THE SURVIVAL PROBABILITY OF A CRITICAL MULTI-TYPE BRANCHING PROCESS IN I.I.D. RANDOM ENVIRONMENT

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We study the asymptotic behavior of the probability of non-extinction of a critical multi-type Galton-Watson process in i.i.d. random environments by using limits theorems for products of positive random matrices. Under suitable assumptions, the survival probability is proportional to $1/\sqrt{n}$ [*].

1. Introduction.

Branching processes in random environments (BPREs) were introduced in the 1960s (see for instance [20]) to describe the development of populations whose evolution may be affected by environmental factors; they have been a central topic of research.

In the single-type case, the behaviour of these processes is mainly determined by the 1-dimensional random walk generated by the logarithms of the expected population sizes $m_k, k \geq 0$, of the respective generations. The theory of fluctuations of random walks on $\mathbb{R}$ with i.i.d. increments allows one to classify BPREs in three classes - supercritical, critical or subcritical - of single-type BPREs, according to the fact that the associated random walk tends to $+\infty$, oscillates or tends to $-\infty$ (see for instance the fundamental papers [2] and [3]). In particular, when $\mathbb{E}(|\log m_k|) < +\infty$, the BPRE is supercritical (resp. critical or subcritical) when $\mathbb{E}(\log m_k) > 0$ (resp. $\mathbb{E}(\log m_k) = 0$ or $\mathbb{E}(\log m_k) < 0$). There exist numerous statements concerning the asymptotic behaviour of the probability of non-extinction up to time n, the distribution of the population size conditionally to survival up to moment n, large deviation type results (see for instance [1], [4], [7]). In the critical case, the branching process is degenerate with probability one [2]; M. V. Kozlov [17] (for BPREs with linearly fractional generating functions) and J. Geiger and G. Kersting [10] in the general case strengthened

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this result and proved that the probability of non-extinction up to time n is equivalent to $c_1/\sqrt{n}$ as $n \to +\infty$, for some explicit constant $c_1 > 0$. Let us recall that this probability of non-extinction up to time n is equivalent to $1/n$ when the offspring distribution is fixed, i.e. it does not vary randomly; in other words, branching processes in random environment die more slowly. In the supercritical and subcritical cases, similar studies have been done (see for instance [2], [3] and [11]), we do not go into detail about these cases since they are outside the scope of this paper.

It is of interest to prove analogues of the above statements for the multi-type BPREs $(Z_n)_{n \geq 0}$. The main difficulty which appears is that the role of the random walk associated to the BPRE in this case is played by the logarithm of the norm of some $\mathbb{R}^p$-valued Markov chain whose increments are governed by i.i.d. random $p \times p$-matrices $M_k, k \geq 0$; the coefficients of these matrices M_k are non negative and correspond to the expected population sizes of the respective generations, according to the types of the particles and their direct parents. Products of random matrices have been the object of many investigations and many limit theorems do exist in this context: for instance, the law of large numbers, the central limit theorem, the large deviations principle (see [5] and references therein). Unfortunately, the theory of their fluctuations remains a field in which practically no research has been done and the multi-type BPREs constitute a relevant application area for this study. Nevertheless, as in the single-type case, the set of multi-type BPREs may be divided into three classes: they are supercritical (resp. critical or subcritical) when the upper Lyapunov exponent of the associated random matrices is positive, null or negative. [16].

A problem of particular importance is to specify the asymptotic behaviour of their probability of non-extinction up to time n. As for single-type BPREs, the case has been first studied when the generating functions defining the random environment are linear fractional; it yields to explicit formulas which are easier to tackle.

In [6], E. E. Dyakonova obtained in the critical case an equivalence of the survival probability at time n, under the restrictive assumption that the mean matrices M_k have a concordant deterministic Perron-Frobenius eigenvector. It happens for instance when the M_k are upper-triangular matrices with strictly decreasing coefficients on the diagonal. In this case, the behaviour of the BPRE is governed by the 1-dimensional random walk whose increments are the logarithm of the spectral radius of the M_k; this allows one to apply the same techniques as in the single-type case.

To tackle the "general" case when the action of the random matrices M_k is strongly irreducible (see hypothesis H2 below), limit theorems on the

fluctuations of the norm of products of random matrices were required; they have been recently achieved in [12] and [13] with asymptotic results on the tails of certain hitting time distribution.

In this paper, we investigate the asymptotic behaviour of the probability of non-extinction up to time n of critical multi-type BPREs and obtain an optimal result in the case of linear fractional generating functions. To formulate our main results, we first introduce some standard notations and definitions.

We fix an integer $p \geq 2$ and denote $\mathbb{R}^p$ (resp. $\mathbb{N}^p$) the set of p-dimensional column vectors with real (resp. non-negative integers) coordinates; for any column vector $x = (x_i)_{1 \leq i \leq p} \in \mathbb{R}^p$, we denote $\tilde{x}$ the row vector $\tilde{x} := (x_1, \ldots, x_p)$. Let $\mathbf{1}$ (resp. $\mathbf{0}$) be the column vector of $\mathbb{R}^p$ whose all coordinates equal 1 (resp. 0). We fix a basis $\{e_i, 1 \leq i \leq p\}$ in $\mathbb{R}^p$ and denote $|.|$ the corresponding L^1 norm. We also consider the general linear semi -group S^+ of $p \times p$ matrices with non-negative coefficients. We endow S^+ with the L^1-norm denoted also by $|.|$.

The multi-type Galton-Watson process is a temporally homogeneous vector Markov process $(Z_n)_{n \geq 0}$ whose states are column vectors in $\mathbb{N}^p$. We always assume that Z_0 is non-random. For any $1 \leq i \leq p$, the i-th component $Z_n(i)$ of Z_n may be interpreted as the number of objects of type i in the n-th generation.

We consider a measurable function $\xi \mapsto f_\xi$ from $\mathbb{R}$ to the set of multi-variate probability generating functions $f_\xi = (f_\xi^{(i)})_{1 \leq i \leq p}$ defined by:

$$f_\xi^{(i)}(s) = \sum_{\alpha \in \mathbb{N}^p} p_\xi^{(i)}(\alpha)s^\alpha,$$

for any $s = (s_i)_{1 \leq i \leq p} \in [0, 1]^p$, where

(i) $\alpha = (\alpha_i)_i \in \mathbb{N}^p$ and $s^\alpha = s_1^{\alpha_1} \ldots s_p^{\alpha_p}$;

(ii) $p_\xi^{(i)}(\alpha) = p_\xi^{(i)}(\alpha_1, \ldots, \alpha_p)$ is the probability that an object of type i in environment ξ has α_1 children of type $1, \ldots, \alpha_p$ children of type p.

Let $(\xi_n)_{n \geq 0}$ be a sequence of real valued i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Galton-Watson process with p types of particles in a random environment $(\xi_n)_{n \geq 0}$ describes the evolution of a particle population $Z_n = (Z_n(i))_{1 \leq i \leq p}$ for $n \geq 0$.

We assume that for any $n \geq 0, i = 1, \ldots, p$ and $\xi \in \mathbb{R}$, if $\xi_n = \xi$ then each of the $Z_n(i)$ particles of type i existing at time n produces offspring in accordance with the p-dimensional generating function $f_\xi^{(i)}(s)$ independently of the reproduction of other particles of all types.

If $Z_0 = \tilde{e}_i$ then Z_1 has the generating function:

$$f_{\xi_0}^{(i)}(s) = \sum_{\alpha \in \mathbb{N}^p}^{+\infty} p_{\xi_0}^{(i)}(\alpha) s^{\alpha}.$$

In general, if $Z_n = (\alpha_i)_{1 \leq i \leq p}$, then Z_{n+1} is the sum of $\alpha_1 + \ldots + \alpha_p$ independent random vectors, where α_i particles of type i have the generating function $f_{\xi_n}^{(i)}$ for $i = 1, \ldots, p$. It is obvious that if $Z_n = \tilde{\mathbf{0}}$, then $Z_{n+1} = \tilde{\mathbf{0}}$.

Denote $f_n = f_{\xi_n}$. By the above descriptions, (written in equation 2.1 in [16]) for any $s = (s_i)_{1 \leq i \leq p}, 0 \leq s_i \leq 1$

$$\mathbb{E}\left(s^{Z_n} | Z_0, \ldots, Z_{n-1}, f_0, \ldots, f_{n-1}\right) = f_{n-1}(s)^{Z_{n-1}}$$

which yields (lemma 2.1 in [16])

$$
\begin{aligned}
\mathbb{E}\left(s^{Z_n} | f_0^{(i)}, f_1, \ldots, f_{n-1}\right) &:= \mathbb{E}\left(s^{Z_n} | Z_0 = \tilde{e}_i, f_0, \ldots, f_{n-1}\right) \\
&= f_0^{(i)}(f_1(\ldots f_{n-1}(s) \ldots)).
\end{aligned}
$$

In particular, the probability of non-extinction $q_n^{(i)}$ at generation n given the environment $f_0^{(i)}, f_1, \ldots f_{n-1}$ is

$$
\begin{aligned}
q_n^{(i)} &:= \mathbb{P}(Z_n \neq \tilde{\mathbf{0}} | f_0^{(i)}, f_1, \ldots, f_{n-1}) \\
(1.1) \quad &= 1 - f_0^{(i)}(f_1(\ldots f_{n-1}(\tilde{\mathbf{0}}) \ldots)) = \tilde{e}_i(\mathbf{1} - f_0(f_1(\ldots f_{n-1}(\tilde{\mathbf{0}}) \ldots))),
\end{aligned}
$$

so that

$$\mathbb{E}[q_n^{(i)}] = \mathbb{E}[\mathbb{P}(Z_n \neq \tilde{\mathbf{0}} | f_0^{(i)}, f_1, \ldots, f_{n-1})] = \mathbb{P}(Z_n \neq \tilde{\mathbf{0}} | Z_0 = e_i).$$

As in the classical one-type case, the asymptotic behaviour of the quantity above is controlled by the mean of the offspring distributions. We assume that the offspring distributions have finite first and second moments; the generating functions $f_{\xi}^{(i)}, \xi \in \mathbb{R}, 1 \leq i \leq p$, are thus C^2-functions on $[0,1]^p$ and we introduce

(i) the random mean matrices $M_{\xi_n} = (M_{\xi_n}(i,j))_{1 \leq i,j \leq p} = \left(\dfrac{\partial f_{\xi_n}^{(i)}(\mathbf{1})}{\partial s_j}\right)_{i,j}$

   taken from the vector-valued random generating function $f_{\xi_n}(s)$ at $s = \mathbf{1}$, namely

$$
M_{\xi_n} = \begin{pmatrix} \dfrac{\partial f_{\xi_n}^{(1)}(\mathbf{1})}{\partial s_1} & \cdots & \dfrac{\partial f_{\xi_n}^{(1)}(\mathbf{1})}{\partial s_p} \\ \vdots & & \\ \dfrac{\partial f_{\xi_n}^{(p)}(\mathbf{1})}{\partial s_1} & \cdots & \dfrac{\partial f_{\xi_n}^{(p)}(\mathbf{1})}{\partial s_p} \end{pmatrix}.
$$

(ii) the random Hessian matrices $B_{\xi_n}^{(i)} = (B_{\xi_n}^{(i)}(k,l))_{1 \le k,l \le p} = \left(\frac{\partial^2 f_{\xi_n}^{(i)}}{\partial s_k \partial s_l}(\mathbf{1}) \right)_{k,l}$, $1 \le i \le p$, taken from the real-valued random generating function $f_{\xi_n}^{(i)}(s)$ at $s = \mathbf{1}$.

