Iterated function systems with place dependent probabilities and application to the Diaconis-Freedman's chain on $[0,1]$

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Fetima Ladjimi [(1)] & Marc Peigné [(2)] [(3)]

Abstract

We study Markov chains generated by iterated Lipschitz functions systems with possibly place dependent probabilities. Under general conditions, we prove uniqueness of the invariant probability measure for the associated Markov chain, by using quasi-compact linear operators technics. We use the same approach to describe the behavior of the Diaconis-Freedman's chain on $[0,1]$ with possibly place dependent probabilities.

Keywords: Iterated function systems, quasi-compact linear operators, invariant probability measure, absorbing compact set

1 Introduction

We are interested in the Markov chain $(Z_n)_{n\geq 0}$ on $[0,1]$ introduced by P. Diaconis and D. Freedman in [4]. As it is described there, if the chain is at x at time n, it selects at time $n+1$ one of the two intervals, $[0,x]$ or $[x,1]$ with equal probability $\frac{1}{2}$, and then moves to a random point y in the chosen interval.

For $x\in]0,1[$, the transition probability of the chain $(Z_n)_{n\geq 0}$ has a density $k(x,\cdot)$ with respect to the Lebesgue measure on $]0,1[$ given by

$$\forall y\in]0,1[\qquad k(x,y)=\frac{1}{2}\times\frac{1}{x}\mathbf{1}_{]0,x[}(y)+\frac{1}{2}\times\frac{1}{1-x}\mathbf{1}_{]x,1[}(y).$$

Starting from 0 (resp. 1), the chain stays in 0 (resp. 1) with probability $\frac{1}{2}$ or moves with probability $\frac{1}{2}$ to a (uniformly chosen) random point in $]0,1[$.

It is shown in [4] that it possesses a unique invariant probability measure ν on $]0,1[$; this measure is the famous "arcsine law" which admits the density $f_{\frac{1}{2}}$ with respect to the Lebesgue measure on $]0,1[$ given by

$$\forall x\in]0,1[,\quad f_{\frac{1}{2}}(x)=\frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{]0,1[}(x).$$

The same applies when the intervals $]0,x[$ and $]x,1[$ are chosen with the respective probabilities $p\in]0,1[$ and $q=1-p$. In this case, the invariant probability measure is the Beta distribution $\mathcal{B}(q,p)$ of parameters q and p whith density f_p defined by:

$$\forall x\in]0,1[,\quad f_p(x)=\frac{1}{\Gamma(p)\Gamma(q)}x^{q-1}(1-x)^{p-1}\mathbf{1}_{]0,1[}(x).$$

[1]Fetima Ladjimi, Département de Mathématiques, Université Mouloud MAMMERI Tizi-Ouzou (UMMTO), 15000, Algérie – email : ladjimi-fetima@ummto.dz

[2]Marc Peigné, LMPT, UMR 7350, Faculté des Sciences et Techniques, Parc de Grandmont, 37200 Tours, France – email : peigne@univ-tours.fr

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The transition operator Q of the chain $(Z_n)_{n\geq 0}$ is defined by: for any bounded Borel function $\varphi : [0,1] \to \mathbb{C}$,

$$Q\varphi(0) = p\varphi(0) + q\int_0^1 \varphi(y)\mathrm{d}y, \qquad Q\varphi(1) = p\int_0^1 \varphi(y)\mathrm{d}y + q\varphi(1)$$

and

$$\forall x \in]0,1[\qquad Q\varphi(x) = \frac{p}{x}\int_0^x \varphi(y)\mathrm{d}y + \frac{q}{1-x}\int_x^1 \varphi(y)\mathrm{d}y.$$

We may rewrite shortly Q as follows: for any $x \in [0,1]$,

$$Q\varphi(x) = p\int_0^1 \varphi(tx)\mathrm{d}t + q\int_0^1 \varphi(tx+1-t)\mathrm{d}t. \tag{1}$$

This last expression shows that the chain $(Z_n)_{n\geq 0}$ fits into the framework of iterated random continuous functions. For any $t \in [0,1]$, let H_t be the homothety $x \mapsto tx$ and A_t be the affine transformation $x \mapsto tx + 1 - t$ and denote by μ the probability measure on the space $C([0,1],[0,1])$ of continuous functions from $[0,1]$ to $[0,1]$ defined by

$$\mu(\mathrm{d}T) = p\int_0^1 \delta_{H_t}(\mathrm{d}T)\mathrm{d}t + q\int_0^1 \delta_{A_t}(\mathrm{d}T)\mathrm{d}t,$$

where δ_T is the Dirac masse at T. Equality (1) may be restated as

$$Q\varphi(x) = \int_{C([0,1],[0,1])} \varphi(T(x))\mu(\mathrm{d}T).$$

Thus we may introduce a sequence $(T_n)_{n\geq 1}$ of independent random variables defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ with law μ on $C([0,1],[0,1])$. We have $Z_n = T_n \cdots T_1 \cdot Z_0$; in other words, the chain $(Z_n)_{n\geq 0}$ is generated by iterating random functions and its behavior is strongly connected to the contraction properties of the maps H_t and $A_t, 0 \leq t \leq 1$. We refer to $(Z_n)_{n\geq 0}$ as *the Diaconis-Freedman's chain.*

In [4], the authors focus on the case when the weights p and q are depending on the position x, and in particular when $p(x) = 1-x$. In the sequel, we propose a systematic examination of the general situation addressed by the two authors.

The study of Markov processes generated by composition products of random independent functions T_n has been the object of numerous works for 50 years. When the probabilities that govern the choice of these transformations are spatially varying, the study of these processes escapes the random walks framework. We refer the reader to [8], [13], [14] or [17] and references there in, and to [10] or [15] for the approach via the theory of quasi-compact operators. We use the terminology on Markov chains as stated in [18].

2 Iterated function systems

Let (E,d) be a metric compact space and denote $C(E,E)$ the space of continuous functions from E to E endowed with the norm $|\cdot|_\infty$ of the uniform convergence on E. Let $(T_n)_{n\geq 1}$ be a sequence of i.i.d random continuous functions from E to E with distribution μ. The case when the T_n are Lipschitz continuous from E to E is fruitful, in particular to use the so-called "spectral gap property", based on the properties of contraction of the closed semi-group T_μ generated by the support of μ.

