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Random walks in Weyl chambers and crystals

Cédric Lecouvey, Emmanuel Lesigne and Marc Peigné

Abstract

We use Kashiwara crystal basis theory to associate a random walk $\mathcal{W}$ to each irreducible representation V of a simple Lie algebra. This is achieved by endowing the crystal attached to V with a (possibly non-uniform) probability distribution compatible with its weight graduation. We then prove that the generalized Pitman transform defined for similar random walks with uniform distributions yields yet another Markov chain. When the representation is minuscule, and the associated random walk has a drift in the Weyl chamber, we establish that this Markov chain has the same law as $\mathcal{W}$ conditioned to never exit the cone of dominant weights. For the defining representation V of $\mathfrak{gl}_n$, we notably recover the main result of O'Connell [*Trans. Amer. Math. Soc.* 355 (2003) 3669–3697]. At the heart of our proof is a quotient version of a renewal theorem which holds for a general random walk on a lattice. This theorem also has applications in representation theory since it permits the behaviour of some outer multiplicities for large dominant weights to be described.

1. *Introduction*

The purpose of this article is to study some interactions between representation theory of simple Lie algebras over $\mathbb{C}$ and certain random walks defined on lattices in Euclidean spaces. We provide both results on random walks conditioned to never exit a cone and identities related to asymptotic representation theory.

The well-known ballot walk in $\mathbb{R}^n$ appears as a particular case of the random walks we consider here. Let $B = (\varepsilon_1, \ldots, \varepsilon_n)$ be the standard basis of $\mathbb{R}^n$. The ballot walk can be defined as the Markov chain $(\mathcal{W}_\ell = X_1 + \ldots + X_\ell)_{\ell \geqslant 1}$, where $(X_k)_{k \geqslant 1}$ is a sequence of independent and identically distributed random variables taking values in the base B. Our main motivation is to generalize results due to O'Connell [**18**, **19**] given the law of the random walk $\mathcal{W} = (\mathcal{W}_\ell)_{\ell \geqslant 1}$ conditioned to never exit the Weyl chamber $\overline{C} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geqslant \ldots \geqslant x_n\}$. This is achieved in [**18**] by considering first a natural transformation $\mathfrak{P}$ which associates to any path with steps in B a path in the Weyl chamber $\overline{C}$, next by checking that the image of the random walk $\mathcal{W}_\ell$ by this transformation is a Markov chain and finally by establishing that this Markov chain has the same law as $\mathcal{W}$ conditioned to never exit $\overline{C}$. The transformation $\mathfrak{P}$ is based on the Robinson–Schensted correspondence which maps the words on the ordered alphabet $\mathcal{A}_n = \{1 < \ldots < n\}$ (regarded as finite paths in $\mathbb{R}^n$) on pairs of semistandard tableaux. It can be reinterpreted in terms of Kashiwara's crystal basis theory [**12**] (or equivalently in terms of the Littelmann path model). Each path of length ℓ with steps in B is then interpreted as a vertex in the crystal $B(\omega_1)^{\otimes \ell}$ (see § 5.1 for basics on crystals) corresponding to the ℓ-tensor power of the defining representation of $\mathfrak{sl}_n$. The transformation $\mathfrak{P}$ associates to each vertex $b \in B(\omega_1)^{\otimes \ell}$ the highest weight vertex of $B(b)$, where $B(b)$ denotes the connected component of $B(\omega_1)^{\otimes \ell}$ containing b.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with weight lattice $P \subset \mathbb{R}^N$ and dominant weights P_+ (see § 2 for some background on root systems and representation theory). Write also $\overline{C}$ for the corresponding Weyl chamber. For any dominant weight $\delta \in P_+$, let $V(\delta)$ be the

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(finite-dimensional) irreducible representation of $\mathfrak{g}$ of highest weight δ and denote by $B(\delta)$ its Kashiwara crystal graph. One can then consider the transformation $\mathfrak{P}$ which maps any $b \in B(\delta)^{\otimes \ell}$ on the highest weight vertex $\mathfrak{P}(b)$ of the connected component $B(b)$ of the graph $B(\delta)^{\otimes \ell}$ containing the vertex b. This simply means that $\mathfrak{P}(b)$ is the source vertex of $B(b)$ considered as an oriented graph. This transformation was introduced in [**1**]. As described in [**1**], it can be interpreted as a generalization of the Pitman transform for one-dimensional paths.

