x,y\in E \quad \mathbb{P}(X_{n+1}=y/X_n=x)=p_{x,y}.$$

Coefficients $p_{x,y}$ are all non-negative and $\sum_{y\in E} p_{x,y}=1$. For any $k\geq 1$ and $x,y\in\mathbb{X}$, we set

$$p_{x,y}^{(k)}=\sum_{x_1\in\mathbb{X}}\sum_{x_2\in\mathbb{X}}\cdots\sum_{x_{k-1}\in\mathbb{X}} p_{x,x_1}p_{x_1,x_2}\cdots p_{x_{k-1},y}.$$

The chain $(X_n)_{n\geq 0}$ is said to be **irreducible** when for all $x,y\in\mathbb{X}$, there exists $k\geq 1$ such that $p_{x,y}^{(k)}>0$.

For such a chain, there exists on $\mathbb{X}$ (at least) one invariant measure m; it is unique, up to a multiplicative constant, if and only if $(X_n)_{n\geq 0}$ is recurrent. For any $y\in\mathbb{X}$, we set $\tau_y:=\inf\{n>0: X_n=y\}$. By irreducibility, we obtain $\mathbb{P}_x(\tau_y<+\infty)=1$ for any $x,y\in\mathbb{X}$. Furthermore,

- when m is finite, we may normalize it in such a way it is a probability measure and obtain

$$\forall x\in\mathbb{X}\quad \mathbb{E}_x(\tau_x)=\frac{1}{m(x)}.$$

One says that $(X_n)_{n\geq 0}$ is positive recurrent;

- when m is infinite,

$$\forall x\in\mathbb{X}\quad \mathbb{E}_x(\tau_x)=+\infty.$$

One says that $(X_n)_{n\geq 0}$ is null recurrent.

9.2 Ergodicity and Kac's formula

We consider a measurable space $(X,\mathcal{T})$, a measurable map T from X to X and a positive (finite or infinite) Radon measure m which is T-invariant, i.e. $m(T^{-1}A)=m(A)$ for any $A\in\mathcal{T}$. We say that $(X,\mathcal{T},T,m)$ is a *measurable dynamical system.* We state Kac's formula for both cases (see [1], Theorem 1.5.5). Let us recall that m is *ergodic* if and only if the T-invariant subsets of X either have 0 measure or a complementary set of 0 measure.

Theorem 9.1. *Let $(X,\mathcal{T},T,\mu)$ be an ergodic dynamical system. For any $A\in\mathcal{T}$ such that $0<m(A)<+\infty$, we set $\tau_A:=\inf\{n>0: T^n(x)\in A\}$. Then*

$$\int_A \tau_A(x)m(\mathrm{d}x)=m(X).$$

Let m_A denote the probability measure on A defined by $m_A(\cdot):=\dfrac{m(A\cap\cdot)}{m(A)}$. Then, we may rewrite this statement as follows

- $\displaystyle\int_X \tau_A(x)m_A(\mathrm{d}x)=\frac{1}{m(A)}<+\infty$ when m is a probability measure on X,
- $\displaystyle\int_X \tau_A(x)m_A(\mathrm{d}x)=+\infty$ when m is infinite.

9.3 On the conservativity of Feller's chains

We consider here a Markov chain $(X_n)_{n\geq 0}$ on a topological space E whose transition operator P is Feller, i.e.

$$\forall f\in C(E),\quad Pf\in C(E).$$

For such a Markov chain $(X_n)_{n\geq 0}$, M. Lin has proved the following statement ([23], Theorem 5.1).

Theorem 9.2. *Assume that P is a Feller Markov operator and there exists a positive continuous function with compact support g such that $\displaystyle\sum_{n=0}^{+\infty}P^n g(x)=+\infty$ for all $x\in E$. Then, there exists on E a P-invariant Radon measure m.*