2.1 Iterated function systems with place independent probabilities

We denote $\mathbb{L}\mathrm{ip}(E,E)$ the space of Lipschitz continuous functions from E to E, i.e. of functions $f : E \to E$ such that

$$[f] = \sup_{\substack{x,y\in E\\ x\neq y}} \frac{d(f(x),f(y))}{d(x,y)} < \infty,$$

and we endow $\mathbb{L}\mathrm{ip}(E,E)$ with the norm $\|\cdot\| = |\cdot|_\infty + [\cdot]$. Let $(T_n)_{n\geq 1}$ be a sequence of independent random functions defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$, with values in $\mathbb{L}\mathrm{ip}(E,E)$ and common distribution μ. We consider the Markov chain $(X_n)_{n\geq 0}$ on E, defined by: for any $n \geq 0$,

$$X_{n+1} := T_{n+1}(X_n)$$

where X_0 is a fixed random variable with values in E. One says that the chain $(X_n)_{n\geq 0}$ is generated by the *iterated function system* $(T_n)_{n\geq 1}$. Its transition operator P is defined by: for any bounded Borel function $\varphi : E \to \mathbb{C}$ and any $x \in E$

$$P\varphi(x) = \int_{\mathbb{L}\mathrm{ip}(E,E)} \varphi(T(x))\mu(\mathrm{d}T).$$

The chain $(X_n)_{n\geq 0}$ has the "Feller property", i.e. the operator P acts on the space $C(E)$ of continuous functions from E to $\mathbb{C}$. The maps T_n being Lipschitz continuous on E, the operator P acts also on the space of Lipschitz continuous from E to $\mathbb{C}$ and more generally on the space $\mathcal{H}_\alpha(E), 0 < \alpha \leq 1$, of α-Hölder continuous functions from E to $\mathbb{C}$, defined by

$$\mathcal{H}_\alpha(E) := \{f \in C(E) \mid \|f\|_\alpha := |f|_\infty + m_\alpha(f) < +\infty\}$$

where $m_\alpha(f) := \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty$. Endowed, with the norm $\|\cdot\|_\alpha$, the space $\mathcal{H}_\alpha(E)$ is a Banach space and the identity map from $C(E)$ to $\mathcal{H}_\alpha(E)$ is compact.

The behavior of the chain $(X_n)_{n\geq 0}$ is closely related to the spectrum of P on these spaces; under some "contraction in mean" assumption on the T_n, the restriction of the operator P to $\mathcal{H}_\alpha(E)$ satisfies some spectral gap property. We first cite the following theorem, due to Diaconis & Freedman [4]; we detail the proof for the sake of completeness.

Theorem 2.1 *Assume that there exists $\alpha \in]0,1]$ such that*

$$r := \sup_{\substack{x,y\in E\\ x\neq y}} \int_{\mathbb{L}\mathrm{ip}(E,E)} \Big(\frac{d(T(x),T(y))}{d(x,y)}\Big)^\alpha \mu(\mathrm{d}T) < 1. \tag{2}$$

Then there exists on E a unique P-invariant probability measure ν. Furthermore, there exists constants $\kappa > 0$ and $\rho \in]0,1[$ such that

$$\forall \varphi \in \mathcal{H}_\alpha(E),\ \forall x \in E \quad |P^n\varphi(x) - \nu(\varphi)| \leq \kappa\rho^n. \tag{3}$$

Proof. The Feller operator P is Markovian, thus its spectral radius $\rho_\infty(P)$ in $C(E)$ equals 1. Furthermore, P acts on $\mathcal{H}_\alpha(E)$ and for any function $\varphi \in \mathcal{H}_\alpha(E)$, it holds

$$m_\alpha(P\varphi) \leq r\ m_\alpha(\varphi), \tag{4}$$

which yields

$$\forall \varphi \in \mathcal{H}_\alpha(E), \qquad \|P\varphi\|_\alpha \leq r\|\varphi\|_\alpha + |\varphi|_\infty. \tag{5}$$

Inequality (5) allows us to use the Ionescu-Tulcea and Marinescu theorem for quasi-compact operators. By Hennion's work [9], it implies that the essential spectral radius of P on $\mathcal{H}_\alpha(E)$ is less than r; in other words, any spectral values with modulus strictly larger then r is an eigenvalue of P with finite multiplicity and is isolated in the spectrum of P.

To prove the theorem, it is sufficient to control the peripheral spectrum of P on $\mathcal{H}_\alpha(E)$. Let λ an eigenvalue of P of modulus 1 and consider an eigenfunction f associated to λ. For any $n \geq 1$, the equality $P^n f = \lambda^n f$ combined with (5) yields

$$m_\alpha(f) = m_\alpha(\lambda^n f) = m_\alpha(P^n f) \leq r^n m_\alpha(f)$$

which implies $m_\alpha(f) = 0$, since $0 \le r < 1$. Consequently, the function f is constant on E and $\lambda = 1$. Thus, the operator P on $\mathcal{H}_\alpha(E)$ can be decomposed as,

$$P = \Pi + R \tag{6}$$

where

(i) the operator Π is the projector from $\mathcal{H}_\alpha(E)$ to the eigenspace $\mathbb{C} \cdot \mathbf{1}$ associated to the eigenvalue 1,

(ii) R is an operator with spectral radius ρ for some $\rho \in [0, 1[$,

(iii) $\Pi R = R\Pi = 0$.

In particular, for any $\varphi \in \mathcal{H}_\alpha(E)$, the sequence $(P^n\varphi)_{n\ge0}$ converges to $\Pi(\varphi)\mathbf{1}$; thus, there exists on E a unique invariant probability measure ν and the projector Π may be written as $\Pi : \varphi \mapsto \nu(\varphi)\mathbf{1}$. Inequality (3) follows from decomposition (6).

□

Application to the Diaconis-Freedman's chain for p fixed in $]0, 1[$.

Inequality (2) holds with $r = \frac{1}{1+\alpha}$ since in this case $m(H_t) = m(A_t) = t$ for any $0 \le t \le 1$. Hence

$$\begin{aligned} \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \int_{\mathbb{L}\mathrm{ip}([0,1],[0,1])} \Big(\frac{d(T(x),T(y))}{d(x,y)}\Big)^\alpha \mu(\mathrm{d}T) &\le p\int_0^1 m(H_t)^\alpha \mathrm{d}t + q\int_0^1 m(A_t)^\alpha \mathrm{d}t \\ &= \int_0^1 t^\alpha \mathrm{d}t = \frac{1}{1+\alpha}. \end{aligned}$$

Thus, the chain $(Z_n)_{n\ge0}$ admits an unique invariant probability measure on $[0, 1]$, this measure being the Beta distribution $\mathcal{B}(q, p)$.

