Stochastic dynamical systems Exercises

October 2017

by

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Exercise 1. We consider a sequence of i.i.d. $\mathbb{R}$-valued random variables $(Y_n)_{n\geq 1}$ with distribution μ. We are interested in the fluctuations of the random walk $(S_n)_{n\geq 0}$ defined by

$$S_0 = 0 \quad \text{and} \quad S_n = Y_1 + \ldots + Y_n.$$

We introduce the *ascending ladder epochs* $T_n^+, n \geq 0$, defined by $T_0^+ = 0$ and, for $n \geq 1$,

$$T_n^+ := \inf\{k > T_{n-1}^+ : S_k > S_{T_{n-1}^+}\}$$

with the convention $\inf \emptyset = +\infty$ (in particular, it yields $T_n^+ = +\infty \Longrightarrow T_k^+ = +\infty$ for any $k \geq n$).
These random variables take values in the set $\{1, 2, \ldots\} \bigcup \{+\infty\}$ and are stopping times with respect to the filtration $(\sigma(Y_1, \ldots, Y_n))_{n\geq 1}$ associated with the sequence $(Y_n)_{n\geq 1}$. When the random variables $T_n^+, n \geq 1$, are finite $\mathbb{P}$-a.s. (which occurs under some suitable conditions detailed in the course), we may consider the random variable $S_{T_1^+}$ defined by

$$S_{T_1^+} := \sum_{n=1}^{+\infty} S_n \ \mathbf{1}_{[T_1^+ = n]}.$$

For any $n \geq 1$, we set $\tau_n^+ := T_n^+ - T_{n-1}^+$ and $A_n := S_{T_n^+} - S_{T_{n-1}^+}$; consequently,

$$T_n^+ = \tau_1^+ + \ldots + \tau_n^+ \quad \text{and} \quad S_{T_n^+} = A_1 + \ldots + A_n.$$

1. Prove that, for any $k \geq 1$, it holds $\quad \mathbb{P}(\tau_1^+ = k) = \mathbb{P}[S_1 \leq 0, \ldots, S_{k-1} \leq 0, S_k > 0]$.

2. Prove that, for any $k, l \geq 1$, it holds $\quad \mathbb{P}[\tau_1^+ = k, \tau_2^+ = l] = \mathbb{P}[\tau_1^+ = k] \times \mathbb{P}[\tau_1^+ = l]$.

3. Conclude that τ_1^+ and τ_2^+ are i.i.d. random variables (with values in $\mathbb{N}^* \cup \{+\infty\}$).
By the same argument, it can be proved that $(\tau_n^+)_{n\geq 1}$ is a sequence of i.i.d. random variables with values in $\mathbb{N}^* \cup \{+\infty\}$.

4. Assume that $\mathbb{P}[\tau_1^+ < +\infty] = 1$.

   (a) Check that, for any $s \in \mathbb{R}$,

   $$\mathbb{P}[A_1 \leq s] = \begin{cases} 0 & \text{when} \quad s \leq 0 \\ \displaystyle\sum_{n=1}^{+\infty} \mathbb{P}[T_1^+ = n, 0 < S_n \leq s] & \text{when} \quad t > 0. \end{cases}$$

   (b) Prove that, for any $s, t \in \mathbb{R}$,

   $$\mathbb{P}[A_1 \leq s, A_2 \leq t] = \mathbb{P}[A_1 \leq s] \times \mathbb{P}[A_2 \leq t].$$

   (c) Conclude that A_1 and A_2 are i.i.d. random variables with values in $\mathbb{R}^{*+}$.

   By the same argument, it can be proved that $(A_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with values in $\mathbb{R}^*$.

5. Assume now that $\mathbb{E}[|Y_1|] < +\infty$.

   (a) Prove that $\mathbb{P}[\tau_1^+ < +\infty] = 1$ if and only if $\mathbb{E}[Y_1] \geq 0$.

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(b) Under the condition $\mathbb{E}[Y_1] \geq 0$, check that $\mathbb{P}[T_n^+ < +\infty] = 1$ for any $n \geq 1$ and that the so called "Wald's formula" holds:

$$\mathbb{E}[A_1] = \mathbb{E}[Y_1] \times \mathbb{E}[\tau_1^+].$$

Hint: we may consider the ratio $\dfrac{A_1 + \ldots + A_n}{n} = \dfrac{S_{T_n^+}}{T_n^+} \times \dfrac{T_n^+}{n}$ *and apply the strong law of large numbers.*

(c) Prove that $\mathbb{E}[\tau_1^+] = +\infty$ when the Y_i are centered.

