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Lecture in hyperbolic geometry -Mathematics - November 2009 -

On some aspects of discrete groups of isometries of the hyperbolic plane

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1. Introduction

We consider a finitely generated group Γ, we fix a (symmetric) set S of generators of Γ and we denote by $\tilde{X} = \tilde{X}(S,\Gamma)$ the Cayley graph of Γ associated with S : the vertices of $\tilde{X}$ are the elements of Γ and two such vertices g and h are connected by an edge $[g,h]$ if and only if $g^{-1}h \in S$.

For instance, the Cayley graph associated to the group $\mathbb{Z}^2$, with the set of generators $S := \{(1,0),(0,1),(-1,0),(0,-1)\}$ may be represented as the set $\{(x,y) \in \mathbb{R}^2 / x \in \mathbb{Z} \text{ } or \text{ } y \in \mathbb{Z}\}$.

Another classical example is the **free group** $\mathbb{F}_2$ generated by two elements a and b and the symmetric set S of generators is equal to $\{a, a^{-1}, b, b^{-1}\}$ in this case[3]. The Cayley graph associated with $\mathbb{F}_2$ is thus the regular tree with 4 edges starting from each vertex.

The set of vertices of $\tilde{X}$ is a metric space when it is endowed with the following distance : $d_S(g,h)$ is equal to the minimal number n of letters such that $g^{-1}h = a_1 \cdots a_1$ with $a_i \in S$ (if one assumes that the length of any edge is equal to 1, then $d_S(g,h)$ is also the length of the shortest curve in $\tilde{X}$ joigning the vertices g and h). In particular, we get $d_S(g,h) = d_S(e, g^{-1}h)$ and so $d_S(g,h) = d_S(\gamma g, \gamma h) = d_S(e, g^{-1}h)$ for any $\gamma \in \Gamma$.

In the example of the group $\mathbb{Z}^2$, we thus have $d((0,0),(k,l)) = |k| + |l|$ (this is the l^1-distance) ; on the other hand, in the example of $\mathbb{F}_2$, one gets $d(e, a_1 \cdots a_n) = n$ when $a_1 \cdots a_n$ is a reduced word.

We now study the function $N_{\Gamma,d}(R) := \sharp\{\gamma \in \Gamma / d_S(e,\gamma) \leq R\}$; we get, for any $R \in \mathbb{N}$

$$N_{\mathbb{Z}^2}(R) = 2R + 1 + 2\Big((2(R-1)+1) + (2(R-2)+1) \cdots + (2 \times 0 + 1)\Big) = 2R^2 + 2R + 1$$

and

$$N_{\mathbb{F}_2}(R) = 4 \times 3^{R-1}.$$

There is another classical orbital function associated with $\mathbb{Z}^2$ whose study is the classical **Gauss-Minkowsky circle problem** ; the set $\mathbb{Z}^2$ is considered as a subset of the euclidean space $(\mathbb{R}^2, \|.\|_2)$ and the orbital function is

$$N(R) = N_{\mathbb{Z}^2, \|.\|_2}(R) := \sharp\{(x,y) \in \mathbb{Z}^2 : x^2 + y^2 \leq R^2\}.$$

One gets $N(0) = 1, N(1) = 5, N(\sqrt{10}) = 37, N(\sqrt{100}) = 317, N(\sqrt{1000}) = 3149$; N. Gauss has proved that

$$N(R) = \pi R^2 + O(R).$$

Proof. For $M \in \mathbb{Z}^2$ let C_M be the unit square with M as south west corner. Denote by $B(0,R)$ the euclidean disc with center 0 and radius R; one gets

- $M \in B(0,R) \cap \mathbb{Z}^2 \Rightarrow C_M \subset B(0, R+\sqrt{2})$

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[3]this is the set of "finite reduced words" $a_1 \cdots a_n$ of length n and letters $a_i \in \{a^{\pm 1}, b^{\pm 1}\}$ (with the convention that the "empty word", ie the case when $n = 0$ corresponds to the neutral element e) where the significance of "reduced" is the following one : $a_{i+1} \neq a_i^1$ for any $1 \leq i \leq n-1$. The law $*$ on $\mathbb{F}^2$ is defined inductively by $a_1 * a_1^{-1} = e$ for any a_1 in S and

$-a_1 \cdots a_n * b_1 \cdots b_m = a_1 \cdots a_n b_1 \cdots b_m$ when $a_n \neq b_1^{\pm 1}$ and

$-a_1 \cdots a_n * b_1 \cdots b_m = a_1 \cdots a_{n-1} * b_2 \cdots b_m$.

- $C_M \cap B(0, R-\sqrt{2}) \neq \emptyset \Rightarrow C_M \subset B(0,R)$

with leads to the following double inequality

$$\pi(R-\sqrt{2})^2 \leq N(R) \leq \pi(R+\sqrt{2})^2$$

and gives $E(R) := |N(R) - \pi R^2| \leq 2\pi(1+\sqrt{2}R)$. It is conjectured that the error term $E(R)$ is $O(R^{\frac{1}{2}+\epsilon})$ for any $\epsilon > 0$; it is proved that $E(R) = O(R^{\frac{46}{93}+\epsilon}$ (Husley (1993)) and $\limsup \frac{E(R)}{\sqrt{R \ln R}} > 0$ (Hardy Landau).

The circle problem is connected to the theory of groups as follows : $\mathbb{Z}^2$ is a discrete subgroup of $\mathbb{R}^2$ (which means that the orbit of any point is discrete in $\mathbb{R}^2$) and one gets

$$N(R) = \sharp\Big(\mathbb{Z}^2 \cdot 0 \cap B(0,R)\Big) \sim \frac{\text{area } B(0,R)}{\text{area } (\mathbb{R}^2/\mathbb{Z}^2)}$$

where $\mathbb{R}^2/\mathbb{Z}^2$ is the quotient space and corresponds to the classical flat torus.

The main argument is that the area of the annulus $B(0, R+\Delta)\setminus B(0,R)$ has linear growth and is thus negligeable with respect to the area of $B(0,R)$. This allows to obtain a precise estimate of the orbital function in this case.

This will not the case on the Cayley graph of $\mathbb{F}^2$ where the growth is exponential ; there is a similar example in Riemannian geometry, the hyperbolic space $\mathbb{H}^2$, and we will focuse our attention on this subject now. We will introduce the main tools we need, we will look at co-compact subgroups of isometries of $\mathbb{H}^2$ but also consider non uniform lattices and another large class of discrete subgroups, called the Schottky groups.

We will prove with elementary tools the following :

THEOREM 1.1. *Let Γ a discrete subgroup of isometries of $\mathbb{H}$ which is either a co-compact group (called also uniform lattice), either a non uniform lattice or either a Schottky subgroup of $Iso^+(\mathbb{H})$. Then, there exists $\delta_\Gamma > 0$ (with $\delta_\Gamma = 1$ in the case of lattices) and, for any $\mathbf{x}, \mathbf{y} \in \mathbb{H}$, a constant $C := C(\Gamma, \mathbf{x}, \mathbf{y}) > 0$) such that for any $R > 0$ one gets*

$$\frac{e^{\delta_\Gamma R}}{C} \leq \sharp\{\gamma \in \Gamma : d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leq R\} \leq C e^{\delta_\Gamma R}. \quad [4]$$

To obtain a precise estimate of the orbital function in this case is much more harder than in the euclidean one ; it is due to the fact that the area of the annulus is not negligeable with respect to the one of the disc, we say that there is a *boundary effect*. To avoid this difficulty, it is necessary to introduce a quite subtle notion : *the mixing property of the geodesic flow* of $\mathbb{H}^2/\Gamma$ with respect to some measure, which leads to the *equidistribution of the spheres* on $\mathbb{H}^2/\Gamma$. This is much more difficult to prove and it goes out of the goals of this course ; we will suggest an approach to solve this problem in the euclidean case, giving thus another proof of Gauss' result.

As a corollary of the previous theorem, and in fact of the stronger result which states that we have in fact $N_\Gamma(R) \sim C_\Gamma e^R$ we may for instance obtain the following result which clearly appears as a generalisation of the Gauss-Minkowsky's circle problem :

COROLLARY 1.2. *One gets*

$$\sharp\Big\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : a^2+b^2+c^2+d^2 \leq R^2\Big\} \sim 6R^2.$$

We fix here once and for all some notation about asymptotic behavior of functions :

Notations. *We shall sometimes write $f \overset{c}{\preceq} g$ (or simply $f \preceq g$) when $f(R) \leq cg(R)$ for some constant $c > 0$ and R large enough. The notation $f \overset{c}{\asymp} g$ (or simply $f \asymp g$) means $f \overset{c}{\preceq} g \overset{c}{\preceq} f$.*

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[4]Note that it suffices to prove this theorem for $\mathbf{x} = \mathbf{y} = \mathbf{x}_0$ for some $\mathbf{x}_0 \in \mathbb{H}$, since, as γ is an isometry of $\mathbb{H}$ we have by the triangle inequality

$$d(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0) - d(\mathbf{x}, \mathbf{x}_0) - d(\mathbf{y}, \mathbf{x}_0) \leq d(\mathbf{x}, \gamma\cdot\mathbf{y}) \leq d(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0) + d(\mathbf{x}, \mathbf{x}_0) + d(\mathbf{y}, \mathbf{x}_0).$$

2. Introduction to the hyperbolic plane

2.1. The hyperbolic space and its metric. We denote by $\mathbb{H}$ the hyperbolic plane defined by :

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} / y > 0\}.$$

This space is equipped with the hyperbolic Riemannian structure [5] : the scalar product $\langle\ .\ ,\ .\ \rangle_z$ on the tangent plane of $\mathbb{H}$ at the point $z = x + iy$ is defined by:

$$\forall \vec{u}, \vec{v} \in \mathbb{R}^2 \quad \langle \vec{u}, \vec{v} \rangle_z = \frac{\langle \vec{u}, \vec{v} \rangle}{y^2}$$

where $\langle\ .\ ,\ .\ \rangle$ denotes the standart scalar product on $\mathbb{R}^2$. It implies that the norm on the tangent space $T_z\mathbb{H} \simeq \mathbb{R}^2$ at $z = x + iy \in \mathbb{H}$ is $\|\vec{u}\|_z = \frac{\|\vec{u}\|}{y}$. Intuitively, this means that the infinitesimal element of length in the neighbourhood of the point $z = x + iy$ is $ds := \dfrac{\sqrt{dx^2 + dy^2}}{y}$.

In the same way, the infinitesimal element of area near $x + iy$ is given by $dv = \dfrac{dx\ dy}{y^2}$.

The length $L(\gamma)$ of a curve $\gamma : [0, 1] \to \mathbb{H}, t \mapsto \gamma(t) = z(t) := x(t) + iy(t)$ is

$$L(\gamma) := \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt = \int_0^1 \frac{1}{y(t)} |z'(t)| dt.$$

The distance between two points z and z' in $\mathbb{H}$ is defined by $d_{\mathbb{H}}(z, z') := \inf L(\gamma)$ where the infimum is taken over all smooth curves γ connecting z and z'.

2.2. The unit disc model. There exists another model of the hyperbolic 2-dimensionnal geometry in the *unit disc* $\mathbb{D} := \{z \in \mathbb{C} / |z| < 1\}$. Note that the map

$$f : z \mapsto Z := \frac{iz + 1}{z + i} \tag{1}$$

is 1 to 1 from $\mathbb{H}$ to $\mathbb{D}$. [6]

The unit disc $\mathbb{D}$ is endowed with the distance d^* given by $d^*(Z, W) = d(f^{-1}Z, f^{-1}W)$. A simple calculous leads to the following equality

$$|f'(z)| = \frac{1 - |f(z)|^2}{2\text{Im}(z)} = \frac{1 - |Z|^2}{2y}$$

so that

$$|dZ| = \left|\frac{dZ}{dz}\right| \times |dz| = |f'(z)| \times |dz|$$

which may be also written as

$$\sqrt{dX^2 + dY^2} = |f'(x + iy)| \sqrt{dx^2 + dy^2} = \frac{1 - (X^2 + Y^2)}{2} \frac{\sqrt{dx^2 + dy^2}}{y}.$$

The infinitesimal element of hyperbolic length in $\mathbb{H}$ and $\mathbb{D}$ is

$$ds := \frac{\sqrt{dx^2 + dy^2}}{y} = 2 \frac{\sqrt{dX^2 + dY^2}}{1 - (X^2 + Y^2)}$$

and the infinitesimal element of area is

$$dv := \frac{dx\ dy}{y^2} = \frac{4dX\ dY}{\Big(1 - (X^2 + Y^2)\Big)^2}.$$

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[5] We say that a manifold M is equipped with a Riemannian structure g if for any $p \in M$ we have choosen a scalar product g_p on the tangent space of M at p which depends smoothly on p : for instance, the standart Riemannian structure on $\mathbb{R}^2$ corresponds to the choice of the same scalar product for any tangent plane.

[6] Note that $f(x + iy) = X + iY$ with $X = \frac{2x}{x^2 + (1+y)^2}$ and $Y = \frac{1 - x^2 - y^2}{x^2 + (1+y)^2}$ so that $X^2 + Y^2 < 1$. The inverse map f^{-1} is defined by $Z \mapsto \frac{-iZ + 1}{Z - i}$ so that $f^{-1}(X + iY) = \frac{2X + i(1 - X^2 - Y^2)}{X^2 + (1 - Y)^2}$ and one easily cheks that f^{-1} maps $\mathbb{D}$ to $\mathbb{H}$.

The same calculous show that the scalar product $\langle\ .\ ,\ .\ \rangle_Z$ on the tangent plane of $\mathbb{D}$ at the point Z is given by

$$\langle \vec{u}, \vec{v}\rangle_Z = \frac{4}{(1-|Z|^2)^2}\langle \vec{u}, \vec{v}\rangle.$$

We shall refer to these two models of hyperbolic geometry as the Poincaré models, and we shall change from one model to the other as each of them has its own particular advantage.

2.3. The compactification of the hyperbolic plane. There is a natural compactification of the above two models, since these models are open subsets of the complex plane. The topology induced by the hyperbolic metric on $\mathbb{H}$ and $\mathbb{D}$ is the one induced by the classical topology on $\mathbb{C}$; the compactification of $\mathbb{H}$ and $\mathbb{D}$ will thus be the closure in $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ of these subsets :

- the compactification of $\mathbb{H}$ is $\bar{\mathbb{H}} := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\} \cup \{\infty\}$ and its boundary is $\partial\mathbb{H} := \mathbb{R} \cup \{\infty\}$.
- the compactification of $\mathbb{D}$ is the closed unit disc and its boundary is the unit circle $\partial\mathbb{D}$.

2.4. The isometries of $\mathbb{H}$. In the sequel $SL(2, \mathbb{R})$ is the group of 2×2 matrices with real coefficients and determinant 1. Let us consider the group G of **Möbius transformations** g (or **linear fractional maps** or **homographies**) defined by : for any $z \in \mathbb{H}$

$$g \cdot z = \frac{az+b}{cz+d} \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}). \tag{2}$$

Note that $g \cdot z \in G$ belongs to $\mathbb{H}$ since

$$\mathrm{Im}(g \cdot z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Furthermore, for any $g, g' \in G$ and $z \in \mathbb{C}$ one gets $gg' \cdot z = g \cdot (g' \cdot z)$ (where gg' denotes the product of the matrices g and g') ; this means that the composition of two homographies is an homography whose associated matrice if the product of the two initial matrices, one says that G **acts on** $\mathbb{H}$.

In fact, **the group G we will consider is $PSl(2, \mathbb{R})$, ie the quotient of $SL(2, \mathbb{R})$ by its center** $\{\pm I\}$

$$G = PSL(2, \mathbb{R}) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})\right\} \Big/ \{\pm I\}.$$

Indeed, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ satisfies $\forall z \in \mathbb{H} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = z$, one gets equivalently $cz^2 + (d-a)z + b = 0$ for any $z \in \mathbb{H}$ and so $c = b = 0$ and $a = d = \pm 1$; it follows that if $g, h \in SL(2, \mathbb{R})$ satisfy $g \cdot z = h \cdot z'$ for any $z \in \mathbb{H}$, one gets equivalently $g^{-1}h \cdot z = z$ and so $h = \pm g$.

The main result of this paragraph is the following

Lemma 2.1. *The action of $G = PSL(2, \mathbb{R})$ on $\mathbb{H}$ is isometric and transitive.*

Proof. The action is transitive since G contains all the transformations $z \mapsto az + b$ with $a > 0$ and $b \in \mathbb{R}$.

The differential $D_z g$ of the diffeomorphism $g \in G$ at the point $z \in \mathbb{H}$ is given by

$$\forall \vec{u} \in T_z\mathbb{H} \quad D_z g(\vec{u}) = \frac{1}{(cz+d)^2}\vec{u}\ ; \quad [7]$$

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[7] To compute the differential of g we have derivated the fonction $z \mapsto g \cdot z$ with respect to the variable z ; we may give some explanations about this "identification".

Any function $f : \mathbb{C} \to \mathbb{C}$ may be decomposed between real and imaginary parts : for any $z \in \mathbb{C}$, one gets $f(z) = u(z) + iv(z)$ where u and v are real valued functions defined on $\mathbb{C}$. One may thus associate to f the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ setting

$$\forall (x, y) \in \mathbb{R}^2 \quad f(x, y) = (U(x, y), V(x, y))$$

with $u(x + iy) = U(x, y)$ and $v(x + iy) = V(x, y)$. The functions u, v are holomorphic on $\mathbb{C}$ and the functions U, V continuously differentiable on $\mathbb{R}^2$; the local expansion of order 1 leads to the following equations : for any $x, y, a, b \in \mathbb{R}^2$, setting $z := x + iy$ and $h := a + ib$, one gets

$$f(z + h) = f(z) + hf'(z) + ho(h)$$

and

$$F(x + a, y + b) = F(x, y) + DF_{(x,y)}(a, b) + \sqrt{a^2 + b^2}\,o(\sqrt{a^2 + b^2}),$$

It follows $\|D_z g(\vec{u})\| = \dfrac{\|\vec{u}\|}{|cz+d|^2}$ and so

$$\|D_z g(\vec{u})\|_{g\cdot z} = \frac{\|D_z g(\vec{u})\|}{\mathrm{Im}(g\cdot z)} = \frac{\|\vec{u}\|}{|cz+d|^2} \times \Big(\frac{\mathrm{Im}(z)}{|cz+d|^2}\Big)^{-1} = \frac{\|\vec{u}\|}{\mathrm{Im}(z)} = \|\vec{u}\|_z.$$

So g preserves the length of any curve γ on $\mathbb{H}$: it is an isometry ; more precisely, one gets, for any $z \in \mathbb{H}$ and $\vec{u}, \vec{v} \in \mathbb{R}^2$ $\langle Dg_z(\vec{u}), Dg_z(\vec{v})\rangle_{g\cdot z} = \langle \vec{u}, \vec{v}\rangle_z$ which proves that the function is a preserving orientation isometry. Consequently, **the group G is included in the group of isometries of $\mathbb{H}$ which preserve the orientation.**

□

In fact, one may check that $PSL(2,\mathbb{R})$ is exactly the set of isometries of $\mathbb{H}$ which respect the orientation (see [**4**]).