□

2.2 Iterated function systems with spacial dependant increments probabilities

It this section, we replace the measure μ by a collection $(\mu_x)_{x\in E}$ of probability measures on E, depending continuously on x. We consider the Markov chain $(X_n)_{n\ge0}$ on E whose transition kernel P is given by: for any bounded Borel function $\varphi : E \to \mathbb{C}$ and any $x \in E$,

$$P\varphi(x) = \int_{\mathbb{L}\mathrm{ip}(E,E)} \varphi(T(x))\mu_x(\mathrm{d}T).$$

First, we introduce the following definition.

Definition 2.2 *A sequence $(\xi_n)_{n\ge0}$ of continuous functions from E to E is a contracting sequence if there exist $x_0 \in E$ such that*

$$\forall x \in E \qquad \lim_{n\to+\infty} \xi_n(x) = x_0.$$

The following statement is a generalization of Theorem 2.1.

Theorem 2.3 *Assume that there exists $\alpha \in (0, 1]$ such that*

H1. $r := \sup_{\substack{x,y\in E\\ x\neq y}} \int_{\mathbb{L}\mathrm{ip}(E,E)} \Big(\frac{d(T(x),T(y))}{d(x,y)}\Big)^\alpha \mu_x(\mathrm{d}T) < 1,$

H2. $R_\alpha := \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|\mu_x - \mu_y|}{d(x,y)^\alpha} < +\infty,$

H3. *there exists $\delta > 0$ and a probability measure μ on E such that*

$$\forall x \in E \qquad \mu_x \ge \delta\mu \tag{7}$$

and the closed semi-group T_μ generated by the support S_μ of μ possesses a contracting sequence.

Then, there exists on E a unique P-invariant probability measure ν; furthermore, for some constants $\kappa > 0$ and $\rho \in]0,1[$, it holds

$$\forall \varphi \in \mathcal{H}_\alpha(E),\ \forall x \in E \quad |P^n\varphi(x) - \nu(\varphi)| \leq \kappa \rho^n. \tag{8}$$

Remark. Hypothesis **H1** means that the maps T satisfy some contraction property "in mean", with respect to each measure μ_x. Nevertheless, the measures μ_x may be singular versus another; this implies that, starting from two different points, the maps which govern the transition may be totally different and it becomes quite impossible to control their common evolution. Thus, hypothesis **H3** is useful to fill up this gap.

Proof. The operator P acts on $C(E)$, with spectral radius 1 since it is Markovian. It also acts on $\mathcal{H}_\alpha(E)$; indeed, for any function $\varphi \in \mathcal{H}_\alpha(E)$ and any $x, y \in E$, it holds

$$|P\varphi(x) - P\varphi(y)| \quad \leq \quad \int_{\mathbb{L}\mathrm{ip}(E,E)} |\varphi(T(x)) - \varphi(T(y))|\mu_x(\mathrm{d}T) + |\varphi|_\infty \int_{\mathbb{L}\mathrm{ip}(E,E)} |\mu_x - \mu_y|(\mathrm{d}T).$$

Hence

$$m_\alpha(P\varphi) \leq r m_\alpha(\varphi) + R_\alpha |\varphi|_\infty \tag{9}$$

which readily yields

$$\|P\varphi\|_\alpha \leq r\|\varphi\|_\alpha + (1 + R_\alpha)|\varphi|_\infty. \tag{10}$$

Thus, by [9], the operator P is quasi-compact on $\mathcal{H}_\alpha(E)$; its spectral radius on $\mathcal{H}_\alpha(E)$ equals the modulus of a dominant eigenvalue, thus is less than the one of P on $C(E)$, that is 1. To control the peripheral spectrum, the argument differs then from the one used to prove Theorem 2.1: property (4) does not hold here and inequality (9) is much weaker. We get use of the two following lemmas, valid under hypotheses **H1**, **H2** and **H3**.

Lemma 2.4 *Let $h \in \mathcal{H}_\alpha(E)$ such that $Ph = h$. For any $x \in E$, the sequence $(h(X_n))_{n\geq 0}$ is a bounded martingale on the space $(\Omega, \mathcal{F}, \mathbb{P}_x)$, where $\mathbb{P}_x$ denotes the conditional probability $\mathbb{P}(\cdot/X_0 = x)$. It converges $\mathbb{P}_x$-a.s. and in $\mathbb{L}^1(\Omega, \mathbb{P}_x)$ to a random variable H^∞ and it holds*

$$\forall n \geq 0 \quad h(x) = \mathbb{E}_x(h(X_n)) = \mathbb{E}_x(H^\infty). \tag{11}$$

Furthermore, for any $\xi \in T_\mu$,

$$H^\infty = \lim_{n\to+\infty} h(\xi \cdot X_n) \quad \mathbb{P}_x - \text{a.s.} \tag{12}$$

Proof of Lemma 2.4. The function h is P-harmonic and bounded; the first assertion and equality (11) follow. Let us now prove (12). First, let us fix positive integers n an q and set

$$u_{n,q}(x) := \mathbb{E}_x\Big(\big|h(X_{n+q}) - h(X_n)\big|^2\Big).$$

From the martingale equality, for any $N \geq 1$, it holds

$$\sum_{n=1}^N u_{n,q}(x) = \sum_{n=1}^N \mathbb{E}_x\Big(\big|h(X_{n+q})\big|^2\Big) - \sum_{n=1}^N \mathbb{E}_x\Big(\big|h(X_n)\big|^2\Big) \leq 2q\big|h\big|_\infty^2.$$

Hence, $\sum_{n=1}^{+\infty} u_{n,q}(x) < +\infty$ and

$$\sum_{n=1}^{+\infty} \mathbb{E}_x\Big(\int_{\mathbb{L}\mathrm{ip}(E,E)^q} \big|h(T_q \cdots T_1 \cdot X_n) - h(X_n)\big|^2 \mu_{X_n}(\mathrm{d}T_1) \cdots \mu_{T_{q-1}\cdots T_1 \cdot X_n}(\mathrm{d}T_q)\Big) < +\infty. \tag{13}$$

Consequently, using H3,

$$\sum_{n=1}^{+\infty} \mathbb{E}_x\Big(\int_{\mathbb{L}\mathrm{ip}(E,E)^q} \big|h(T_q \cdots T_1 \cdot X_n) - h(X_n)\big|^2 \mu(\mathrm{d}T_1) \cdots \mu(\mathrm{d}T_q)\Big) < +\infty$$

and

$$\int_{\mathbb{L}\mathrm{ip}(E,E)^q} \mathbb{E}_x\Big(\sum_{n=1}^{+\infty} \big|h(T_q\cdots T_1\cdot X_n)-h(X_n)\big|^2\Big)\mu(\mathrm{d}T_1)\cdots\mu(\mathrm{d}T_q) < +\infty.$$

For any $q \geq 1$ and $\mu^{\otimes q}$-almost all $T_1, T_2, \cdots, T_q$, the sequence $(h(T_q\cdots T_1\cdot X_n)-h(X_n))_{n\geq 1}$ converges $\mathbb{P}_x$-a.s. to 0. We conclude by a density argument.