(d) Now, we assume $\mathbb{E}[Y_1] > 0$ and introduce the random variable $\tau_1^- := \inf\{k \geq 1 : S_k \leq 0\}$. Using the strong law of large number, check that $\mathbb{P}[\tau_1^- = +\infty] > 0$ and deduce that $\mathbb{E}[\tau_1^+] < +\infty$. Conclude that $\mathbb{E}[A_1] < +\infty$.

Exercise 2. We consider a queue with one server and denote $\mathcal{A}_1, \mathcal{A}_2, \ldots$ the inter-arrival times between two successive customers; the arrival times of the customers are $0, \mathcal{A}_1, \mathcal{A}_1 + \mathcal{A}_2, \ldots$. We denote $\mathcal{S}_1, \mathcal{S}_2, \ldots$ the service time of the different customers.

We assume that $(\mathcal{A}_n)_{n\geq 1}$ and $(\mathcal{S}_n)_{n\geq 1}$ are two independent sequences of i.i.d random variables and that the distribution of the $\mathcal{A}_n$ (resp. the $\mathcal{B}_n$) is exponential with parameter $\mathfrak{a} > 0$ (resp. with parameter $\mathfrak{s} > 0$).
We set $W_0 = 0$, and, for any $n \geq 1$, we denote W_n the waiting time of the n^{th} customer in the queue.

If the n^{th} customer arrives at time t, he is served at time $t + W_n$ and leaves the queue at time $t + W_n + \mathcal{S}_n$; the customer $n+1$ arrives at time $t + \mathcal{A}_{n+1}$ and his waiting time W_{n+1} in the queue equals

$$W_{n+1} := \begin{cases} 0 & \text{when} \qquad \mathcal{A}_{n+1} \geq W_n + \mathcal{S}_n \\ W_n + \mathcal{S}_n - \mathcal{A}_{n+1} & \text{otherwise.} \end{cases}$$

In other words, setting $Y_{n+1} := \mathcal{A}_{n+1} - \mathcal{S}_n$, it holds $W_{n+1} := \max(W_n - Y_{n+1}, 0)$.

1. Check that $(W_n)_{n\geq 0}$ is a stochastic dynamical system (SDS) as defined in the course (give explicitly the family of random maps which defines this SDS).

2. Check that the distribution of the Y_n has the following density $t \mapsto h(t)$ with respect to the Lebesgue measure on $\mathbb{R}$:

$$\forall t \in \mathbb{R} \qquad h(t) := \frac{\mathfrak{a}}{\mathfrak{a}+\mathfrak{s}} \mathfrak{s} e^{\mathfrak{s}t} 1_{\mathbb{R}^-}(t) + \frac{\mathfrak{s}}{\mathfrak{a}+\mathfrak{s}} \mathfrak{a} e^{-\mathfrak{a}t} 1_{\mathbb{R}^+}(t).$$

3. Compute $\mathbb{E}[\mathcal{A}_n]$ and $\mathbb{E}[\mathcal{S}_n]$. Deduce that

   (a) If $\mathfrak{a} > \mathfrak{s}$, then the process $(W_n)_{n\geq 0}$ is positive recurrent.

   (b) If $\mathfrak{a} = \mathfrak{s}$, then the process $(W_n)_{n\geq 0}$ is null recurrent.

   (c) If $\mathfrak{a} < \mathfrak{s}$, then the sequence $(W_n)_{n\geq 0}$ tends $\mathbb{P}$-a.s. towards $+\infty$.

4. We denote τ the first time at which the queue is empty. In which cases is the random variable τ finite $\mathbb{P}$-a.s.? has finite expectation?

Exercise 3. We consider the reflected random walk $(R_n)_{n\geq 0}$ on $\mathbb{R}^+$ defined by: $R_0 = x$ for some fixed $x \in \mathbb{R}^+$ and, for any $n \geq 0$,

$$R_{n+1} = |R_n - Y_{n+1}|$$

where $(Y_k)_{k\geq 1}$ is a sequence of i.i.d. $\mathbb{R}$-valued random variables whose distribution is adapted on $\mathbb{R}$. We assume that $\mathbb{E}[|Y_1|] < +\infty$ and $\mathbb{E}[Y_1] > 0$.