9.4 Fluctuations of random walks on $\mathbb{R}$

Throughout this section, we consider a sequence of i.i.d. $\mathbb{R}$-valued random variables $(Y_n)_{n\geq 1}$ with law μ. We are interested in the fluctuations of the random walk $(S_n)_{n\geq 0}$ defined by

$$S_0 = 0 \quad \text{and} \quad S_n = Y_1 + \ldots + Y_n.$$

We introduce the *ascending and descending ladder epochs* T_n^+ and $T_n^-, n \geq 0$, defined by $T_0^+ = T_0^- = 0$ and, for $n \geq 1$,

$$T_n^+ := \inf\{k > T_{n-1}^+ : S_k > S_{T_{n-1}^+}\} \quad \text{and} \quad T_n^- := \inf\{k > T_{n-1}^+ : S_k < S_{T_{n-1}^+}\}$$

with the convention $\inf \emptyset = +\infty$ and $T_n^\pm = +\infty \Longrightarrow T_k^\pm = +\infty$ for any $k \geq n$.

These random variables take values in the set $\{1, 2, \ldots\} \bigcup \{+\infty\}$ and are stopping times with respect to the filtration $(\sigma(Y_1, \ldots, Y_n))_{n\geq 1}$ associated with the sequence $(Y_n)_{n\geq 1}$.

When these random variables are $\mathbb{P}$-a.s. finite (which occurs under some suitable conditions to be detailed later), we may consider the random variables $S_{T_1^+}$ and $S_{T_1^-}$ defined by

$$S_{T_1^+} := \sum_{n=1}^{+\infty} S_n \ \mathbf{1}_{(T_1^+ = n)} \quad \text{and} \quad S_{T_1^-} := \sum_{n=1}^{+\infty} S_n \ \mathbf{1}_{(T_1^- = n)}.$$

Warning ! We should also introduce the ascending and descending ladder epochs

$$\inf\{n \geq 1 \mid S_n \geq 0\} \quad \text{and} \quad \inf\{n \geq 1 \mid S_n \leq 0\}$$

corresponding to the first entrance times in $\mathbb{R}^+$ and $\mathbb{R}^-$. In order to simplify the notations, we assume that

$$\mathbb{P}\Big(\bigcup_{n\geq 1} (S_n = 0)\Big) = 0,$$

which implies that the first entrance times in $\mathbb{R}^+$ and $\mathbb{R}^-$ coincide with the first entrance times in $\mathbb{R}^{*+}$and $\mathbb{R}^{*-}$ respectively. Otherwise, some additive terms appear in the formula which follows, the strategy being exactly the same. We refer to [12] for the details.

The random walk $(S_n)_{n\geq 0}$ after time T_1^+ is a copy of the initial random walk, starting from $S_{T_1^+}$; en particular, its first ascending ladder epoch equals the second one of $(S_n)_{n\geq 0}$.

For any $n \geq 1$, we set $\tau_n^+ := T_n^+ - T_{n-1}^+, A_n := S_{T_n^+} - S_{T_{n-1}^+}, \tau_n^- := T_n^- - T_{n-1}^-$ and $D_n := S_{T_n^-} - S_{T_{n-1}^-}$. Consequently $T_n^+ = \tau_1^+ + \ldots + \tau_n^+, S_{T_n^+} = A_1 + \ldots + A_n, T_n^- = \tau_1^- + \ldots + \tau_n^-$ and $S_{T_n^-} = D_1 + \ldots + D_n$.

We first state the following fact.

Fact 9.3. *The sequences $(A_n)_{n\geq 1}, (\tau_n^+)_{n\geq 1}, (D_n)_{n\geq 1}$ and $(\tau_n^-)_{n\geq 1}$ are sequences of i.i.d. random variables.*

The following alternative holds

Proposition 9.4. *The random walk $(S_n)_{n\geq 0}$ has one of the three following mutually exclusive types:*

1. $\mathbb{P}(\tau^+ < +\infty) < 1$*; in this case* $\mathbb{E}(\tau^-) < +\infty$ *(and in particular* $\mathbb{P}(\tau^- < +\infty) = 1$*) and the random walk* $(S_n)_{n\geq 0}$ *converges* $\mathbb{P}$*-a.s. towards* $-\infty$*;*
2. *or* $\mathbb{P}(\tau^- < +\infty) < 1$*; in this case* $\mathbb{E}(\tau^+) < +\infty$ *(and in particular* $\mathbb{P}(\tau^+ < +\infty) = 1$*) and the random walk* $(S_n)_{n\geq 0}$ *converges* $\mathbb{P}$*-a.s. towards* $+\infty$*;*
3. $\mathbb{P}(\tau^+ < +\infty) = 1$ *and* $\mathbb{P}(\tau^- < +\infty) = 1$*; in this case* $\mathbb{E}(\tau^+) = \mathbb{E}(\tau^-) = +\infty$ *and the random walk . $(S_n)_{n\geq 0}$ oscillates* $\mathbb{P}$*-a.s. between* $-\infty$ *and* $+\infty$*.*