The random variables M_{ξ_n} and $B_{\xi_n}^{(i)}$ are i.i.d.. The common law of the M_{ξ_n} is denoted by μ and for the sake of brevity, we write M_n instead of M_{ξ_n}.

Let R_n and L_n denote the right and the left product of random matrices $M_k, k \ge 0$, respectively $R_n = M_0 M_1 \ldots M_{n-1}$ and $L_n = M_{n-1} \ldots M_1 M_0$.

By [9], if $\mathbb{E}(\max(0, \log |M_0|)) < +\infty$, then the sequence $\left(\frac{1}{n} \log |R_n| \right)_{n \ge 1}$ converges $\mathbb{P}$-almost surely to some constant limit $\pi_\mu := \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}[\log |R_n|]$.

Furthermore, by [16], if there exists a constant $A > 0$ such that $\frac{1}{A} \le M_{\xi_n}(i,j) \le A$ and $0 \le B_{\xi_n}^{(i)}(k,l) \le A$ $\mathbb{P}$-almost surely for any $1 \le i, j, k, l \le p$, then the process $(Z_n)_{n \ge 0}$ becomes extinct $\mathbb{P}$-almost surely if and only if $\pi_\mu \le 0$.

In the present work, we focus our attention on the so-called **critical case**, that is $\pi_\mu = 0$, and specify the speed of extinction of the Galton-Watson process.

We define the cone $\mathcal{C}$ and the space $\mathbb{X}$ respectively as follows:

$$\mathcal{C} := \{ \tilde{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p \mid \forall i = 1, \ldots, p, \ x_i \ge 0 \},$$

and

$$\mathbb{X} := \{ \tilde{x} \in \mathcal{C} \mid |\tilde{x}| = 1 \}.$$

In the sequel, we consider

- the right and the left linear actions of S^+ on $\mathcal{C}$ defined by:

$$(\tilde{x}, g) \mapsto \tilde{x}g \quad \text{and} \quad (\tilde{x}, g) \mapsto gx$$

for any $\tilde{x} \in \mathcal{C}$ and $g \in S^+$,
- the right and the left projective actions of S^+ on $\mathbb{X}$ defined by:

$$(\tilde{x}, g) \mapsto \tilde{x} \cdot g = \frac{\tilde{x}g}{|\tilde{x}g|} \quad \text{and} \quad (\tilde{x}, g) \mapsto g \cdot x = \frac{gx}{|gx|}$$

for any $\tilde{x} \in \mathbb{X}$ and $g \in S^+$.

For any $g = (g(i,j))_{1 \le i,j \le d} \in S^+$, let $v(g) := \min\limits_{1 \le j \le d} \left(\sum\limits_{i=1}^{d} g(i,j) \right)$. Then, for any $x \in \mathcal{C}$,

$$(1.2) \qquad 0 < v(g) \, |x| \le |gx| \le |g| \, |x|.$$

We set $N(g) := \max \left(\frac{1}{v(g)}, |g| \right)$. We also introduce some proper subset of S^+ which is of interest in the sequel: for any constant $B \ge 1$, let $S^+(B)$ denote the set of $p \times p$ matrices $g = (g(i,j))_{1 \le i,j \le p}$ with positive coefficients such that for any $1 \le i, j, k, l \le p$,

$$(1.3) \qquad \frac{1}{B} \le \frac{g(i,j)}{g(k,l)} \le B.$$

On the product space $\mathbb{X} \times S^+$ we define the function ρ by setting $\rho(\tilde{x}, g) := \log |\tilde{x}g|$ for $(\tilde{x}, g) \in \mathbb{X} \times S^+$. This function satisfies the cocycle property, namely for any $g, h \in S^+$ and $\tilde{x} \in \mathbb{X}$,

$$(1.4) \qquad \rho(\tilde{x}, gh) = \rho(\widetilde{x} \cdot g, h) + \rho(\tilde{x}, g).$$

Under hypothesis H3 introduced below, there exists a unique μ-invariant measure ν on $\mathbb{X}$ such that for any continuous function φ on $\mathbb{X}$,

$$(\mu * \nu)(\varphi) = \int_{S^+} \int_{\mathbb{X}} \varphi(\tilde{x} \cdot g) \nu(d\tilde{x}) \mu(dg) = \int_{\mathbb{X}} \varphi(\tilde{x}) \nu(d\tilde{x}) = \nu(\varphi).$$

Moreover, the upper Lyapunov exponent π_μ defined above coincides with the quantity $\int_{\mathbb{X} \times S^+} \rho(\tilde{x}, g) \mu(dg) \nu(d\tilde{x})$ and is finite [5].

In the sequel, we first focus our attention on "linear-fractional multidimensional generating functions" $f = f_{\gamma, M}$ of the form:

$$(1.5) \qquad f(s) = f_{\gamma, M}(s) = \mathbf{1} - \frac{1}{1 + \tilde{\gamma}(\mathbf{1} - s)} M(\mathbf{1} - s),$$

for any $s \in [0,1]^p$, where $\tilde{\gamma} = (\gamma, \ldots, \gamma) \in \mathbb{R}^p$ with $\gamma > 0$ and $M \in S^+$. For such a function $f = f_{\gamma, M}$, we set $\gamma = \gamma(f)$ and $M = M(f)$ and notice that $M(f)$ equals the mean matrix $\left(\frac{\partial f^{(i)}}{\partial s_j}(\mathbf{1}) \right)_{1 \le i,j \le p}$.

Here, we specify hypotheses concerning the distribution μ of the mean matrices $M_{\xi_n} = M(f_{\xi_n}), n \ge 1$.

H1. *There exists $\epsilon_0 > 0$ such that $\int_{S^+} N(g)^{\epsilon_0} \mu(dg) < \infty$.*

H2. *(Strong irreducibility).* *There exists no affine subspaces $\mathcal{A}$ of $\mathbb{R}^d$ such that $\mathcal{A} \cap \mathcal{C}$ is non-empty, bounded and invariant under the action of all elements of the support of μ.*

H3. *There exists $B \geq 1$ such that the support of μ is included in $S^+(B)$.*

H4. *The upper Lyapunov exponent π_μ of μ is equal to 0.*

H5. *There exists $\delta > 0$ such that $\mu(E_\delta) > 0$, where*

$$E_\delta := \{g \in S^+ \mid \forall \tilde{x} \in \mathbb{X}, \ \log |\tilde{x}g| \geq \delta\}.$$

If the variables f_{ξ_n} are linear fractional generating functions, then we introduce an additional hypothesis.

H6. *There exists $B' \geq 1$ such that $\dfrac{1}{B'} \leq \gamma(f_{\xi_n}) \leq B' \quad \mathbb{P} - a.s.$*

Here comes the main result of this paper.

THEOREM 1.1. *Assume that the random variables f_{ξ_n} are linear fractional generating functions and that hypotheses* H1-H6 *hold. Then, for any $i \in \{1, \ldots, p\}$, there exists a real number $\beta_i \in (0, +\infty)$ such that*

$$(1.6) \qquad \lim_{n \to +\infty} \sqrt{n} \mathbb{P}(Z_n \neq \tilde{\mathbf{0}} | Z_0 = \tilde{e}_i) = \beta_i.$$

For general generating functions, we obtain the following weaker result.

THEOREM 1.2. *Assume that the random variables f_{ξ_n} are C^2-functions on $[0,1]^p$ such that*

1. there exists $A > 0$ such that for any $i, k, l \in \{1, \ldots, p\}$,

$$\frac{\partial^2 f_{\xi_n}^{(i)}}{\partial s_k \partial s_l}(\mathbf{1}) \leq A \frac{\partial f_{\xi_n}^{(i)}}{\partial s_k}(\mathbf{1}) \quad \mathbb{P} - \text{a.s.}$$

2. the distribution μ of the matrices $M_{\xi_n} = \left(\frac{\partial f_{\xi_n}^{(i)}}{\partial s_j}(\mathbf{1}) \right)_{1 \leq i,j \leq p}$ satisfies hypotheses H1-H5.

Then, there exist real constants $0 < c_1 < c_2 < +\infty$ such that for any $i \in \{1, \ldots, p\}$, and $n \geq 1$,

$$(1.7) \qquad \frac{c_1}{\sqrt{n}} \leq \mathbb{P}(Z_n \neq \tilde{\mathbf{0}} | Z_0 = \tilde{e}_i) \leq \frac{c_2}{\sqrt{n}}.$$

In particular, under weaker assumptions than in [16], this theorem states that the process $(Z_n)_{n \geq 0}$ becomes extinct $\mathbb{P}$-a.s. in the critical case. Notice that by a recent work [8], equivalence (1.6) holds true even when the f_{ξ_n}

are not assumed to be linear fractional generating functions. The authors in [8] apply the proof in the present paper with two new ingredients: first, an extension to the multitype case of Geiger & Kersting's decomposition of the extinction probability ([10], formula (2.2)) and second, a variation of Lemma 3.1 below, taking into account the residual term which appears in this expression of the extinction probability.

Notations. *Let $c > 0$ and ϕ, ψ be two functions of some variable x; we shall write $\phi \overset{c}{\preceq} \psi$ (or simply $\phi \preceq \psi$) when $\phi(x) \le c\psi(x)$ for any value of x. The notation $\phi \overset{c}{\asymp} \psi$ (or simply $\phi \asymp \psi$) means $\phi \overset{c}{\preceq} \psi \overset{c}{\preceq} \phi$.*

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2. Preliminary results.

2.1. *Product of matrices with non-negative coefficients.* We describe in this section some properties of the set S^+. We first endow $\mathbb{X}$ with a distance d which is a variant of the Hilbert metric; it is bounded on $\mathbb{X}$ and any element $g \in S^+$ acts on $(\mathbb{X}, d)$ as a contraction. We summarize here its construction and its major properties.

For any $\tilde{x} = (x_i)_{1 \le i \le p}, \tilde{y} = (y_i)_{1 \le i \le p} \in \mathbb{X}$, we write

$$m(\tilde{x}, \tilde{y}) = \min\left\{\frac{x_i}{y_i}\Big| i = 1, \ldots, p \text{ such that } y_i > 0\right\}$$

and we set

$$d(\tilde{x}, \tilde{y}) := \varphi\Big(m(\tilde{x}, \tilde{y})m(\tilde{y}, \tilde{x})\Big),$$

where φ is the one-to-one function on $[0, 1]$ defined by $\varphi(s) := \dfrac{1-s}{1+s}$. For $g \in S^+$, we set

$$c(g) := \sup\{d(\tilde{x} \cdot g, \tilde{y} \cdot g) \mid \tilde{x}, \tilde{y} \in \mathbb{X}\}.$$

We present some crucial properties of d.

PROPOSITION 2.1. *[15] The function d is a distance on $\mathbb{X}$ wich satisfies the following properties.*

1. *$\sup\{d(\tilde{x}, \tilde{y}) \mid \tilde{x}, \tilde{y} \in \mathbb{X}\} = 1$.*
2. *For any $g = (g(i,j))_{i,j} \in S^+$*

$$c(g) = \max_{i,j,k,l \in \{1,\ldots,p\}} \frac{|g(i,j)g(k,l) - g(i,l)g(k,j)|}{g(i,j)g(k,l) + g(i,l)g(k,j)}.$$

In particular, there exists $\kappa_B \in [0,1)$ such that $c(g) \leq \kappa_B < 1$ for any $g \in S^+(B)$.

3. $d(\tilde{x} \cdot g, \tilde{y} \cdot g) \leq c(g)d(\tilde{x}, \tilde{y}) \leq c(g)$ for any $\tilde{x}, \tilde{y} \in \mathbb{X}$ and $g \in S^+(B)$.

4. $c(gg') \leq c(g)c(g')$ for any $g, g' \in S^+(B)$.