In order to define a random walk from the tensor powers $B(\delta)^{\otimes \ell}$, we need first to endow $B(\delta)$ with a probability distribution $(p_a)_{a \in B(\delta)}$. Contrary to [**1**] where the random walks considered are discrete versions of Brownian motions, we will consider non-uniform distributions on $B(\delta)$ and use random walks with drift in C. Once we have defined a probability on $B(\delta)$, it suffices to consider the product probability on $B(\delta)^{\otimes \ell}$: the probability associated to the vertex $b = a_1 \otimes \ldots \otimes a_\ell \in B(\delta)^{\otimes \ell}$ is then $p_b = p_{a_1} \ldots p_{a_\ell}$. In order to use Kashiwara crystal basis theory, it is also natural to impose that the distributions we consider on the crystal $B(\delta)^{\otimes \ell}$ are compatible with their weight graduation wt, that is $p_b = p_{b'}$ whenever $\mathrm{wt}(b) = \mathrm{wt}(b')$. Since $\mathrm{wt}(b) = \mathrm{wt}(a_1) + \ldots + \mathrm{wt}(a_\ell)$, the two conditions $p_b = p_{a_1} \ldots p_{a_\ell}$ and $p_b = p_{b'}$ whenever $\mathrm{wt}(b) = \mathrm{wt}(b')$ essentially impose that our initial distribution on $B(\delta)$ should be exponential with respect to the weight graduation. To be more precise, recall that $B(\delta)$ is an oriented and coloured graph with arrows $a \xrightarrow{i} a'$ labelled by the simple roots α_i, $i = 1, \ldots, n$, of $\mathfrak{g}$. The arrow $a \xrightarrow{i} a'$ indicates, in particular, that $\mathrm{wt}(a') = \mathrm{wt}(a) - \alpha_i$. We thus associate to each simple root α_i a positive real number t_i and consider only probability distributions on $B(\delta)$ such that $a \xrightarrow{i} a'$ implies $p_{a'} = p_a \times t_i$. These distributions and their corresponding product on $B(\delta)^{\otimes \ell}$ are then compatible with the weight graduation as required. The uniform distribution corresponds to the case when $t_1 = \ldots = t_n = 1$.

The tensor power $B(\delta)^{\otimes \mathbb{N}}$ (defined as the projective limit of the crystals $B(\delta)^{\otimes \ell}$) can then also be endowed with a probability distribution of product type. We define the random variable $\mathcal{W}_\ell$ on $B(\delta)^{\otimes \mathbb{N}}$ by $\mathcal{W}_\ell(b) = \mathrm{wt}(b(\ell))$ where $b(\ell) = \bigotimes_{k=1}^{\ell} a_k \in B(\delta)^{\otimes \ell}$ for any $b = \bigotimes_{k=1}^{+\infty} a_k \in B(\delta)^{\otimes \mathbb{N}}$. This yields a random walk $\mathcal{W} = (\mathcal{W}_\ell)_{\ell \geqslant 1}$ on P whose transition matrix $\Pi_{\mathcal{W}}$ can be easily computed (see (5.10)). Our next step is to introduce the random process $\mathcal{H} = (\mathcal{H}_\ell)_{\ell \geqslant 1}$ on P_+ such that $\mathcal{H}_\ell(b) = \mathrm{wt}(\mathfrak{P}(b(\ell)))$ for any $b \in B(\delta)^{\otimes \mathbb{N}}$. We prove in Theorem 6.2 that $\mathcal{H}$ is a Markov chain and compute its transition matrix $\Pi_{\mathcal{H}}$ in terms of the Weyl characters of the irreducible representations $V(\lambda), \lambda \in P_+$.