Note also that the hyperbolic plane is homogeneous in the sense that any two points z, w may be sent one on the other by an isometry ; in some sense, the group of isometries of $\mathbb{H}$ is quite big.[8]

Remark 1. *It is important to recall that, like any holomorphic map on the complex plane, the homography $z \mapsto \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc = 1$, is a "conformal" map on $\mathbb{C} \setminus \{\frac{-d}{c}\}$: that means that it preserves the angles, in particular if two curves are orthogonal at a point z, then they images by g are orthogonal at $g \cdot z$.*

Remark 2. *Since the isometries of $\mathbb{H}$ are of the form (2.4), it is quite simple to describe the isometries in the unit disc model ; note that g is an isometry of $\mathbb{D}$ if and only if $f^{-1} \circ g \circ f$ is an isometry of $\mathbb{H}$, where f is given by (1), and is thus a linear fractional transformation of the form (). Since f and f^{-1} are also linear fractional transformations, associated respectively to the matrices $\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$ and $\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$, the map g is of the same type, with associated matrice $\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} A & \bar{C} \\ C & \bar{A} \end{pmatrix}$ with $A = a + d + i(b - c)$ and $C = b + c + (d - a)i$, so that $A\bar{A} - C\bar{C} = 1$. This is the general form of the isometries in the unit disc model.*

2.5. The boundary at infinity. There is a natural way to compactify the set $\mathbb{H}$; its boundary at infinity is the set of extremities of its geodesics, we set $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. In the same way the boundary at infinity of the hyperbolic unit disc is the unit circle.

The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ may be continuously extended to the boundary $\mathbb{R} \cup \{\infty\}$ if we consider that ∞ is mapped on $\frac{a}{c}$ and $\frac{-d}{c}$ on ∞.

2.6. The geodesics of $\mathbb{H}$. Recall that the distance d between two points z and w in $\mathbb{H}$ is given by

$$d(z, w) = \inf_{\gamma} L(\gamma)$$

where the infimum is over the set of smooth curves joigning z and z'. We have the

PROPOSITION 2.2. *For any two points z, w in $\mathbb{H}$, there exists a unique differentiable curve c such that whose length $l(c)$ is equal to $d(z, w)$. It is called the* **geodesic between z and w.**

If Re(z) = Re(w), this curve is the vertical segment of line between these two points (and the distance is $d(z, w) = \big|\ln(\mathrm{Im}(z)) - \ln(\mathrm{Im}(w))|$.

Otherwise, c is the arc between z and w of the unique (euclidean) circle passing through z and w and which is orthogonal to the real axis.

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with $f'(z) = u'(z) + iv'(z)$ and $DF_{(x,y)}(a, b) = \begin{pmatrix} U'_x(x,y) & U'_y(x,y) \\ V'_x(x,y) & V'_y(x,y) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} U'_x a + U'_y b \\ V'_x a + V'_y b \end{pmatrix}$ The equality $(u'(z)+iv'(z))(a+ib) = U'_x a + U'_y b + i(V'_x a + V'_y b)$, valid for any $a, b \in \mathbb{R}^2$ leads to the following well known *Cauchy equations* :

$$f'(x + iy) = U'_x - iU'_y = V'_y + iV'_x.$$

When f is given by $z \mapsto \frac{az+b}{cz+d}$, one gets $f'(z) = \frac{1}{(cz+d)^2} = \frac{e^{2i\varphi}}{|cz+d|^2}$ and so $DF_{(x,y)} = \frac{1}{|cz+d|^2}\begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{pmatrix}$.

[8]Note that given two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of points in $\mathbb{H} \cup \{\infty\}$ there exists a unique complex Möbius transformation T with $T(z_i) = w_i$ for $i = 1, 2, 3$. The proof is left to the reader.

Proof. Assume first that $z = ia$ and $w = ib$ with $b > a > 0$. Let $\gamma : [0,1] \to \mathbb{H}$ be a C^1 arc with extremities z and w. One gets

$$L(\gamma) = \int_0^1 \frac{\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}}{y(t)} dt \geq \int_0^1 \frac{|y'(t)|}{y(t)} dt = \ln b - \ln a$$

with equality when $\gamma(t) = (1-t)ia + tib$; so the imaginary axis is a geodesic and $d_{\mathbb{H}}(ia, ib) = \left|\ln \frac{b}{a}\right|$.

Suppose now that z and w are arbitrary. One first choose $g \in SL(2,\mathbb{R})$ which maps these two points on the imaginary axis, which can be done as follows :

- if Re(z)=Re(w), one take $g \cdot u \mapsto u - \mathrm{Re}(z)$, and so $g = \begin{pmatrix} 1 & -\mathrm{Re}(z) \\ 0 & 1 \end{pmatrix}$

- otherwise, let $\mathcal{C}$ the circle passing through z and w and which is orthogonal to the real axis; let s and s, with $s < t$ be the intersection of this circle with $\mathbb{R}$. Consider now the linear fractional transformation

$$f : u \mapsto -\frac{1}{u-s} + \frac{1}{t-s} = \frac{u-t}{(t-s)(u-s)}.$$

for any $u \in \mathbb{C}$, which corresponds to the matrice $g = \begin{pmatrix} \frac{1}{t-s} & -\frac{t}{t-s} \\ 1 & -s \end{pmatrix}$ This transformation f maps s to $+\infty$ and t to 0; so, by classical properties of homographies, the half circle $\mathcal{C}$ is send on the imaginary axis $i\mathbb{R}$. Since f is an isometry, the half circle $\mathcal{C}$ is a geodesic of $\mathbb{H}$.□

When Re(z)=Re(w), then the distance between these two points is equal to $d(z,w) = |\ln(\mathrm{Im}(z)) - \ln(\mathrm{Im}(w))|$. More generally, when z and w are arbitrary, one gets

$$\text{(3)} \qquad \sinh \frac{d(z,w)}{2} = \frac{|z-w|}{2\sqrt{\mathrm{Im}(z)\mathrm{Im}(w)}}. \qquad [9]$$

In the unit disc model $\mathbb{D}$, the distance $d(0,r)$ between the origin 0 and the point $r \in]0,1[$ is equal to $2\displaystyle\int_0^r \frac{dx}{1-x^2} = \ln \frac{1+r}{1-r}$ so that $\tanh \frac{d(0,r)}{2} = r$ which implies $\sinh \dfrac{d(0,r)}{2} = \dfrac{r}{\sqrt{1-r^2}}$. More generally, the distance between any two points Z and W is given by

$$d(Z,W) = \ln \frac{|1 - Z\bar{W}| + |Z - W|}{|1 - Z\bar{W}| - |Z - W|};$$

which can be expressed also as follows

$$\sinh \frac{d(Z,W)}{2} = \frac{|Z-W|}{\sqrt{1-|Z|^2}\sqrt{1-|W|^2}}. \qquad [10]$$

2.7. Hyperbolic area. For any open subset $A \subset \mathbb{H}$, the hyperbolic area is defined by

$$\text{area } (A) = \int_A \frac{dx\ dy}{y^2},$$

when the integral exists. In the unit disc model, the area of an open subset $B \subset$ is given by

$$\text{area } (B) = 4 \int_B \frac{dx\ dy}{(1 - x^2 - y^2)^2}.$$

It can be easily shown that the hyperbolic area is invariant under all transformations in $PSL(2,\mathbb{R})$: for any open subset $A \subset \mathbb{H}$ with finite area and any $g \in PSL(2,\mathbb{R})$, one gets area $(g(A))$ = area (A).

Using either one of the two models of the hyperbolic plane, one may compute the area of classical domains.

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[9]We give some hints to obtain this formula. One checks first it is valid when$z = ia$ and $w = ib$: in this case $d(z,w) = \left|\ln \frac{b}{a}\right|$ so that $\sinh(d(z,w)/2) = |b-a|/\sqrt{ab}$. Furthermore, since elements of G are isometries of $\mathbb{H}$, the left-hand side of (3) is invariant under the action of any $g \in G$; on the other hand, by the Property **??**, the right hand side is also invariant under g. It thus suffices to choose g in such a way the geodesic segment $[z,w]$ is sent on a segment of the imaginary axis (see the proof of Proposition 2.2

[10]one uses the map $z \mapsto Z := \frac{iz+1}{z+i}$ which is an isometry between the two models.

- For instance, if one considers a hyperbolic geodesic triangle Δ with angles α, β, γ, one gets the famous Gauss Bonnet Formula

$$\text{area } (\Delta) = \pi - \alpha - \beta - \gamma.$$

In particular, the area of **any** triangle cannot exced π, and it is exactly equal to π for the *ideal triangles*, that is the triangles whose vertices belong to $\mathbb{R} \cup \{\infty\}$! (see [**4**] Theorem 1.4.2 for the proof of this formula, in the half plane model ; this is also the subject of some exercice)

- One may also compute the area of the discs in $\mathbb{H}^2$. It is easier to identify the discs of the hyperbolic plane and to compute their area in the unit disc model. Fix $0 < r < 1$; one gets, for any $\theta \in [0, 2\pi[$

$$d(0, r) = d(0, re^{i\theta}) = \int_0^r \frac{2dt}{1-t^2} = \ln\Big(\frac{1+r}{1-r}\Big).$$

So the disc $B(0, R) \subset \mathbb{D}$ with center the origin 0 and (hyperbolic) radius R coincides with the (Euclidean) disc with same center 0 and Euclidean radius $r = \frac{e^R-1}{e^R+1} = \tanh \frac{R}{2}$. Its area does not depend on the center of the disc and is equal to

$$\begin{aligned}
\text{area } (B(0,R)) &= 4\int_{B(0,R)} \frac{dx\ dy}{(1-x^2-y^2)^2} \\
&= 4\int_0^{2\pi}\int_0^{\tanh\frac{R}{2}} \frac{r\ dr\ d\theta}{(1-r^2)^2} \\
&= 4\pi\frac{(\tanh\frac{R}{2})^2}{1-(\tanh\frac{R}{2})^2} \\
&= 2\pi(\cosh R - 1) \\
&\simeq \pi e^R.
\end{aligned}$$

It is why one says that the growth of the hyperbolic plane is exponential, with exponential rate 1.

2.8. The curvature of $\mathbb{H}^2$. Let $x \in \mathbb{H}$ and $R > 0$; the circle with center x and radius R is the set $\mathcal{C}(x, R) := \{y \in \mathbb{H}/d(x, y) = R\}$. In the unit disc model, one may see immediately that $\mathcal{C}(0, R)$ coincides with the euclidean circle with center 0 and (euclidean) radius $\tanh \frac{R}{2}$. We parametrize this circle as follows : $\gamma : [0, 2\pi[\to \mathcal{C}(0, R), t \mapsto (x(t), y(t)) = (\tanh \frac{R}{2} \cos t, \tanh \frac{R}{2} \sin t)$; let us compute its circumference $l(R)$, one gets

$$l(R) = 2\int_0^{2\pi} \frac{\sqrt{x'(t)^2 + y'(t)^2}}{1-\tanh^2\frac{R}{2}} dt = \frac{4\pi \tanh\frac{R}{2}}{1-\tanh^2\frac{R}{2}} = 2\pi \sinh R.$$

On thus gets

$$l(R) = 2\pi R + \pi\frac{R^3}{3} + o(R^3).$$

In particular, the length of the circles is greater in hyperbolic geometry than in Euclidean one ; more precisely, the *Gauss curvature* $K(z)$ at a point $z \in H$ measures the difference between these lengthes is given by

$$K(z) = -\frac{3}{\pi}\lim_{R\to 0}\frac{1}{R^3}(l(R) - 2\pi R);$$

in the present case it is equal to -1.

2.9. Some basic hyperbolic trigonometry. As in Euclidean geometry, there exists a lot of formulae of trigonometry in hyperbolic geometry. We will focus our attention on the following one and its corollary, which will be usefull i the sequel :

PROPOSITION 2.3. *For any hyperbolic triangle Δ with sides of hyperbolic length a, b, c and opposite angles α, β, γ one gets*

$$\textbf{Cosine Rule I}: \qquad \cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

Proof. Let us denote the vertices opposite the sides a, b, c by v_a, v_b, v_c respectively ; up to an isometry, one may assume that $v_c = 0, \text{Im } v_a = 0$ and $\text{Re } v_a > 0$. We have $v_a = \tanh \frac{d(0,v_a)}{2} = \tanh \frac{b}{2}$, and similarly $v_b = e^{i\gamma} \tanh \frac{a}{2}$.

We have $c = d(v_a, v_b)$ so that $\cosh c = 1 + 2\sinh^2 \frac{c}{2} = 1 + 2\frac{|v_a - v_b|^2}{(1-|v_a|^2)(1-|v_b|^2)}$.

Applying the identities $\cosh 2\theta = \frac{1+\tanh^2\theta}{1-\tanh^2\theta}$ and $\sinh 2\theta = \frac{2\tanh\theta}{1-\tanh^2\theta}$ for $\theta = \frac{a}{2}$ and $\theta = \frac{b}{2}$ one checks that $\cosh a \cosh b - \sinh a \sinh b \cos \gamma = 1 + 2\frac{|v_a - v_b|}{(1-|v_a|^2)(1-|v_b|^2)}$ and the proof is complete.

□

COROLLARY 2.4. *For any $\epsilon > 0$, there exists a constant $\kappa_\epsilon > 0$ such that, for any triangle Δ with sides of hyperbolic length a, b, c and opposite angles α, β, γ with $\gamma \geq \epsilon$, one gets*

$$a + b - \kappa_\epsilon \leq c \leq a + b.$$

Proof. The second inequality is just the triangle inequality. To prove the left hand side one, we use the hyperbolic law of cosines :

$$\begin{aligned} c \geq \ln(\cosh c) &= \ln\Big(\cosh a \ \cosh b - \sinh a \sinh b \cos\gamma\Big) \\ &\geq \ln\Big((1 - |\cos\gamma|)\cosh a \ \cosh b\Big) \\ &\geq \ln\Big((1 - |\cos\gamma|)\frac{e^a}{2}\ \frac{e^b}{2}\Big) = a + b + \ln(1 - |\cos\gamma|) - 2\ln 2. \end{aligned}$$

□

Remark. There exists another "geometrical" argument to prove this theorem. Without loss of generality, we may assume that the vertice of Δ corresponding to the angle γ is 0 and that the others vertices are on the geodesic segments $[0, e^{-i\frac{\gamma}{2}}]$ and $[0, e^{i\frac{\gamma}{2}}]$; furthermore, the geodesic edge c joigning these two vertices lies inside the domain surounded by the two above segments and the circle $\mathcal{C}_\gamma$ orthogonal to the unit circle at points $e^{i\frac{\gamma}{2}}$ and $e^{-i\gamma/2}$. So the hyperbolic distance from 0 to c is less than the one between 0 and $\mathcal{C}_\gamma$, that is to say to $\ln\frac{1+r}{1-r}$ with $r = \frac{1}{\cos\frac{\gamma}{2}} - \tan\frac{\gamma}{2} = \frac{1-\sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2}} \geq \frac{1-\sin\frac{\epsilon}{2}}{\cos\frac{\epsilon}{2}} > 0$.

This corollary will be very usefull in the sequel, since we have the following

PROPOSITION 2.5. *For any $\kappa > 0$, there exits $C := C(\kappa) > 0$ such that, for any $\mathbf{x}, \mathbf{y}$ and $\mathbf{z} \in \mathbb{H}$ satisfying the inequality $d(\mathbf{x}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) - \kappa$, one gets*

$$d(\mathbf{y}, [\mathbf{x}, \mathbf{z}]) \leq C. \tag{4}$$

Proof. Let $\mathbf{y}^\perp$ be the projection of $\mathbf{y}$ on $[\mathbf{x}, \mathbf{z}]$: the geodesic segment $[\mathbf{y}, \mathbf{y}^\perp]$ and $[\mathbf{x}, \mathbf{z}]$ are thus orthogonal at $\mathbf{y}^\perp$. and by Corollary 2.4, one gets

$$d(\mathbf{x}, \mathbf{y}) \geq d(\mathbf{x}, \mathbf{y}^\perp) + d(\mathbf{y}^\perp, \mathbf{y}) - \kappa(\frac{\pi}{2}) \quad \text{and} \quad d(\mathbf{y}, \mathbf{z}) \geq d(\mathbf{y}, \mathbf{y}^\perp) + d(\mathbf{y}^\perp, \mathbf{z}) - \kappa(\frac{\pi}{2}).$$

Since $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{x}, \mathbf{y}^\perp) + d(\mathbf{y}^\perp, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) - \kappa$ and $d(\mathbf{y}, [\mathbf{x}, \mathbf{z}]) = d(\mathbf{y}, \mathbf{y}^\perp)$, inequality (4) follows with $C = \frac{\kappa + 2\kappa(\frac{\pi}{2})}{2}$.□

3. Classification and decomposition of the preserving orientation isometries

We first introduce the 3 following subgroups of G :

- $K := \Big\{ r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} / \theta \in \mathbb{R} \Big\}$
- $A := \Big\{ a_\lambda = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} / \lambda > 0 \Big\}$

- $N := \left\{ n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} / t \in \mathbb{R} \right\}$

Note that for any $\lambda > 0, t \in \mathbb{R}$ and $z \in \mathbb{H}$ one gets $a_\lambda \cdot z = \lambda z$ and $n_t \cdot z = z + t$; the homographies a_λ are thus homotheties with center 0 and the maps n_t are horizontal translations.

Furthermore, these subgroups are caracterized by their fixed points in $\mathbb{H}^2$:

- the group K is exactly the set of $g \in G$ which fixes i,
- the group A is exactly the set of $g \in G$ which fixes both 0 and $+\infty$
- the group N is exactly the set of $g \in G$ which fixes $+\infty$ and no other points in $\bar{\mathbb{H}}$.

Indeed, let $g : z \mapsto \frac{az+b}{cz+d}, g \neq \pm Id$:

- the equation $g \cdot i = i$ implies $a = d$ and $b = -c$ and one concludes $g \in K$ using the fact that $ad - bc = 1$;

- the two equations $g \cdot \infty = \infty$ and $g \cdot 0 = 0$ imply $b = c = 0$ so that $d = a^{-1}$ and $g \in A$.

- the equation $g \cdot \infty = \infty$ implies $c = 0$. If $a \neq 1$ then $a \neq d$ and $\frac{b}{d-a}$ is another fixed point of g; consequently, ∞ is the unique fixed point of g if and only if $g \in N$. More generally, any isometry $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $G \setminus \{Id\}$ can be classified with its fixed points ; this is the content of the following subsection.