The following lemma follows from [9] Lemma 2. It is the key argument in the sequel to control the asymptotic behaviour of the norm of products of matrices of $S^+(B)$.

LEMMA 2.1. *For any $g \in S^+(B)$ and $1 \leq i, j, k, l \leq p$,*

$$(2.1) \qquad\qquad g(i,j) \stackrel{B^2}{\asymp} g(k,l).$$

In particular, there exist $c > 1$ such that for any $g, h \in S^+(B)$ and $\tilde{x}, \tilde{y} \in \mathbb{X}$

1. *$|gx| \stackrel{c}{\asymp} |g|$ and $|\tilde{y}g| \stackrel{c}{\asymp} |g|$,*
2. *$|\tilde{y}gx| \stackrel{c}{\asymp} |g|$,*
3. *$|gh| \stackrel{c}{\asymp} |g||h|$.*

2.2. *Conditioned product of random matrices.* Recall that $(M_n)_{n \geq 0}$ is a sequence of i.i.d. matrices whose law μ satisfies hypotheses H1-H5 and $R_n = M_0 \ldots M_{n-1}$ and $L_n = M_{n-1} \ldots M_0$ for $n \geq 1$. Consider the homogenous Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{X}$, with initial value $X_0 = \tilde{x} \in \mathbb{X}$, defined by

$$X_n = \tilde{x} \cdot R_n, n \geq 1.$$

Its transition probability P is given by: for any $\tilde{x} \in \mathbb{X}$ and any bounded Borel function $\varphi : \mathbb{X} \to \mathbb{R}$,

$$P\varphi(\tilde{x}) := \int_{S^+} \varphi(\tilde{x} \cdot g)\mu(dg).$$

The chain $(X_n)_{n \geq 0}$ has been the object of many studies, in particular there exists on $\mathbb{X}$ a unique P-invariant probability measure ν.

Indeed, by Proposition 2.1, for any $\tilde{x}, \tilde{y} \in \mathbb{X}$ and **any sequence** $(g_n)_{n \geq 1}$ **of elements** in $S^+(B)$, it follows that

$$(2.2) \qquad\qquad d(\tilde{x} \cdot g_n \ldots g_1, \tilde{y} \cdot g_n \ldots g_1) \leq \kappa_B^n.$$

This yields $\sup_{k \geq 0} d(\tilde{x} \cdot g_{n+k} \cdots g_n \cdots g_1, \tilde{x} \cdot g_n \ldots g_1) \to 0$ as $n \to +\infty$; the sequence $(\tilde{x} \cdot g_1 \cdots g_n)_{n \geq 0}$ thus converges in $\mathbb{X}$.

In particular, under hypothesis H3, the sequence $(\tilde{x} \cdot L_n)_{n \geq 0}$ converges $\mathbb{P}$-a.s. towards some $\mathbb{X}$-valued random variable X_∞. It follows that the Markov

chain $(\tilde{x} \cdot R_n)_{n \geq 0}$ converges in distribution to the law ν of X_∞, which is the unique P-invariant probability measure on $\mathbb{X}$. Property 2.2 allows one to prove that the restriction of P to some suitable space of continuous functions from $\mathbb{X}$ to $\mathbb{C}$ is quasi-compact, which is a crucial ingredient to study the asymptotic behavior of $(\tilde{x} \cdot R_n)_{n \geq 0}$ ([14], [5], [15]).

In the sequel, we are interested in the left linear action $\tilde{x} \mapsto \widetilde{x}R_n$ of the right products R_n, for any $\tilde{x} \in \mathbb{X}$. Since it holds

$$\widetilde{x}R_n = e^{\log |\widetilde{x}R_n|} \ \widetilde{x} \cdot R_n,$$

we consider the random process $(S_n)_{n \geq 0}$ defined by: for any $\tilde{x} \in \mathbb{X}, a \in \mathbb{R}$ and $n \geq 1$,

$$S_0 = S_0(\tilde{x}, a) := a \quad \text{and} \quad S_n = S_n(\tilde{x}, a) := a + \log |\widetilde{x}R_n|.$$

Iterating the cocycle property (1.4), the basic decomposition of $S_n(\tilde{x}, a)$ arrives:

$$(2.3) \qquad S_n(\tilde{x}, a) = a + \log |\tilde{x}R_n| = a + \sum_{k=0}^{n-1} \rho(X_k, M_k).$$

Let us emphasize that for any $a \in \mathbb{R}$ the sequence $(X_n, S_n)_{n \geq 0}$ is a Markov chain on $\mathbb{X} \times \mathbb{R}$ whose transition probability $\widetilde{P}$ is defined by: for any $(\tilde{x}, a) \in \mathbb{X} \times \mathbb{R}$ and any bounded Borel function $\psi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$,

$$\widetilde{P}\psi(\tilde{x}, a) = \int_{S^+} \psi(\tilde{x} \cdot g, a + \rho(\tilde{x}, g))\mu(dg).$$

We set $\mathbb{R}_*^+ := \mathbb{R}^+ \backslash \{0\}$ and denote $\widetilde{P}_+$ the restriction of $\widetilde{P}$ to $\mathbb{X} \times \mathbb{R}^+$ defined by: for $a \geq 0$ and any $\tilde{x} \in \mathbb{X}$,

$$\widetilde{P}_+((\tilde{x}, a), \cdot) = 1_{\mathbb{X} \times \mathbb{R}^+}(\cdot)\widetilde{P}((\tilde{x}, a), \cdot).$$

Fix $a \geq 0$ and denote by τ the first time greater than 1 the random process $(S_n)_{n \geq 0}$ becomes non-positive:

$$(2.4) \qquad\qquad\qquad \tau := \min\{n \geq 1 \mid S_n \leq 0\}.$$

For any $\tilde{x} \in \mathbb{X}$ and $a \geq 0$, let us denote $\mathbb{P}_{\tilde{x},a}$ the probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned to the event $[X_0 = \tilde{x}, S_0 = a]$ and $\mathbb{E}_{\tilde{x},a}$ the corresponding expectation; we omit the index a when $a = 0$ and denote $\mathbb{P}_{\tilde{x}}$ the corresponding probability. For any $n \geq 0$, let $m_n = m_n(\tilde{x}, a)$ be the minimum of the values $S_1(\tilde{x}, a), \ldots, S_n(\tilde{x}, a)$ and set

$$\mathbf{m}_n(\tilde{x}, a) := \mathbb{P}_{\tilde{x},a}[m_n > 0] = \mathbb{P}_{\tilde{x},a}(\tau > n).$$

We present a general result concerning the behavior of the tail distribution of the random variable τ.

The asymptotic behaviour of the probability $\mathbb{P}_{\tilde{x},a}(\tau > n)$ is established in [13] when the matrices M_n are invertible and in [19] when the M_n has non negative entries, under several conditions P1-P5; the first step is to establish the existence of a $\widetilde{P}_+$-harmonic function h on $\mathbb{X} \times \mathbb{R}^+$. Our hypotheses H1, H2, H4 and H5 are exactly P1, P2, P4 and P5 in [19] and H3 is obviously stronger than P3. Proposition 2.2 concerns the existence of the function h and its properties.

PROPOSITION 2.2. *[19] Under hypotheses* H1-H5, *for any* $x \in \mathbb{X}$ *and* $a \geq 0$, *the sequence* $\left(\mathbb{E}_{x,a}[S_n; \tau > n] \right)_{n \geq 0}$ *converges to a function* $h(\tilde{x}, a)$ *which satisfies the following properties.*

1. *For any* $x \in \mathbb{X}$, *the function* $h(\tilde{x}, a)$ *is increasing on* $\mathbb{R}^+$.
2. *There exists* $c > 0$ *and* $A > 0$ *such that for any* $x \in \mathbb{X}$ *and* $a \geq 0$,

$$(2.5) \qquad \frac{1}{c} \vee (a - A) \leq h(\tilde{x}, a) \leq c(1 + a).$$

3. *For any* $x \in \mathbb{X}$, *the function* $h(\tilde{x}, a)$ *satisfies* $\lim_{a \to +\infty} \dfrac{h(\tilde{x}, a)}{a} = 1$.
4. *The function* h *is* $\widetilde{P}_+$-*harmonic.*

This statement yields to the following theorem about the limit behaviour of $\mathbb{P}_{\tilde{x},a}(\tau > n)$ as $n \to +\infty$; the relation $u_n \sim v_n$ defines $\lim_{n \to +\infty} \dfrac{u_n}{v_n} = 1$.

THEOREM 2.1. *[19] Under hypotheses* H1-H5, *for any* $\tilde{x} \in \mathbb{X}$ *and* $a \geq 0$,

$$(2.6) \qquad \mathbb{P}_{\tilde{x},a}(\tau > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} h(\tilde{x}, a) \ as \ n \to \infty,$$

where $\sigma^2 > 0$ *is the variance of the Markov walk* $(S_n)_{n \geq 0}$, *given in [13]. Moreover, there exists a constant* $c > 0$ *such that for any* $\tilde{x} \in \mathbb{X}$, $a \geq 0$ *and* $n \geq 0$,

$$(2.7) \qquad \sqrt{n}\mathbb{P}_{\tilde{x},a}(\tau > n) \leq c \ h(\tilde{x}, a).$$

Remark. The fact that $\sigma^2 > 0$ is a direct consequence of hypotheses H2 and H5 (which implies in particular that the semi-group generated by the support of μ is unbounded); see [5], chap 6, Lemmas 5.2 and 5.3 and section 8 for more details.

3. Proof of Theorem 1.1.

3.1. *Expression of non-extinction probability.* For any $0 \leq k < n$, set $R_{k,n} := M_k \ldots M_{n-1}$ and $R_{k,n} := I$ otherwise. Let $Y_{k,n} := R_{k,n} \cdot \mathbf{1}$. It is known (see Lemma 1 in [6]) that

$$(3.1) \quad (q_n^{(i)})^{-1} = \frac{1 + \tilde{\gamma}_0 M_1 \ldots M_{n-1}\mathbf{1} + \tilde{\gamma}_1 M_2 \ldots M_{n-1}\mathbf{1} + \ldots + \tilde{\gamma}_{n-1}\mathbf{1}}{\tilde{e}_i R_n \mathbf{1}}.$$

Since $\tilde{e}_i R_k R_{k,n}\mathbf{1} = \tilde{e}_i M_0 \ldots M_{n-1}\mathbf{1}$ for any $1 \leq k \leq n$, we may rewrite (3.1) as

$$(3.2) \quad \left(q_n^{(i)}\right)^{-1} = \frac{1}{\tilde{e}_i R_n \mathbf{1}} + \sum_{k=0}^{n-1} \frac{\tilde{\gamma}_k Y_{k+1,n}}{\tilde{e}_i R_k Y_{k+1,n}} = \frac{1}{\tilde{e}_i R_n \mathbf{1}} + \sum_{k=0}^{n-1} \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,n}}.$$

Notice that, for any fixed $k \geq 0$, the sequence $(Y_{k,n})_{n \geq 0}$ converges to some limit $Y_{k,\infty}$ ([9], [15]). Indeed, the matrices $M_l, l \geq 1$, all belong to $S^+(B)$, so do their transposes and we may write as in (2.2)

$$\sup_{m \geq 1} d(\widetilde{Y}_{k,n}, \widetilde{Y}_{k,n+m}) \leq \kappa_B^{n-k} \overset{n \to +\infty}{\longrightarrow} 0.$$

The sequence of row vectors $(\widetilde{Y}_{k,n})_{n \geq 0}$ is thus a Cauchy sequence in $\mathbb{X}$, it converges and so does $(Y_{k,n})_{n \geq 0}$; we denote $Y_{k,\infty}$ the limit of this last sequence. Using this key fact, we prove in Lemma 3.2 that the sequence $\left(q_n^{(i)}\right)_{n \geq 1}$ converges to some positive limit

$$(3.3) \qquad\qquad q_\infty^{(i)} = \left(\sum_{k=0}^{+\infty} \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}}\right)^{-1}$$

in the L^1-norm with respect to a probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ introduced in the following subsection.