Write $\Pi_{\mathcal{W}}^{\overline{C}}$ for the restriction of $\Pi_{\mathcal{W}}$ to the Weyl chamber $\overline{C}$. Proposition 7.1 states that $\Pi_{\mathcal{H}}$ can be regarded as a Doob h-transform of $\Pi_{\mathcal{W}}^{\overline{C}}$ if and only if δ is a minuscule weight. As in [**18**, **19**], we use a theorem of Doob (Theorem 3.1) to compare the law of $\mathcal{W}$ conditioned to stay in $\overline{C}$ with the law of $\mathcal{H}$. Nevertheless the proofs in [**18**, **19**] use, as a key argument, asymptotic behaviour of the outer multiplicities in $V(\mu) \otimes V(\delta)^{\otimes \ell}$ when δ is the defining representation of $\mathfrak{sl}_n$. These limits seem very difficult to obtain for any minuscule representation by purely algebraic means. To overcome this problem, we establish Theorems 4.3 and 4.6 which can be seen as quotient Local Limit Theorem and quotient Renewal Theorem for large deviations of a random walk conditioned to stay in a cone. They hold in the general context of random walks in a lattice with drift m in the interior C of a fixed closed cone $\overline{C}$. This yields Theorem 7.6 equating the law of $\mathcal{W}$ conditioned to never exit $\overline{C}$ with the one of $\mathcal{H}$ provided δ is a minuscule representation and $m = \mathbb{E}(\mathcal{W}_1)$ belongs to C.

The quotient Local Limit Theorem also has some applications in representation theory since it provides the asymptotic behaviour of the outer multiplicities in $V(\mu) \otimes V(\delta)^{\otimes \ell}$ when δ is a minuscule representation (Theorem 8.1).

The paper is organized as follows. Sections 2 and 3 are, respectively, devoted to basics on representation theory of Lie algebras and Markov chains. In § 4, we state and prove the

probability results (Theorems 4.3 and 4.6). Section 5 details the construction of the random walk $\mathcal{W}$. The transition matrix of the Markov chain $\mathcal{H}$ is computed in § 6. The main results of § 7 are Theorem 7.6 and Corollary 7.7. This corollary gives an explicit formula for the probability that $\mathcal{W}$ remains forever in $\overline{C}$. Finally, in § 8, we study random walks defined from non irreducible representations and prove Theorem 8.1.

2. *Background on representation theory*

We recall in the following paragraphs some classical background on representation theory of simple Lie algebras that we shall need in the sequel. For a complete review, the reader is referred to [3] or [9].

2.1. *Root systems*

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ be a triangular decomposition. We shall follow the notation and conventions of Bourbaki [3]. According to the Cartan-Killing classification, $\mathfrak{g}$ is characterized (up to isomorphism) by its root system. This root system is realized in an Euclidean space $\mathbb{R}^N$ with standard basis $B = (\varepsilon_1, \ldots, \varepsilon_N)$. We denote by $\Delta_+ = \{\alpha_i \mid i \in I\}$ the set of simple roots of $\mathfrak{g}$ and by R_+ the (finite) set of positive roots; let $n = \mathrm{card}(I)$ be the rank of $\mathfrak{g}$. The root lattice of $\mathfrak{g}$ is the integral lattice $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$. Write $\omega_i,\ i = 1, \ldots, n$ for the fundamental weights associated to $\mathfrak{g}$. The weight lattice associated to $\mathfrak{g}$ is the integral lattice $P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$. It can be regarded as an integral sublattice of $\mathfrak{h}^*_{\mathbb{R}}$. We have $\dim(P) = \dim(Q) = n$ and $Q \subset P$.

The cone of dominant weights for $\mathfrak{g}$ is obtained by considering the positive integral linear combinations of the fundamental weights, that is

$$P_+ = \bigoplus_{i=1}^n \mathbb{N}\omega_i.$$

The corresponding Weyl chamber is the cone $C = \bigoplus_{i=1}^n \mathbb{R}_{>0}\omega_i$. We also introduce its closure $\overline{C} = \bigoplus_{i=1}^n \mathbb{R}_{\geqslant 0}\omega_i$. In type A, we shall use, for simplicity, the weight lattice of $\mathfrak{gl}_n$ rather than the one of $\mathfrak{sl}_n$. We also introduce the Weyl group W of $\mathfrak{g}$ which is the group generated by the reflections s_i through the hyperplanes perpendicular to the simple root $\alpha_i,\ i = 1, \ldots, n$; each $w \in W$ may be decomposed as a product of the $s_i,\ i = 1, \ldots, n$. All the minimal length decompositions of w have the same length $l(w)$. For any weight β, the orbit $W \cdot \beta$ of β under the action of W intersects P_+ in a unique point. We define a partial order on P, setting $\mu \leqslant \lambda$ if $\lambda - \mu$ belongs to $Q_+ = \bigoplus_{i=1}^n \mathbb{N}\alpha_i$.