3.1. Classification of the the isometries. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to G, we denote by $tr(g) = a + d$ the trace of the matrice g ; we get

Proposition 3.1. *Let $g \in G \setminus \{Id\}$.*

1. *if $|tr(g)| < 2$ then g fixes exactly one point in $\bar{\mathbb{H}}$, this point belongs to $\mathbb{H}$ and g is conjugated to some element of K*
2. *if $|tr(g)| > 2$, then g fixes exactly two points in $\mathbb{R} \cup \{\infty\}$ and is conjugated to some element of A,*
3. *if $|tr(g)| = 2$ then g fixes exactly one point in $\mathbb{H}$, this point belongs to $\partial\mathbb{H}$ and g is conjugated to some element of N*

Proof. The point $z \in \bar{\mathbb{H}}$ is fixed by g if and only if $cz^2 + (d-a)z - b = 0$; the discriminant of this equation is $\Delta := (d-a)^2 + 4bc = (d+a)^2 - 4 = tr(g)^2 - 4$.

- if $|tr(g)| < 2$, then $\Delta < 0$ and there exists two distinct solutions which are conjugated each to the other ; there exists thus a unique solution $z_0 \in \mathbb{H}$; if $h \in G$ is such that $h(z_0) = i$, then hgh^{-1} fixes i and thus belongs to K.

- if $|tr(g)| > 2$ then $\Delta > 0$ and there exist two distinct solutions α, β in $\mathbb{R} \subset \partial\mathbb{H}$; if $h \in G$ is such that $h(\alpha) = 0$ and $h(\beta) = \infty$ then hgh^{-1} fixes 0 and ∞ and thus belongs to A.

- if $|tr(g)| = 2$ then $\Delta = 0$ and there exists a unique solution $\alpha \in \mathbb{R}$; if $h \in G$ is such that $h(\alpha) = \infty$ then hgh^{-1} fixes ∞ and no other points in $\bar{\mathbb{H}}$, it thus belongs to N.□

Let us introduce some definitions :

- In the first case, one says that g is **elliptic** ; it has a unique fixed point in $\bar{\mathbb{H}}$, and this point belongs to $\mathbb{H}$,.
- In the second case , one says that g is an **hyperbolic isometry**. It fixes two points in $\mathbb{H}$ and these points belong to $\partial\mathbb{H}$:
  - one of these two points (denoted by ξ_g^+) is such that, for any $z \in \mathbb{H}$, one gets $g^n \cdot z \to \xi_g^+$ as $n \to +\infty$. It is called the **attractive fixed point** of g.
  - the other one (denoted by ξ_g^-) satisfies $g^n \cdot z \to \xi_g^-$ as $n \to -\infty$. It is called the **reppeling fixed point** of g.
- Atlast, when g fixes only one point in $\mathbb{H}$ and this point (denoted by ξ_g) belongs to $\partial\mathbb{H}$, one says that g is **parabolic** ; g acts by translation on the horospheres centered at ξ_g.

3.2. The action of G on the unit tangent bundle. Any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ acts naturally on $T\mathbb{H}$ as follows :

$$\forall z \in \mathbb{H}, \forall \vec{u} \in T_z\mathbb{H} \quad g \cdot (z\vec{u}) = (g \cdot z, Dg_z(\vec{u})) = \Big(\frac{az+b}{cz+d}, \frac{1}{(cz+d)}\vec{u}\Big)$$

Since g is an isometry, one gets $\|\vec{u}\| = \|Dg_z \vec{u}\|$ so that G acts also on the unit tangent bundle $T^1\mathbb{H}$ of $\mathbb{H}$ defined by $T^1\mathbb{H} = \{(z, \vec{u})/z \in \mathbb{H}, \vec{u} \in T_z\mathbb{H}, \|\vec{u}\| = 1\}$; this is this action which contains the global information on G.

For any $z \in \mathbb{H}$, we denote by $\vec{u}_z$ the unit vector in $(T_z\mathbb{H}, \langle, \rangle_z)$ pointing to ∞.

- Let $k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K$ with $0 \leq \theta < \pi($ since we are in $PSL(2, \mathbb{R}))$; one has $k_\theta \cdot i = i$ and $Dk_\theta \cdot \vec{u}_i = \frac{1}{(\sin\theta + i\cos\theta)^2}\vec{u}_i = -e^{2i\theta}\vec{u}_i$.

  Then for any unit vector $\vec{u}$ in $T_i\mathbb{H}$, there exists a unique $k \in K$ such that $k \cdot (i, \vec{u}_i) = (i, \vec{u})$.
- For $a_\lambda \in A$, one has $a_\lambda \cdot i = \lambda i$ and $Da_\lambda \cdot \vec{u}_i = \frac{1}{(1/\sqrt{\lambda}))^2}\vec{u}_i = \lambda\vec{u}_i = \vec{u}_{\lambda \cdot i}$.
- For $n_t \in N$, one has $n_t \cdot i = t + i$ and $Dn_t \cdot \vec{u}_i = \vec{u}_i = \vec{u}_{t+i}$.

Fix now $z \in \mathbb{H}$ and $\vec{u} \in T^1_z\mathbb{H}$ and set $z = t + i\lambda$.
One has $z = n_t a_\lambda \cdot i$ ie $i = (n_t a_\lambda)^{-1} \cdot z$; the vector $D_z(n_t a_\lambda)^{-1} \cdot \vec{u}$ is a unit vector in $T_i\mathbb{H}$, there thus exists an unique $\theta \in [0, \pi[$ such that $D_z(n_t a_\lambda)^{-1} \cdot \vec{u} = -e^{2i\theta}\vec{u}_i$ so that $\vec{u} = D_i(n_t a_\lambda k_\theta) \cdot \vec{u}$. Finally $(z, \vec{u}) = n_t a_\lambda k_\theta \cdot (i, \vec{u}_i)$.

By this argument, we have also prove that any $g \in G$ may be decomposed as nak with $n \in N, a \in A$ and $k \in K$, and the decomposition is unique.

We have thus prove the

Proposition 3.2. *The action of G on $T^1\mathbb{H}$ is simply transitive : for any $(z, \vec{u}) \in T^1\mathbb{H}$, there exists a unique element $g \in G$ such that $(z, \vec{u}) = g \cdot (i, \vec{u}_i)$.*

Any element $g \in G$ may be decomposed in a unique way as $g = nak$ with $n \in N, a \in A$ and $k \in K$. This is the so-called **Iwasawa decomposition of** *G.*

We may also obtain by a similar way a geometrical construction of the **Cartan decomposition** of elements g of G which states that there exists $k, k' \in K$ and $a \in A$ such that $g = kak'$:

Fix $g \in G$, set $z = g \cdot i$ and $\lambda = d(i, z)$; the point z belongs to the (hyperbolic) circle $\mathcal{C}$ centered at i with (hyperbolic) radius λ ; set $w = i\mathbb{R} \cap \mathcal{C}$, one gets $w = a_\lambda \cdot i$. Let $k \in K$ such that $k \cdot w = z$. One thus gets $z = g \cdot i = ka_\lambda \cdot i$ so that $(ka_\lambda)^{-1}g \cdot i = i$; it follows that $(ka_\lambda)^{-1}g \in K$, in other words there exists $k' \in K$ such that $g = ka_\lambda k'$.

4. Fuchsian groups

Definition 4.1. *A subgroup Γ of $Isom(\mathbb{H})$ is discrete if it is a discrete set in the topological space $Isom(\mathbb{H}) \simeq PSL(2, \mathbb{R})$.*

Equivalently, Γ is discrete if and only if $\gamma_n \to Id, \gamma_n \in \Gamma$ implies $\gamma_n = Id$ for sufficiently large n.

Proposition 4.2. *A discrete group Γ acts properly discontinuously on $\mathbb{H}$: that means that, for any $\mathbf{x} \in \mathbb{H}$, one gets* $\mathrm{Card}(\Gamma \cdot \mathbf{x} \cap C) < +\infty$ *for any compact subset $C \subset \mathbb{H}$.*

Proof. Let $\mathbf{x} \in \mathbb{H}$ and C be a compact subset of $\mathbb{H}$, then $\{\gamma \in \Gamma : \gamma \cdot \mathbf{x} \in C\} = \{g \in PSL(2, \mathbb{R}) : g \cdot \mathbf{x} \in K\} \cap \Gamma$ is a finite set, since it is the intersection of a compact set [(11)] and a discrete set, and hence Γ acts properly and discontinuously.$\square$

One says that a discrete group Γ is and **elementary group** when it is a cyclic group generated by some element ; otherwise one says that Γ is **a Fuchsian group**.

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[11] one has to check that the set $\{g \in PSL(2, \mathbb{R}) : g \cdot i \in C\}$ is a compact subset of $PSL(2, \mathbb{R})$; using a Cartan decomposition $ka_{\lambda_g}k'$ of $g == \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one gets $a^2 + b^2 + c^2 + d^2 = \lambda_g + \frac{1}{\lambda_g}$ with $d(i, g \cdot i) = d(i, a_{\lambda_g} \cdot i) = 2|\ln\lambda_g|$. The point $g \cdot i$ lives in a compact set in $\mathbb{C}$ if and only if the reals λ_g lives in a compact subset of $\mathbb{R}^{*+}$ which is equivalent to the fact that g lives in a compact subset of G.

4.1. The notion of fundamental domain. We are going to be concerned with fundamental regions of mainly Fuchsian groups :

DEFINITION 4.3. *Let Γ be a discrete subgroup of $Iso^+(\mathbb{H})$. A closed region $F \subset \mathbb{H}$ with non-empty interior is a fundamental region for Γ if*

(1) $\cup_{\gamma\in\Gamma}\gamma \cdot F = \mathbb{H}^2$
(2) $\mathring{F} \cap \gamma \cdot \mathring{F} = \emptyset$ *for any* $\gamma \in \Gamma \setminus \{Id\}$

The family $(\gamma \cdot F)_{\gamma\in\Gamma}$ is a tesselation of $\mathbb{H}$.

It is not always easy to construct a fundamental domain of a Fuchsian group . Since Γ is discrete, there exists in $\mathbb{H}$ a point $\mathbf{x}$ which is not fixed by any element of $\Gamma \setminus \{Id\}$ (see Lemma 2.2.5 in [**4**]) ; it is thus possible to defined the **Dirichlet region for Γ centered at $\mathbf{x}$**:

$$D_{\mathbf{x}}(\Gamma) := \Big\{\mathbf{y} \in \mathbb{H}/d(\mathbf{y}, \mathbf{x}) \le d(\mathbf{y}, \gamma \cdot \mathbf{x}) \ for\ all\ \gamma \in \Gamma\Big\}.$$

Clearly, one gets $\mathbf{x} \in D_{\mathbf{x}}(\Gamma)$ and one may check that $D_{\mathbf{x}}(\Gamma)$ contains a neighbourhood of $\mathbf{x}$ (see [**4**], chap.3 for the details)

Note that for any $g \in G$, the set $H_{\mathbf{x}}(g) = \Big\{\mathbf{y} \in \mathbb{H}/d(\mathbf{y}, \mathbf{x}) \le d(\mathbf{y}, g\cdot \mathbf{x})\Big\}$ is the hyperbolic half-plane which contains $\mathbf{x}$ and with boundary the geodesic $\Big\{\mathbf{y} \in \mathbb{H}/d(\mathbf{y}, \mathbf{x}) = d(\mathbf{y}, g \cdot \mathbf{x})\Big\}$ orthogonal to the geodesic segment $[\mathbf{x}, g \cdot \mathbf{x}]$ through its mid-point [(12)]. The Dirichlet domain $D_{\mathbf{x}}(\Gamma)$ is thus

$$D_{\mathbf{x}}(\Gamma) = \bigcap_{\substack{\gamma\in\Gamma\\ \gamma\neq Id}} H_{\mathbf{x}}(\gamma).$$

4.2. Examples of co-compacts groups, or uniform lattices. The existence of co-compact Fuchsian group is not easy to check and is related to the theory of Riemann surfaces.

When Γ contains no elliptic elements, one may prove that $area(\mathbb{H}/\Gamma) = 2\pi(2g - 2)$ where g is the genus of the surface X/Γ ; in particular $g \geq 2$! One may find in [**4**] an explicit construction of a Fuchsian group Γ without elliptic elements and such that $\mathbb{H}/\Gamma$ is a surface of genus 2 (see Example C, chap 4.3.).

When Γ contains elliptic elements, these are necessarily of finite order since Γ is discrete ; the volume of the quotient surface may be computed in terms of the genus and the order of the elliptic elements (see [**4**] chap.4.).

4.3. An example of non uniform lattices. In this paragraph, we show that the subgroup $\Gamma = PSL(2, \mathbb{Z})$ of G (which is of course discrete!) is such that the quotient manifold $\mathbb{H}/\Gamma$ is non-compact but has finite area : one says that Γ is a **non-uniform lattice of G**.

One has $PSL(2, \mathbb{Z}) = \Big\{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}/a, b, c, d \in \mathbb{Z}\ and\ ad - bc = 1\}$. We will prove the

PROPOSITION 4.4. *Fix $k > 1$. The Dirichlet region of $\Gamma = PSL(2, \mathbb{Z})$ centered at ki is the set*

$$F = \{z \in \mathbb{H}/|z| \geq 1\ and\ |\mathrm{Re}\ z| \leq \frac{1}{2}\}.$$

In particular Γ is a lattice and $area(\mathbb{H}/\Gamma) = \frac{\pi}{3}$.

Proof. The isometries $T : z \mapsto z + 1$ and $S : z \mapsto \frac{-1}{z}$ are in Γ ; the 3 geodesics sides of F are $\partial H_{ki}(T), \partial H_{ki}(T^{-1})$ and $\partial H_{ki}(S)$. Thus $D_{ki}(\Gamma) \subset F$.

If $D_{ki}(\Gamma) \neq F$, there exists $z \in int(F)$ and $\gamma \in \Gamma$ such that $\gamma \cdot z \in int(F)$; this cannot happen! Indeed, suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then

$$|cz + d|^2 = c^2|z|^2 + d^2 + 2cd\mathrm{Re}(z) > c^2 + d^2 - [|cd| = (|c| - |d|)^2 + |cd|,$$

[12] To see that the set $\Big\{\mathbf{y} \in \mathbb{H}/d(\mathbf{y}, \mathbf{x}) = d(\mathbf{y}, \mathbf{x}')\Big\}$ is the geodesic orthogonal to the geodesic segment $[\mathbf{x}, \mathbf{x}']$ through its mid-point w, it suffices to check that it is true when $\mathbf{x} = i$ and $\mathbf{x}' = r^2 i$; in this case, one gets $w = ir$ and, by (3), the equation $d(\mathbf{y}, \mathbf{x}) = d(\mathbf{y}, \mathbf{x}')$ is equivalent to $\frac{|\mathbf{y}-i|^2}{\mathrm{Im}(\mathbf{y})} = \frac{|\mathbf{y}-r^2 i|^2}{r^2\mathrm{Im}(\mathbf{y})}$ ie $|y| = r$.

since $|z| > 1$ and $\mathrm{Re}(z) > -\frac{1}{2}$. But $(|c| - |d|)^2 + |cd|$ is a non negative integer which cannot be equal to 0 (otherwise $c = d = 0$ which contradicts $ad - bc = 1$). Therefore $(|c| - |d|)^2 + |cd| \geq 1$ and $|cz + d| > 1$. It follows that

$$\mathrm{Im}\ \gamma \cdot z = \frac{\mathrm{Im}\ z}{|cz + d|^2} < \mathrm{Im}\ z.$$

the same arguments holds with z, γ replaced by $\gamma \cdot z, \gamma^{-1}$ which leads to a contradiction. Then $D_i(\Gamma) = F$.□

Note that the orbit $PSL(2, \mathbb{Z})$ orbit of i remains under the line $\mathrm{Im}(z) = 1$, ie outside the horosphere $\mathcal{H} := \{z \in \mathbb{H} : \mathrm{Im}(z) \geq 1\}$: indeed, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R})$, one gets

$$\mathrm{Im}\ \gamma \cdot i = \frac{1}{\sqrt{c^2 + d^2}} \leq 1$$

since $c^2 + d^2 \geq 1$.

More precisely, for any $r_0 > 1$ and $z \in \mathbb{H}$ with $\mathrm{Im}(z) \geq r_0$, the inequality $\mathrm{Im}\ \gamma \cdot z \geq r_0$ implies $c = 0$ [13] ; in otherwords, if $c \neq 0$, the $\gamma \cdot z$ for any $z \in \mathbb{H}$ with $\mathrm{Im}(z) \geq r_0 > 1$ are strictly below the horocycle $\mathrm{Im}(z) = r_0$. We have then the

LEMMA 4.5. *Fix $r_0 > 1$ and denote by $\mathcal{H}_{r_0}$ the horosphere $\{z \in H : \mathrm{Im}(z) \geq r_0\}$. The images of $\mathcal{H}_{r_0}$ by the elements of $PSL(2, \mathbb{Z})$ are pairwise disjoint.*

Proof. If $\gamma \cdot \mathcal{H}_{r_0} \cap \gamma' \cdot \mathcal{H}_{r_0} \neq \emptyset$, then, setting $g = \gamma^{-1}\gamma$, one gets $g \cdot \mathcal{H}_{r_0} \cap \mathcal{H}_{r_0} \neq \emptyset$. By the above, this implies that g is an upper triangular matrice and so $\gamma \cdot \mathcal{H}_{r_0} \neq \gamma' \cdot \mathcal{H}_{r_0}$ and so $\gamma \cdot \mathcal{H}_{r_0} = \gamma' \cdot \mathcal{H}_{r_0}$.□

4.4. The Schottky groups. In this subsection, we introduce the notion of Schottky group and give an important property they satisfy and which will be usefull in the sequel. These Schottky groups will be typical examples of groups satisfying the following *property* :

Property C : *There exist a closed and proper subset F in the boundary of the hyperbolic space such that any element of Γ different from the identity maps $\partial X - F$ into F.*

We now consider a group Γ satisfying Property C. We have the two following facts :

Fact 1- *The group Γ is discrete and its limit set is contained in F.*

Proof of Fact 1- If Γ was not discrete, one would get in Γ a sequence of elements acting on ∂X and converging to the identity so that any point outside F would be the limit of a sequence of points in F. That the limit set of Γ is contained in F follows from the fact that fixed points of isometries in $\Gamma^* = \Gamma - \{Id\}$ belong to F.

We can now state the definition of a Schottky group :

DEFINITION 4.6. *Let $\mathfrak{h}_i$, $i = 1, \ldots L$, be a family of hyperbolic isometries in $Iso^+(\mathbb{H})$ and $\theta > 0$ a constant ; one says that $\mathfrak{h}_1, \cdots, \mathfrak{h}_L$ are in θ-Schottky position if there exist closed and pairewise disjoint sets $F_i^{\pm}$ in $\mathbb{H}$ such that*

- *$\mathfrak{h}_i(\mathbb{H} \setminus F_i^-) \subset F_i^+$ for any $i \in \{1, \cdots, L\}$*
- *$\mathfrak{h}_i^-(\mathbb{H} \setminus F_i^+) \subset F_i^-$ for any $i \in \{1, \cdots, L\}$*
- *for any $\mathbf{x}$ and $\mathbf{y}$ which belong to distinct sets $F_i^{\pm}$, the angle between the geodesic segments $[\mathbf{o}, \mathbf{x}]$ and $[\mathbf{o}, \mathbf{y}]$ is greater than θ.*

The group Γ generated by the isometries $\mathfrak{h}_1, \cdots, \mathfrak{h}_L$ is called the Schottky subgroup generated by the $\mathfrak{h}_i$'s.