□

Similarly, one may prove the following lemma, which is of interest to control the other modulus 1 eigenvalues of P in $\mathcal{H}_\alpha(E)$.

Lemma 2.5 *Let $\phi \in \mathcal{H}_\alpha(E)$ such that $P\phi = \lambda\phi$ where λ is a complex number of modulus 1. For any $x \in E$, the sequence $(\lambda^{-n}\phi(X_n))_{n\geq 0}$ is a bounded martingale; it converges $\mathbb{P}_x$-a.s and in $\mathbb{L}^1(\Omega,\mathbb{P}_x)$ to a random variable Φ^∞ and we have*

$$\forall n \geq 0 \quad \phi(x) = \mathbb{E}_x(\lambda^{-n}\phi(X_n)) = \mathbb{E}_x(\Phi^\infty) \tag{14}$$

Furthermore, for any $q \geq 1$ and any transformations $T_1, \cdots, T_q$ on the support S_μ of μ, one has

$$\Phi^\infty = \lim_{n\to+\infty} \lambda^{-(n+q)}\phi(T_q\cdots T_1\cdot X_n) \quad \mathbb{P}_x - \text{a.s.} \tag{15}$$

Let us first prove that the P-harmonic functions in $\mathcal{H}_\alpha(E)$ are constant. Let $h \in \mathcal{H}_\alpha(E)$ such that $Ph = h$. According to Lemma 2.4, for any $x \in E$, there exists a set $\Omega_x \subset \Omega$ of full measure with respect to $\mathbb{P}_x$ such that, for any $\omega \in \Omega_x$ and any transformation $\xi \in T_\mu$, the sequences $(h(X_n(\omega)))_{n\geq 0}$ and $(h(\xi\cdot X_n(\omega)))_{n\geq 0}$ converge to $H^\infty(\omega)$.

Let $(\xi_k)_{k\geq 0}$ be a contracting sequence in T_μ, with limit point $x_0 \in E$. Since h is continuous on E, for any $\omega \in \Omega_x$, any cluster value x_ω of $(X_n(\omega))_{n\geq 0}$ and any $k \geq 0$,

$$H^\infty(\omega) = h(x_\omega) = h(\xi_k(x_\omega)).$$

Letting $k \to +\infty$, it yields $H^\infty(\omega) = h(x_0)$ and thus $h(x) = h(x_0)$, by (11). Finally, the bounded P-harmonic functions in $\mathcal{H}_\alpha(E)$ are constant.

Using Lemma 2.5, we prove that the peripheral spectrum of P is reduced to 1. Let $(n_l)_{l\geq 0}$ be a fixed sequence of integers such that $\lim_{l\to+\infty} \lambda^{-n_l} = 1$ and $(\xi_k)_{k\geq 0}$ be a contracting sequence on T_μ, with limit point x_0.

For all integer $q \geq 1$, the set $\{T = T_q\cdots T_1 \ | \ T_1\ldots, T_q \in S_\mu\}$ is dense in T_μ. Without loss of generality, we assume that any function ξ_k can be decomposed as a product $T_{q_k}\cdots T_1$, with $T_i \in S_\mu, 1 \leq i \leq q_k$. By Lemma 2.5, there exists $\Omega_x \subset \Omega$, $\mathbb{P}_x(\Omega_x) = 1$, such that, for any $\omega \in \Omega_x$ and $k \geq 0$, the sequences $\Big(\lambda^{-n_l}\phi(X_{n_l}(\omega)\Big)_{l\geq 0}$ and $\Big(\lambda^{-(n_l+q_k)}\phi(\xi_k\cdot X_{n_l}(\omega))\Big))_{l\geq 0}$ converge to the same limit $\Phi^\infty(\omega)$. Let us choose sequences of integers $(\varphi(l))_{l\geq 0}$ (depending on ω) and $(\psi(k))_{k\geq 0}$ (which does not depend on ω) such that $(\lambda^{-n_{\varphi(l)}}X_{n_{\varphi(l)}}(\omega))_{l\geq 0}$ and $(\lambda^{-q_{\psi(k)}})_{k\geq 0}$ converge resp. to $x_\omega \in E$ and $e^{i\beta}, \beta \in \mathbb{R}$. Equalities (14) and (15) yield

$$\Phi^\infty(\omega) = \phi(x_\omega) = e^{i\beta}\phi(x_0) \quad \text{and} \quad \phi(x) = \mathbb{E}_x(\Phi^\infty) = e^{i\beta}\phi(x_0).$$

Eventually, the function ϕ is constant on E and $\lambda = 1$.

□

3 The Diaconis-Freedman's chain

This section deals with the Diaconis-Freedman's chain $(Z_n)_{n\geq 0}$ on $E = [0,1]$ described in the introduction; we assume that the weights p and q vary with $x \in [0,1]$. The transition operator Q of $(Z_n)_{n\geq 0}$ is given by: for any bounded Borel function $\varphi : [0,1] \to \mathbb{C}$,

$$Q\varphi(x) = p(x)\int_0^1 \varphi(tx)\mathrm{d}t + q(x)\int_0^1 \varphi(tx+1-t)\mathrm{d}t.$$

For $x \in [0,1]$, let μ_x be the probability measure on the space $\mathbb{L}\mathrm{ip}([0,1],[0,1])$ of Lipschitz continuous functions from $[0,1]$ into $[0,1]$, defined by

$$\mu_x(\mathrm{d}T) = p(x)\int_0^1 \delta_{H_t}(\mathrm{d}T)\mathrm{d}t + q(x)\int_0^1 \delta_{A_t}(\mathrm{d}T)\mathrm{d}t. \tag{16}$$

Then the transition operator Q may be rewritten as

$$Q\varphi(x) = \int_{\mathbb{L}\mathrm{ip}([0,1],[0,1])} \varphi(T(x))\mu_x(\mathrm{d}T) = p(x)\int_0^1 \varphi(tx)\mathrm{d}t + q(x)\int_0^1 \varphi(tx+1-t)\mathrm{d}t.$$

Let us first consider explicit examples.