Example 1. (i) For $\mathfrak{g} = \mathfrak{gl}_n,\ P_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0\}$ and W coincides with the permutation group on $\mathcal{A}_n = \{1, \ldots, n\}$. We have $N = n$. For any $\sigma \in W$ and any $v = (v_1, \ldots, v_n)$, we have $\sigma \cdot v = (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$.

(ii) For $\mathfrak{g} = \mathfrak{sp}_{2n},\ P_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0\}$ and W coincides with the group of signed permutations on $\mathcal{B}_n = \{\overline{n}, \ldots, \overline{1}, 1, \ldots, n\}$, that is, permutations w of $\mathcal{B}_n$ such that $\overline{w(x)} = w(\overline{x})$ for any $x \in \mathcal{B}_n$ (here, we write $\overline{x}$ for $-x$ so that $\overline{\overline{x}} = x$). We have $N = n$. For any $\sigma \in W$ and any $v = (v_1, \ldots, v_n)$, we have $\sigma \cdot v = (\theta_1 v_{|\sigma(1)|}, \ldots, \theta_n v_{|\sigma(n)|})$ where $\theta_i = 1$ if $\sigma(i)$ is positive and $\theta_i = -1$ otherwise.

(iii) For $\mathfrak{g} = \mathfrak{so}_{2n+1},\ P_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \sqcup (\mathbb{Z} + \frac{1}{2})^n \mid \lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0\}$. The Weyl group W is the same as for $\mathfrak{sp}_{2n}$.

(iv) For $\mathfrak{g} = \mathfrak{so}_{2n},\ P_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \sqcup (\mathbb{Z} + \frac{1}{2})^n \mid \lambda_1 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant |\lambda_n| \geqslant 0\}$. The Weyl group is the subgroup of the signed permutations on $\mathcal{B}_n$ switching an even number of signs.

2.2. *Highest weight modules*

Let $U(\mathfrak{g})$ be the enveloping algebra associated to $\mathfrak{g}$. Each finite-dimensional $\mathfrak{g}$ (or $U(\mathfrak{g})$)-module M admits a decomposition in weight spaces

$$M = \bigoplus_{\mu \in P} M_\mu,$$

where $M_\mu := \{v \in M \mid h(v) = \mu(h)v \text{ for any } h \in \mathfrak{h}\}$ and P is identified with a sublattice of $\mathfrak{h}^*_{\mathbb{R}}$. In particular, $(M \oplus M')_\mu = M_\mu \oplus M'_\mu$. The Weyl group W acts on the weights of M and for any $\sigma \in W$, we have $\dim M_\mu = \dim M_{\sigma\cdot\mu}$. The character of M is the Laurent polynomial in the variables $x_1, \dots, x_N$

$$\mathrm{char}(M)(x) := \sum_{\mu \in P} \dim(M_\mu) x^\mu,$$

where $\dim(M_\mu)$ is the dimension of the weight space M_μ and $x^\mu = x_1^{\mu_1} \dots x_N^{\mu_N}$ with $(\mu_1, \dots, \mu_N)$ the coordinates of μ on the standard basis $\{\varepsilon_1, \dots, \varepsilon_N\}$.

The irreducible finite-dimensional representations of $\mathfrak{g}$ are labelled by the dominant weights. For each dominant weight $\lambda \in P_+$, let $V(\lambda)$ be the irreducible representation of $\mathfrak{g}$ associated to λ. The category $\mathcal{C}$ of finite-dimensional representations of $\mathfrak{g}$ over $\mathbb{C}$ is semisimple: each module decomposes into irreducible components. The category $\mathcal{C}$ is equivariant to the (semisimple) category of finite-dimensional $U(\mathfrak{g})$-modules (over $\mathbb{C}$). Any finite-dimensional $U(\mathfrak{g})$-module M decomposes as a direct sum of irreducible

$$M = \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus m_{M,\lambda}},$$

where $m_{M,\lambda}$ is the multiplicity of $V(\lambda)$ in M. Here, we slightly abuse the notation, denoting by $V(\lambda)$ the irreducible finite-dimensional $U(\mathfrak{g})$-module associated to λ.