Since any element in Γ^* maps the complement of the closed set $F = \cup F_i$ into F, the group Γ is discrete by Fact 1. Moreover, by the Klein's tennis table criteria, it is the free group generated by the $\mathfrak{h}_1^{\pm 1}, \cdots, \mathfrak{h}_L^{\pm 1}$: any element in Γ can be uniquely written as the product $\gamma = a_1 \cdots a_k$ where $a_i \in \{\mathfrak{h}_1^{\pm 1}, \cdots, \mathfrak{h}_L^{\pm 1}\}$ with the property that $a_j \neq a_{j+1}^{-1}$ for any $1 \leq j \leq k - 1$.

Notations The set $\mathcal{A} := \{\mathfrak{h}_1^{\pm 1}, \cdots, \mathfrak{h}_L^{\pm 1}\}$ is called the *alphabet* of Γ. A sequence $a_1 \cdots a_k$ where $a_i \in \{\mathfrak{h}_1^{\pm 1}, \cdots, \mathfrak{h}_L^{\pm 1}\}$ with $a_j \neq a_{j+1}^{-1}$ for any $1 \leq j \leq k - 1$ is said *admissible*. If $\gamma = a_1 \ldots a_k$, the $a_1, \cdots, a_k$ are called the *letters* of γ and the number k of letters is the *symbolic length* of γ.

[13] Set $z = x + iy$ with $y \geq r_0$; one gets $\mathrm{Im}\ \gamma \cdot z \geq r_0 \Leftrightarrow \frac{y^2}{(cx+d)^2 + c^2y^2} \geq r_0^2 \Leftrightarrow y^2 \geq r_0^2(cx + d)^2 + r_0^2c^2y^2$ which cannot be satisfied when $c \neq 0$ since in this case $c^2 \geq 1$.

PROPOSITION 4.7. *Let Γ be a Schottky group generated by the set $\{\mathfrak{h}_1^{\pm 1}, \cdots, \mathfrak{h}_L^{\pm 1}\}$ of hyperbolic isometries of $\mathbb{H}$. Then*

- *there exists a constant $\kappa_\Gamma > 0$ such that for any $\gamma = a_1 \cdots a_k \in \Gamma$ with symbolic length $k \geq 1$ one gets*
$$d(\mathbf{o}, \gamma \cdot \mathbf{o}) \geq d(\mathbf{o}, a_1 \cdots a_i \cdot \mathbf{o}) + d(\mathbf{o}, a_{i+1} \cdots a_k \cdot \mathbf{o}) - \kappa_\Gamma.$$
- *there exists a constant $C_\Gamma > 0$ such that*
$$d(a_1 \cdots a_i \cdot \mathbf{o}, [\mathbf{o}, \gamma \cdot \mathbf{o}]) \leq C_\Gamma.$$
- *for any $\gamma \in \Gamma$ and any $\mathbf{x} \in [\mathbf{o}, \gamma \cdot \mathbf{o}]$, one gets*
$$d(\mathbf{x}, \Gamma \cdot \mathbf{o}) \leq C_\Gamma + \Delta_\Gamma$$
with $\Delta_\Gamma := \max_{i \leq 1 \leq L} d(\mathbf{o}, \mathfrak{h}_i \cdot \mathbf{o})$.

Proof. By the assumption over the subsets $F_i^{\pm} \subset \mathbb{H}$ associated with the generators $\mathfrak{h}_i$, there exists $\theta > 0$ such that for any $\gamma := a_1 \cdots a_k$ and $\gamma' := b_1 \cdots b_l$ in Γ with $a_1 \neq b_1$ the angle between the geodesic segments $[\mathbf{o}, \gamma \cdot \mathbf{o}]$ and $[\mathbf{o}, \gamma' \cdot \mathbf{o}]$ is greater than θ ; in particular, for any $1 \leq i \leq k-1$, the angle between $[\mathbf{o}, a_i^{-1} \cdots a_1^{-1} \cdot \mathbf{o}]$ and $[\mathbf{o}, a_{i+1} \cdots a_k \cdot \mathbf{o}]$ is greater than θ since $a_{i+1} \neq a_i^{-1}$. It readily follows that, for some constant $\kappa = \kappa(\theta) > 0$ one gets, for any admissible sequence $(a_i)_{1 \leq i \leq k}$
$$d(\mathbf{o}, a_1 \cdots a_k \cdot \mathbf{o}) \geq d(\mathbf{o}, a_1 \cdots a_i \cdot \mathbf{o}) + d(\mathbf{o}, a_{i+1} \cdots a_k \cdot \mathbf{o}) - \kappa_\Gamma.$$
By Proposition 2.5, it follows that the point $\gamma_i \cdot \mathbf{o}$ lies in some $C_\Gamma = C(K_\Gamma)$-neighbourhood of the geodesic segment $[\mathbf{o}, \gamma \cdot \mathbf{o}]$.

For any $\gamma = a_1 \cdots a_k \in \Gamma$ with symbolic length $k \geq 1$ and any $1 \leq i \leq k$, one sets $\gamma_i = a_1 \cdots a_i$. One sets $d(\gamma_i \cdot \mathbf{o}, \gamma_{i+1} \cdot \mathbf{o}) \leq \Delta_\Gamma$ for any $1 \leq i \leq k-1$; if $\mathbf{x}_i$ denotes the orthogonal projection of $\gamma_i \cdot \mathbf{o}$ on $[\mathbf{o}, \gamma \cdot \mathbf{o}]$ (with the convention $\mathbf{x}_0 = \mathbf{o}$ and $\mathbf{x}_k = \gamma \cdot \mathbf{o}$), one also gets $d(\mathbf{x}_i, \mathbf{x}_{i+1}) \leq \Delta_\Gamma$ for any $0 \leq i \leq k-1$ since the projection is a contraction. Now, fix $\mathbf{x}$ in $[\mathbf{o}, \gamma \cdot \mathbf{o}]$; there exists $i \in \{0, \cdots, k-1\}$ such that $\mathbf{x}$ lies in the interval $[\mathbf{x}_i, \mathbf{x}_{i+1}]$; so $d(\mathbf{x}, \Gamma \cdot \mathbf{o}) \leq d(\mathbf{x}, \mathbf{x}_i) + d(\mathbf{x}_i, \gamma_i \cdot \mathbf{o}) \leq \Delta_\Gamma + C_\Gamma$. □

5. On the orbital fonction of some discrete groups of isometries of $\mathbb{H}$

5.1. The orbital function of a discrete subgroup of $Iso(\mathbb{H})$. We assume that Γ is a discrete subgroup of $Iso^+(\mathbb{H})$. We denote by N_Γ its orbital function defined by :
$$\forall \mathbf{x}, \mathbf{y}) \in H, \forall R > 0 \quad N_\Gamma(\mathbf{x}, \mathbf{y}, R) := \mathrm{Card}\{\gamma \in \Gamma : d(\mathbf{x}, \gamma \cdot \mathbf{y}\} \leq R\}.$$
The aim of the present section is to prove that for a large class of "non elementary " discrete subgroups Γ of isometries of $\mathbb{H}$, one gets : there exists $\delta_\Gamma > 0$ and $C_\Gamma > 0$ such that
$$\forall \mathbf{x}, \mathbf{y}) \in H, \forall R > 0 \quad \frac{e^{\delta_\Gamma R}}{C_\Gamma} \leq N_\Gamma(\mathbf{x}, \mathbf{y}, R) \leq C_\Gamma e^{\delta_\Gamma R}. \tag{5}$$
This estimation will be valid for uniform and non uniform lattices with $\delta_\Gamma = 1$, and also for Schottky groups, with δ_Γ to be precised.

5.1.1. *The critical exponent of* Γ. By the triangle inequality, one gets, for any $\mathbf{x}, \mathbf{y} \in H$ and any isometry $\gamma \in \mathrm{Iso}^+(\mathbb{H})$
$$d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leq R \Longrightarrow d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leq R + d(\mathbf{o}, \mathbf{x}) + d(\mathbf{o}, \mathbf{y})$$
and
$$d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leq R \Longrightarrow d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leq R + d(\mathbf{o}, \mathbf{x}) + d(\mathbf{o}, \mathbf{y}).$$
It readily follows that for any discrete subgroup of isometries of $\mathbb{H}$, and any $R \geq d(\mathbf{o}, \mathbf{x}) + d(\mathbf{o}, \mathbf{y})$, one gets
$$N\Big(\mathbf{o}, \mathbf{o}, R - d(\mathbf{o}, \mathbf{x}) - d(\mathbf{o}, \mathbf{y})\Big) \leq N(\mathbf{x}, \mathbf{y}, R) \leq N\Big(\mathbf{o}, \mathbf{o}, R + d(\mathbf{o}, \mathbf{x}) + d(\mathbf{o}, \mathbf{y})\Big)$$

so that the exponential growth $\limsup\limits_{R\to+\infty} \dfrac{\log\{\gamma\in\Gamma : d(\mathbf{x},\gamma.\mathbf{y})\le R\}}{R}$ of the orbital function of Γ does not depend on the choice of the points $\mathbf{x}$ and $\mathbf{y}$. By definition, **the critical exponent of** Γ is equal to

$$\delta_\Gamma := \limsup_{R\to+\infty} \frac{\log\{\gamma\in\Gamma : d(\mathbf{x},\gamma.\mathbf{y})\le R\}}{R}$$

and does not depend on $\mathbf{x}$ and $\mathbf{y}$.

This exponent δ_Γ is also called **the Poincaré exponent of** Γ and is defined as the exponent of convergence of the so-called *Poincaré series* of Γ defined by

$$\mathcal{P}(\mathbf{x},\mathbf{y},s) := \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{x},\gamma\cdot\mathbf{y})}.$$

Note that if $\mathcal{P}(\mathbf{x},\mathbf{y},s_0) < +\infty$ then $\mathcal{P}(\mathbf{x},\mathbf{y},s) < +\infty$ for any $s \ge s_0$; conversely, if $\mathcal{P}(\mathbf{x},\mathbf{y},s_0) = +\infty$ then $\mathcal{P}(\mathbf{x},\mathbf{y},s) = +\infty$ for any $s \le s_0$. So, there exists a value $s_c \ge 0$, called *the critical exponent of* Γ such that $\mathcal{P}(\mathbf{x},\mathbf{y},s) = +\infty$ if $s < s_c$ and $\mathcal{P}(\mathbf{x},\mathbf{y},s) < +\infty$ if $s > s_c$. Note that this critical exponent does not depend on $\mathbf{x}$ and $\mathbf{y}$ since for any $s\ge 0$ and any $\mathbf{x},\mathbf{y}\in\mathbb{H}$ one gets

$$e^{-sd(\mathbf{o},\mathbf{x})-sd(\mathbf{o},\mathbf{y})}\mathcal{P}(\mathbf{o},\mathbf{o},s) \le \mathcal{P}(\mathbf{x},\mathbf{y},s) \le e^{sd(\mathbf{o},\mathbf{x})+sd(\mathbf{o},\mathbf{y})}\mathcal{P}(\mathbf{o},\mathbf{o},s).$$

Furthermore, denoting by $\mathcal{A}_\Gamma(\mathbf{x},\mathbf{y},R,D)$ the annulus defined by

$$\mathcal{A}_\Gamma(\mathbf{x},\mathbf{y},R,D) := \Big\{\gamma\in\Gamma / R \le d(\mathbf{x},\gamma\cdot\mathbf{y}) < R + D\Big\},$$

one gets, for any $s>0$

$$\sum_{n\ge 0} e^{-s(n+1)}\mathrm{Card}\mathcal{A}_\Gamma(\mathbf{x},\mathbf{y},n,1) \le \mathcal{P}(\mathbf{x},\mathbf{y},s_0) \le \sum_{n\ge 0} e^{-sn}\mathrm{Card}\mathcal{A}_\Gamma(\mathbf{x},\mathbf{y},n,1).$$

so that the critical exponent of Γ is equal to $\limsup\limits_{n\to+\infty} \dfrac{\log\Big(\mathrm{Card}\mathcal{A}_\Gamma(\mathbf{x},\mathbf{y},n,1)\Big)}{n}$. A straightforward argument leads to the fact that this coincides with the critical exponent of Γ.

5.1.2. *The critical exponent of the cyclic group $\langle\mathfrak{h}\rangle$ generated by an hyperbolic element $\mathfrak{h}$.* Let $\mathfrak{h}$ be an hyperbolic element of $\mathrm{Iso}^+(\mathbb{H})$ with fixed points ξ^+ and ξ^- at infinity ; the geodesic $(\xi^-\ \xi^+)$ is globaly invariant by $\mathfrak{h}$ and for any $n\in\mathbb{Z}$ and $\mathbf{x}\in(\xi^-\ \xi^+)$, one gets $d(\mathbf{x},\mathfrak{h}^n\cdot(\mathbf{x}) = |n|\times l_{\mathfrak{h}}$ where $l_{\mathfrak{h}} = d(\mathbf{x},\mathfrak{h}\cdot\mathbf{x})$ is the length of the closed geodesic associated with $\mathfrak{h}$. The Poincaré series of the group $\langle\mathfrak{h}\rangle$ generated by $\mathfrak{h}$ is given by

$$\forall s>0 \quad \mathcal{P}(\mathbf{x},\mathbf{x},s) = \sum_{n\in\mathbb{Z}} e^{-s|n|l_{\mathfrak{h}}} = \frac{2}{1-e^{-sl_{\mathfrak{h}}}} - 1$$

and its critical exponent is equal to 0.

5.1.3. *The critical exponent of the cyclic group $\langle p\rangle$ generated by a parabolic element $\mathfrak{p}$.* Let $\mathfrak{p}$ be an parabolic element of $\mathrm{Iso}^+(\mathbb{H})$ with fixed points ξ at infinity and consider $\gamma\in PSL(2,\mathbb{R})$ such that $\gamma\cdot\infty = \xi^{(14)}$ so that the parabolic element $\gamma^{-1}\circ\mathfrak{p}\circ\gamma$ has ∞ as unique fixed point ; there thus exists $a\in\mathbb{R}$ such that $\gamma^{-1}\circ\mathfrak{p}\circ\gamma = \begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}$. To compute the critical exponent of $\mathfrak{p}$, we will consider the Poincaré series $\mathcal{P}_{\langle\mathfrak{p}\rangle}(\gamma\cdot\mathbf{x},\mathfrak{p}\gamma\cdot\mathbf{x})$ for $\mathbf{x}=i$; one gets

$$d(\gamma\cdot i,\mathfrak{p}\gamma\cdot i) = d(i,\gamma^{-1}\mathfrak{p}\gamma\cdot i) = d(i,a+i) = 2a+\epsilon(a). \quad ^{(15)}$$

It follows $d(\gamma\cdot i,\mathfrak{p}^n\gamma\cdot i) = 2(\ln|n|)(1+\epsilon(n))$ and so $\mathcal{P}_{\langle\mathfrak{p}\rangle}(\gamma\cdot i,\mathfrak{p}\gamma\cdot i,s) \asymp \sum\limits_{n\in\mathbb{Z}^*} \dfrac{1}{|n|^{2s}}$. The critical exponent of $\langle\mathfrak{p}\rangle$ thus equals $\frac{1}{2}$ and the Poincaré series $\mathcal{P}_{\langle\mathfrak{p}\rangle}$ diverges at $s=\frac{1}{2}$.

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[14] one may take $\gamma = Id$ when $\xi=\infty$, $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ when $\xi=0$ and $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & -\xi\end{pmatrix}$ when $\xi\in\mathbb{R}^*$

[15] applying formula (3), one gets $\sinh\frac{d(i,a+i)}{2} = \frac{a}{2}$ and thus $d(i,a+i) = 2\ln(a+\sqrt{a^2+1}) = 2(\ln a + \ln 2 + \epsilon(a))$

5.1.4. *The critical gap property.* In this subsection, we consider a non elementary discrete subgroup $\Gamma \subset Iso^+(\mathbb{H})$ without finite order elements and which is not generated, either by an hyperbolic element or a parabolic one.

Let $\mathfrak{h}_1$ be an hyperbolic element of Γ, with fixed points ξ_1^+ and ξ_1^- ; since the limit set Λ_Γ of Γ contains infinitely many points and the set of attractive fixed points of hyperbolic elements of Γ is dense in Λ_Γ, there exists an open set $U \subset \partial\mathbb{H}$ which does contain neither ξ_1^+ nor ξ_1^- and an (hyperbolic) isometry $\gamma \in \Gamma$ with the attractive fixed point x_γ in U ; the other fixed point of γ may not be equal to ξ_1^+ or ξ_1^- [(16)] , so that one may enlarge U in such a way U contains the two fixed points of γ.

We choose now two open subsets U_1^- and U_1^+ of $\partial\mathbb{H}$ such that

- for any $n \geq 1$ one gets $\mathfrak{h}_1^n(\partial\mathbb{H} \setminus U_1^-) \subset U_1^+$ and $\mathfrak{h}_1^{-n}(\partial\mathbb{H} \setminus U_1^+) \subset U_1^-$
- $U_1^- \cap U = U_1^+ \cap U = \emptyset$.

Since U contains the two fixed point of γ, there exists $N \geq 1$ such that for any $n \geq N$, one gets $\gamma^n(\partial\mathbb{H} \setminus U) \subset U$; we fix such a n and set $\mathfrak{h}_2 := \gamma^n$.

It follows that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are in Schottky position and that they generate a free (discrete) subgroup of Γ; in particular $\left\{\mathfrak{h}_1^{n_1}\mathfrak{h}_2\mathfrak{h}_1^{n_2}\cdots\mathfrak{h}_2\mathfrak{h}_1^{n_k}/k \geq 1, n_1 \cdots, n_k \in \mathbb{Z}^*\right\} \subset \Gamma$; so

$$\begin{aligned}
\mathcal{P}_\Gamma(\mathbf{o},\mathbf{o},s) &\geq \sum_{k\geq 1}\sum_{n_1,\cdots,n_k\in\mathbb{Z}^*} e^{-sd(\mathbf{o},\mathfrak{h}_1^{n_1}\mathfrak{h}_2\mathfrak{h}_1^{n_2}\cdots\mathfrak{h}_2\mathfrak{h}_1^{n_k}\cdot\mathbf{o})}\\
&\geq \sum_{k\geq 1}\sum_{n_1,\cdots,n_k\in\mathbb{Z}^*} e^{-sd(\mathbf{o},\mathfrak{h}_1^{n_1}\cdot\mathbf{o})-sd(\mathbf{o},\mathfrak{h}_2\cdot\mathbf{o})-\cdots-sd(\mathbf{o},\mathfrak{h}_1^{n_k}\cdot\mathbf{o})}\\
&\geq \sum_{k\geq 1}\Big(e^{-sd(\mathbf{o},\mathfrak{h}_2\cdot\mathbf{o})}\sum_{n\in\mathbb{Z}^*} e^{-sd(\mathbf{o},\mathfrak{h}_1^{n}\cdot\mathbf{o})}\Big)^k
\end{aligned}$$

Since $\lim\limits_{s\to 0} e^{-sd(\mathbf{o},\mathfrak{h}_2\cdot\mathbf{o})}\sum\limits_{n\in\mathbb{Z}^*} e^{-sd(\mathbf{o},\mathfrak{h}_1^{n}\cdot\mathbf{o})} = +\infty$, one gets $e^{-\delta d(\mathbf{o},\mathfrak{h}_2\cdot\mathbf{o})}\sum\limits_{n\in\mathbb{Z}^*} e^{-\delta d(\mathbf{o},\mathfrak{h}_1^{n}\cdot\mathbf{o})} > 1$ when $\delta > 0$ is quite small ; consequently $\mathcal{P}_\Gamma(\mathbf{o},\mathbf{o},\delta) = +\infty$ which gives $\delta_\Gamma \geq \delta > 0$.