Proof. The main argument is the following one: for any $n \geq 1$,

$$\mathbb{P}(S_n > 0, S_n > S_1, \ldots, S_n > S_{n-1}) = \mathbb{P}(S_n > 0, S_{n-1} > 0, \ldots, S_1 > 0),$$

since $(X_1, \ldots, X_n)$ and $(X_n, , \ldots, X_1)$ have the same distribution. The left side term in this equality means that n is an ascending ladder epoch of the random walk $(S_n)_{n\geq 0}$, it thus equals

$$\mathbb{P}\Big(\bigcup_{k=1}^{+\infty}(T_k^+ = n)\Big) = \sum_{k=1}^{+\infty}\mathbb{P}(T_k^+ = n).$$

The right side term is nothing else but $\mathbb{P}(\tau^- > n)$. Summing over $n \geq 1$, it yields

$$\sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty}\mathbb{P}(T_k^+ = n) = \sum_{n=1}^{+\infty}\mathbb{P}(\tau^- > n) = \mathbb{E}(\tau^-).$$

Now, observe that

$$\begin{aligned}
\sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty}\mathbb{P}(T_k^+ = n) &= \sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty}\mathbb{P}(T_k^+ = n) \\
&= \sum_{k=1}^{+\infty}\mathbb{P}(T_k^+ < +\infty) \\
&= \sum_{k=1}^{+\infty}\mathbb{P}(\tau_1^+ < +\infty) \times \ldots \times \mathbb{P}(\tau_k^+ < +\infty) \\
&= \frac{1}{1 - \mathbb{P}(\tau_1^+ < +\infty)}.
\end{aligned}$$

Finally,

$$\frac{1}{1 - \mathbb{P}(\tau^+ < +\infty)} = \mathbb{E}(\tau^-), \tag{9.1}$$

and similarly

$$\frac{1}{1 - \mathbb{P}(\tau^- < +\infty)} = \mathbb{E}(\tau^+). \tag{9.2}$$

There are 3 cases to consider.

1. Assume $\mathbb{P}(\tau^+ < +\infty) < 1$; thus $\mathbb{E}(\tau^-) < +\infty$ so that $\mathbb{P}(\tau^- < +\infty) = 1$.

   The random walk $(S_n)_{n\geq 0}$ converges $\mathbb{P}$-a.s. towards $-\infty$. Indeed, since $T_k^- < +\infty$ $\mathbb{P}$-a.s. for any $k \geq 1$, the strong law of large numbers applied to the random walk $(D_1 + \ldots D_n)_{n\geq 1}$ implies that the sequence $(S_{T_k^-})_{k\geq 0}$ converges $\mathbb{P}$-a.s. towards $-\infty$.

   Furthermore, for any $k \geq 1$, it holds $\mathbb{P}(T_k^+ < +\infty) = (\mathbb{P}(\tau^+ < +\infty))^k$ with $\mathbb{P}(\tau^+ < +\infty) < 1$. Consequently $\mathbb{P}\Big(\bigcup_k (T_k^+ = +\infty)\Big) = 1$, which means $\sup_{n\geq 1} S_n < +\infty$ $\mathbb{P}$-a.s. It follows that $(S_n)_{n\geq 0}$ is transient. Indeed, a recurrent random walk visits infinitely often and $\mathbb{P}$-a.s. any (great enough) compact subset of $\mathbb{R}$, so that $\limsup_n S_n = +\infty$ $\mathbb{P}$-a.s. Hence $(S_n)_{n\geq 0}$ visits only finitely many times any compact subset of $\mathbb{R}$; since $\sup_{n\geq 1} S_n < +\infty$ $\mathbb{P}$-a.s., it visits only finitely many times any half line $[t, +\infty[, t \in \mathbb{R}$, thus it converges towards $-\infty$.