3.2. *Construction of a new probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ conditioned to the environment.* Since the function h is $\widetilde{P}_+$-harmonic on $\mathbb{X} \times \mathbb{R}^+$, it gives rise to a Markov kernel $\widetilde{P}_+^h$ on $\mathbb{X} \times \mathbb{R}^+$ defined by

$$\widetilde{P}_+^h \phi = \frac{1}{h} \widetilde{P}_+(h\phi)$$

for any bounded measurable function ϕ on $\mathbb{X} \times \mathbb{R}^+$. The kernels $\widetilde{P}_+$ and $\widetilde{P}_+^h$ are related to the stopping times τ by the following identity: for any $\tilde{x} \in \mathbb{X}$,

$a \geq 0$ and $n \geq 1$,

$$
\begin{aligned}
(\widetilde{P}_+^h)^n \phi(\tilde{x}, a) &= \frac{1}{h(\tilde{x}, a)} \widetilde{P}_+^n (h\phi)(\tilde{x}, a) \\
&= \frac{1}{h(\tilde{x}, a)} \mathbb{E}_{\tilde{x}, a} \left(h\phi(X_n, S_n); \tau > n \right) \\
&= \frac{1}{h(\tilde{x}, a)} \mathbb{E}_{\tilde{x}, a} \left(h\phi(X_n, S_n); m_n > 0 \right).
\end{aligned}
$$

This new Markov chain with kernel $\widetilde{P}_+^h$ allows one to change the measure on the canonical path space $((\mathbb{X} \times \mathbb{R})^{\otimes \mathbb{N}}, \sigma(X_n, S_n : n \geq 0), \theta)$ of the Markov chain $(X_n, S_n)_{n \geq 0}$ [1] from $\mathbb{P}$ to the measure $\widehat{\mathbb{P}}_{\tilde{x}, a}$ characterized by the property that

(3.4)
$$
\widehat{\mathbb{E}}_{\tilde{x}, a}[\varphi(X_0, S_0, \ldots, X_k, S_k)] = \frac{1}{h(\tilde{x}, a)} \mathbb{E}_{\tilde{x}, a}[\varphi(X_0, \ldots, S_k) h(X_k, S_k); \ m_k > 0]
$$

for any positive Borel function φ on $(\mathbb{X} \times \mathbb{R})^{k+1}$.

For any $0 \leq k \leq n$,

$$
\begin{aligned}
& \mathbb{E}_{\tilde{x}, a}[\varphi(X_0, S_0, \ldots, X_k, S_k); m_n > 0] \\
={} & \mathbb{E}_{\tilde{x}, a}[\varphi(X_0, \ldots, S_k); S_0 > 0, S_1 > 0, \ldots, S_n > 0] \\
={} & \mathbb{E}_{\tilde{x}, a}[\varphi(X_0, \ldots, S_k); S_0 > 0, a + \rho(X_0, M_0) > 0, \\
& \qquad \ldots, a + \sum_{i=0}^{k-1} \rho(X_i, M_i) + \sum_{i=k}^{n-1} \rho(X_i, M_i) > 0] \\
={} & \mathbb{E}_{\tilde{x}, a}[\mathbb{E}[\varphi(X_0, \ldots, S_k); S_0 > 0, \ldots, S_k > 0, \\
& \qquad S_0 \circ \theta^k > 0, \ldots, S_k + S_{n-k} \circ \theta^k > 0 | M_0, \ldots, M_{k-1}]] \\
={} & \mathbb{E}_{\tilde{x}, a}\Big[\varphi(X_0, \ldots, S_k) \\
& \qquad \times \mathbb{E}[S_k > 0, \ldots, S_k + S_{n-k} \circ \theta^k > 0 | M_0, \ldots, M_{k-1}]; m_k > 0 \Big] \\
={} & \mathbb{E}_{\tilde{x}, a}\Big[\varphi(X_0, \ldots, S_k) \\
& \qquad \times P_{X_k, S_k}(S_0 \circ \theta^k > 0, \ldots, S_{n-k} \circ \theta^k > 0); m_k > 0 \Big] \\
={} & \mathbb{E}_{\tilde{x}, a}\Big[\varphi(X_0, \ldots, S_k) \mathbb{P}_{X_k, S_k}(\tau > n - k); m_k > 0 \Big].
\end{aligned}
$$

---

[1] θ denotes the shift operator on $(\mathbb{X} \times \mathbb{R})^{\otimes \mathbb{N}}$ defined by $\theta\big((x_k, s_k)_{k \geq 0}\big) = (x_{k+1}, s_{k+1})_{k \geq 0}$ for any $(x_k, s_k)_{k \geq 0}$ in $(\mathbb{X} \times \mathbb{R})^{\otimes \mathbb{N}}$

Hence,
(3.5)
$$\mathbb{E}_{\tilde{x},a} \left(\varphi(X_0, \ldots, S_k); m_n > 0 \right) = \mathbb{E}_{\tilde{x},a}[\varphi(X_0, \ldots, S_k)\mathbf{m}_{n-k}(X_k, S_k); m_k > 0].$$

Moreover, in view of Theorem 2.1, the dominated convergence theorem and (3.5), we obtain for any bounded function φ with compact support,

$$\lim_{n \to +\infty} \mathbb{E}_{\tilde{x},a}[\varphi(X_0, \ldots, S_k)|m_n > 0]$$

$$= \frac{1}{h(\tilde{x},a)}\mathbb{E}_{\tilde{x},a}[\varphi(X_0, \ldots, S_k)h(X_k, S_k); m_k > 0]$$

(3.6) $$= \widehat{\mathbb{E}}_{\tilde{x},a}[\varphi(X_0, \ldots, S_k)],$$

which clarifies the interpretation of $\widehat{\mathbb{P}}_{\tilde{x},a}$.

We formalize in three steps the construction of a new probability measure, denoted again $\widehat{\mathbb{P}}_{\tilde{x},a}$, for each $\tilde{x} \in \mathbb{X}$ and $a \geq 0$, but defined this time on the bigger σ-algebra $\sigma(f_n, Z_n : n \geq 0)$. Retaining the notations from the previous parts, the measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ is characterized by properties (3.4), (3.7) and (3.8).

Step 1. The marginal distribution of $\widehat{\mathbb{P}}_{\tilde{x},a}$ on $\sigma(X_n, S_n : n \geq 0)$ is $\widehat{\mathbb{P}}_{\tilde{x},a}$ characterized by the property (3.4).

Step 2. For any $n \geq 0$, the conditional distribution of $(f_0, \cdots, f_n)$ under $\widehat{\mathbb{P}}_{\tilde{x},a}$ given $X_0 = \tilde{x}_0 = \tilde{x}, \ldots, X_n = \tilde{x}_n, S_0 = s_0 = a, \ldots, S_n = s_n$ is given by:

$$\widehat{\mathbb{P}}_{\tilde{x},a}(f_k \in A_k, 0 \leq k \leq n | X_i = \tilde{x}_i, S_i = s_i, 0 \leq i \leq n)$$
(3.7) $$= \mathbb{P}(f_k \in A_k, 0 \leq k \leq n | X_i = \tilde{x}_i, S_i(\tilde{x}) = s_i, 0 \leq i \leq n),$$

for any measurable sets $A_0, \ldots, A_n$ in $\mathcal{H}_B$ and almost all $(\tilde{x}_i)_{0 \leq i \leq n}$ and $(s_i)_{0 \leq i \leq n}$ with respect to the law of $(X_0, \ldots, X_n, S_0, \ldots, S_n)$ under $\mathbb{P}$ (and also under $\widehat{\mathbb{P}}_{\tilde{x},a}$ since, by formula (3.4), the probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ is absolutely continuous with respect to $\mathbb{P}$ on each σ-algebra $\sigma(X_0, \ldots, S_n)$).

Step 3. The conditional distribution of $(Z_n)_{n \geq 0}$ under $\widehat{\mathbb{P}}_{\tilde{x},a}$ given $f_0^{(i)}, f_1, \ldots$ is the same as under $\mathbb{P}$, namely

$$\widehat{\mathbb{E}}_{\tilde{x},a} \left(s^{Z_n} | Z_0, \ldots, Z_{n-1}, f_0^{(i)}, f_1, \ldots, f_{n-1} \right)$$
(3.8) $$= f_{n-1}(s)^{Z_{n-1}} = \mathbb{E} \left(s^{Z_n} | Z_0, \ldots, Z_{n-1}, f_0^{(i)}, f_1, \ldots, f_{n-1} \right).$$

3.3. *Proof of Theorem 1.1.* We follow J. Geiger and G. Kersting's approach. We fix $1 \leq i \leq p$. For any $\tilde{x} \in \mathbb{X}$ and $a \geq 0$ let us denote $\mathbb{P}_{\tilde{x},a}^{(i)}$ the probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned to the event $[X_0 = \tilde{x}, S_0 = a, Z_0 = \tilde{e}_i]$ and $\mathbb{E}_{\tilde{x},a}^{(i)}$ the corresponding expectation.

First, notice that the quantity $\mathbb{P}(Z_n \neq \tilde{\mathbf{0}} \mid Z_0 = \tilde{e}_i)$ equals $\mathbb{E}_{\tilde{x},a}(q_n^{(i)})$ for any $(x,a) \in \mathbb{X} \times \mathbb{R}^+$; thus we fix $(\tilde{x},a) \in \mathbb{X} \times \mathbb{R}^+, \rho > 1$ and $0 \leq k \leq n$ and decompose $\mathbb{P}(Z_n \neq \tilde{\mathbf{0}} \mid Z_0 = \tilde{e}_i)$ as

(3.9)
$$\mathbb{P}^{(i)}(Z_n \neq \tilde{\mathbf{0}}) = A_n + B_n + C_n - D_n,$$

where

- $A_n = A_n(\tilde{x},a) = \mathbb{P}_{\tilde{x},a}^{(i)}(Z_n \neq \tilde{\mathbf{0}}, m_n < 0)$;
- $B_n = B_n(\tilde{x},a,\rho) = \mathbb{P}_{\tilde{x},a}^{(i)}(Z_n \neq \tilde{\mathbf{0}}, m_n \geq 0) - \mathbb{P}_x(Z_n \neq \tilde{\mathbf{0}}, m_{\rho n} \geq 0)$;
- $C_n = C_n(\tilde{x},a,\rho,k) = \mathbb{P}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}, m_{\rho n} \geq 0)$;
- $D_n = D_n(\tilde{x},a,\rho,k) = \mathbb{P}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}, Z_n = \tilde{\mathbf{0}}, m_{\rho n} \geq 0)$.

We decompose the proof in 5 steps and then conclude, letting first $k \to +\infty$, then $\rho \to 1$ and at last $a \to +\infty$.

1. The quantity $A = A(\tilde{x},a) := \limsup_{n \to +\infty} \sqrt{n} A_n(\tilde{x},a)$ tends to 0 when $a \to +\infty$.
2. The quantity $B = B(\tilde{x},a,\rho) := \limsup_{n \to +\infty} \sqrt{n} B_n(\tilde{x},a,\rho)$ tends to 0 when $\rho \to +1$.
3. As $n \to +\infty$, the sequence $\left(\sqrt{n} C_n(\tilde{x},a,\rho,k)\right)_{n \geq 0}$ converges to

$$C = C(\tilde{x},a,\rho,k) := c_1 \frac{h(\tilde{x},a)}{\sqrt{\rho}} \widehat{\mathbb{P}}_x(Z_k \neq \tilde{\mathbf{0}}).$$

4. For any $\rho > 1$ the quantity $D = D(\tilde{x},a,\rho,k) := \limsup_{n \to +\infty} \sqrt{n} D_n(\tilde{x},a,\rho,k)$ tends to 0 when $k \to +\infty$.
5. The sequence $(\widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_k \neq 0))_{k \geq 0}$ converges to some limit $v^{(i)}(\tilde{x},a) > 0$.