Let us emphasize that the fact that the series $\sum\limits_{n\in\mathbb{Z}^*} e^{-sd(\mathbf{o},\mathfrak{h}_1^{n}\cdot\mathbf{o})}$ diverges at its critical exponent (which is evident since this critical exponent is 0) is crucial in the above argument. The same holds when $\mathfrak{h}_1$ is replaced by a parabolic element $\mathfrak{p}_1$ since the Poincaré series $\mathcal{P}_{\langle\mathfrak{p}\rangle}(\mathbf{o},\mathbf{o},s)$ diverges at its critical exponent $\delta_{\langle\mathfrak{p}\rangle} = \frac{1}{2}$. We have thus prove the

PROPOSITION 5.1. **The critical gap property** *The critical exponent of a non elementary discrete subgroup Γ of $Iso^+(\mathbb{H})$ is > 0 ; furthermore, if Γ contains a parabolic transformation, one gets $\delta_\Gamma > \frac{1}{2}$.*

5.2. The compact case. We assume here that Γ is a discrete subgroup of $Iso^+(\mathbb{H})$ such that $M := H/\Gamma$ is compact. Let Δ be the diameter of the Dirichlet domain D of Γ centered at 0.

If $\gamma \in \Gamma$ is such that $d(\mathbf{o},\gamma\cdot\mathbf{o}) \leq R - \Delta$, then for any $R > \Delta$, one gets $\gamma\cdot D \subset B(\mathbf{o},R)$.

On the other hand, if $\gamma\cdot D \cap B(\mathbf{o},R) \neq \emptyset$, then $d(\mathbf{o},\gamma\cdot\mathbf{o}) \leq R + \Delta$.

Consequently

$$\bigcup_{\substack{\gamma\in\Gamma\\ d(\mathbf{o},\gamma\cdot\mathbf{o})\leq R-\Delta}} \gamma\cdot D \subset B(\mathbf{o},R) \subset \bigcup_{\substack{\gamma\in\Gamma\\ d(\mathbf{o},\gamma\cdot\mathbf{o})\leq R+\Delta}} \gamma\cdot D,$$

so that

$$N_\Gamma(\mathbf{o},\mathbf{o},R-\Delta)\times \text{area }(M) \leq \text{area }(B(\mathbf{o},R)) \leq N_\Gamma(\mathbf{o},\mathbf{o},R+\Delta)\times\text{area }(M).$$

The estimation $N_\Gamma(\mathbf{o},\mathbf{o},R) \asymp e^R$ follows ; in particular, the critical exponent of any uniform lattice of $\mathbb{H}$ is equal to 1.

Remark. The same argument works in the case when $\mathbb{H}$ is replaced by a general simply connected Riemannian manifold X whose group of isometries admits co-compact subgroups Γ. In the general case we have no precise estimates of the volume of balls, but the following weaker result may

[16]the fact that Γ is discrete implies that two elements in Γ have a commun fixed point if and only if they both belongs to the cyclic group generated by some $g \in \Gamma$; this is not the case here since $x_\gamma \notin \{\xi_1^+, \xi_1^-\}$.

be obtained : if $B_X(\mathbf{x}, R)$ is the ball of X with center $\mathbf{x}$ and radius $R > 0$, then the quantity $\dfrac{vol(B_X(\mathbf{x}, R))}{R}$ admits a limit as $R \to +\infty$ which is called the **volume entropy of X** and denoted by $h_{vol}(X)$; furthermore, for any discrete uniform lattice $\Gamma \subset Iso^+(X)$, one gets $\delta_\Gamma = h_{vol}(X)$.

5.3. The case of Schottky groups. In this section, Γ is a Schottky group generated by the isometries $\mathfrak{h}_1, \cdots, \mathfrak{h}_L$ with $L \geq 2$. By
From now on, we will use the following notations : *for any* $\mathbf{x} \in \mathbb{H}$ *and any* $R, \alpha \in \mathbb{R}^{*+}$*, we denote by*

- $\mathcal{A}_\Gamma(\mathbf{x}, \alpha, R)$ *the "annulus " of width* α *defined by*
$$\mathcal{A}_\Gamma(\mathbf{x}, \alpha, R) := \{\mathbf{y} \in \Gamma \cdot \mathbf{x} / R - \alpha < d(\mathbf{x}, \mathbf{y}) \leq R + \alpha\}$$
- $v_\Gamma(\mathbf{x}, \alpha, R) := \sharp \mathcal{A}_\Gamma(\mathbf{x}, \alpha, R)$

5.3.1. *We first prove the following.*

Lemma 5.2. *There exist* $\Delta > 1$ *and* $C > 0$ *such that for any* $\mathbf{x} \in \mathbb{H}$ *and* $A, B \geq 2\Delta$*, one gets*
$$v_\Gamma(\mathbf{o}, 2\Delta, A + B) \leq C \times v_\Gamma(\mathbf{o}, 2\Delta, A) \times v_\Gamma(\mathbf{o}, 2\Delta, B). \tag{6}$$

Proof. We fix $\Delta \geq 1$ greater than the constant $C + \Delta_\Gamma$ which appears in Proposition 4.7. If $\mathbf{y} \in \mathcal{A}_\Gamma(\mathbf{o}, 2\Delta, A + B)$, one sets $d(\mathbf{o}, \mathbf{y}) = A + B + 2\lambda$ with $-\Delta \leq \lambda \leq \Delta$, one considers the point $\mathbf{x}$ on the geodesic segment $[\mathbf{o}, \mathbf{y}]$ such that $d(\mathbf{o}, \mathbf{x}) = A + \lambda$, and one chooses $\mathbf{x}' \in \Gamma \cdot \mathbf{o}$ such that $d(\mathbf{x}, \mathbf{x}') \leq \Delta$; one thus gets $\mathbf{x}' \in \mathcal{A}_\Gamma(\mathbf{x}, 2\Delta, A)$ and $\mathbf{y} \in \mathcal{A}_G(\mathbf{x}', 2\Delta, B)$. Inequality (6) follows. □

We apply the following elementary Lemma to the sequence $v_n := v_\Gamma(\mathbf{o}, 2\Delta, n)$, which satisfies the submultiplicative condition (6) above:

Lemma 5.3. *(Fekete [5]) Let* $(v_n)_{n \geq 1}$ *be a sequence of positive numbers such that* $v_{n+m} \leq v_n v_m \quad \forall n, m \geq n_0$.
Then, $\lim\limits_{n \to \infty} \dfrac{1}{n} \ln v_n = L \in \mathbb{R} \cup \{-\infty\}$ *and* $v_n \geq e^{Ln}$ *for all* $n \geq n_0$.

As $\lim\limits_{R \to \infty} \dfrac{1}{R} \ln v_\Gamma(\mathbf{x}, \Delta, R) = \delta_\Gamma$, we thus get $e^{\delta_\Gamma R} \preceq v_\Gamma(\mathbf{x}, \Delta, R)$.
Remark. We must emphasize that the function $v_\Gamma(\mathbf{x}, 2\Delta, R)$ has the same exponential growth than $N_\Gamma(\mathbf{x}, \mathbf{x}, R)$ since
$$v_\Gamma(\mathbf{x}, 2\Delta, n) \leq N_\Gamma(\mathbf{x}, \mathbf{x}, n + [2\Delta]) \leq \sum_{k=0}^{n+[2\Delta]} v_\Gamma(\mathbf{x}, \mathbf{x}, k).$$
(we may for instance use assertion (ii) of Lemma 5.6, which will proved in the next section).

5.3.2. *We now prove the converse inequality.* For any $i \in \{1, \cdots, L\}$, we denote by Γ^i the subsets of Γ whose elements have first and last letter equal to $\mathfrak{h}_i^{\pm 1}$: note that if the index of the last letter of some $\gamma \in \Gamma$ is distinct from the index of the first letter (say i), then setting $\gamma' := \gamma \mathfrak{h}_i$, on gets $|d(\mathbf{o}, \gamma \cdot \mathbf{o}) - d(\mathbf{o}, \gamma' \cdot \mathbf{o})| \leq \Delta_\Gamma$ so that
$$\max_{1 \leq i \leq L} N_{\Gamma^{(i)}}(\mathbf{o}, \mathbf{o}, R) \leq N_\Gamma(\mathbf{o}, \mathbf{o}, R) \leq \sum_{1 \leq i \leq L} N_{\Gamma^{(i)}}(\mathbf{o}, \mathbf{o}, R + \Delta_\Gamma).$$
It is thus sufficient to prove that for any $1 \leq i \leq L$ one gets $N_{\Gamma^{(i)}}(\mathbf{o}, \mathbf{o}, R) \preceq e^{\delta_\Gamma R}$.

We thus fix $i \in \{1, \cdots, L\}$, we set
$$\mathcal{A}^{(i)}{}_\Gamma(\mathbf{x}, \alpha, R) := \{\gamma \cdot \mathbf{x} \in \Gamma^{(i)} \cdot \mathbf{x} / R - \alpha < d(\mathbf{x}, \mathbf{y}) \leq R + \alpha\}$$
and
$$v_\Gamma^{(i)}(\mathbf{x}, \alpha, R) := \sharp \mathcal{A}_\Gamma^{(i)}(\mathbf{x}, \alpha, R).$$

We fix $j \neq i$ and $\alpha > 0$ and we choose N large enough in such a way that $d(\mathbf{o}, \mathfrak{h}_j^n \cdot \mathbf{o}) > \kappa_\Gamma + 2\alpha$ for any $n \geq N$, where κ_Γ is the constant which appears in Proposition 4.7.

For any $\gamma, \gamma' \in \Gamma^{(i)}$ with $\gamma \cdot \mathbf{o} \in \mathcal{A}_\Gamma^{(i)}(\mathbf{o}, \alpha, A)$ and $\gamma' \cdot \mathbf{o} \in \mathcal{A}_\Gamma^{(i)}(\mathbf{o}, \alpha, B)$, we set
$$\Phi(\gamma, \gamma') := \gamma'' = \gamma \mathfrak{h}_j^N \gamma'.$$

We thus get $d(\mathbf{o},\gamma''\cdot\mathbf{o})\leq A+B+2\alpha+N\Delta_\Gamma$; on the other hand

$$\begin{aligned} d(\mathbf{o},\gamma''\cdot\mathbf{o}) &\geq d(\mathbf{o},\gamma\cdot\mathbf{o})+d(\mathbf{o},\gamma'\cdot\mathbf{o})+d(\mathbf{o},\mathfrak{h}_j^N\cdot\mathbf{o})-2\kappa_\Gamma \\ &\geq A+B-\kappa_\Gamma \end{aligned}$$

so that, assuming $\alpha\geq\max(\Delta_\Gamma,\kappa_\Gamma)$, one gets

$$A+B-\alpha\leq d(\mathbf{o},\gamma''\cdot\mathbf{o})\leq A+B+(N+2)\alpha. \tag{7}$$

Note that Φ is one-to-one ; otherwise one should have $\gamma''=\gamma_1\mathfrak{h}_j^N\gamma_1'=\gamma_2\mathfrak{h}_j^N\gamma_2'$ for some $\gamma_1,\gamma_2\in\Gamma^{(i)}$ such that $\gamma_1\cdot\mathbf{o},\gamma_2\cdot\mathbf{o}\in\mathcal{A}_\Gamma^{(i)}(\mathbf{o},\alpha,A)$ and $\gamma_1'\cdot\mathbf{o},\gamma_2'\cdot\mathbf{o}\in\mathcal{A}_\Gamma^{(i)}(\mathbf{o},\alpha,B)$. Without loss of generality, one may assume that $l(\gamma_1)\leq l(\gamma_2)$. Note that the words $\gamma_1\mathfrak{h}_j^N\gamma_1'$ and $\gamma_2\mathfrak{h}_j^N\gamma_2'$ are reduced :

- if $l(\gamma_1)=l(\gamma_2)$, we have in fact $\gamma_1=\gamma_2$ and $\gamma_1'=\gamma_2'$ follows immediately,.
- suppose that $l(\gamma_1)\neq l(\gamma_2)$; since the last letter of γ_2 is $\mathfrak{h}_i^{\pm1}$, one gets $\gamma_2=\gamma_1\mathfrak{h}_j^N\cdots$ so that

$$\begin{aligned} d(\mathbf{o},\gamma_2\cdot\mathbf{o}) &\geq d(\mathbf{o},\gamma_1\cdot\mathbf{o})+d(\mathbf{o},\mathfrak{h}_j^N\cdot\mathbf{o})-\kappa_\Gamma \\ &\geq A+d(\mathbf{o},\mathfrak{h}_j^N\cdot\mathbf{o})-\alpha-\kappa_\Gamma \\ &> A+\alpha \end{aligned}$$

which contradicts the fact that $\gamma_2\cdot\mathbf{o}\in\mathcal{A}_\Gamma^{(i)}(\mathbf{o},\alpha,A)$. Consequently, Φ is one-to-one and (7) leads to

$$v_\Gamma^{(i)}(\mathbf{o},\alpha,A)\times v_\Gamma^{(i)}(\mathbf{o},\alpha,B)\leq\sum_{k=0}^{N+1}v_\Gamma^{(i)}(\mathbf{o},\alpha,A+B+k\alpha). \tag{8}$$

$\square$

The inequality (6) may be extended as follows for elements of $\Gamma^{(i)}$: for $\Delta>1$ large enough and $A,B\geq2\Delta$

$$v_\Gamma^{(i)}(\mathbf{o},2\Delta,A+B)\leq C\times v_\Gamma(\mathbf{o},2\Delta,A)\times v_\Gamma^{(i)}(\mathbf{o},2\Delta,B) \tag{9}$$

so that , for $\alpha\geq2\Delta$ and $k\geq0$ one gets

$$v_\Gamma^{(i)}(\mathbf{o},\alpha,A+B+k\alpha)\leq C\Big(v_\Gamma(\mathbf{o},\alpha,\alpha)\Big)^k\times v_\Gamma^{(i)}(\mathbf{o},\alpha,A+B). \tag{10}$$

Combining (8) and (10), one obtains :

$$v_\Gamma^{(i)}(\mathbf{o},\alpha,A)\times v_\Gamma^{(i)}(\mathbf{o},\alpha,B)\leq C_iv_\Gamma^{(i)}(\mathbf{o},\alpha,A+B)$$

for some $C_i>0$. One may thus replace $v_\Gamma^{(i)}(\mathbf{o},\alpha,A)$ with $C_iv_\Gamma^{(i)}(\mathbf{o},\alpha,A)$ and note that the sequence $\Big(\dfrac{1}{v_\Gamma^{(i)}(\mathbf{o},\alpha,n\alpha)}\Big)_{n\geq1}$ is submultiplicative. As $\lim\limits_{A\to\infty}\dfrac{1}{R}\ln v_\Gamma^{(i)}(\mathbf{x},\Delta,R)=\delta_\Gamma$, we thus get $e^{-\delta_\Gamma R}\preceq\dfrac{1}{v_\Gamma^{(i)}(\mathbf{x},\Delta,R)}$, which is the expected inequality.

5.4. The case of non uniform lattices. In this section, we consider a discrete subgroup $\Gamma\in Iso^+(\mathbb{H})$ such that $\mathbb{H}/\Gamma$ is not compact but has finite area (for instance $\Gamma=PSL(2,\mathbb{Z})$).

Th above argument does not work in this case since the diameter of $\mathbb{H}^2/\Gamma$ is infinite. The first step is to identify the value of δ_Γ and to check, in this case, that it is equal to the "volume entropy " 1 of $\mathbb{H}$. It is contained in the following subsection.

5.4.1. **The Poincaré exponent of lattices.** We state here the

PROPOSITION 5.4. *Let Γ be a non uniform lattice of $PSL(2,\mathbb{R})$; its critical exponent δ_Γ is equal to* 1*, that is the exponential growth of the area of the discs in $\mathbb{H}$ of radius $R>0$. Furthermore, for any $\mathbf{x},\mathbf{y}\in\mathbb{H}$, one gets $N_\Gamma(\mathbf{x},\mathbf{y},R)\preceq e^R$.*

To prove this Proposition, we first need to decompose the Dirichlet domain of Γ in suitables subsets. We will admit that the quotient manifold $\mathbb{H}^2/\Gamma$ is the disjoint union of a compact part C_0 and of finitely many *cusps* $C_1,\cdots,C_l$, each C_i being isometric to the quotient of an horosphere $\mathcal{H}_i$ by a parabolic group $\mathcal{P}_i=\langle\mathfrak{p}_i\rangle$.

To simplify the notations we assume $l=1$ in the sequel, the proof being similar in the general case.

Let D be the Dirichlet domain of Γ centerd at $\mathbf{o}$ and denote by $D \mapsto \mathbb{H}/\Gamma \mathbf{x} \mapsto \overline{\mathbf{x}}$ the canonical identification between D and $\mathbb{H}/\Gamma$. Assume that $\mathbf{o}$ belongs to the interior of $\mathcal{D}$ and let $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) be the preimage by this map of the set C_0 (resp. C_1). We set $\tilde{\mathcal{C}}_0 = \cup_{\gamma\in\Gamma}\gamma\cdot\mathcal{C}_0$ and $\tilde{\mathcal{C}}_1 = \cup_{\gamma\in\Gamma}\gamma\cdot\mathcal{C}_1$.

One may choose C_1 as a cylinder whose boundary is an horocycle of the surface; in other words, C_1 is isometric to the quotient of an horoball $\mathcal{H}_1$ (centered at a point $\xi_1 \in \partial\mathbb{H}$) by a parabolic subgroup $P_1 := \langle \mathfrak{p}_1\rangle$ whith fixed point ξ_1 ; in particular, $\mathcal{C}_1$ may be choosen as a fundamental domain for the action of P_1 on $\mathcal{H}_1$.