2. Assume now $\mathbb{P}(\tau^- < +\infty) < 1$; in this case $\mathbb{E}(\tau^+) < +\infty$ (and in particular $\mathbb{P}(\tau^+ < +\infty) = 1$) and the random walk $(S_n)_{n\geq 0}$ converges $\mathbb{P}$-a.s. to $+\infty$.

3. Assume at last $\mathbb{P}(\tau^+ < +\infty) = 1$ and $\mathbb{P}(\tau^- < +\infty) = 1$. By (9.1) and (9.2), we may write $\mathbb{E}(\tau^+) = \mathbb{E}(\tau^-) = +\infty$; furthermore, the random walk $(S_n)_{n\geq 0}$ visits $\mathbb{P}$-a.s. infinitely often $\mathbb{R}^{*+}$ and $\mathbb{R}^{*-}$. By the strong law of large numbers, the subsequences $(A_1+\ldots+A_n)_{n\geq 0}$ and $(D_1+\ldots+D_n)_{n\geq 0}$ converge $\mathbb{P}$-a.s. towards $+\infty$ and $-\infty$ respectively and $(S_n)_{n\geq 0}$ oscillates $\mathbb{P}$-a.s. between $-\infty$ and $+\infty$.

□

Remark. The random walk of type (1) or (2) are clearly transient; in contrast, there exist recurrent and transient random walk of type (3) [30].

Now, let us describe more precisely the case when $\mathbb{E}(|Y_i|) < +\infty$ and $\mathbb{E}(Y_1) > 0$.

Proposition 9.5. *If $\mathbb{E}(|Y_i|) < +\infty$ and $\mathbb{E}(Y_1) > 0$, then $\mathbb{E}(\tau^+) < +\infty$ and $\mathbb{E}(A_1) < +\infty$ (in particular the variables τ_n^+ and A_n are $\mathbb{P}$-a.s. finite).*

Proof. First $S_n \to +\infty$ $\mathbb{P}$-a.s., and the random walk has type (2) of Proposition 9.4. Hence $\mathbb{P}(\tau^+) < +\infty$ and $(A_1 + \ldots A_n)_{n\geq 0}$ is a sub-sequence of $(S_n)_{n\geq 0}$. By Fact 9.3 and applying the strong law of large numbers, it holds simultaneously

$$\frac{S_{T_n^+}}{T_n^+} = \frac{A_1 + \ldots + A_n}{T_n^+} \to \mathbb{E}(Y_1) \quad \text{and} \quad \frac{T_n^+}{n} \to \mathbb{E}(\tau^+) \quad \mathbb{P}\text{-a.s.}.$$

Hence $\dfrac{A_1 + \ldots + A_n}{n} \to \mathbb{E}(Y_1) \times \mathbb{E}(\tau^+)$ $\mathbb{P}$-a.s. The limit $\mathbb{E}(Y_1) \times \mathbb{E}(\tau^+)$ is finite and equals $\mathbb{E}(A_1)$. This readily implies that

$$\mathbb{E}(A_1) = \mathbb{E}(Y_1) \times \mathbb{E}(\tau^+) \qquad (\text{Wald's equality}). \tag{9.3}$$

Similarly, when $\mathbb{E}(|Y_i|) < +\infty$ and $\mathbb{E}(Y_1) < 0$,

$$\mathbb{E}(\tau^-) < +\infty \quad \text{and} \quad \mathbb{E}(|D_1|) < +\infty.$$

Assume now $\mathbb{E}(Y_i) = 0$ and let us first check that $\mathbb{P}(\tau^- < +\infty) = 1$; otherwise, by Proposition 9.4, we obtain $\mathbb{E}(\tau^+) < +\infty$ and equality (9.3) implies $\mathbb{E}(A_1) = 0 \times \mathbb{E}(\tau^+) = 0$, contradiction. Similarly $\mathbb{P}(\tau^+ < +\infty) = 1$. Thus, we have established the following statement.