Step 1. We may write

$$A_n(\tilde{x},a) = \mathbb{E}_{\tilde{x},a}\left(\mathbb{E}\left(Z_n \neq \tilde{\mathbf{0}}|f_0^{(i)}, f_1, \ldots, f_{n-1}\right); m_n \leq 0\right)$$

(3.10)
$$= \mathbb{E}_{\tilde{x},a}\left(q_n^{(i)}; m_n \leq 0\right).$$

To control the quantity $q_n^{(i)}$, we use the expression (3.2); Lemma 2.1 yields

$$
\begin{aligned}
\frac{1}{q_n^{(i)}} &= \frac{1}{\tilde{e}_i R_n \mathbf{1}} + \sum_{k=0}^{n-1} \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,n}} \\
&\geq \max_{0 \leq k \leq n-1}\left\{\frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,n}}\right\} \asymp \max_{0 \leq k \leq n-1}\left\{\frac{1}{|\tilde{x} R_k|}\right\} \\
&\geq \frac{1}{\exp\left\{\min_{0 \leq k \leq n-1}(a + \ln|\tilde{x} R_k|)\right\}}.
\end{aligned}
$$

so that $q_n^{(i)} \preceq \exp(m_n(\tilde{x}, a))$ and by applying Theorem 2.1 equation (2.7), we obtain

$$
\begin{aligned}
A_n(\tilde{x}, a) &= \mathbb{P}_{\tilde{x}}^{(i)}(Z_n \neq \tilde{\mathbf{0}}, m_n \leq -a) \\
&\preceq \mathbb{E}_{\tilde{x}}[\exp(m_n); m_n \leq -a] \\
&\leq \sum_{k=a}^{+\infty} e^{-k} \mathbb{P}_{\tilde{x}}(-k < m_n \leq -k+1) \\
&\leq \sum_{k=a}^{+\infty} e^{-k} \mathbb{P}_{\tilde{x},k}(\tau > n) \\
&\preceq \frac{1}{\sqrt{n}} \sum_{k=a}^{+\infty} (k+1)e^{-k}.
\end{aligned}
$$

(3.11)

Notice that the sum $\displaystyle\sum_{k=a}^{+\infty} (k+1)e^{-k}$ becomes arbitrarily small for sufficiently great a. Consequently $A(\tilde{x}, a) = \limsup_{n \to +\infty} \sqrt{n} A_n(\tilde{x}, a) \longrightarrow 0$ as $x \to +\infty$.

Step 2. Theorem 2.1 yields

$$
\begin{aligned}
0 \leq B_n(\tilde{x}, a, \rho) &= \mathbb{P}_{\tilde{x},a}^{(i)}(Z_n \neq 0, m_n > 0) - \mathbb{P}_{\tilde{x},a}^{(i)}(Z_n \neq 0, m_{\rho n} > 0) \\
&\sim c_1 \frac{h(\tilde{x}, a)}{\sqrt{n}} \left(1 - \frac{1}{\sqrt{\rho}}\right) \quad \text{as} \quad n \to +\infty
\end{aligned}
$$

with $c_1 = \dfrac{2}{\sigma\sqrt{2\pi}}$. Hence

$$
B(\tilde{x}, a, \rho) := \limsup_{n \to +\infty} \sqrt{n} B_n(\tilde{x}, a, \rho) \leq c_1 h(\tilde{x}, a) \left(1 - \frac{1}{\sqrt{\rho}}\right) \xrightarrow{\rho \to 1} 0.
$$

Step 3. Fix $0 \leq k \leq n$. Using (3.5) and the fact that $1_{[m_k>0]}$ and $\mathbf{m}_{\rho n - k}(X_k, S_k)$ are measurable with respect to the σ-algebra $\sigma(f_0^{(i)}, f_1, \ldots, f_{k-1})$, we may write

$$
\begin{aligned}
C_n(\tilde{x}, a, \rho, k) &= \mathbb{P}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}, m_{\rho n} > 0) \\
&= \mathbb{P}_{\tilde{x},a}^{(i)} \left(Z_k \neq \tilde{\mathbf{0}}, m_k > 0, \mathbf{m}_{\rho n - k}(X_k, S_k)\right) \\
&= \mathbb{E}_{\tilde{x},a}^{(i)} \left(\mathbb{E}(1_{Z_k \neq \tilde{\mathbf{0}}} 1_{[m_k>0]} \mathbf{m}_{\rho n - k}(X_k, S_k) \mid f_0^{(i)}, \ldots, f_{k-1})\right) \\
&= \mathbb{E}_{\tilde{x},a}^{(i)} \left(\mathbb{P}(Z_k \neq \tilde{\mathbf{0}} | f_0^{(i)} \ldots, f_{k-1}) 1_{[m_k \geq 0]} \mathbf{m}_{\rho n - k}(X_k, S_k)\right) \\
&= \mathbb{E}_{\tilde{x},a} \left(q_k^{(i)}, m_{\rho n} > 0\right) \\
&= \mathbb{E}_{\tilde{x},a} \left(q_k^{(i)} \mid m_{\rho n} > 0\right) \mathbb{P}_{\tilde{x},a}(m_{\rho n} > 0).
\end{aligned}
$$

By (3.6), since k is fixed

$$\lim_{n\to+\infty} \mathbb{E}_{\tilde{x},a}\left(q_k^{(i)}|m_{\rho n}>0\right) = \widehat{\mathbb{E}}_{\tilde{x},a}\left(q_k^{(i)}\right) = \widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}).$$

Hence, by Theorem 2.1

$$\lim_{n\to+\infty} \sqrt{n}C_n(\tilde{x},a,\rho,k) = c_1 \frac{h(\tilde{x},a)}{\sqrt{\rho}}\widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}).$$

Step 4. As in the previous step, we may write

$$
\begin{aligned}
D_n(\tilde{x},a,\rho,k) &= \mathbb{P}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}, Z_n = \tilde{\mathbf{0}}, m_{\rho n}>0) \\
&= \mathbb{P}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}, m_{\rho n}>0) - \mathbb{P}_{\tilde{x},a}^{(i)}(Z_n \neq \tilde{\mathbf{0}}, m_{\rho n}>0) \\
&= \mathbb{E}_{\tilde{x},a}\left(q_k^{(i)}, m_{\rho n}>0\right) - \mathbb{E}_{\tilde{x},a}\left(q_n^{(i)}, m_{\rho n}>0\right) \\
&= \mathbb{E}_{\tilde{x},a}\left(\left(q_k^{(i)} - q_n^{(i)}\right)\mathbf{m}_{(\rho-1)n}(X_n, S_n); m_n > 0\right) \\
&\preceq \frac{1}{\sqrt{(\rho-1)n}}\frac{1}{h(\tilde{x},a)}\mathbb{E}_{\tilde{x},a}\left((\bar{q}_k - \bar{q}_n)h(X_n,S_n); m_n>0\right).
\end{aligned}
$$

Since $1_{[m_n>0]}$ and $h(X_n,S_n)$ are $\sigma(S_0,\ldots,S_n)$-measurable, we observe that

$$
\begin{aligned}
\sqrt{n}D_n(\tilde{x},a,\rho,k) &\preceq \frac{1}{\sqrt{\rho-1}}\widehat{\mathbb{E}}_{\tilde{x},a}\left(q_k^{(i)} - q_n^{(i)}\right) \\
&= \frac{1}{\sqrt{\rho-1}}\left(\widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}) - \widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_n \neq \tilde{\mathbf{0}})\right).
\end{aligned}
$$

Hence

$$D(\tilde{x},a,\rho,k) = \limsup_{n\to+\infty} \sqrt{n}D_n(\tilde{x},a,\rho,k) \preceq \frac{1}{\sqrt{\rho-1}}\left(\widehat{\mathbb{P}}_x(Z_k \neq \tilde{\mathbf{0}}) - v^{(i)}(\tilde{x},a)\right)$$

so that $D(\tilde{x},a,\rho,k) \longrightarrow 0$ as $k \to +\infty$.

Step 5. We first state the following lemmas whose proofs follow in the next section.

LEMMA 3.1.   *For any $\tilde{x} \in \mathbb{X}$ and $a \geq 0$,*

$$\widehat{\mathbb{E}}_{\tilde{x},a}\sum_{n=0}^{+\infty} e^{-S_n} < +\infty.$$

This allows one to identify the limit of the sequence $(\widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}))_{k\geq 0}$.

LEMMA 3.2.   *For any $\tilde{x} \in \mathbb{X}$ and $a \geq 0$, the sequence $\left(q_n^{(i)}\right)_{n \geq 1}$ converges to $q_\infty^{(i)}$ in $L^1(\widehat{\mathbb{P}}_{\tilde{x},a})$; in particular*

$$(3.12) \qquad \lim_{k \to +\infty} \widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_k \neq \tilde{\mathbf{0}}) = \widehat{\mathbb{E}}_{\tilde{x},a} q_\infty^{(i)},$$

where $q_\infty^{(i)}$ is given by the expression (3.3).

From Lemma 3.2, it is obvious that $\widehat{\mathbb{E}}_{\tilde{x},a} q_\infty^{(i)} \leq 1$. On the other hand, the expression of $q_\infty^{(i)}$ combined with Lemma 3.1 and the fact that $|\tilde{x} R_n| \asymp \tilde{e}_i R_n Y_{n+1,\infty}$ (see Lemma 2.1 property 2) yields $\widehat{\mathbb{E}}_{\tilde{x},a} q_\infty^{(i)} > 0$. In otherwords, for any $a \geq 0$

$$0 < v^{(i)}(\tilde{x}, a) := \widehat{\mathbb{E}}_{\tilde{x},a} q_\infty^{(i)} < +\infty.$$

This achieves the proof of Step 5.

Let us complete the proof of Theorem 1.1. Letting $n \to +\infty$ in expression (3.9) yields, for any $\tilde{x} \in \mathbb{X}, a \geq 0, \rho > 1$ and $k \geq 1$,

$$C - D \leq \liminf_{n \to +\infty} \sqrt{n}\mathbb{P}^{(i)}(Z_n \neq \tilde{\mathbf{0}}) \leq \limsup_{n \to +\infty} \sqrt{n}\mathbb{P}^{(i)}(Z_n \neq \tilde{\mathbf{0}}) \leq A + B + C + D.$$

Letting first $k \to +\infty$ then $\rho \to 1$ yields

$$
\begin{aligned}
c_1 h(\tilde{x}, a) v^{(i)}(\tilde{x}, a) &\leq& \liminf_{n \to +\infty} \sqrt{n}\mathbb{P}(Z_n \neq \tilde{\mathbf{0}}) \\
&\leq& \limsup_{n \to +\infty} \sqrt{n}\mathbb{P}(Z_n \neq \tilde{\mathbf{0}}) \\
&\leq& c_1 h(\tilde{x}, a) v^{(i)}(\tilde{x}, a) + A(\tilde{x}, a).
\end{aligned}
$$

Since $v^{(i)}(\tilde{x}, a) > 0, h(\tilde{x}, a) > 0$ and $A(\tilde{x}, a) < +\infty$, we obtain in particular

$$0 < \liminf_{n \to +\infty} \sqrt{n}\mathbb{P}(Z_n \neq \tilde{\mathbf{0}}) \leq \limsup_{n \to +\infty} \sqrt{n}\mathbb{P}(Z_n \neq \tilde{\mathbf{0}}) < +\infty.$$

Finally, letting $a \longrightarrow +\infty$, we conclude that both limits

$$\lim_{n \to +\infty} \sqrt{n}\mathbb{P}^{(i)}(Z_n \neq \tilde{\mathbf{0}}) \quad \text{and} \quad \lim_{a \to +\infty} c_1 h(\tilde{x}, a) v^{(i)}(\tilde{x}, a)$$

exist, coincide and belong to $]0, +\infty[$.