1. Does there exists an invariant probability measure on $\mathbb{R}^+$ for $(R_n)_{n\geq 0}$? Is it unique? Why?

2. Is $(R_n)_{n\geq 0}$ transient? positive recurrent? null recurrent?

3. We assume that the Y_i are uniformly distributed on the interval $[0, C]$ for some $C > 0$. Compute explicitely the invariant probability measure. What is its support?

4. We assume that the Y_i have the exponential distribution with parameter $\lambda > 0$. Compute explicitly the invariant probability measure of $(R_n)_{n\geq 0}$ in this case. What is its support?

Exercise 4. We consider the evolution over the time of a life insurance contract. For any $n \geq 1$, a payment b_n is made in the life insurance portfolio at the beginning of the n-th period; we set $X_0 = 0$ and denote X_n the amount which has been accumulated during the n first periods. The amount X_{n-1} is subject to interest, given by the random variables $a_n = 1 + \delta_n$ (the coefficient δ_n is the interest rate announced by the insurance company). Thus, the value of the portfolio at the beginning of the n-th period is given by the equation

$$\begin{cases} X_0 &= 0 \\ X_n &= a_n X_{n-1} + b_n = (1+\delta_n)X_{n-1} + b_n \quad \text{when } n \geq 1. \end{cases}$$

Thus, it holds $\quad X_n = F_n \circ \ldots \circ F_1(0) = L_n(0) \quad$ where $F_n(x) := a_n x + b_n = (1+\delta_n)x + b_n$ and $L_n = F_n \circ \ldots \circ F_1$.

The coefficients a_n and b_n may vary randomly over time, with some homogeneous behavior; we assume that the random variables (a_n, b_n) are i.i.d. with distribution μ on $\mathbb{R}^+ \times \mathbb{R}$.

• during each period, the interest rate may be equal to $\alpha > 0$ with probability $0 < p < 1$ and to $-\beta \in]-1, 0[$ with probability $q = 1-p$; in other words, the distribution of the random variable a_n equals $p\delta_{1+\alpha} + q\delta_{1-\beta}$, with $p, q > 0, p+q = 1$ and $0 < \alpha, \beta < 1$.

• the deposits b_n are non negative (i.e. the customer is not allowed to make withdrawals) and $\mathbb{E}[|\log b_n|] < +\infty$.

We assume that

$$p\log(1+\alpha) + q\log(1-\beta) < 0. \tag{1}$$

1. Prove that, under the condition (1), the sequence $(F_1 \circ \ldots \circ F_n(x))_{n\geq 0}$ converges $\mathbb{P}$-a.s. towards a finite random variable Z_∞, which can be given explicitly and which does not depend on the value of x.

2. Deduce that the sequence $(X_n)_{n\geq 0}$ converges in distribution to Z_∞.

3. The customer has signed the insurance contract because the announced mean interest rate $p(1+\alpha) + q(1-\beta)$ satisfies the inequality

$$p(1+\alpha) + q(1-\beta) > 1. \tag{2}$$

   Check that, for any $0 < \beta < 1$ and $\alpha > 0$, both conditions (1) and (2) may hold simultaneously for some suitable choice of p and q to be specified.

Exercise 5. The nearest neighbourd (reflected or not) random walks on $\mathbb{N}$

Let $(Y_n)_{n\geq 0}$ be a sequence of independent $\mathbb{Z}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with commun distribution $\mu = q\delta_{-1} + r\delta_0 + p\delta_1$ where $p, q, r \geq 0, p+q+r = 1$ and $p \times q \times r \neq 0$.

1. We first consider the **classical random walk** $(S_n)_{n\geq 0}$ on $\mathbb{Z}$ defined by

$$S_0 := x \in \mathbb{Z} \quad \text{and} \quad S_n := x + Y_1 + \ldots + Y_n \quad \text{for} \quad n \geq 1.$$

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   (a) Check that $(S_n)_{n\geq 0}$ is a stochastic dynamical system as defined in the course (give explicitly the family of random maps which defines this SDS).