One chooses a representant $\bar\gamma$ of a coset $\gamma P_1 \in \Gamma/P_1$, so one gets

$$\tilde{\mathcal{C}}_1 = \bigcup_{\gamma\in\Gamma}\gamma\cdot\mathcal{C}_1 = \bigcup_{\overline{\gamma}\in\Gamma/P_1}\cup_{n\in\mathbb{Z}}\bar\gamma\mathfrak{p}_1^n\cdot\mathcal{C}_1 = \bigcup_{\overline{\gamma}\in\Gamma/P_1}\bar\gamma\cdot\mathcal{H}_1.$$

Proof of Proposition 5.4. Since Γ is discrete, the Γ-orbit of any point (say $\mathbf{o}$) is discrete in $\mathbb{H}$, so there exists $r>0$ such that $B(\mathbf{o},r)\cap\gamma\cdot B(\mathbf{o},r)=\emptyset$. It follows that

$$\bigcup_{\substack{\gamma\in\Gamma\\ d(\mathbf{o},\gamma\cdot\mathbf{o})\le R-r}} B(\gamma\cdot\mathbf{o},r)\subset B(\mathbf{o},R).$$

Since the discs $B(\gamma\cdot\mathbf{o},r)$ are pairewise disjoints and have the same area, it follows that

$$N_\Gamma(\mathbf{o},\mathbf{o},R-r)\times \text{area }(B(\gamma\cdot\mathbf{o},r))\le \text{area}(B(\mathbf{o},R))\preceq e^R. \tag{11}$$

The last assertion of Proposition 5.4 ; in particular one gets $\delta_\Gamma\le 1$.

The converse inequality $\delta_\Gamma\ge 1$ is more subtle (and not always satisfied in the variable curvature case, where 1 is replaced with the volume entropy of the universal covering). We write decompose $B(\mathbf{o},R)$ into the disjoint union $\Big(B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_0\Big)\cup\Big(B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_1\Big)$ and deals with the two subsets separately :

(1) let Δ_0 be the diameter of $\mathcal{C}_0$; one gets

$$B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_0\subset\bigcup_{\gamma/d(\mathbf{o},\gamma\cdot\mathbf{o})\le R+\Delta_0}\gamma\cdot\mathcal{C}_0$$

so that area $B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_0\le N_\Gamma(\mathbf{o},\mathbf{o},R+\Delta_0)\times\text{area}(C_0)$.

(2) on the other hand $B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_1\subset\bigcup_{\substack{\overline{\gamma}\in\Gamma/P_1\\ d(\mathbf{o},\overline{\gamma}\cdot\mathcal{H}_1)\le R}}B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1$, which can be also written into

$$B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_1\subset\bigcup_{n=0}^{[R]}\bigcup_{\substack{\overline{\gamma}\in\Gamma/P_1\\ n\le d(\mathbf{o},\overline{\gamma}\cdot\mathcal{H}_1)\le n+1}}B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1. \tag{12}$$

Let us compute the area of $B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1$ setting $\mathbf{x}_{\bar\gamma}$ the point of $\mathcal{H}_1$ which achieves the distance $d(\mathbf{o},\overline{\gamma}\cdot\mathcal{H}_1)$ (this point is unique by strict convexity of the horocycles) ; for any $\mathbf{y}\in B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1$, the angle at $\mathbf{x}_{\bar\gamma}$ of the triangle $\mathbf{o},\mathbf{x}_{\bar\gamma},\mathbf{y}$ is greater than $\frac{\pi}{2}$, so that there exists $c>0$, which does not depend on γ such that $d(\mathbf{o},\mathbf{y})\ge d(\mathbf{o},\mathbf{x}_{\bar\gamma})+d(\mathbf{x}_{\bar\gamma},\mathbf{y})-c$ which readily implies that

$$B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1\subset B\Big(\mathbf{x}_{\bar\gamma},R+c-d(\mathbf{o},\overline{\gamma}\cdot\mathcal{H}_1)\Big)\cap\bar\gamma\cdot\mathcal{H}_1$$

since $d(\mathbf{o},\mathbf{x}_{\bar\gamma})=d(\mathbf{o},\overline{\gamma}\cdot\mathcal{H}_1))$. In particular, when $n\le d(\mathbf{o},\overline{\gamma}\cdot\mathcal{H}_1)\le n+1$, one gets

$$B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1\subset B\Big(\mathbf{x}_{\bar\gamma},R+c-n\Big)\cap\bar\gamma\cdot\mathcal{H}_1.$$

so that the area of $B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1$ is less than $e^{\frac{R-n}{2}}$, up to a multiplicative constant. By (12), it follows

$$\begin{aligned}
\mathrm{area}(B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_1) &\leq \sum_{n=0}^{[R]}\sum_{\substack{\bar\gamma\in\Gamma/P_1\\ n\leq d(\mathbf{o},\bar\gamma\cdot\mathcal{H}_1)\leq n+1}}\mathrm{area}\Big(B(\mathbf{o},R)\cap\bar\gamma\cdot\mathcal{H}_1\Big)\\
&\preceq \sum_{n=0}^{[R]}\mathrm{Card}\{\gamma\in\Gamma/P_1\ s.t.\quad n\leq d(\mathbf{o},\mathbf{x}_{\bar\gamma})\leq n+1\}\times e^{\frac{R-n}{2}}\\
&\preceq \sum_{n=0}^{[R]}\mathrm{Card}\{\gamma\in\Gamma/d(\mathbf{o},\gamma\cdot\mathbf{o})\leq n+1+\Delta_0\}\times e^{\frac{R-n}{2}}
\end{aligned}$$

the last inequality following from the fact that $\mathbf{x}_{\bar\gamma}$ lies on the boundary of some $\gamma\cdot\mathcal{C}_0$ so at a distance less than Δ_0 from $\gamma\cdot\mathbf{o}$. One thus obtains, for any $\epsilon>0$

$$\mathrm{area}(B(\mathbf{o},R)\cap\tilde{\mathcal{C}}_1)\preceq\sum_{n=0}^{[R]}e^{n(\delta_\Gamma+\epsilon)}e^{\frac{R-n}{2}}\preceq e^{(\delta_\Gamma+\epsilon)R}$$

since $\delta_\Gamma\geq\delta_{P_1}=\frac{1}{2}$ by the critical gap property.

Finaly one gets

$$\mathrm{area}(B(\mathbf{o},R))\preceq N_\Gamma(\mathbf{o},\mathbf{o},R+\Delta_0)\times\mathrm{area}(C_0)+e^{(\delta_\Gamma+\epsilon)R}\preceq e^{(\delta_\Gamma+\epsilon)R}$$

which is the expected inequality.

□

5.4.2. **The asymptotic behavior of the orbital function of lattices.** The aim of this paragraph is to prove Theorem 1.1 in the case when Γ is a lattice of $\mathbb{H}$.

In this section, we will also consider the following function $w_\Gamma:\mathbb{H}\times\mathbb{H}\times\mathbb{R}\to\mathbb{R}$ defined

$$w_\Gamma(\mathbf{x},\alpha,R):=e^{-\delta R}v_\Gamma(\mathbf{x},\alpha,R)$$

where δ is some fixed real in $]\frac{1}{2},1[$.

We will first prove the following

Lemma 5.5. *There exist $\Delta>1$ and $C>0$ such that for any $\mathbf{x}\in\mathbb{H}$ and $A,B>0$, one gets*

$$w_\Gamma(\mathbf{x},2\Delta,A+B)\leq C\times\Big(\sum_{0\leq n\leq A+3\Delta+2}w_\Gamma(\mathbf{x},2\Delta,n)\Big)\times\Big(\sum_{0\leq n\leq B+3\Delta+2}w_\Gamma(\mathbf{x},2\Delta,n)\Big).\tag{13}$$

Proof. We fix $\Delta\geq1$ greater than the diameter of $\mathcal{C}_0$, two constants $A,B\geq2\Delta$ and $\mathbf{y}\in\mathcal{A}_\Gamma(\mathbf{x},2\Delta,A+B)$. Set $d(\mathbf{x},\mathbf{y})=A+B+2\lambda$ with $-\Delta\leq\lambda\leq\Delta$ and let $\mathbf{z}$ on the geodesic segment $[\mathbf{x},\mathbf{y}]$ such that $d(\mathbf{x},\mathbf{z})=A+\lambda$; there are two cases :

1. the point $\mathbf{z}$ belongs to $\Gamma\cdot\mathcal{C}_0$. There thus exists $\mathbf{z}'\in\Gamma\cdot\mathbf{x}$ such that $d(\mathbf{z},\mathbf{z}')\leq\Delta$; one gets $\mathbf{z}'\in\mathcal{A}_\Gamma(\mathbf{x},2\Delta,A)$ and $\mathbf{y}\in\mathcal{A}_\Gamma(\mathbf{z}',2\Delta,B)$.

2. the point $\mathbf{z}$ belongs to $\Gamma\cdot\mathcal{C}_1$. Namely, $\mathbf{z}$ belongs to some horoball $\mathcal{H}\in\{\gamma\cdot\mathcal{H}_1/\gamma\in\Gamma\}$ fixed by a parabolic subgroup $P\subset\Gamma$ (more precisely, if $\mathcal{H}=\gamma\cdot\mathcal{H}_1$, then $P=\gamma P_1\gamma^{-1}$).

Since $\Gamma\cdot\mathbf{x}\cap\mathcal{H}=\emptyset$, we have $[\mathbf{x},\mathbf{y}]\cap\mathcal{H}=[\mathbf{u},\mathbf{v}]\subset]\mathbf{x},\mathbf{y}[$. Let $\gamma\in\Gamma$ such that $\mathbf{u}\in\gamma\cdot\mathcal{C}_0$ and $p\in P_1$ satisfying $\mathbf{v}\in\gamma p\cdot\mathcal{C}_0$.

Setting $\alpha:=d(\mathbf{x},\mathbf{u})$ and $\beta:=d(\mathbf{v},\mathbf{y})$, with $\alpha\leq A+\lambda$ and $\beta\leq B+\lambda$, one gets

(1) $\gamma\cdot\mathbf{x}\in\mathcal{A}_\Gamma(\mathbf{x},\Delta,\alpha)$

(2) $\mathbf{y}\in\mathcal{A}_\Gamma(\gamma p\cdot\mathbf{x},\Delta,\beta)$

(3) $d(\gamma\cdot\mathbf{x},\gamma p\cdot\mathbf{x})\leq A+B+2\lambda+2\Delta-\alpha-\beta$

If one replaces α and β by their integer part $a=[\alpha]$ and $b=[\beta]$, one gets

(1') $\gamma\cdot\mathbf{x}\in\mathcal{A}_\Gamma(\mathbf{x},\Delta+1,a)$

(2') $\mathbf{y}\in\mathcal{A}_\Gamma(\gamma p\cdot\mathbf{x},\Delta+1,b)$

(3') $d(\gamma\cdot\mathbf{x},\gamma p\cdot\mathbf{x})=d(\mathbf{x},p\cdot\mathbf{x})\leq A+B+4\Delta+2-a-b$.

Summing over integers $a \le A+\Delta$ and $b \le B+\Delta$, one gets

$$v_\Gamma(\mathbf{x}, 2\Delta, A+B) \quad \le \sum_{\substack{a\le A+\Delta\\ b\le B+\Delta}} v_\Gamma(\mathbf{x}, \Delta+1, a) \times v_\Gamma(\mathbf{x}, \Delta+1, b)$$
$$\times \quad \sharp\{p \in P : d(\mathbf{x}, p\cdot\mathbf{x}) \le A+B+4\Delta+2-a-b\}.$$

Since $\delta_P = \frac{1}{2} < \delta$, one gets $\sharp\{p \in P : d(\mathbf{x}, p\cdot\mathbf{x}) \le R\} \preceq e^{\delta R}$ and inequality (13) follows.$\square$

In the case when Γ is a Schottky group (and more generally in the convex-cocompact case) we may use a submultiplicative argument (the so-called Fekete's Lemma) , which we have to relaxed in the case of non uniform lattices and which can be obtained in the same spirit :

Lemma 5.6. *Let $(w_n)_{n\ge 1}$ a sequence of positive numbers such that*

$$w_{n+m} \le c\cdot\left(\sum_{k=1}^n w_k\right)\cdot\left(\sum_{k=1}^m w_k\right) \qquad \forall n, m \ge n_0$$

for some positive constant c. Then, setting $W_n = \sum_{k=1}^n w_k$ and $\tilde W_n := 1 + W_1 + ... + W_n$, we get :

(i) $\lim\limits_{n\to\infty} \frac{1}{n}\ln W_n = L \in \mathbb{R}\cup\{-\infty\}$ and $\tilde W_n \ge \frac{e^{Ln}}{c}$ for all $n \ge 1$; in particular $W_n \ge \frac{e^{Ln}}{2cn}$ for all $n \gg 0$.

(ii) if $L > 0$ then the series $\sum_n W_n e^{-sn}$ and $\sum_n w_n e^{-sn}$ have the same critical exponent L ; furthermore, they both diverge for $s = L$.

(iii) Furthermore, if $W_n \preceq e^{Ln}$ one gets in fact $\qquad e^{Ln} \asymp W_n$.

Let us assume for a moment Lemma 5.6 and conclude the proof of Theorem 1.1 in the case of lattices. By Proposition 5.5, the sequence $w_n := e^{-\delta n} v_\Gamma(\mathbf{x}, \Delta, n)$ satisfies the assumption of the previous lemma, with $L = \lim\limits_{n\to\infty}\frac{1}{n}\ln w_n = \delta_\Gamma - \delta = 1-\delta > 0$; consequently, the series $\sum\limits_n W_n e^{-Ln}$ diverges, so does $\mathcal{P}_\Gamma(\mathbf{x}, \delta_\Gamma)$ since $\sum\limits_n W_n e^{-Ln} \asymp \mathcal{P}_\Gamma(\mathbf{x}, \delta_\Gamma)$.

Furthermore, by Proposition 5.4, we get $N_\Gamma(\mathbf{x}, \mathbf{x}, R) \preceq e^R$, so $W_n \preceq e^{LR}$ and the last assertion of the lemma leads to the estimation $W_n \asymp e^{LR}$ i.e. $N_\Gamma(\mathbf{x}, \mathbf{x}, R) \asymp e^R$.

$\square$

Proof of Lemma 5.6. We prove the Lemma when $n_0 = 1$ (the argument for $n_0 > 1$ being analogous), and we may assume $c > 1$. We set

$$\tilde W_n := 1 + W_1 + ... + W_n$$

and $v_n := cw_n$, $V_n := \sum_{k=1}^n v_k = cW_n$, $\tilde V_n := c\tilde W_n \ge 1 + \sum_{k=1}^n V_k$, so we obtain

$$v_{n+m} \le V_n V_m \quad \forall n, m \ge 1.$$

This in turn yields $V_{n+m} \le V_n \tilde V_m$ and, consequently,

$$\tilde V_{n+m} = \tilde V_n + V_{n+1} + \cdots + V_{n+m} \le \tilde V_n \tilde V_m \quad \forall n, m \ge 1.$$

So, $(\tilde V_n)_{n\ge1}$ satisfies the assumption of Fekete's lemma. Hence the sequences $\frac{\ln \tilde V_n}{n}$ and $\frac{\ln \tilde W_n}{n}$ converge to some $L \in \mathbb{R}\cup\{-\infty\}$, and $\tilde V_n \ge e^{Ln}$, $\tilde W_n \ge \frac{1}{c}e^{Ln}$ for $n \ge 1$. As $\frac{1}{n}(\tilde W_n - 1) \le W_n \le \tilde W_n$, we obtain the same limit L for the sequence $\frac{\ln W_n}{n}$, and the series $\sum_n e^{-sn} W_n$ has critical exponent equal to L. Moreover, we find

$$W_n \ge 1 + \frac{1}{n}(\tilde W_n - 1) \ge \frac{1}{2cn}e^{Ln}$$

so the series diverges at $s = L$. Now

$$\sum_{n\ge1} e^{-sn} W_n = \sum_{k\ge1}\left(\sum_{n=k}^{\infty} e^{-sn}\right) w_k = \sum_{k\ge1}\left(\frac{e^{-sk}}{1-e^{-s}}\right) w_k = \left(\frac{1}{1-e^{-s}}\right)\sum_{k\ge1} e^{-sk} w_k$$

so, the series $\sum_k e^{-sk}w_k$ has the same critical exponent L as $\sum_n e^{-sn}W_n$ and, if $L>0$, it diverges at $s=L$.

In particular, one gets $W_n \succeq e^{Ln}/n$. To obtain the inequality $W_n \succeq e^{Ln}$, we make the additionnal condition $W_n \preceq e^{Ln}$; in otherwords, there exists a constant $A\geq 1$ such that $W_n \leq Ae^{Ln}$ for any $n\geq 1$.

Assume that the estimation from below $W_n \succeq e^{Ln}$ does not hold ; there thus exists a sequence of integers $(k_n)_n$ which tends to $+\infty$ such that

$$\epsilon_n := W_{k_n} e^{-Lk_n} \to 0 \quad \text{as} \quad n\to+\infty.$$

Without loss of generality, one may suppose that $k_n - k_{n-1} \to +\infty$ and one set $l_n := \inf\Big(\dfrac{k_n-k_{n-1}}{2}, \Big[\dfrac{-\ln\epsilon_n}{L}\Big]\Big)$; consequently, $k_n - l_n \geq k_{n-1}$ and $\epsilon_n \leq e^{-Ll_n}$ for any $n\geq 1$ and the sequence $(l_n)_n$ tends to $\to +\infty$ as $n\to+\infty$. We thus set, for any $n,m\geq 1$

$$V_n = \begin{cases} Ae^{Ln} & \text{if} \quad n\in\{k_{m-1},\cdots,k_m-l_m-1\} \\ \epsilon_m e^{Lk_m} & \text{if} \quad n\in\{k_m-l_m,\cdots,k_m\}. \end{cases}$$

Note that $W_n \leq V_n \leq Ae^{Ln}$ for any $n\geq 1$ (the right inequality require the fact that $A\geq 1$). By the above, there exists $a>0$ such that $\mathfrak{W}_n \geq ae^{Ln}$ for any $n\geq 1$; it follows

$$\begin{aligned} ae^{Lk_m} \leq \quad \mathfrak{W}_{k_m} \quad &\leq \sum_{k=1}^{k_m} V_k \\ &\leq \sum_{k=1}^{k_m-l_m-1} Ae^{Lk} + l_m\epsilon_m e^{Lk_m} \\ &\leq \frac{A}{e^L-1}e^{L(k_m-l_m)} + l_m e^{L(k_m-l_m)} \end{aligned}$$

so that $a \leq (\frac{A}{e^L-1}+l_m)e^{-Ll_m}$, which contradicts the fact that $l_m \to +\infty$. We thus have $\liminf_{k\to+\infty} W_k e^{-Lk} > 0$, which achieves the proof of the lemma. ■

6. Exercices

Exercice 1 - Prove that the isometries of the hyperbolic plane in the unit disc model are fractional transformations of the complex plane corresponding to matrices of the form $\begin{pmatrix} A & \bar{C} \\ C & \bar{A} \end{pmatrix}$ with $A\bar{A} - C\bar{C} = 1$.

Exercice 2 - Prove that for any $z,w\in\mathbb{C}$ and any $g\in PSL(2,\mathbb{C})$ one gets

$$|g\cdot z - g\cdot w|^2 = |g'(z)|\times|g'(w)|\times|z-w|^2.$$

Exercice 3 - We study here the action of the fractional transformations of the complex plane (i.e. transformations of the form $z\mapsto \dfrac{az+b}{cz+d}$, with $a,d,b,c\in\mathbb{C}$ and $ad-bc=1$) on the set $\mathfrak{C}$ of circles and lines.