3.4. *Proof of Theorem 2.2.* First, for any $n \geq 1$ and $s = (s_1, \ldots, s_p)$, we denote $F_n(s) = f_{\xi_0}(f_{\xi_1}(\ldots(f_{\xi_{n-1}}(s))\ldots))$. By definition of $q_n^{(i)}$, we have for any $0 \leq k < n$,

$$q_n^{(i)} = \tilde{e}_i(F_k(\mathbf{1}) - F_k(z)),$$

where $z = z(k, n) = f_{\xi_k}(\ldots(f_{\xi_{n-1}}(0))\ldots)$; the Mean Value Theorem yields

$$q_n^{(i)} = \tilde{e}_i(F_k(\mathbf{1}) - F_k(z)) \leq \tilde{e}_i M_0 \ldots M_{k-1}\mathbf{1} \asymp |\tilde{x}R_k| = \exp(S_k(\tilde{x}, 0))$$

so that $q_n^{(i)} \preceq \exp(m_n(\tilde{x}, 0))$ and $\mathbb{E}[q_n^{(i)}] \preceq \mathbb{E}[e^{m_n(\tilde{x},0)}] = \mathbb{E}_{\tilde{x}}[e^{m_n}]$.

Using the same trick like in (3.11), we can deduce that there exists a constant c_2 such that

$$\mathbb{E}_{\tilde{x}}[e^{m_n}] = \mathbb{E}_{\tilde{x}}[e^{m_n}; m_n \leq 0] \sim \frac{c_2}{\sqrt{n}},$$

and thus the upper estimate in equation (1.7) arrives.

To obtain the lower estimate in (1.7), for any $\mathbb{R}$-valued multi-dimensional generating function $f(s)$, $s = (s_1, \ldots, s_p)^T$, we obtain (see for instance formulas (64) and (65) in [21])

(3.13)

$$f(s) \leq 1 - \left(\sum_{i=1}^p \frac{\partial f}{\partial s_i}(\mathbf{1})(1 - s_i) \right) \left(1 + \frac{\sum\limits_{i,j=1}^p \frac{\partial^2 f}{\partial s_i \partial s_j}(\mathbf{1})(1 - s_j)(1 - s_i)}{\sum\limits_{l=1}^p \frac{\partial f}{\partial s_l}(\mathbf{1})(1 - s_l)} \right)^{-1}.$$

We set $g_{\xi_n}(s) = \mathbf{1} - \dfrac{M_{\xi_n}(\mathbf{1} - s)}{1 + \widetilde{\gamma}_{\xi_n}(\mathbf{1} - s)}$, where M_{ξ_n} is the mean matrix of $f_{\xi_n}(s)$ and $\widetilde{\gamma}_{\xi_n} = (A, \ldots, A)$; thus, hypothesis H6 holds for g_{ξ_n} with $B' = A$. Applying inequality (3.13) with $f = f_{\xi_n}^{(i)}$, we may write

$$f_{\xi_n}^{(i)}(s) \leq g_{\xi_n}^{(i)}(s), \quad i = 1, \ldots, p,$$

which yields

(3.14) $\mathbb{E}[\mathbf{1} - g_{\xi_0}(g_{\xi_1}(\ldots(g_{\xi_{n-1}}(0))\ldots))] \leq \mathbb{E}[\mathbf{1} - f_0(f_1(\ldots(f_{n-1}(0))\ldots))].$

The lower estimate in equation (2.2) appears by applying Theorem 1.1 to the left side of equation (3.14). Therefore, the assertion of the Theorem 2.2 arrives.

4. Proof of lemmas. We first give some hints for the proof of Lemma 2.1 and describe later the proofs of Lemmas 3.1 and 3.2.

4.1. *Proof of Lemma 2.1.* First, using (2.1), we obtain

$$(4.1) \qquad |g| = \sum_{i,j=1}^{p} g(i,j) \overset{p^2 B^2}{\asymp} g(k,l).$$

Further properties can be easily deduced from (4.1). Indeed, the assertions we need are obvious by noticing that

$$|gx| = \sum_{i,j=1}^{p} g(i,j) x_j \overset{p^3 B^2}{\asymp} |g|,$$

$$\tilde{y}gx = \sum_{i,j=1}^{p} y_i g(i,j) x_j \overset{p^2 B^2}{\asymp} |g|,$$

$$|gh| = \sum_{i,j,k=1}^{p} g(i,j) h(j,k) \overset{p^7 B^4}{\asymp} |g||h|.$$

4.2. *Proof of Lemma 3.1.* Before going into the proof, we first claim that in the critical case, for any $\delta > 0$ and c given from Lemma 2.1, there exists an integer $\kappa = \kappa(\delta, c) \geq 1$ such that

$$(4.2) \qquad \mu^{*\kappa}(E_\delta) := \mu^{*\kappa}\{g \mid \forall \tilde{x} \in \mathbb{X}, \ \log|\tilde{x}g| \geq \delta\} > 0.$$

Indeed, let $\tau' := \inf\{n \geq 1 \mid \log|R_n| \geq \log c + \delta\}$. The random variable τ' is a stopping time with respect to the natural filtration $(\sigma(M_0, \ldots, M_k))_{k \geq 0}$; furthermore $\mathbb{P}(\tau' < +\infty) = 1$ since the Lyapunov exponent π_μ equals 0, which yields

$$\limsup_{n \to +\infty} \log|R_n| = \limsup_{n \to +\infty} \log|\tilde{x} \cdot R_n| = +\infty$$

(this last property is a direct consequence of the ergodic theorem applied here to the Birkhof sums $\log|\tilde{x} \cdot R_n|, n \geq 0$ [5]).

Therefore, for any $\delta > 0$ and c given from Lemma 2.1, there exists $\kappa \geq 1$ such that $\mathbb{P}(\tau' = \kappa) > 0$. Moreover, we also have

$$\begin{aligned} \mathbb{P}(\log|R_\kappa| \geq \log c + \delta) &\geq \mathbb{P}(\log|R_\kappa| \geq \log c + \delta, \tau' = \kappa) \\ &= \mathbb{P}(\log|R_{\tau'}| \geq \log c + \delta, \tau' = \kappa) \\ &= \mathbb{P}(\tau' = \kappa) > 0. \end{aligned}$$

Since for any $\tilde{x} \in \mathbb{X}$, $g \in S^+$, $|gx| \geq \frac{|g|}{c}$, it follows that

$$\{g \mid \log|g| \geq \log c + \delta\} \subset \{g \mid \forall \tilde{x} \in \mathbb{X}, \log|\tilde{x}g| \geq \delta\}.$$

Thus,

$$0 < \mathbb{P}(\log |R_\kappa| \geq \log c + \delta) = \mu^{*\kappa}\{g \mid \log |g| \geq \log c + \delta\} \leq \mu^{*\kappa}\{g \mid \forall \tilde{x} \in \mathbb{X}, \ \log |\tilde{x}g| \geq \delta\},$$

which is the assertion of the claim (4.2).

Let us go into the proof of Lemma 3.1. For any $\tilde{x} \in \mathbb{X}$, $a \geq 0$ and $\lambda \in (0,1)$, there exists some constant $C(\lambda) > 0$ such that $(t+1)e^{-t} \leq C(\lambda)e^{-\lambda t}$ for any $t > 0$. Hence

$$
\begin{aligned}
\widehat{\mathbb{E}}_{\tilde{x},a}\left[\sum_{n=0}^{+\infty} e^{-S_n}\right] &\leq 1 + \frac{1}{h(\tilde{x},a)}\sum_{n=1}^{+\infty}\mathbb{E}_{\tilde{x},a}\left(e^{-S_n}h(X_n,S_n); S_0 > 0, \ldots, S_n > 0\right) \\
&\leq 1 + \frac{c}{h(\tilde{x},a)}\sum_{n=1}^{+\infty}\mathbb{E}_{\tilde{x},a}\left(e^{-S_n}(1+S_n); S_0 > 0, \ldots, S_n > 0\right) \\
&\leq 1 + \frac{c\,C(\lambda)}{h(\tilde{x},a)}\sum_{n=1}^{+\infty}\mathbb{E}_{\tilde{x},a}\left(e^{-\lambda S_n}; S_0 > 0, \ldots, S_n > 0\right) \\
&\leq 1 + \frac{c\,C(\lambda)}{h(\tilde{x},a)}\sum_{n=1}^{+\infty}\mathbb{E}_{\tilde{x},a}\left(e^{-\lambda S_n}; S_1 > 0, \ldots, S_n > 0\right),
\end{aligned}
$$

where c is introduced in inequality (2.5).

We define a function Φ for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$ as follow:

$$\Phi(\tilde{x},a) := \sum_{n=1}^{+\infty}\mathbb{E}_{\tilde{x},a}\left(e^{-\lambda S_n}; S_1 > 0, \ldots, S_n > 0\right)$$

Notice that for any $\tilde{x} \in \mathbb{X}$, the function $\Phi(\tilde{x},.)$ increases on $\mathbb{R}$. To prove Lemma 3.1, it is sufficient to check that for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$

$$(4.3) \qquad\qquad\qquad \Phi(\tilde{x},a) < +\infty.$$

Let us explain briefly the strategy of our proof, inspired by Lemma 3.2 in [10]. J. Geiger and G. Kersting first use in a crucial way the so-called "duality principle" (that is the fact that for a classical random walk $(S_n)_{n\geq 1}$ on $\mathbb{R}$ the vectors $(S_1, S_2, \cdots, S_n)$ and $(S_n - S_{n-1}, S_n - S_{n-2}, \ldots, S_n)$ have the same distribution) and they prove in this context of random walks with i.i.d. increments that the quantity $\sum_{n=1}^{+\infty}\mathbb{E}_a[e^{-\lambda S_n}; S_1 > 0, \ldots, S_n > 0]$ is finite when $a = 0$. Second, they extend this property for any $a \geq 0$.

In the multi-dimensional case, it is more complicated to apply the duality principle and we can only prove at the beginning that the quantity $\Phi(\tilde{x}, a_0)$

is finite for some $a_0 < 0$ (without any control on the value of a_0). To extend this property to $\Phi(\tilde{x}, a)$ for $a > a_0$, we use crucially hypothesis H5 ; we refer to Step 2 below, especially inequality (4.4), and emphasize that the argument holds only when $-a_0 \leq \delta$). To avoid this difficulty, we modify the function Φ by introducing the functions Φ_κ associated with the κth power of convolution $\mu^{*\kappa}$ of μ. For any $\tilde{x} \in \mathbb{X}, a \in \mathbb{R}$, let

$$\Phi_\kappa(\tilde{x}, a) := \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x}, a}\left(e^{-\lambda S_{n\kappa}}; S_\kappa > 0, \dots, S_{n\kappa} > 0\right).$$

The relation is that $\Phi(\tilde{x}, a) \preceq \Phi_\kappa(\tilde{x}, a)$ for any $\tilde{x} \in \mathbb{X}, a \in \mathbb{R}$. Then, by using the duality principle, we bound from above $\Phi_\kappa(\tilde{x}, a)$ by a new quantity $\Psi_\kappa(\tilde{x})$ defined below for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$. Finally, we prove $\Psi_\kappa(\tilde{x}) < +\infty$ by using the ascending ladder epochs associated to the Markov walk $(L_n \cdot x, \log|L_n x|)_{n \geq 0}$ and the Elementary Renewal Theorem.