1. When $p(x) = x$, the chain $(Z_n)_{n\geq 0}$ is a sequence of independent random variables of uniform distribution on $[0,1]$; thus, its unique invariant measure is the uniform distribution on $[0,1]$.
2. When $p(x) = 1-x$, the points 0 and 1 are absorbing points for $(Z_n)_{n\geq 0}$. Hence, the Dirac measures at 0 and 1 are Q-invariant. The following theorem states that these two measures are the only ergodic probability measures on $[0,1]$ and that $(Z_n)_{n\geq 0}$ converges $\mathbb{P}_x$-a.s. to a random variable Z_∞ with values in $\{0,1\}$.
3. Assume that $p \in \mathcal{H}_\alpha[0,1]$ satisfies
$$\forall x \in [0,1], \quad p(x) > 0 \tag{17}$$
(or in a symmetric way, $p(x) < 1$ for any $x \in [0,1]$). In this case, the chain $(Z_n)_{n\geq 0}$ admits a unique Q-invariant probability measure on $[0,1]$. This is a direct consequence of Theorem 2.3. Indeed, hypotheses **H1**, **H2** and **H3** of Theorem 2.3 hold:
   (a) Hypothesis **H1**. For any $x, y \in [0,1], x \neq y$,
   $$\sup_{\substack{x,y\in[0,1]\\ x\neq y}} \int_{\mathbb{L}\mathrm{ip}([0,1],[0,1])} \left(\frac{|T(x)-T(y)|}{|x-y|}\right)^\alpha \mu_x(\mathrm{d}T) \leq \frac{1}{1+\alpha}.$$
   (b) Hypothesis **H2**. For any $x, y \in [0,1], x \neq y$,
   $$\frac{|\mu_x - \mu_y|}{|x-y|^\alpha} \leq \frac{|p(x)-p(y)|}{|x-y|^\alpha}\Big|\int_0^1 \delta_{H_t}\mathrm{d}t\Big| + \frac{|q(x)-q(y)|}{|x-y|^\alpha}\Big|\int_0^1 \delta_{A_t}\mathrm{d}t\Big| \leq 2m_\alpha(p).$$
   (c) Hypothesis **H3**. For any $x \in [0,1]$ it holds $\mu_x \geq \delta\mu$ with $\delta := \inf_{x\in[0,1]} p(x) > 0$ and $\mu = \int_0^1 \delta_{H_t}\mathrm{d}t$. The constant function $H_0 : x \mapsto 0$ belongs to the support of μ; hence, the semi-group T_μ contains a contracting sequence, with limit point 0.

   If $p(0) = 1$, the Dirac mass at 0 is the unique invariant probability measure for $(Z_n)_{n\geq 0}$. When $p(0) < 1$, one can prove that the unique invariant probability measure for $(Z_n)_{n\geq 0}$ is absolutely continuous with respect to the Lebesgue measure (see Theorem 3.1 below).

If p and q are both strictly positive on $[0,1]$, by using the approach developed in [4], we may prove that the unique invariant probability measure for $(Z_n)_{n\geq 0}$ is absolutely continuous with respect to the Lebesgue measure. This property holds as soon as $p(0) < 1$ and $q(1) < 1$. Let us emphasize that the strict positivity of p or q is sufficient to ensure the unicity of an invariant probability measure but it is a too strong condition. These remarks lead to the following statement, which is not a direct consequence of Theorem 2.3 but whose proof is strongly inspired.

Theorem 3.1 *Let $(Z_n)_{n\geq 0}$ be the Diaconis-Freedman's chain on $[0,1]$ with weight functions p and q in $\mathcal{H}_\alpha[0,1]$. Then, one of the 3 following options holds.*

1. *If $p(0) < 1$ and $q(1) < 1$, then there exists on $[0,1]$ an unique Q-invariant probability measure ν_p. Furthermore, this measure is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ with density f_p given by:*

$$\forall x \in [0,1] \quad f_p(x) = C \exp\Big(\int_x^{\frac{1}{2}} \frac{p(y)}{y}\mathrm{d}y + \int_{\frac{1}{2}}^x \frac{q(y)}{1-y}\mathrm{d}y\Big)$$

where C is a normalization constant. At last, there exist constants $\kappa > 0$ and $\rho \in [0,1[$ such that

$$\forall \varphi \in \mathcal{H}_\alpha[0,1],\ \forall x \in [0,1] \quad |Q^n\varphi(x) - \nu_p(\varphi)| \leq \kappa\rho^n \|\varphi\|_\alpha.$$

2. *If $p(0) = 1$ and $q(1) < 1$, then the Dirac measure δ_0 is the unique Q-invariant probability measure on $[0,1]$. Furthermore, there exist constants $\kappa > 0$ and $\rho \in [0,1[$ such that*

$$\forall \varphi \in \mathcal{H}_\alpha[0,1],\ \forall x \in [0,1] \quad |Q^n\varphi(x) - \varphi(0)| \leq \kappa\rho^n \|\varphi\|_\alpha.$$

(A similar statement holds when $p(0) < 1$ and $q(1) = 1$).