(1) Prove that the set of fractional transformations is a group generated by the maps $h_A : z\mapsto Az, \tau_B : z\mapsto z+B$ and $\mathcal{I} : z\mapsto \dfrac{1}{z}$ with $A,B\in\mathbb{C}$.

(2) Prove that the maps h_A and τ_B preserve both the set of circles and the set of lines.

(3) Let $\mathcal{C}=\mathcal{C}(z_0,r)$ be the circle with center z_0 and radius $r>0$.

   (a) Assume first $z_0=0$; prove that $\mathcal{I}(\mathcal{C})$ is the circle with center 0 and radius $\dfrac{1}{r}$.

   (b) Assume now $z_0\neq 0$ and $|z_0|\neq r$; prove that $\mathcal{I}(\mathcal{C})$ is the circle with center $Z_0 := \dfrac{\bar{z}_0}{|z_0|^2-r^2}$ and radius $R := \Big|\dfrac{r}{|z_0|^2-r^2}\Big|$.

   (c) Assume atlast $|z_0| = r>0$; prove that $\mathcal{I}(\mathcal{C})$ is the line $\Big\{\dfrac{\bar{z}_0}{2|z_0|^2}(1+i\lambda)/\lambda\in\mathbb{R}\Big\}$ (that is the line passing through $\dfrac{\bar{z}_0}{2|z_0|^2}$ and orthogonal to $\bar{z}_0$.)

(4) Let $\mathcal{L}$ be a line in $\mathbb{C}$ and z_1 the orthogonal projection of 0 on $\mathcal{L}$.
  (a) Assume $z_1 = 0$: identify $\mathcal{I}(\mathcal{L})$ and $\mathcal{I}(\mathcal{L} \cup \{\infty\})$.
  (b) Assume $z_1 \neq 0$; using the question 3.c, prove that $\mathcal{I}(\mathcal{L})$ is the circle with center $\dfrac{\bar{z}_1}{2|z_1|^2}$ and radius $\dfrac{1}{2|z_1|}$.

(5) *Application 1.* Prove that hyperbolic circles (in the two models $\mathbb{H}$ and $\mathbb{D}$) are Euclidean circles (with different center and radius) and vice-versa.

(6) *Application 2.* Let $\mathcal{C}$ be a circle in $\mathbb{H}^2$, orthogonal to the real axis at points α and β in $\mathbb{R}$. Check that the homography $z \mapsto \frac{1}{z-\alpha} - \frac{1}{\beta-\alpha} = \frac{\beta-z}{(\beta-\alpha)(z-\alpha)}$ maps $\mathcal{C}$ on the imaginary axis $i\mathbb{R}$.

(7) *Application 3.* Prove that geodesics in $\mathbb{D}$ are segments either of Euclidean circles orthogonal to the unit circle or of its diameters.

Exercice 4 - On the cross-ratio Let z_1, z_2, z_3, z_4 be 4 distinct points in $\mathbb{C} \cup \{\infty\}$; their **cross ratio** is defined by $[z_1, z_2, z_3, z_4] := \dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$, with the convention $\frac{\infty}{\infty} = 1$.

In order to understand the geometrical meaning of the cross-ratio we will write

$$[z_1, z_2, z_3, z_4] := \frac{z_2 - z_1}{z_4 - z_1} \Big/ \frac{z_2 - z_3}{z_4 - z_3}.$$

(1) Check that if $z_1, z_2, z_4 \in \mathbb{C}$ belong to the same line $\mathcal{L}$ of the complex plane then $\dfrac{z_2 - z_1}{z_4 - z_1} \in \mathbb{R}$; conclude that $[z_1, z_2, z_3, z_4] \in \mathbb{R}$ when $z_3 \in \mathcal{L} \cup \{\infty\}$.

(2) One supposes that $z_1, z_2, z_3, z_4 \in \mathbb{C}$ belong to the same circle $\mathcal{C}$ in the complex plane ; prove that $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.
Hint. Without loss of generality, one may assume that $\mathcal{C}$ is the unit circle centered at 0 and $z_2 = e^{i\theta}, z_4 = 1$; check that in this case any $z = e^{it} \in \mathcal{C} \setminus \{1, e^{i\theta}\}$ satisfies $\dfrac{z_2 - z}{z_4 - z} = re^{i\theta/2}$ with $r \in \mathbb{R}$. Conclude.
For $i \in \{1, \cdots, 4\}$, let p_i the point on the complex plane corresponding to z_i ; what is the interpretation in term of the angles $\widehat{p_2 p_1 p_4}$ and $\widehat{p_2 p_3 p_4}$?

(3) Conversely, fix z_1, z_2 and z_4 distinct elements in $\mathbb{C} \cup \{\infty\}$; show that there exists either a (unique) line $\mathcal{L}$ either a (unique) circle $\mathcal{C}$ such that these 3 points belong to $\mathcal{L}$ or to $\mathcal{C}$.
  (a) Assume that z_1, z_2 and z_4 belong to a line $\mathcal{L}$; check that if $z_3 \notin \mathcal{L}$ then $[z_1, z_2, z_3, z_4] \notin \mathbb{R}$.
  (b) Assume that z_1, z_2 and z_4 belong to the unit circle $\mathcal{C}$ and set $z_2 = e^{i\theta}$ and $z_4 = 1$; prove that if $z = \rho e^{it}$ with $\rho \in \mathbb{R}^+, \rho \neq 1$, then $e^{-i\theta/2} \dfrac{e^{i\theta} - z}{1 - z} \notin \mathbb{R}$. Conclude.
  Hint. We will use the fact that $e^{-i\theta/2} \dfrac{z_2 - z_1}{z_4 - z_1} \in \mathbb{R}$ since $z_1 \in \mathcal{C}$.

(4) Prove that for any $g \in PSL(2, \mathbb{R})$ and any 4 distinct points z_1, z_2, z_3, z_4 in $\mathbb{C} \cup \{\infty\}$, one gets $[z_1, z_2, z_3, z_4] = [g \cdot z_1, g \cdot z_2, g \cdot z_3, g \cdot z_4]$; using the fact that g is a conformal map on $\mathbb{H}$, deduce that g maps the family of lines and circle orthogonal to the real line into itself.

(5) For any $z, w \in \mathbb{H}$ let $z^*, w^* \in \mathbb{R} \cup \{\infty\}$ the points at infinity of the geodesic of $\mathbb{H}^2$ passing through z and w, such that $z \in]z^*, w[$ and $w \in]z, w^*[$; prove that the hyperbolic distance between z and w is equal to $\Big| \log |[z^*, z, w^*, w]| \Big|$.
Hint. One first checks this formula when $z = ia, w = ib$ with $a < b$.

Exercice 5 - Let z_1, z_2, z_3 and w_1, w_2, w_3 be in $\bar{\mathbb{H}}$ such that $z_i \neq z_j$ and $w_i \neq w_j$ for $i, j \in \{1, 2, 3\}, i \neq j$.

(1) Prove that there exists a unique fractional map $g : z \mapsto \frac{az+b}{cz+b}$ with $a, b, c, d \in \mathbb{C}, ad - bc = 1$, such that $g \cdot z_i = w_i$ for $i \in \{1, 2, 3\}$.

(2) Check that $a, b, c, d \in \mathbb{R}$ when the z_i, w_i belong to $\partial \mathbb{H}$.

(3) We suppose that $z_1 \in \mathbb{H}$. Denote by $\vec{u}$ (resp. $\vec{v}$, $\vec{u}'$ and $\vec{v}'$) the initial unit tangent vetctor of the geodesic segment $[z_1, z_2]$ (resp. $[z_1, z_3]$, $[w_1, w_2]$ and $[w_1, w_3]$).

   (a) Explain why there exists a unique $g \in PSL(2,\mathbb{R})$ such that $g \cdot (z_1, \vec{u}) = (w_1, \vec{u}')$ and $g \cdot (z_1, \vec{v}) = (w_1, \vec{v}')$.
   (b) Check that, if $h \in PSl(2,\mathbb{R})$ is such that $h : z_i \to w_i$ for $i = 1, 2, 3$, then $h = g$ necessarily.
   (c) Prove that g maps each z_i to w_i if and only if $d(z_i, z_j) = d(w_i, w_j)$ for any $i \neq j$.
   (d) Conclude that the unique solution $g \in PSl(2,\mathbb{C})$ of the question (1) belongs to $PSL(2,\mathbb{R})$ if and only if $d(z_i, z_j) = d(w_i, w_j)$ for any $i \neq j$.

(4) *Consequence.* A triangle is said to be **ideal** if its 3 vertices belong to $\mathbb{R} \cup \{\infty\}$. Explain, without any computation, why all the ideal triangles have the same area.

Exercice 6 - Let $g \in A$ and $\Gamma = \langle g \rangle = \{g^n / n \in \mathbb{Z}\}$; draw a (natural) fondamental domain of Γ and of the quotient manifold $\mathbb{H}/\Gamma$ (idem when $g \in N$).

Exercice 7 - Fix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G.

(1) Compute the elements $n \in N, \alpha \in A$ and $k \in K$ such that $n\alpha k = g$. Prove that this decomposition is unique.
(2) Let $\alpha \in A$ such that $g = k\alpha k'$ with $k, k' \in K$. Is the Cartan decomposition unique ?

Exercice 8 - Let Γ be a discrete subgroup of isometries of $Iso(\mathbb{H})$ which is **torsion free**, i.e. without elliptic elements.

(1) Let g and h be two elements in Γ ; prove that if g and h have a common fixed point, then there exists $\gamma \in \Gamma$ and $n, m \in \mathbb{Z}^*$ such that $g = \gamma^n$ and $h = \gamma^m$.
(2) Prove that Γ is either generated by an element or contains a Schottky subgroup with two generators.
(3) Prove that the limit set of Γ contains either 1 or 2 points or is not countable.

Exercice 9 - We consider a hyperbolic triangle Δ with vertices v_a, v_b and v_c and corresponding angles α, β and γ ; we want to prove the

$$\textit{Gauss-Bonnet formula}: \qquad \text{area}(\Delta) = \pi - \alpha - \beta - \gamma.$$

(1) Assume first that at most one of the vertices (say v_c) belongs to $\partial\mathbb{H}$.
   (a) Explain why one may assume that $v_c = \infty$ (consequently, two sides of Δ are vertical geodesics (say $\text{Re}(z) = a$ for the side $[v_a, v_c]$ and $\text{Re}(z) = b$ for the side $[v_b, v_c]$) (eventually one or two of the others vertices belong to $\partial\mathbb{H}$, so that v_a and v_b may be real)
   (b) The base of Δ is a segment of a Euclidean semicircle orthogonal to the real axis. Why one may assume that this semicircle has center 0 and radius 1 ?
   (c) Prove that $a = \cos\alpha$ and $b = \cos\beta$.
   (d) Compute the area of Δ.
(2) Assume now that Δ has no vertices in $\mathbb{R} \cup \{\infty\}$.
   (a) Explain why one may assume that $v_a, v_b \in i\mathbb{R}$ (we will assume $\text{Im}(v_a) < \text{Im}(v_b)$ and denote by Δ_{ac} (resp. Δ_{bc}) the triangle (v_a, v_c, ∞) (resp. (v_b, v_c, ∞)) and θ the angle $\widehat{v_b v_c \infty}$).
   (b) Compute the angles of Δ_{ac} and Δ_{bc}.
   (c) Prove the Gauss-Bonnet formula in this case, using the previous question and the fact that $\text{area}(\Delta_{ac}) = \text{area}(\Delta) + \text{area}(\Delta_{bc})$.
(3) *Application.* Compute the area of the Dirichlet domain $\mathcal{D}_{2i} := \{z \in \mathbb{H} / |z| \geq 1 \textit{ and } |\text{Re}\, z| \leq \frac{1}{2}\}$ of $PSL(2,\mathbb{Z})$, centered at $2i$.

Exercice 10 - Let $\mathcal{H}$ be the horosphere $\{z = x + iy \in H : y > 1\}$. Prove that there exists $c > 0$ such that

$$\text{area}(B(i, R) \cap \mathcal{H}) \sim c e^{R/2} \quad \text{as} \qquad R \to +\infty.$$

7. Solutions

Exercice 1 - Note that g is an isometry of $\mathbb{D}$ if and only if $f^{-1} \circ g \circ f$ is an isometry of $\mathbb{H}$, where f is the map from $\mathbb{H}$ to $\mathbb{D}$ defined by $z \mapsto Z := \frac{iz+1}{z+i}$. We know that the isometries of

$\mathbb{H}$ are the homographies $z \mapsto \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$ so $f^{-1} \circ g \circ f$ is a linear fractional transformation of this form. Since f and f^{-1} are also linear fractional transformations, associated respectively to the matrices $\frac{-1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$ and $\frac{-1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$, the map g is of the same type, with matrice $\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} A & \bar{C} \\ C & \bar{A} \end{pmatrix}$ where $A = \frac{a+d+i(b-c)}{2}$ and $C = \frac{b+c+i(d-a)}{2}$, so that $A\bar{A} - C\bar{C} = 1$.

Conversely, let g be a fractional map of the form $Z \mapsto \frac{AZ+\bar{C}}{CZ+\bar{A}}$ with $A, C \in \mathbb{C}, |A|^2 - |C|^2 = 1$; it maps $\mathbb{D}$ on $\mathbb{D}$ and, setting $a = \mathrm{Re}(A) - \mathrm{Im}(C), b = \mathrm{Re}(C) + \mathrm{Im}(A), c = \mathrm{Re}(C) - \mathrm{Im}(A)$ and $d = \mathrm{Re}(A) + \mathrm{Im}(C)$, one gets $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} A & \bar{C} \\ C & \bar{A} \end{pmatrix}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$ with $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$; then $f^{-1} \circ g \circ f \in PSL(2,\mathbb{R})$. This is the general form of the isometries in the upper half space model.

Exercice 2 - It is a straightforward computation.

Exercice 3 - 1.If $c=0$ one gets necessarily $d \neq 0$ and $\frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d} = \tau_{b/d} \circ h_{a/d}(z)$. Otherwise $\frac{az+b}{cz+d} = -\frac{1}{c^2} \times \frac{1}{z+\frac{d}{c}} + \frac{a}{c} = \tau_{a/c} \circ h_{-1/c^2} \circ \mathcal{I} \circ \tau_{d/c}(z)$

2. The circle $\mathcal{C}(z_0, r)$ with center z_0 and radius r is the set of complex numbers z such that $|z-z_0| = r$; if $z \in \mathcal{C}(z_0, r)$ then $|h_A(z) - h_A(z_0)| = |A|r$ and $|\tau_B(z) - \tau_B(z_0)| = r$ ie $h_A(z) \in \mathcal{C}(h_A(z_0), |A|r)$ and $\tau_B(z) \in \mathcal{C}(\tau_B(z_0), r)$. A line $\mathcal{L}$ orthogonal to $n \in \mathbb{C}^*$ is the set of complex numbers z such that $\bar{n}z + n\bar{z} = c$ where c is a constant in $\mathbb{R}$; if $z \in \mathcal{L}$ one has $\bar{n}z + n\bar{z} = c$ which can be written $\frac{\bar{n}}{A}Az + \frac{n}{A}\overline{Az} = c$ (resp. $\bar{n}(z+B) + n(\bar{z}+\bar{B}) = c + \bar{n}B + n\bar{B}$) so that $h_A(z)$ (resp. $\tau_B(z)$) belong to a line orthogonal to $\frac{n}{A}$ (resp. orthogonal to n).

3.(a) One has $|z| = r \Rightarrow |\mathcal{I}(z)| = |1/z| = 1/r$.

(b) Set $Z = \frac{1}{z}$ with $|z - z_0| = r$, ie $z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$; then $|Z - Z_0| = \left|\frac{|z_0|^2 - r^2 - z\bar{z}_0}{z(|z_0|^2 - r^2)}\right| = \left|\frac{z - z_0}{|z_0|^2 - r^2}\right| = R$.

(c) The complex number $\mathcal{I}(z)$ belongs to the line orthogonal to $\bar{z}_0$ and passing through $\frac{\bar{z}_0}{2|z_0|^2}$ iff $\frac{1}{z}z_0 + \frac{1}{\bar{z}}\bar{z}_0 = \frac{\bar{z}_0}{2|z_0|^2}z_0 + \overline{\frac{\bar{z}_0}{2|z_0|^2}}\bar{z}_0$ ie $\frac{1}{z}z_0 + \frac{1}{\bar{z}}\bar{z}_0 = 1$. We know that $z \in \mathcal{C}(z_0, |z_0|)$, ie $|z - z_0| = |z_0|$ ie $z\bar{z} = \bar{z}z_0 + z\bar{z}_0$; it follows $\frac{1}{z}z_0 + \frac{1}{\bar{z}}\bar{z}_0 = \frac{\bar{z}z_0 + z\bar{z}_0}{z\bar{z}} = 1$.

4. (a) If $\mathcal{L} = \{\lambda z_0 / \lambda \in \mathbb{R}\}$ with $z_0 \neq 0$, then $\mathcal{I}(\mathcal{L}) = \{\lambda \bar{z}_0 / \lambda \in \mathbb{R}^*\} \cup \{\infty\}$ and $\mathcal{I}(\mathcal{L} \cup \{\infty\}) = \{\lambda \bar{z}_0 / \lambda \in \mathbb{R}\}$.

(b) We use directly the hint combined with the fact that $\mathcal{I} \circ \mathcal{I} = Id$ on $\bar{\mathbb{C}}$.

5. In the lecture, we have seen that, in $\mathbb{D}$, the euclidean circle $\mathcal{C}_{euc}(0, R)$ with center 0 and radius $r > 0$ is the hyperbolic one (denoted by $\mathcal{C}_{hyp}(0, R)$) with radius $R = \log \frac{1+r}{1-r}$. If the center is $z_0 \neq 0$, one chooses an isometry g of $\mathbb{D}$ which sends 0 onto z_0 and thus $\mathcal{C}_{hyp}(0, R)$ onto $\mathcal{C}_{hyp}(z_0, r)$; $\mathcal{C}_{hyp}(z_0, r)$ is also an euclidean circle, since it is the image of $\mathcal{C}_{euc}(0, R)$ by the homography g .

To obtain the result in the upper half plane model, one notes that the map f which sends $\mathbb{H}$ onto $\mathbb{D}$ is an homography.

6. Note that $z \mapsto \frac{1}{z-\alpha} - \frac{1}{\beta-\alpha}$ is an homography ; it maps $\mathcal{C}$ onto a circle or a line of $\mathbb{H}$. The image of α is ∞, the one of β is 0 so $\mathcal{C}$ is mapped onto $i\mathbb{R}$.

7. The map $f : \mathbb{H} \to \mathbb{D}$ is an homography which sends $\mathbb{R} \cup \{\infty\}$ onto the unite circle $\mathbb{S}^1$; this is also a conformal map, so it sends circles or lines orthogonal to $\mathbb{R}$ onto circles/lines orthogonal to $\mathbb{S}^1$.