We set $L_0 = 0$ and denote $L_n := M_{n-1} \dots M_0$ the left product of the matrices $M_0, \dots, M_n$ when $n \geq 1$. Set, for any $\tilde{x} \in \mathbb{X}$,

$$\Psi_\kappa(\tilde{x}) := \sum_{n=1}^{+\infty} \mathbb{E}\left[|L_{n\kappa}x|^{-\lambda}; |L_{n\kappa}x| > |L_{(n-1)\kappa}x|, \dots, |L_{n\kappa}x| > 1\right].$$

Property (4.3) is a direct consequence of the four following steps :

1. For any $\kappa \geq 1$, there exists $C(\kappa) > 0$ such that, for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$,
$$\Phi(\tilde{x}, a) \leq C(\kappa)(1 + \Phi_\kappa(\tilde{x}, a)).$$

2. If there exist some $\kappa \geq 1, \tilde{x}_0 \in \mathbb{X}$ and $a_0 < 0$ such that $0 < \Phi_\kappa(\tilde{x}_0, a_0) < +\infty$, then
$$\forall \tilde{x} \in \mathbb{X}, \forall a \in \mathbb{R} \qquad \Phi_\kappa(\tilde{x}, a) < +\infty.$$

3. There exist $C_1 > 0$ and $a_1 < 0$ such that for any $\kappa \geq 1$, $\tilde{x} \in \mathbb{X}$ and $a < a_1$
$$\Phi_\kappa(\tilde{x}, a) \overset{C_1}{\preceq} \Psi_\kappa(\tilde{x}).$$

4. For any $\kappa \geq 1$ and $\tilde{x} \in \mathbb{X}$
$$\Psi_\kappa(\tilde{x}) < +\infty.$$

With these steps at hand, we may prove (4.3) as follows. First, for any $a_0 \leq a_1 < 0$, we choose some δ_0 such that $\delta_0 > -a_0 > 0$. For each δ_0, there exists $\kappa_0 \geq 1$ such that $\mathbb{P}(\log|\tilde{x}R_{\kappa_0}| \geq \delta_0) > 0$ (see (4.2) above). Since $\delta_0 > -a_0$, we have $\mathbb{P}_{\tilde{x}, a}(S_{\kappa_0} > 0) > 0$, which implies $\Phi_{\kappa_o}(\tilde{x}_0, a_0) > 0$. On

the other hand, since $a_0 \leq a_1$, step 3 and step 4 yield $\Phi_{\kappa_o}(\tilde{x}_0, a_0) < +\infty$. Therefore, we can apply step 2 and it yields $\Phi_\kappa(\tilde{x}, a) < +\infty$ for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$. Finally, thanks to Step 1, (4.3) arrives.

Step 1. It is easy to see that

$$
\begin{aligned}
\Phi(\tilde{x}, a) \quad &\leq \quad \sum_{r=1}^{\kappa-1} \mathbb{E}_{\tilde{x},a}[e^{-\lambda S_r}] + \sum_{n=1}^{+\infty} \sum_{r=0}^{\kappa-1} \mathbb{E}_{\tilde{x},a}[e^{-\lambda S_{n\kappa+r}}; S_\kappa > 0, \ldots, S_{n\kappa} > 0] \\
&\leq \quad \sum_{r=1}^{\kappa-1} \mathbb{E}_{\tilde{x},a}\left(e^{-\lambda S_r}\right) \\
&\quad + \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x},a}\left(e^{-\lambda S_{n\kappa}}; S_\kappa > 0, \ldots, S_{n\kappa} > 0\right) \times \sum_{r=0}^{\kappa-1} \sup_{\tilde{y}\in\mathbb{X}} \mathbb{E}_{\tilde{y},a}\left(e^{-\lambda S_r}\right) \\
&\leq \quad \left(\sum_{r=0}^{\kappa-1} \sup_{\tilde{y}\in\mathbb{X}} \mathbb{E}_{\tilde{y},a}\left(e^{-\lambda S_r}\right)\right)(1 + \Phi_\kappa(\tilde{x}, a)),
\end{aligned}
$$

which yields to the expected result with

$$
0 < C(\kappa) = \sum_{r=0}^{\kappa-1} \sup_{\tilde{y}\in\mathbb{X}} \mathbb{E}_{\tilde{y},a}[e^{-\lambda S_r}] < +\infty.
$$

Step 2. The inequality $\Phi_\kappa(\tilde{x}_0, a_0) > 0$ implies that $\mathbb{P}(\log|\tilde{x}_0 R_\kappa| > -a_0) > 0$; we thus fix $\delta > -a_0 > 0$ and $\kappa \geq 1$ such that $\mu^{*\kappa}(E_\delta) > 0$. Since $a_0 < 0$, this property may hold only when κ is large enough; this happens for instance when the support of μ is bounded.

To simplify the notations, we assume that $-a_0 < \delta$ where δ is given by H5. We set $\kappa = 1$ and write

$$
\begin{aligned}
\Phi(\tilde{x}_0, a_0) \quad &= \quad \sum_{n=1}^{+\infty} \mathbb{E}\left[|\tilde{x}_0 R_n|^{-\lambda}; |\tilde{x}_0 R_1| > e^{-a_0}, \ldots, |\tilde{x}_0 R_n| > e^{-a_0}\right] \\
&\geq \quad \int_{\{g\in S^+:|\tilde{x}_0 g|\geq e^{-a_0}\}} \sum_{n=2}^{+\infty} \mathbb{E}\left[|\tilde{x}_0 g R_{1,n}|^{-\lambda}; |\tilde{x}_0 g| > e^{-a_0}, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \ldots, |\tilde{x}_0 g R_{1,n}| > e^{-a_0}\right] \mu(dg) \\
&\geq \quad \int_{E_\delta} \sum_{n=2}^{+\infty} \mathbb{E}\left[|\tilde{x}_0 g R_{1,n}|^{-\lambda}; |\tilde{x}_0 g| \geq e^\delta > e^{-a_0}, |\tilde{x}_0 g R_{1,2}| > e^{-a_0}, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \ldots, |\tilde{x}_0 g R_{1,n}| > e^{-a_0}\right] \mu(dg)
\end{aligned}
$$

$$= \int_{E_\delta} |\tilde{x}_0 g|^{-\lambda} \sum_{m=1}^{+\infty} \mathbb{E}\Big[|(\tilde{x}_0 \cdot g)R_m|^{-\lambda}; |(\tilde{x}_0 \cdot g)R_1| > e^{-a_0 - \log|\tilde{x}_0 g|},$$

$$\ldots, |(\tilde{x}_0 \cdot g)R_k| > e^{-a_0 - \log|\tilde{x}_0 g|}\Big]\mu(dg)$$

$$= \int_{E_\delta} |\tilde{x}_0 g|^{-\lambda} \Phi(\tilde{x}_0 \cdot g, a_0 + \log|\tilde{x}_0 g|)\mu(dg),$$

so that

$$(4.4) \qquad \Phi(\tilde{x}_0, a_0) \geq \int_{E_\delta} |\tilde{x}_0 g|^{-\lambda} \Phi(\tilde{x}_0 \cdot g, a_0 + \delta)\mu(dg).$$

Consequently, if $\Phi(\tilde{x}_0, a_0) < +\infty$ then $\Phi(\tilde{x}_0 \cdot g, a_0 + \delta) < +\infty$ for μ-almost all $g \in E_\delta$ and by iterating this argument, there thus exists a sequence $(g_k)_{k \geq 1}$ of elements of E_δ such that

$$\forall k \geq 1, \quad \Phi(\tilde{x}_0 \cdot g_1 \cdots g_k, a_0 + k\delta) < +\infty.$$

By Lemma 2.1, for any $\tilde{x}, \tilde{y} \in \mathbb{X}$ and $a \in \mathbb{R}$

$$\Phi(\tilde{x}, a - \log c) \leq c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}[|R_n|^{-\lambda}; |R_1| > e^{-a}, \ldots, |R_n| > e^{-a}] \leq c^{2\lambda}\Phi(\tilde{y}, a + \log c);$$

it follows that, by choosing k sufficiently great such that $a_0 + k\delta > a + 2\log c$, we have

$$\Phi(\tilde{x}, a) \leq \Phi(\tilde{x}_0 \cdot g_1 \cdots g_k, a + 2\log c) \leq \Phi(\tilde{x}_0 \cdot g_1 \cdots g_k, a_0 + k\delta) < +\infty.$$

Step 3. For any $0 \leq k < n$, denote $L_{n,k} := M_{n-1} \ldots M_k$ and $L_{n,k} = I$ otherwise. Let $c > 1$ be the constant given by Lemma 2.1. For any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$, by using Lemma 2.1, we may write

$$\Phi_\kappa(\tilde{x}, a) = \sum_{n=1}^{+\infty} \mathbb{E}\left(|\tilde{x}R_{n\kappa}|^{-\lambda}; |\tilde{x}R_\kappa| > e^{-a}, \ldots, |\tilde{x}R_{n\kappa}| > e^{-a}\right)$$

$$\leq c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}\left(|R_{n\kappa}|^{-\lambda}; |R_\kappa| > \frac{e^{-a}}{c}, \ldots, |R_{n\kappa}| > \frac{e^{-a}}{c}\right)$$

so that, by duality principle and Lemma 2.1,

$$
\begin{aligned}
\Phi_\kappa(\tilde{x}, a) &\leq c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}\left(|L_{n\kappa}|^{-\lambda}; |L_{n\kappa,(n-1)\kappa}| > \frac{e^{-a}}{c}, \ldots, |L_{n\kappa}| > \frac{e^{-a}}{c}\right) \\
&= c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}[|L_{n\kappa}|^{-\lambda}; |L_{n\kappa,(n-1)\kappa}| \times |L_{(n-1)\kappa}| > |L_{(n-1)\kappa}|\frac{e^{-a}}{c}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots, |L_{n\kappa}| > \frac{e^{-a}}{c}] \\
&\leq c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}\left(|L_{n\kappa}|^{-\lambda}; |L_{n\kappa}| > |L_{(n-1)\kappa}|\frac{e^{-a}}{c^2}, \ldots, |L_{n\kappa}| > \frac{e^{-a}}{c^2}\right) \\
&\leq c^{2\lambda} \sum_{n=1}^{+\infty} \mathbb{E}\left(|L_{n\kappa}x|^{-\lambda}; |L_{n\kappa}x| > |L_{(n-1)\kappa}x|\frac{e^{-a}}{c^4}, \ldots, |L_{n\kappa}x| > \frac{e^{-a}}{c^4}\right)
\end{aligned}
$$

Consequently, setting $a_1 := -4\log c$ and using the fact that the map $a \mapsto \Phi_\kappa(\tilde{x}, a)$ is non decreasing for any $a \in \mathbb{R}$, one may write $\Phi_\kappa(\tilde{x}, a) \leq \Psi(\tilde{x})$ as long as $a < a_1$.