3. *If $p(0) = 1$ and $q(1) = 1$, then the invariant probability measures of $(Z_n)_{n\geq 0}$ are the convex combinations of δ_0 and δ_1. Furthermore, for any $x \in [0,1]$, the chain $(Z_n)_{n\geq 0}$ converges $\mathbb{P}_x$-a.s. to a random variable Z_∞ with values in $\{0,1\}$; the law of Z_∞ is given by*

$$\mathbb{P}_x(Z_\infty = 0) = 1 - h(x) \quad \text{and} \quad \mathbb{P}_x(Z_\infty = 1) = h(x),$$

where h is the unique function in $\mathcal{H}_\alpha[0,1]$ such that $Qh = h$ and $h(0) = 0,\ h(1) = 1$. At last, there exist $\kappa > 0$ and $\rho \in [0,1[$ such that

$$\forall \varphi \in \mathcal{H}_\alpha[0,1],\ \forall x \in [0,1] \quad |Q^n\varphi(x) - (1-h(x))\varphi(0) - h(x)\varphi(1)| \leq \kappa\rho^n \|\varphi\|_\alpha.$$

Proof. First, let us consider the adjoint operator Q^* of Q in $\mathbb{L}^2[0,1]$, defined by: for any $\varphi, \psi \in \mathbb{L}^2[0,1]$,

$$\int_0^1 \varphi(x)Q\psi(x)\mathrm{d}x = \int_0^1 Q^*\varphi(x)\psi(x)\mathrm{d}x.$$

A straightforward computation yields to the following expression:

$$\forall \varphi \in \mathbb{L}^2[0,1], \forall x \in [0,1] \qquad Q^*\varphi(x) := \int_0^x \frac{q(t)}{1-t}\varphi(t)\mathrm{d}t + \int_x^1 \frac{p(t)}{t}\varphi(t)\mathrm{d}t. \tag{18}$$

Notice that (18) is valid for any Borel function $\varphi \in \mathbb{L}^1[0,1]$. Furthermore, if $\varphi \in \mathbb{L}^1[0,1]$ is non negative and satisfies the equality $Q^*\varphi = \varphi$, then the measure with density φ with respect to the Lebesgue measure on $[0,1]$ is Q-invariant.

Assume for a while that φ is differentiable on $]0,1[$; the equation $Q^*\varphi = \varphi$ yields

$$\forall x \in]0,1[\quad \varphi'(x) = \Big(\frac{q(x)}{1-x} - \frac{p(x)}{x}\Big)\varphi(x),$$

hence $\varphi(x) = \exp\Big(\int_x^{1/2} \frac{p(t)}{t}\mathrm{d}t + \int_{1/2}^x \frac{q(t)}{1-t}\mathrm{d}t\Big)$ up to a multiplicative constant. This function φ is integrable with respect to the Lebesgue measure on $[0,1]$ if and only if $p(0) < 1$ and $q(1) < 1$; in this case, we set

$$f_p : x \mapsto \frac{1}{C_p}\exp\Big(\int_x^{1/2} \frac{p(t)}{t}\mathrm{d}t + \int_{1/2}^x \frac{q(t)}{1-t}\mathrm{d}t\Big) \tag{19}$$

with $C_p := \int_0^1 \exp\Big(\int_x^{1/2} \frac{p(t)}{t}\mathrm{d}t + \int_{1/2}^x \frac{q(t)}{1-t}\mathrm{d}t\Big)\mathrm{d}x$. The probability measure ν_p on [0, 1] with density f_p with respect to the Lebesgue measure on $[0,1]$ is Q-invariant.

Now, we come back to the proof of Theorem 3.1 and decompose the argument into 3 steps.

Step1- Quasi-compacity of the operator Q on $\mathcal{H}_\alpha[0,1]$

The operator Q is non negative, bounded on $\mathcal{H}_\alpha[0,1]$ with spectral radius 1. Furthermore,

$$\forall \varphi \in \mathcal{H}_\alpha[0,1], \qquad \|Q\varphi\|_\alpha \leq \frac{1}{\alpha+1}\|\varphi\|_\alpha + (1+2m_\alpha(p))|\varphi|_\infty.$$

Hence, by [9], the operator Q is quasi-compact on $\mathcal{H}_\alpha[0,1]$.

Step 2- Description of the characteristic space of Q corresponding to $\lambda = 1$

We use here a general result of [11], based on the notion of *absorbing compact set.* A compact subset K of $[0,1]$ is said to be Q-*absorbing* if $Q1_{[0,1]\setminus K}(x) = 0$ for any $x \in K$. It is *minimal* when it does not contain any proper absorbing compact subset. The condition $p(x) > 0$ ensures that

$$Q(x, I) > 0 \quad \text{for any closed interval } I \subset [0, x] \text{ not reduced to a single point.} \tag{20}$$

Similarly, the condition $q(x) > 0$ implies

$$Q(x, I) > 0 \quad \text{for any closed interval } I \subset [x, 1] \text{ not reduced to a single point.} \tag{21}$$

There are four cases to explore.

1. $q(0) > 0$ *and* $p(1) > 0$

   In this case, the interval $[0,1]$ is the unique (and thus minimal) Q-absorbing compact set. To prove this, we fix a compact and proper subset K of $[0,1]$; we have to find a point $x_0 \in K$ such that $Q(x_0, [0,1] \setminus K) > 0$. There are 3 sub-cases to consider.

   (a) $0 \notin K$

   Assume that $q(x) = 1$ for any $x \in K$. The condition $p(1) > 0$ implies $q(1) < 1$, so that $1 \notin K$; thus, there exist $\epsilon > 0$ such that $K \subset [0, 1-\epsilon]$. Consequently, for any $x \in K$,

   $$Q(x, [0,1] \setminus K) \geq Q(x,]1-\epsilon, 1]) > 0$$

   which means that K is not absorbing. Contradiction.

   Consequently, there exists $x_0 \in K$ such that $p(x_0) > 0$; if $\epsilon > 0$ is such that $K \subset [\epsilon, 1]$, then,

   $$Q(x_0, [0,1] \setminus K) \geq Q(x_0, [0, \epsilon[) > 0.$$

   (b) $1 \notin K$

   The same argument holds, exchanging the role of 0 and 1.

   (c) $0 \in K$ and $1 \in K$

   In this case, we can set $x_0 = 0$. Indeed, let us fix $x' \in]0,1[\setminus K$ and $\epsilon' > 0$ such that $]x'-\epsilon', x'+\epsilon'[\subset [0,1] \setminus K$ and notice that

   $$Q(0, [0,1] \setminus K) \geq Q(0,]x'-\epsilon', x'+\epsilon'[) > 0.$$

2. $q(0) = 0$ and $p(1) > 0$ In this case, the set $\{0\}$ is invariant and is the unique Q-absorbing minimal compact set. Indeed, there exists $x \in K$ such that $p(x) > 0$. Otherwise, the function q equals 1 on K; by (21), it follows that $[y, 1] \subset K$ for any $y \in K$. Consequently $1 \in K$ and $q(1) = 1$, which contradicts the condition $p(1) > 0$. Applying (20), it yields $[0, x] \subset K$ and in particular $\{0\} \subset K$.