   (b) What is the transition matrix of this Markov chain?

   (c) Show that $(S_n)_{n\geq 0}$ is irreducible on $\mathbb{Z}$. Is this Markov chain aperiodic?

   (d) Let $\lambda = (\lambda_k)_{k\in\mathbb{Z}}$ be a measure on $\mathbb{Z}$ (i.e. $\lambda_k \in \mathbb{R}$ for any $k \in \mathbb{Z}$) which is invariant for $(S_n)_{n\geq 0}$; prove that, for any $k \in \mathbb{Z}$,

$$q\lambda_{k+1} + r\lambda_k + p\lambda_{k-1} = \lambda_k. \tag{3}$$

   Rewriting (3) as $q\lambda_{\ell+2} - (p+q)\lambda_{\ell+1} + p\lambda_\ell = 0$ for any $\ell \in \mathbb{Z}$, deduce that there exist constants $a, b \in \mathbb{R}$ such that $\lambda_\ell = a\left(\frac{p}{q}\right)^\ell + b$ for any $\ell \in \mathbb{Z}$.

   Check that, in particular, the measure m on $\mathbb{Z}$ such that $m(k) = 1$ for any $k \in \mathbb{Z}$ is always invariant for the random walk $(S_n)_{n\geq 0}$, whatever the values of p, q, r are.

   (e) Assume $p > q > 0$ (the case when $q > p > 0$ can be studied similarly).

   • Prove that $\lim_{n\to+\infty} S_n = +\infty$ and that $(S_n)_{n\geq 0}$ is transient on $\mathbb{Z}$.

(f) Assume now $p = q > 0$. Set $\tau_1^+ = \inf\{n \geq 1 \mid S_n = 1\}$ and $\tau_1^- = \inf\{n \geq 1 \mid S_n = -1\}$.
- Using the fact that the law μ is symmetric, check that $\mathbb{P}[\tau_1^+ < +\infty] = \mathbb{P}[\tau_1^- < +\infty] = 1$.
- Prove that $\mathbb{P}[T_n^+ < +\infty] = \mathbb{P}[T_n^- < +\infty] = 1$ for any $n \geq 0$ (we use the same notations as in the course).
- Check that $\mathbb{P}[S_{\tau_1^+} = 1] = \mathbb{P}[S_{\tau_1^-} = -1] = 1$ and that $\mathbb{P}[S_{T_n^+} = n] = \mathbb{P}[S_{T_n^-} = -n] = 1$ for any $n \geq 0$.
- Deduce that $\liminf_{n\to+\infty} S_n = -\infty$ and $\limsup_{n\to+\infty} S_n = +\infty$ $\mathbb{P}$-almost surely and that $(S_n)_{n\geq 0}$ is recurrent on $\mathbb{Z}$.
- What is the unique (up to a multiplicative constant) σ-finite invariant measure for $(S_n)_{n\geq 0}$ in this case?
- Does there exist an unique (up to a multiplicative constant) invariant measure for $(S_n)_{n\geq 0}$ in this case?

2. Now, we consider the **reflected random walk** $(A_n)_{n\geq 0}$ **on** $\mathbb{N}$ **with absorption at** 0 defined by

$$A_0 := x \in \mathbb{N} \quad \text{and} \quad A_{n+1} := \max(0, A_n - Y_{n+1}) \quad \text{for} \quad n \geq 1.$$

(a) Check that $(A_n)_{n\geq 0}$ is a stochastic dynamical system as defined in the course (give explicitly the family of random maps which defines this SDS).

(b) What is the transition matrix of this Markov chain?

(c) Show that $(A_n)_{n\geq 0}$ is irreducible on $\mathbb{N}$. Is this Markov chain aperiodic?

(d) Let $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ be an invariant measure on $\mathbb{N}$ for $(A_n)_{n\geq 0}$; prove that

$$(p + r)\lambda_0 + p\lambda_1 = \lambda_0 \qquad \text{and} \qquad p\lambda_{k+1} + r\lambda_k + q\lambda_{k-1} = \lambda_k \quad \text{for} \quad k \geq 1. \tag{4}$$

Rewriting (4) as $p\lambda_1 - q\lambda_0 = 0$ and $p\lambda_{\ell+2} - (p+q)\lambda_{\ell+1} + q\lambda_\ell = 0$ for any $\ell \geq 0$, deduce that there exists a constant $a \in \mathbb{R}$ such that $\lambda_\ell = a\left(\frac{q}{p}\right)^\ell$ for any $\ell \in \mathbb{N}$.