Step 4. To simplify the notations, we assume here $\kappa = 1$; the proof is the same when $\kappa \geq 2$. For any $\tilde{x} \in \mathbb{X}$ and $n \geq 0$, set $X_n' := L_n \cdot x$ and $S_n' := \log |L_n x|$; the random process $(X_n', S_n')_{n \geq 0}$ is a Markov walk on $\mathbb{X} \times \mathbb{R}$ starting from $(x, 0)$ and whose transitions are governed by the ones of the Markov chain $(X_n')_{n \geq 0}$ on $\mathbb{X}$. To study the quantity $\Psi(\tilde{x})$, we follow the strategy developed in the case of one dimensional random walks on $\mathbb{R}$ with independent increments and we thus introduce the sequence $(\eta_j)_{j \geq 0}$ of ladder epochs of $(S_n')_{n \geq 0}$ defined by

$$
\eta_1 = 0, \; \eta_{j+1} = \eta_{j+1}(x) := \min\left\{n > \eta_j : \log |L_n x| > \log |L_{\eta_j} x|\right\}, \; j \geq 0.
$$

For any $\tilde{x} \in \mathbb{X}$, one may write

$$
\begin{aligned}
\Psi(\tilde{x}) &= \sum_{n=1}^{+\infty} \mathbb{E}\left(|L_n x|^{-\lambda}; \exists j \geq 1 \mid n = \eta_j\right) \\
&= \sum_{j=1}^{+\infty} \mathbb{E}\left(|L_{\eta_j} x|^{-\lambda}\right).
\end{aligned}
$$
(4.5)

Let Q' denote the transition kernel of the Markov walk $(X_n', S_n')_{n \geq 0}$ and $G_{Q'} := \sum_{n=0}^{+\infty} Q'^n$ its Green kernel. The sub-process $(X_{\eta_j}', S_{\eta_j}')_{j \leq 0}$ is also a

Markov chain, its transition kernel Q'_η is given by: for any bounded Borel function $\phi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$ and for any $x \in \mathbb{X}$, $a \in \mathbb{R}$,

$$
\begin{aligned}
Q'_\eta \phi(x, a) &= \mathbb{E}\left(\phi(X'_{\eta_1}, a + S'_{\eta_1}) | X'_0 = x\right) \\
&= \sum_{n=1}^{+\infty} \mathbb{E}\left(\phi(L_n \cdot x, a + \log |L_n x|); \eta_1 = n\right) \\
&= \sum_{n=1}^{+\infty} \mathbb{E}[\phi(L_n \cdot x, a + \log |L_n x|); \\
&\qquad\qquad\qquad |L_1 x| \leq 1, \cdots, |L_{n-1} x| \leq 1, |L_n x| > 1].
\end{aligned}
$$

Let G'_η denote the Green kernel associated with the process $(X'_{\eta_j}, S'_{\eta_j})_{j \geq 0}$; by (4.5)

$$
\begin{aligned}
1 + \Psi(\tilde{x}) &= \sum_{j=0}^{+\infty} \mathbb{E}\left(|L_{\eta_j} x|^{-\lambda}\right) \\
&= \sum_{j=0}^{+\infty} \int_{\mathbb{X}} \int_{\mathbb{R}} e^{-\lambda a} (Q'_\eta)^j((x, 0), dyda) \\
&= \int_{\mathbb{X}} \int_{\mathbb{R}} e^{-\lambda a} G'_\eta((x, 0), dyda).
\end{aligned}
$$

The Markov walk $(X'_n, S'_n)_{n \geq 0}$ has been studied by many authors (see for instance [9], [14] or [13]). All the works are based on the fact that the transition kernel of the chain $(X'_n)_{n \geq 0}$ has some "nice" spectral properties, namely its restriction to the space of Lipschitz functions on $\mathbb{X}$ is quasi-compact. In particular, it allows these authors to prove that the classical renewal theorem remains valid for this Markov walk on $\mathbb{X} \times \mathbb{R}$ as long as it is not centered, that is $\pi_\mu = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}[\log |L_n|] \neq 0$; in this case one may prove in particular that, for any $\tilde{x} \in \mathbb{X}$, the quantity $G_{Q'}((x, 0), \mathbb{X} \times [0, a])$ is equivalent to $\frac{a}{\pi}$ as $a \to +\infty$ [14]. For the behavior as $a \to +\infty$ of $G'_\eta((x, 0), \mathbb{X} \times [0, a])$, the situation is way different. On one hand, it is easier since for any $j \geq 1$ the random variables S'_{η_j} are strictly positive, one might thus expect a similar result; on the other hand, the control of the spectrum of the transition kernel Q'_η remains unfortunately unknown in this circumstance, in particular the transition kernel Q'_η does not even act on the space of continuous functions on $\mathbb{X}$! Nevertheless, we have the following weak result with the postponed proof at the end of this subsection.

FACT 4.1.   *There exists $C > 0$ such that for any $\tilde{x} \in \mathbb{X}$ and $a \geq 0$*

$$G'_\eta((x,0), \mathbb{X} \times [0,a]) = \sum_{j=0}^{+\infty} \mathbb{P}(\log |L_{\eta_j} x| \leq a) \leq Ca.$$

It follows that

$$
\begin{aligned}
1 + \Psi(\tilde{x}) &= \int_{\mathbb{X}} \int_{\mathbb{R}_*^+} e^{-\lambda a} G'_\eta((x,0), dyda) \\
&\leq e^\lambda \sum_{a=1}^{+\infty} e^{-\lambda a} G'_\eta((x,0), \mathbb{X} \times [a-1, a]) \\
&\leq e^\lambda \sum_{a=1}^{+\infty} e^{-\lambda a} G'_\eta((x,0), \mathbb{X} \times [0, a]) \\
&\leq Ce^\lambda \sum_{a=1}^{+\infty} a e^{-\lambda a} < +\infty.
\end{aligned}
$$

To complete the proof of Step 4, it remains to prove Fact 4.1. First, by definition of E_δ, for any $j \geq 0$ and $\tilde{x} \in \mathbb{X}$, we may write $S'_{\eta_{j+1}} - S'_{\eta_j} \geq \delta 1_{E_\delta}(M_{\eta_j})$; setting $\varepsilon_j := 1_{E_\delta}(M_{\eta_j})$, this yields $S'_{\eta_j} \geq \delta(\varepsilon_0 + \ldots + \varepsilon_{j-1})$ so that

$$
\begin{aligned}
G'_\eta((x,0), \mathbb{X} \times [0,a]) &= \sum_{j=0}^{+\infty} \mathbb{P}(X'_{\eta_j} \in \mathbb{X}, S'_{\eta_j} \in [0,a] | X'_0 = x) \\
&\leq \sum_{j=0}^{+\infty} \mathbb{E}\left(1_{[0,a]}(S'_{\eta_j}) | X'_0 = x \right) \\
&\leq \sum_{j=0}^{+\infty} \mathbb{E}\left(1_{[0,a]}(\delta(\varepsilon_0 + \cdots + \varepsilon_{j-1})) \right).
\end{aligned}
$$

To conclude, we use the fact that $(\varepsilon_i)_{i \geq 0}$ is a sequence of i.i.d. random variables; the Elementary Renewal Theorem for the Bernoulli random walk process $[(\varepsilon_0 + \ldots + \varepsilon_{j-1})]_{j \geq 0}$ implies

$$G'_\eta((x,0), X \times [0,a]) \leq \mathbb{E}\left(\sum_{j=1}^{+\infty} 1_{[0,a]}(\delta(\varepsilon_0 + \ldots + \varepsilon_{j-1})) \right) \preceq a.$$

4.3. *Proof of Lemma 3.2.*   We claim that

(4.6) $$\lim_{n \to +\infty} \widehat{\mathbb{E}}_{\tilde{x},a} \left| \frac{1}{q_n^{(i)}} - \frac{1}{q_\infty^{(i)}} \right| = 0.$$

By definition, the quantities $q_n^{(i)}$ are always less than or equal to 1. Therefore, (4.6) implies that the same property holds $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost surely for $q_\infty^{(i)}$. Hence,

$$|q_n^{(i)} - q_\infty^{(i)}| = q_n^{(i)} q_\infty^{(i)} \left| \frac{1}{q_n^{(i)}} - \frac{1}{q_\infty^{(i)}} \right| \leq \left| \frac{1}{q_n^{(i)}} - \frac{1}{q_\infty^{(i)}} \right|.$$

Using (4.6) again, we find that $\lim\limits_{n \to +\infty} \widehat{\mathbb{E}}_{\tilde{x},a} |q_n^{(i)} - q_\infty^{(i)}| = 0$. In particular,

$$\lim\limits_{n \to +\infty} \widehat{\mathbb{P}}_{\tilde{x},a}^{(i)}(Z_n \neq \tilde{\mathbf{0}}) = \lim\limits_{n \to +\infty} \widehat{\mathbb{E}}_{\tilde{x},a} q_n^{(i)} = \widehat{\mathbb{E}}_{\tilde{x},a} q_\infty^{(i)},$$

which is the assertion of (3.12).

Finally, it remains to verify (4.6). From (3.2) and (3.3)

$$\left| \frac{1}{q_n^{(i)}} - \frac{1}{q_\infty^{(i)}} \right| \leq \left| \frac{1}{\tilde{e}_i R_n \mathbf{1}} \right|$$
$$+ \sum_{k=0}^{n-1} \left| \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,n}} - \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}} \right| + \sum_{k=n}^{+\infty} \left| \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}} \right|$$

so that

$$\widehat{\mathbb{E}}_{\tilde{x},a} \left| \frac{1}{q_n^{(i)}} - \frac{1}{q_\infty^{(i)}} \right| \leq I_n + J_n + K_n$$

with

$$I_n = \widehat{\mathbb{E}}_{\tilde{x},a} \left| \frac{1}{\tilde{e}_i R_n \mathbf{1}} \right|,$$
$$J_n = \sum_{k=0}^{n-1} \widehat{\mathbb{E}}_{\tilde{x},a} \left| \gamma_k \frac{\tilde{e}_i R_k (Y_{k+1,n} - Y_{k+1,\infty})}{(\tilde{e}_i R_k Y_{k+1,n})(\tilde{e}_i R_k Y_{k+1,\infty})} \right|,$$
$$K_n = \sum_{k=n}^{+\infty} \widehat{\mathbb{E}}_{\tilde{x},a} \left| \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}} \right|.$$

By using Lemma 2.1, it is obvious that Lemma 3.1 implies

$$(4.7) \qquad \sum_{k=0}^{+\infty} \widehat{\mathbb{E}}_{\tilde{x},a} \left(\frac{1}{|R_k|} \right) < +\infty.$$

Besides, as a direct consequence of Lemma 2.1

$$(4.8) \qquad I_n \preceq \widehat{\mathbb{E}}_{\tilde{x},a} \left(\frac{1}{|R_n|} \right),$$

$$(4.9) \qquad \left| \gamma_k \frac{\tilde{e}_i R_k (Y_{k+1,n} - Y_{k+1,\infty})}{(\tilde{e}_i R_k Y_{k+1,n})(\tilde{e}_i R_k Y_{k+1,\infty})} \right| \preceq \frac{|Y_{k+1,n} - Y_{k+1,\infty}|}{|R_k|} \preceq \frac{1}{|R_k|},$$

$$(4.10) \qquad K_n \preceq \sum_{k=n}^{+\infty} \widehat{\mathbb{E}}_{\tilde{x},a} \left(\frac{1}{|R_k|} \right).$$

Hence

- (4.7) and (4.8) implies $I_n \to 0$ as $n \to \infty$.
- since $Y_{k+1,n}(\omega) \to Y_{k+1,\infty}(\omega)$ for any $\omega \in \Omega$ (see Section 3.1), this convergence holds in particular $\hat{\mathbb{P}}_{\tilde{x},a}$-almost surely; combining (4.7), (4.9) and the the dominated convergence Theorem, we obtain $J_n \to 0$ as $n \to +\infty$.
- by (4.7) and (4.10), the term C_l can be made arbitrarily small by choosing l sufficiently great.

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