3. $q(0) > 0$ and $p(1) = 0$

   In this case, the unique Q-absorbing minimal compact set is $\{1\}$. The proof is similar to the previous case, exchanging the role of 0 and 1.

4. <u>$q(0)=0$ and $p(1)=0$</u>

   The sets $\{0\}$ and $\{1\}$ are the only minimal absorbing compact sets.

We apply Theorem 2.2 in [11] to conclude that the eigenvalue 1 has index 1 in $\mathcal{H}_\alpha[0,1]$: in other words, the characteristic subspace of Q associated to 1 equals $\mathrm{Ker}(Q-Id)$. Therefore, we may apply Theorem 2.3 in [11] to each of the four cases explored above.

1. If $q(0)>0$ and $p(1)>0$, then $\mathrm{Ker}(Q-Id)=\mathbb{C}\cdot\mathbf{1}$; in this case, the unique Q-invariant probability measure on $[0,1]$ is absolutely continuous with respect to the Lebesgue measure on $[0,1]$, with density f_p.
2. If $q(0)=0$ and $p(1)>0$, then $\mathrm{Ker}(Q-Id)=\mathbb{C}\cdot\mathbf{1}$ and the Dirac mass δ_0 is the unique Q-invariant probability measure on $[0,1]$.
3. If $q(0)>0$ and $p(1)=0$, then $\mathrm{Ker}(Q-Id)=\mathbb{C}\cdot\mathbf{1}$ and the Dirac mass δ_1 is the unique Q-invariant probability measure on $[0,1]$.
4. If $q(0)=0$ and $p(1)=0$, there exists a positive harmonic function h such that $h(0)=0$ and $h(1)=1$; the space $\mathrm{Ker}(Q-Id)$ has dimension 2 and equals $\mathbb{C}\cdot\mathbf{1}\oplus\mathbb{C}\cdot h$. The Q-invariant probability measure on $[0,1]$ are the convex combinations of δ_0 and δ_1.

<u>*Step 3- Control of the peripheral spectrum of Q in $\mathcal{H}_\alpha[0,1]$*</u>

We use here Lemma 2.5 and apply the same technics as in the previous discussion.

Let $\lambda\in\mathbb{C}$ with modulus 1 and $\phi\in\mathcal{H}_\alpha[0,1]$ such that $Q\phi=\lambda\phi$. For any $x\in[0,1]$, the sequence $(\lambda^{-n}\phi(X_n))_{n\geq 0}$ is a bounded martingale in

$(\Omega,\mathcal{F},\mathbb{P}_x)$, thus it converges $\mathbb{P}_x$-a.s to a bounded random variable Φ^∞. We use inequality (13) first with $q=1$ and then $q=2$; there exists $\Omega_x\subset\Omega,\mathbb{P}_x(\Omega_x)=1$, and $I_0\subset[0,1]$ of Lebesgue measure 1 such that, for any $\omega\in\Omega_x$ and any $s,t\in I_0$, it holds

$$\lim_{n\to+\infty}\left|\phi(Z_n(\omega))-\lambda^{-1}\phi(H_s\cdot Z_n(\omega))\right|^2p(Z_n(\omega))=0, \tag{22}$$

$$\lim_{n\to+\infty}\left|\phi(Z_n(\omega))-\lambda^{-1}\phi(A_s\cdot Z_n(\omega))\right|^2q(Z_n(\omega))=0, \tag{23}$$

$$\lim_{n\to+\infty}\left|\phi(Z_n(\omega))-\lambda^{-2}\phi(H_tH_s\cdot Z_n(\omega))\right|^2p(Z_n(\omega))p(H_s\cdot Z_n(\omega))=0, \tag{24}$$

and

$$\lim_{n\to+\infty}\left|\phi(Z_n(\omega))-\lambda^{-2}\phi(A_tA_s\cdot Z_n(\omega))\right|^2q(Z_n(\omega))q(A_s\cdot Z_n(\omega))=0. \tag{25}$$

There are two cases to explore.

1. <u>$\phi(0)=\phi(1)$</u>

   Fix $\omega\in\Omega_x$ and a cluster value z_ω of the sequence $(Z_n(\omega))_{n\geq 0}$.

   If $p(z_\omega)\neq 0$, then, applying (22) with s arbitrarily close to 0, it yields

   $$\phi(z_\omega)=\lambda^{-1}\phi(0).$$

   If $p(z_\omega)=0$, we conclude similarly with (23) that $\phi(z_\omega)=\lambda^{-1}\phi(1)$.

   Consequently, since $\phi(0)=\phi(1)$, the sequence $(\phi(Z_n(\omega))_{n\geq 0}$ converges to $\Phi^\infty(\omega)=\lambda^{-1}\phi(0)$ and $\phi(x)=\mathbb{E}_x(\Phi^\infty)=\lambda^{-1}\phi(0)$. Thus, the function ϕ is constant and $\lambda=1$.

2. <u>$\phi(0)\neq\phi(1)$</u>

   Without loss of generality, we assume $\phi(0)\neq 0$; the case $\phi(1)\neq 0$ is treated the same way.

(a) First, assume that there exists $x \in [0,1]$ and $\omega_x \in \Omega_x$ such that the sequence $(Z_n(\omega_x))_{n\geq 0}$ possesses a cluster point z_{ω_x} with $p(z_{\omega_x}) > 0$. Applying first (22) with s arbitrarily close to 0 and second (24) with s arbitrarily close to 1 (so that $p(H_s \cdot z_{\omega_x}) > 0$) and t arbitrarily close to 0, it yields

$$\phi(z_{\omega_x}) = \lambda^{-1}\phi(0) = \lambda^{-2}\phi(0).$$

The condition $\phi(0) \neq 0$ readily implies $\lambda = 1$ and thus $\phi \in \mathbb{C} \cdot \mathbf{1}$.

(b) Assume that $q(z_\omega) = 1$ for any $x \in [0,1]$, any $\omega \in \Omega_x$ and any cluster values z_ω of the sequence $(Z_n(\omega))_{n\geq 0}$.

Applying first (23) with s arbitrarily close to 0 and second (25) with s arbitrarily close to 1 (so that $q(A_s \cdot z_\omega) > 0$) and t arbitrarily close to 0, it yields

$$\phi(z_\omega) = \lambda^{-1}\phi(1) = \lambda^{-2}\phi(1).$$

If $\phi(1) \neq 0$, we deduce as above that $\lambda = 1$ and $\phi \in \mathbb{C} \cdot \mathbf{1}$. If $\phi(1) = 0$, the sequence $(\lambda^{-n}\phi(Z_n(\omega))_{n\geq 0}$ converges to 0 and the martingale equality $\phi(x) = \mathbb{E}_x(\lambda^{-n_l}\phi(Z_{n_l}))$ yields $\phi \equiv 0$.