Proposition 9.6. *If $\mathbb{E}|Y_i| < +\infty$ and $\mathbb{E}(Y_1) = 0$ (with $\mathbb{P}(Y_1 = 0) \neq 1$), then*

$$\mathbb{P}(\tau^+ < +\infty) = \mathbb{P}(\tau^- < +\infty) = 1 \quad \text{and} \quad \mathbb{E}(\tau^+) = \mathbb{E}(\tau^-) = +\infty.$$

The random variables A_1 and D_1 may have finite or infinite expectation; indeed, if $\mathbb{E}(|Y_1|^{1+\delta}) < +\infty$ for some $\delta > 0$, then $\mathbb{E}(A_1^\delta) < +\infty$ (see [8] for the details).

9.5 On the moments of the induced random walk on $\mathrm{Aff}(\mathbb{R})$

Lemma 9.7. *Let $(U_n)_{n\geq 1}$ be a sequence of i.i.d. positive random variables such that $\mathbb{P}(U_n \neq 0) > 0$. Then*

$$\mathbb{E}\Big(\log(1 + U_1)\Big) < +\infty \quad \Leftrightarrow \quad \limsup_{n\to+\infty} U_n^{1/n} = 1 \ a.s.$$
$$\Leftrightarrow \quad \limsup_{n\to+\infty} U_n^{1/n} < +\infty \ a.s.$$

Proof. First, let us notice that $\limsup_{n\to+\infty} U_n^{1/n}$ is $\mathbb{P}$-a.s. constant, by Kolmogorov's $0-1$ law. Furthermore, there exists $\delta > 0$ such that $\mathbb{P}(U_n \geq \delta) > 0$, so that $\sum_{n=1}^{+\infty} \mathbb{P}(U_n \geq \delta) = +\infty$. By Borel-Cantelli's lemma, it yields $\mathbb{P}\Big(\limsup_n (U_n \geq \delta)\Big) = 1$. Hence $\limsup_{n\to+\infty} U_n^{1/n} \geq 1$ $\mathbb{P}$-a.s.

Notice that, for any $a > 0$

$$\begin{aligned}
\mathbb{E}\log(1 + U_1) < +\infty \quad &\Longleftrightarrow \quad \sum_{n=1}^{+\infty} \mathbb{P}(\log(1 + U_n) \geq an) < +\infty \\
&\Longleftrightarrow \quad \mathbb{P}\left(\limsup_{n\to+\infty}\Big(\log(1 + U_n) \geq an\Big)\right) = 0 \\
&\Longleftrightarrow \quad \limsup_{n\to+\infty} (1 + U_n)^{\frac{1}{n}} \leq e^a \qquad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{9.4}$$

Hence, when $\log(1+U_1)$ is integrable, we obtain $\limsup_{n\to+\infty}(1+U_n)^{\frac{1}{n}} \leq e^a \quad \mathbb{P}\text{-a.s.}$ for any $a>0$, i.e. $\limsup_{n\to+\infty}(1+U_n)^{\frac{1}{n}} \leq 1$. Eventually $\limsup_{n\to+\infty} U_n^{\frac{1}{n}} = 1 \quad \mathbb{P}$-a.s. in this case.

Conversely, if the (constant) r.v. $\limsup_{n\to+\infty} U_n^{\frac{1}{n}}$ is $\mathbb{P}$-a.s. finite, there exists $a>0$ such that

$$\limsup_{n\to+\infty} U_n^{\frac{1}{n}} \leq e^a \quad \mathbb{P}-\text{a.s.}$$

This readily implies $\limsup_{n\to+\infty} U_n^{\frac{1}{n}} \leq 1+e^a \quad \mathbb{P}$-a.s. ; by (9.4), it yields $\mathbb{E}\log(1+U_1) < +\infty$.

□

Now, we want to prove that the random variable $\log^+ b(g_{T_1}\circ\cdots\circ g_1))$ has finite expectation. By the above lemma, it is a consequence of the following proposition.