(e) Assume $p > q > 0$. Using the fact that the measure λ is finite in this case, deduce that $(A_n)_{n\geq 0}$ is positive recurrent.

(f) Assume $p = q > 0$. Using the fact that $\limsup_{n\to+\infty} S_n = +\infty$ $\mathbb{P}$-a.s., prove that $\mathbb{P}_x(\exists n \geq 1 \mid A_n = 0\} = 1$ for any $x \in \mathbb{N}$ [2]. Deduce that $\mathbb{P}_x[A_n = 0 \ i.o.] = 1$.
Is $(A_n)_{n\geq 0}$ recurrent or transient in this case? And if it is recurrent, is it positive or null recurrent?

(g) Assume $q > p > 0$. Prove that $A_n \geq -S_n 1_{[S_n \leq 0]}$ for any $n \geq 0$; deduce that $\lim_{n\to+\infty} A_n = +\infty$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{N}$.
Is $(A_n)_{n\geq 0}$ recurrent or transient in this case?

3. Lastly, we consider the **reflected random walk** $(R_n)_{n\geq 0}$ **on** $\mathbb{N}$ **with elastic collision at** 0 defined by

$$R_0 := x \in \mathbb{N} \quad \text{and} \quad R_{n+1} := |R_n - Y_{n+1}| \quad \text{for} \quad n \geq 1.$$

(a) Check that $(R_n)_{n\geq 0}$ is a stochastic dynamical system as defined in the course (give explicitly the family of random maps which defines this SDS).

(b) What is the transition matrix of this Markov chain?

(c) Show that $(R_n)_{n\geq 0}$ is irreducible on $\mathbb{N}$. Is this Markov chain aperiodic?

(d) Let $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ be an invariant measure on $\mathbb{N}$ for $(R_n)_{n\geq 0}$; prove that

$$r\lambda_0 + p\lambda_1 = \lambda_0, \qquad p\lambda_2 + r\lambda_1 + (p+q)\lambda_0 = \lambda_1 \qquad \text{and} \qquad p\lambda_{k+1} + r\lambda_k + q\lambda_{k-1} = \lambda_k \quad \text{for} \quad k \geq 2. \tag{5}$$

Rewriting (5) as $p\lambda_1 - (p+q)\lambda_0 = 0$, $p\lambda_2 - (p+q)\lambda_1 + (p+q)\lambda_0 = 0$ and $p\lambda_{\ell+2} - (p+q)\lambda_{\ell+1} + q\lambda_\ell = 0$ for any $\ell \geq 1$, deduce that there exists a constant $a \in \mathbb{R}$ such that $\lambda_\ell = a\left(\frac{q}{p}\right)^\ell$ for any $\ell \in \mathbb{N}$.

(e) Assume $p > q > 0$. Using the fact that the measure λ is finite in this case, deduce that $(R_n)_{n\geq 0}$ is positive recurrent.

(f) Assume $p = q > 0$. Using the fact that $\limsup_{n\to+\infty} S_n = +\infty$ and that the steps Y_n are ≤ 1 $\mathbb{P}$-a.s., prove that $\mathbb{P}_x(\exists n \geq 1 \mid R_n = 0\} = 1$ for any $x \in \mathbb{N}$. Deduce that $\mathbb{P}_x[R_n = 0 \ i.o.] = 1$.
Is $(R_n)_{n\geq 0}$ recurrent or transient in this case? And if it is recurrent, is it positive or null recurrent?

(g) Assume $q > p > 0$. Prove that $R_n \geq A_n$ for any $n \geq 0$; deduce that $\lim_{n\to+\infty} R_n = +\infty$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{N}$.
Is $(R_n)_{n\geq 0}$ recurrent or transient in this case?

[2] "i.o" means "infinitely often"; the event $[A_n = 0 \ i.o.]$ also equals $\bigcap_{n\geq 1}\bigcup_{k\geq n}[A_k = 0] = \limsup_{n\to+\infty}[A_n = 0]$