Eventually, the operator Q is quasi-compact on $\mathcal{H}_\alpha[0,1]$ with spectral radius equals 1, its peripheral spectrum is reduced to $\{1\}$ and the characteristic subspace associated to 1 equals $\mathrm{Ker}(Q - Id)$. More precisely, we have the 4 following cases.

1. <u>If $q(0) > 0$ and $p(1) > 0$</u>, there exists a bounded linear operator R on $\mathcal{H}_\alpha[0,1]$ with spectral radius $\rho \in [0,1[$ such that, for any $\varphi \in \mathcal{H}_\alpha[0,1]$ and $n \geq 0$,

$$Q^n\varphi = \Big(\int_0^1 \varphi(x) f_p(x)\mathrm{d}x\Big)\mathbf{1} + R^n\varphi.$$

In this case, the chain $(Z_n)_{n\geq 0}$ is recurrent on $[0,1]$.

2. <u>If $q(0) = 0$ and $p(1) > 0$</u>, there exists a bounded operator R on $\mathcal{H}_\alpha(E)$ with spectral radius $\rho \in [0,1[$ such that, for any $\varphi \in \mathcal{H}_\alpha[0,1]$ and $n \geq 0$,

$$Q^n\varphi = \varphi(0)\mathbf{1} + R^n\varphi. \tag{26}$$

In this case, for any $x \in [0,1]$, the chain $(Z_n)_{n\geq 0}$ converges $\mathbb{P}_x$-a.s. to 0; furthermore, for any $\epsilon \in]0,1[$, the set $[\epsilon, 1]$ is transient and there exists $\kappa_\epsilon > 0$ such that

$$\mathbb{P}_x(Z_n \in [\epsilon, 1]) \leq \kappa_\epsilon \rho^n.$$

3. <u>If $q(0) > 0$ and $p(1) = 0$</u>, there exists a bounded operator R on $\mathcal{H}_\alpha[0,1]$ with spectral radius $\rho \in [0,1[$ such that, for any $\varphi \in \mathcal{H}_\alpha[0,1]$ and $n \geq 0$,

$$Q^n\varphi = \varphi(1)\mathbf{1} + R^n\varphi.$$

For any $x \in [0,1]$, the chain $(Z_n)_{n\geq 0}$ converges $\mathbb{P}_x$-a.s. to 1; furthermore, for any $\epsilon \in]0,1[$, the set $[0, 1-\epsilon]$ is transient and there exists $\kappa_\epsilon > 0$ such that

$$\mathbb{P}_x(Z_n \in [0, 1-\epsilon]) \leq \kappa_\epsilon \rho^n.$$

4. <u>If $q(0) = 0$ and $p(1) = 0$</u>, there exists an harmonic function $h : [0,1] \to [0,1]$ such that $h(0) = 0$ and $h(1) = 1$ and a bounded operator R on $\mathcal{H}_\alpha[0,1]$ with spectral radius $\rho \in [0,1[$ such that, for any $\varphi \in \mathcal{H}_\alpha[0,1]$ and $n \geq 0$,

$$Q^n\varphi = \varphi(0)(\mathbf{1} - h) + \varphi(1)h + R^n\varphi.$$

For any $x \in [0,1]$, the chain $(Z_n)_{n\geq 0}$ converges to 0 with probability $1 - h(x)$ and to 1 with probability $h(x)$. Indeed, the bounded martingale $(h(Z_n))_{n\geq 0}$ converges $\mathbb{P}$-a.s. Since $h(0) \neq h(1)$, it follows that $(Z_n)_{n\geq 0}$ converges to a random variable Z_∞ with values in $\{0,1\}$. The martingale property yields $h(x) = \mathbb{E}_x(h(Z_\infty)) = \mathbb{P}_x(Z_\infty = 1)$. Consequently, for any $\epsilon \in]0,1[$, the set $[0, 1-\epsilon]$ is transient and there exists $\kappa_\epsilon > 0$ such that

$$\mathbb{P}_x(Z_n \in [0, 1-\epsilon]) \leq \kappa_\epsilon \rho^n.$$

Example : $p(x) = 1 - x$**.**
We are in the Case 4 above and the harmonic function $h(x) = x$. In particular, the sets $[\epsilon, 1-\epsilon], 0 < \epsilon < 1$, are transient.
This transience property can be obtained in a different way which is also of interest and we present briefly. Let us introduce the quantity Δ defined by: for any x in $[0, 1]$,

$$\Delta(x) := \text{dist}(x, \{0, 1\}) = \inf(x, 1 - x).$$

Let us compute $\mathbb{E}_x(\Delta(Z_1))$. We assume $x \in]0, \frac{1}{2}]$, the case $x \in [\frac{1}{2}, 1[$ can be treated in a similar way.

$$\begin{aligned}\mathbb{E}_x(\Delta(Z_1)) &= \frac{1-x}{x}\int_0^x y \, \mathrm{d}y + \frac{x}{1-x}\int_x^{\frac{1}{2}} y \, \mathrm{d}y + \frac{x}{1-x}\int_{\frac{1}{2}}^1 (1-y)\mathrm{d}y \\ &= \frac{3x - 4x^2}{4(1-x)} \\ &\leq \frac{3}{4}x.\end{aligned}$$

Hence $\mathbb{E}(\Delta(Z_n)|\mathcal{F}_{n-1}) \leq \frac{3}{4}\Delta(Z_{n-1})$ for any $n \geq 1$ and, iterating,

$$\forall n \geq 1, \forall x \in [0, 1] \qquad \mathbb{E}_x(\Delta(Z_n)) \leq \left(\frac{3}{4}\right)^n \Delta(x) \leq \left(\frac{3}{4}\right)^n.$$

Consequently $\mathbb{E}_x\Big(\sum_{n=0}^{+\infty} \Delta(Z_n)\Big) = \sum_{n=0}^{+\infty} \mathbb{E}_x\Big(\Delta(Z_n)\Big) < +\infty$, so that the sequence $(\Delta(Z_n))_{n\geq 1}$ converges $\mathbb{P}_x$-a.s. to 0.

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