Exercice 4 - 1. z_2 belongs to the line passing through z_1 and z_4 iff $z_2 - z_1 = \lambda(z_4 - z_1)$ for some $\lambda \in \mathbb{R}$; the conclusion follows.

2. One gets $\dfrac{e^{i\theta} - e^{it}}{1 - e^{it}} = -e^{i\theta/2}\dfrac{\sin\frac{\theta - t}{2}}{\sin\frac{t}{2}}$. So if z_1 and z_3 belong to the unit circle, the complex numbers $e^{-i\theta/2}\frac{z_2 - z_1}{z_4 - z_1}$ and $e^{-i\theta/2}\frac{z_2 - z_3}{z_4 - z_3}$ belong to $\mathbb{R}$, so does $[z_1, e^{i\theta}, z_3, 1]$.

Geometrical interpretation : the arc z_2z_4 on the circle is seen with the same angle both from z_1 or z_3 ; this is the classical geometrical criteria for cocyclicity of 4 points.

3. Let $\mathcal{L}_{12}$ (resp. $\mathcal{L}_{14}$) be the perpendicular bisector of the geodesic segment $[z_1, z_2]$ (resp. $[z_1, z_4]$) ; then, either $\mathcal{L}_{12} \cap \mathcal{L}_{14} = \{z_0\} \subset \mathbb{C}$ and the 3 points belong to the circle with center z_0 and radius

$d(z_0,z_1)=d(z_0,z_2)=d(z_0,z_4)$, either $\mathcal{L}_{12}\cap\mathcal{L}_{14}=\{\infty\}$ and $\mathcal{L}_{12}=\mathcal{L}_{14}$ in this case.
(a) One has $\frac{z_2-z_1}{z_4-z_1}\in\mathbb{R}$ since $z_1,z_2,z_4\in\mathcal{L}$ but $\frac{z_2-z_3}{z_4-z_3}\notin\mathbb{R}$ since $z_3\notin\mathcal{L}$.
(b) One gets $\frac{z_2-z}{z_4-z}=\frac{e^{i\theta}-\rho e^{it}}{1-\rho e^{it}}=\frac{e^{i\theta/2}}{|1-\rho e^{it}|^2}(e^{i(\theta-t)/2}-\rho e^{-i(\theta-t)/2})(e^{it/2}-\rho e^{-it/2})$ so that the imaginary part of $e^{-i\theta/2}\frac{z_2-z}{z_4-z}$ is $\frac{(1+\rho^2)\sin(\theta/2)}{|1-\rho e^{it}|^2}\neq 0$. A contrario, by question 2, $e^{-i\theta/2}\frac{z_2-z_1}{z_4-z_1}\in\mathbb{R}$ since $z_1\in\mathcal{C}$, consequently the cross-ratio is not real in this case.
4. The invariance of the cross-ratio by fractional transformations is a straightforward calculous. If the cross-ratio is real, its image by a fractional transformation is also real, so $PSL(2,\mathbb{C})$ maps $\mathfrak{C}$ onto itself.
The elements of $PSL(2,\mathbb{R})$ fix globaly the real line and are conformal maps where they are defined, they thus map the set of circles/lines orthogonal to $\mathbb{R}$ onto itself.
5. Assume first $z=ia$ and $w=ib$ with $a<b$; one thus gets $z^*=0, w^*=\infty$ and $\Big|\log|[z^*,z,w^*,w]|\Big|=\Big|\log|[0,ia,\infty,ib]|\Big|=\log(b/a)=d(z,w)$; for general $z,w\in\mathbb{H}$, one chooses $g\in PSL(2,\mathbb{R})$ such that $g\cdot z=ia$ and $g\cdot w=ib$ and one concludes with the previous question.
Exercice 5 - 1. We first prove the result when $z_1=0, z_2=1$ and $z_3=\infty$; we obtain the 4 equations : $\frac{b}{d}=w_1, \frac{a+b}{c+d}=w_2, \frac{a}{c}=w_3$ and $ad-bc=1$ with a unique solution $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in PSL(2,\mathbb{C})$. When z_1,z_2,z_3 are in general position, one denotes by g_z and g_w the elements of $PSL(2,\mathbb{C})$ such that $g_z(0,1,\infty)=(z_1,z_2,z_3)$ and $g_w(0,1,\infty)=(w_1,w_2,w_3)$ and one gets $g_w\circ g_z^{-1}:(z_1,z_2,z_3)\mapsto(w_1,w_2,w_3)$.
2. In the above 4 equations, when $z_i,w_i\in\mathbb{R}\cup\{\infty\}$, the solutions a,b,c,d are real numbers.
3. (a) It is a direct consequence of the fact that the action of $PSL(2,\mathbb{R})$ is simply transitive on $T^1\mathbb{H}^2$.
(b) if h maps each z_i onto w_i it maps the geodesic segment $[z_1,z_2]$ onto $[w_1,w_2]$; then $h=g$ since it maps $(z_1,\vec{u})$ onto $(w_1,\vec{u}')$.
(c) If $g(z_1,z_2,z_3)=(w_1,w_2,w_3)$ then, since g is an isometry, one must get $d(z_i,z_j)=d(w_i,w_j)$ for any $i\neq j$.

Conversely, if the equalities between the distance hold, then by the Cosine Rule I, the angles $\widehat{z_2z_1z_3}$ and $\widehat{w_2w_1w_3}$ are equal. There exists a unique $g\in PSL(2,\mathbb{R})$ such that $g\cdot(z_1,\vec{u})=(w_1,\vec{u}')$ and $g\cdot(z_1,\vec{v})=(w_1,\vec{u}')$ and the equalities between the distances imply $g\cdot z_2=w_2$ and $g\cdot z_3=w_3$.
(d) Direct consequence of the previous questions.
4. One may send any ideal triangle onto the triangle $(-1,1,\infty)$ using an isometry of $\mathbb{H}$, the area is thus unchanged.
Exercice 6 - It has been done during the course.
Exercice 7 - 1. <u>Existence.</u> Set $n=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, \alpha=\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & \frac{1}{\sqrt{\lambda}}\end{pmatrix}$ and $k=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ such that $n\alpha k=g$ and compute ; one gets $\lambda=\frac{1}{c^2+d^2}, \cos\theta=\frac{d}{\sqrt{c^2+d^2}}, \sin\theta=\frac{c}{\sqrt{c^2+d^2}}$ and $t=\frac{ac+bd}{c^2+d^2}$.
<u>Unicity.</u> If $n\alpha k=n'\alpha'k'$, then $(n'\alpha')^{-1}n\alpha=k'k^{-1}$; the matrice $(n'\alpha')^{-1}n\alpha$ is upper triangular and $k'k^{-1}\in K$, so these two matrices are equal to Id and the conclusion follows.
2. Set $\|g\|=\sqrt{a^2+b^2+c^2+d^2}$ and check that $\|g\|=\|gk\|=\|kg\|$ for any $k\in K$. Then, setting $\alpha=\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & \frac{1}{\sqrt{\lambda}}\end{pmatrix}$, one gets $\lambda+\frac{1}{\lambda}=a^2+b^2+c^2+d^2$ and thus $\lambda=\frac{1}{2}(a^2+b^2+c^2+d^2\pm\sqrt{(a^2+b^2+c^2+d^2)^2+4})$. The Cartan decomposition is not unique, one may for instance replace the couple (k,k') by (kx,xk') with $x=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$; it is unique when $\lambda\neq\frac{1}{\lambda}$ and when one imposes $\lambda>1$.
Exercice 8 - (1) If necessary one may conjugate h and g in such a way the commun fixed point is ∞. Consequently g and h belong to the set of matrices $\Big\{\begin{pmatrix}\lambda & b\\ 0 & \frac{1}{\lambda}\end{pmatrix}/\lambda\in\mathbb{R}^{*+}, b\in\mathbb{R}\Big\}$.

- Assume first that $g=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ and $h=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$; one gets $g^nh^m=\begin{pmatrix}1 & na+mb\\ 0 & 1\end{pmatrix}$; since the group Γ is discrete, the set $\{na+mb/n,m\in\mathbb{Z}\}$ must also be discrete in $\mathbb{R}$ so that there exists $c>0$ such that $\{na+mb/n, m\in\mathbb{Z}\}=c\mathbb{Z}$; there thus exists $k,l\in\mathbb{Z}$ such that $a=kc$ and $b=lc$ so that $g=\gamma^k$ and $h=\gamma^l$ with $\gamma=\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}$.

- Assume now $g=\begin{pmatrix}\lambda & a\\ 0 & \frac{1}{\lambda}\end{pmatrix}$ with $\lambda\neq 1$ and $h=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$. Assume $\lambda>1$, otherwise one replaces g by g^{-1}. One gets first $g^n=\begin{pmatrix}\lambda^n & a_n\\ 0 & \frac{1}{\lambda^n}\end{pmatrix}$ with $a_n=a\Big(\lambda^{n-1}+\lambda^{n-3}+\cdots+\frac{1}{\lambda^{n-3}}+\frac{1}{\lambda^{n-1}}\Big)$ and $g^{-n}=\begin{pmatrix}\frac{1}{\lambda^n} & -a_n\\ 0 & \lambda^n\end{pmatrix}$ so that
$$g^{-n}hg^n=\begin{pmatrix}\frac{1}{\lambda^n} & a_n\\ 0 & \lambda^n\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda^n & a_n\\ 0 & \frac{1}{\lambda^n}\end{pmatrix}=\begin{pmatrix}1 & \frac{b}{\lambda^{2n}}\\ 0 & 1\end{pmatrix}$$
so that the sequence $(g^{-n}hg^n)_{n\geq 1}$ tends to I as $n\to+\infty$. This contradicts the discreteness of Γ.
- Assume atlast that $g=\begin{pmatrix}\lambda & a\\ 0 & \frac{1}{\lambda}\end{pmatrix}$ and $h=\begin{pmatrix}\mu & b\\ 0 & \frac{1}{\mu}\end{pmatrix}$ with $\lambda,\mu\neq 1$. The fixed point of g distinct from ∞ is $\frac{\lambda a}{1-\lambda^2}$ so that, setting $\gamma:=\begin{pmatrix}1 & \frac{\lambda a}{1-\lambda^2}\\ 0 & 1\end{pmatrix}$, the conjugate $\gamma^{-1}g\gamma$ fixes 0 and $+\infty$, in otherwords one gets $\gamma^{-1}g\gamma=\begin{pmatrix}\lambda & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix}$. In the sequel one may thus assume $a=0$ and $\lambda>1$. One gets $g^{-n}hg^n=\begin{pmatrix}\frac{1}{\lambda^n} & 0\\ 0 & \lambda^n\end{pmatrix}\begin{pmatrix}\mu & b\\ 0 & \frac{1}{\mu}\end{pmatrix}\begin{pmatrix}\lambda^n & 0\\ 0 & \frac{1}{\lambda^n}\end{pmatrix}=\begin{pmatrix}\mu & \frac{b}{\lambda^{2n}}\\ 0 & \frac{1}{\mu}\end{pmatrix}\to\begin{pmatrix}\mu & 0\\ 0 & \frac{1}{\mu}\end{pmatrix}$ as $n\to+\infty$. Since Γ is discrete, one gets $\begin{pmatrix}\mu & 0\\ 0 & \frac{1}{\mu}\end{pmatrix}\in\Gamma$ so one may thus assume either $b=0$ ie $h=\begin{pmatrix}\mu & 0\\ 0 & \frac{1}{\mu}\end{pmatrix}$. Since $\Big\{g^nh^m/n,m\in\mathbb{Z}\Big\}=\Big\{\begin{pmatrix}\lambda^n\mu^m & 0\\ 0 & \frac{1}{\lambda^n\mu^m}\end{pmatrix}\Big/n,m\in\mathbb{Z}\Big\}$ is also discrete, the set $\{\lambda^n\mu^m/n,m\in\mathbb{Z}\}$ is a discrete subgroup of $\mathbb{R}^{*+}$, it is of the form $\{\kappa^n/\kappa>1, n\in\mathbb{Z}\}$ and there exists $k,l\in\mathbb{Z}^*$ such that $g=\begin{pmatrix}\kappa & 0\\ 0 & \frac{1}{\kappa}\end{pmatrix}^k$ and $h=\begin{pmatrix}\kappa & 0\\ 0 & \frac{1}{\kappa}\end{pmatrix}^l$.

(2) They are 3 cases to consider :

- if the set of fixed points of elements of Γ is reduced to one points, then Γ is generated by a parabolic isometry.
- If the set of fixed points of elements of Γ is reduced to 2 points ξ_1 and ξ_2, they could be the fixed points of two distincts parabolic transformations p_1 and p_2 ; so $p_1(\xi_2)\notin\{\xi_1,\xi_2\}$ and is fixed by the transformation $p_1p_2p_1^{-1}$ which belongs to Γ ; this contradicts the fact that the set of fixed points of elements of Γ is reduced to 2 points. So when the set of fixed points of elements of Γ is reduced to 2 points, these are the fixed points of an hyperbolic isometry and, by the previous question, the group Γ is generated by such an isometry.
- Assume now that set of fixed points of elements of Γ contains at least 3 points. Then
  - it may be the fixed points of an hyperbolic isometry p and of an hyperbolic one h,
  - two of these points may be the fixed points of parabolic maps p_1 and p_2
  - if Γ contains no parabolic transformation, then it contains 2 hyperbolic isometries h_1 and h_2 with distinct fixed points (at most 4 in this case !)

In these 3 cases, Γ contains at most two elements in Schottky position and so the Schottky group they generate.

3. When Γ is generated by a parabolic map or a hyperbolic one, its limit set is reduced to the set of its fixed point ; it thus contains 1 or 2 points. Otherwise, Γ contains a Schottky subgroup Γ' and each limit point of Γ' may be identified with a infinite reduced sequence with letters in the set of generators of Γ' ; this set is infinite and uncountable.

Exercice 9 - 1.(a) If $v_c \in \mathbb{R}$, by the fractional map $z \mapsto \frac{1}{z-v_c}$, the point v_c is sent onto ∞, without changing the area of the triangle since this map is an isometry.
(b) The base of Δ is semicircle orthogonal to the real axis at points $\mathfrak{a}$ and $\mathfrak{b}$ with for instance $\mathfrak{a} < \mathfrak{b}$. The transformation $T : z \mapsto \frac{2}{\mathfrak{b}-\mathfrak{a}|}(z - \frac{\mathfrak{a}+\mathfrak{b}}{2})$ is an isometry of $\mathbb{H}^2$ which sent $\mathfrak{a}$ onto -1 and $\mathfrak{b}$ onto $+1$ (note that $a, b \in [\mathfrak{a}, \mathfrak{b}]$ so that $T(a), T(b) \in [-1, 1]$). So one may assume that the base is a semicircle with center 0 and radius 1.
(c) Let D_a be the diameter of the base semicircle, joigning 0 to v_a ; since D_a is orthogonal to the semicircle at the vertex v_a, the angle from D_a to the real line is equal to α, and the cosine of this angle equals to a. By the same argument , one gets $\cos\beta = b$.
(d) One gets $\text{area}(\Delta) = \displaystyle\int_\Delta \frac{dx\ dy}{y^2} = \int_a^b dx \int_{\sqrt{1-x^2}}^{+\infty} \frac{dy}{y^2} = \int_a^b \frac{dx}{\sqrt{1-x^2}} = \arcsin b - \arcsin a$ and one obtain the expected result using the equalities $\arcsin b = \frac{\pi}{2} - \arccos b$ and $\arcsin(-a) = -\arcsin a$.
2. (a) One may choose $g \in PSL(2, \mathbb{R})$ which maps the segment $[v_a, v_b]$ onto a segment of the imaginary axis $i\mathbb{R}$.
(b) By a straightforward geometrical argument one checks that the angles of Δ_{ac} (resp. Δ_{bc}) are $\alpha, \gamma + \theta, 0$ (resp. $\pi - \beta, \theta, 0$).
(c) The triangle Δ_{ac} is the union of the two disjoint triangles Δ and Δ_{bc}. It follows that $\text{area}(\Delta) = \text{area}(\Delta_{ac}) - \text{area}(\Delta_{bc}) = \pi - (\alpha + \gamma + \theta) - \pi + (\pi - \beta + \theta) = \pi - (\alpha + \beta + \gamma)$.
3. We apply the first question with $a = -\frac{1}{2}$ and $b = \frac{1}{2}$ ie $\alpha = \beta = \frac{\pi}{3}$; one gets $\text{area}(\mathcal{D}_i) = \pi/3$.
Exercice 10 - The hyperbolic disc $B(i, R)$ with center $z = i$ and radius $R > 0$ coincides with the euclidean one with center $i \cosh R$ and radius $\sinh R$; it is thus the set of $z = x + iy$ such that $x^2 + (y - \cosh R)^2 \leq \sinh^2 R$. Note that the intersectiopn of $B(i, R)$ with the horocycle $\text{Im} z = 1$ is the segment $[-\sqrt{2(\cosh R - 1)}; \sqrt{2(\cosh R - 1)}]$. By symetry of the volume area, we get

$$\begin{aligned}
\text{area}(B(i,R) \cap \mathcal{H}) &= \int_{B(i,R)\cap\mathcal{H}} \frac{dx\ dy}{y^2} \\
&= 2\int_0^{\sqrt{2(\cosh R-1)}} \Big(\int_1^{\cosh R + \sqrt{\sinh^2 R - x^2}} \frac{dy}{y^2}\Big) dx \\
&\quad +2\int_{\sqrt{2(\cosh R-1)}}^{\sqrt{\sinh R}} \Big(\int_{\cosh R - \sqrt{\sinh^2 R - x^2 1}}^{\cosh R + \sqrt{\sinh^2 R - x^2 1}} \frac{dy}{y^2}\Big) dx \\
&= 2\int_0^{\sqrt{2(\cosh R-1)}} \Big(1 - \frac{1}{\cosh R + \sqrt{\sinh^2 R - x^2}}\Big) dx \\
&\quad +8\int_{\sqrt{2(\cosh R-1)}}^{\sinh R} \frac{\sqrt{\sinh^2 R - x^2}}{1 + x^2} dx \\
= I_1 + I_2.
\end{aligned}$$

The first integral I_1 is equal to $2\sqrt{2(\cosh R - 1)} + o(R) = 2e^{R/2}(1 + o(R))$; to estimate the integral I_2, one sets $x := t \sinh R$ so that

$$\begin{aligned}
I_2 &= 8\int_{\frac{1}{\cosh(R/2)}}^1 \frac{\sinh^2 R}{1 + t^2 \sinh^2 R} \sqrt{1 - t^2} dt \\
&\sim 8\int_{\frac{1}{\cosh(R/2)}}^1 \frac{\sqrt{1 - t^2}}{t^2} dt \qquad \text{as } R \to +\infty \\
&\sim 8\int_{\frac{1}{\cosh(R/2)}}^1 \frac{1}{t^2} dt \qquad \text{as } R \to +\infty \\
&\sim 4\cosh\frac{R}{2} \sim 2eR/2 \qquad \text{as } R \to +\infty.
\end{aligned}$$

Then, $\text{area}(B(i, R) \cap \mathcal{H}) \sim 4e^{R/2}$ as $R \to +\infty$.

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