Proposition 9.8. *Under hypothesis H2*

$$\mathbb{E}\Big(\log\Big(1+\sum_{k=1}^{T_1} b_k a_{k+1}\dots a_{T_1}\Big)\Big) < +\infty.$$

Proof. By Lemma 9.7, it suffices to check that

$$\limsup_{n\to+\infty}\Big|\sum_{k=T_{n-1}+1}^{T_n} a_{k+1}\dots a_{T_n} b_k\Big|^{\frac{1}{n}} < +\infty \quad \mathbb{P}\text{-a.s.}$$

For any $n\geq 1$, it holds

$$\sum_{k=T_{n-1}+1}^{T_n} a_{k+1}\dots a_{T_n}|b_k| = a_{T_{n-1}+1}\dots a_{T_n}\sum_{k=T_{n-1}+1}^{T_n}\frac{|b_k|}{a_{T_{n-1}+1}\dots a_k} \leq \sum_{k=T_{n-1}+1}^{T_n}\frac{|b_k|}{a_k}$$

since $a_{T_{n-1}+1}\dots a_{k-1}\geq 1 \quad \mathbb{P}$-a.s. for $T_{n-1}+1\leq k\leq T_n$. It remains to prove that

$$\limsup_{n\to+\infty}\Big(\sum_{k=T_{n-1}+1}^{T_n}|b_k|/a_k\Big)^{\frac{1}{n}} < +\infty \quad \mathbb{P}\text{-a.s.}.$$

In fact, it holds $\limsup_{n\to+\infty}\Big(\sum_{k=T_{n-1}+1}^{T_n}|b_k|/a_k\Big)^{\frac{1}{n}} < +\infty \quad \mathbb{P}$-a.s.; namely, the random variables $T_{k+1}-T_k$ are i.i.d. with distribution $\mathcal{L}(T_1)$ and

$$\mathbb{P}(T_1>n)\sim\frac{c}{\sqrt{n}} \quad \text{as} \quad n\to+\infty,$$

(which implies $\mathbb{E}(T_1^\alpha)<+\infty$ for $\alpha<1/2$). It follows $\limsup_{n\to+\infty} T_n^\alpha/n < +\infty$ $\mathbb{P}$-a.s., so that

$$\limsup_{n\to+\infty}\Big(\sum_{k=1}^{T_n}|b_k|/a_k\Big)^{\frac{1}{n}} \leq \limsup_{n\to+\infty}\exp\frac{T_n^\alpha}{n}\Big(\frac{1}{T_n^\alpha}\log\Big(1+\sum_{k=1}^{T_n}|b_k|/a_k\Big)\Big)$$

and it suffices to prove that

$$\limsup_{n\to+\infty}\frac{1}{T_n^\alpha}\log\Big(1+\sum_{k=1}^{T_n}|b_k|/a_k\Big) < +\infty \quad \mathbb{P}\text{-a.s.}$$

Since $\log\Big(1+\sum_{k=1}^{T_n}|b_k|/a_k\Big) \leq \log T_n + \sup_{1\leq k\leq T_n} \log(1+|b_k|/a_k)$, we obtain

$$\begin{aligned}
\limsup_{n\to+\infty} \frac{1}{T_n^\alpha} \log\Big(1+\sum_{k=1}^{T_n}|b_k|/a_k\Big) &\leq \limsup_{n\to+\infty} \frac{1}{T_n^\alpha} \sup_{1\leq k\leq T_n} \log(1+|b_k|/a_k) \\
&= \limsup_{n\to+\infty} \Big(\frac{1}{T_n} \sup_{1\leq k\leq T_n} \Big(\log(1+|b_k|/a_k)\Big)^{\frac{1}{\alpha}}\Big)^\alpha \\
&\leq \limsup_{n\to+\infty} \Big(\frac{1}{T_n} \sum_{k=1}^{T_n} \Big(\log(1+|b_k|/a_k)\Big)^{\frac{1}{\alpha}}\Big)^\alpha.
\end{aligned}$$

By hypothesis H2, if $\alpha \geq \frac{1}{2+\eta}$, the random variable $\log(1+|b_1|/a_1)^{\frac{1}{\alpha}}$ is integrable and the strong law of large numbers implies

$$\limsup_{n\to+\infty} \frac{1}{T_n^\alpha} \log\Big(1+\sum_{k=1}^{T_n}|b_k|/a_k\Big) \leq \mathbb{E}\Big(\Big(\log(1+|b_k|/a_k)\Big)^{\frac{1}{\alpha}}\Big)^\alpha < +\infty.$$

The proof of Proposition 9.8 arrives.

□

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