When $M = V(\lambda)$ is irreducible, we set

$$s_\lambda(x) := \mathrm{char}(M)(x) = \sum_{\mu \in P} K_{\lambda,\mu} x^\mu$$

with $\dim(M_\mu) = K_{\lambda,\mu}$. Then $K_{\lambda,\mu} \neq 0$ only if $\mu \leqslant \lambda$. The characters can be computed from the Weyl character formula

$$s_\lambda(x) = \frac{\sum_{w \in W} \varepsilon(w) x^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+}(1 - x^{-\alpha})}, \tag{2.1}$$

where $\varepsilon(w) = (-1)^{\ell(w)}$ and $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha_i$ is the half sum of positive roots.

Given δ and μ in P_+ and a nonnegative integer ℓ, we define the tensor multiplicities $f^\ell_{\lambda/\mu,\delta}$ by

$$V(\mu) \otimes V(\delta)^{\otimes \ell} \simeq \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus f^\ell_{\lambda/\mu,\delta}}. \tag{2.2}$$

For $\mu = 0$, we set $f^\ell_{\lambda,\delta} = f^\ell_{\lambda/0,\delta}$. When there is no risk of confusion, we simply write $f^\ell_{\lambda/\mu}$ (respectively f^ℓ_λ) instead of $f^\ell_{\lambda/\mu,\delta}$ (respectively $f^\ell_{\lambda,\delta}$). We also define the multiplicities $m^\lambda_{\mu,\delta}$ by

$$V(\mu) \otimes V(\delta) \simeq \bigoplus_{\mu \rightsquigarrow \lambda} V(\lambda)^{\oplus m^\lambda_{\mu,\delta}} \tag{2.3}$$

where the notation $\mu \rightsquigarrow \lambda$ means that $\lambda \in P_+$ and $V(\lambda)$ appears as an irreducible component of $V(\mu) \otimes V(\delta)$. We have, in particular, $m^\lambda_{\mu,\delta} = f^1_{\lambda/\mu,\delta}$.

LEMMA 2.1. *Consider $\mu \in P_+$. Then for any $\beta \in P$, we have*

$$\sum_{\lambda \in P_+, \mu \rightsquigarrow \lambda} m^{\lambda}_{\mu,\delta} K_{\lambda,\beta} = \sum_{\gamma \in P} K_{\mu,\gamma} K_{\delta,\beta-\gamma}.$$

Proof. According to (2.3), we have

$$s_\mu(x) \times s_\delta(x) = \sum_{\mu \rightsquigarrow \lambda} \sum_{\beta \in P} m^{\lambda}_{\mu,\delta} K_{\lambda,\beta} x^\beta.$$

On the other hand,

$$s_\mu(x) \times s_\delta(x) = \sum_{\gamma \in P} K_{\mu,\gamma} x^\gamma \sum_{\xi \in P} K_{\delta,\xi} x^\xi = \sum_{\beta \in P} \sum_{\gamma \in P} K_{\mu,\gamma} K_{\delta,\beta-\gamma} x^\beta,$$

and the lemma follows. □

2.3. *Minuscule representations*

The irreducible representation $V(\delta)$ is said to be minuscule when the orbit $W \cdot \delta$ of the highest weight δ under the action of the Weyl group W contains all the weights of $V(\delta)$. In that case, the dominant weight δ is also called minuscule. The minuscule weights are fundamental weights and each weight space in $V(\delta)$ has dimension 1. They are given by the following table:

Type	Minuscule weights	N	Decomposition on B
A_n	$\omega_i,\ i = 1, \ldots, n$	$n+1$	$\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$
B_n	ω_n	n	$\omega_n = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$
C_n	ω_1	n	$\omega_1 = \varepsilon_1$
D_n	$\omega_1, \omega_{n-1}, \omega_n$	n	$\omega_1 = \varepsilon_1, \omega_{n+t} = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n) + t\varepsilon_n$, $t \in \{-1, 0\}$
E_6	ω_1, ω_6	8	$\omega_1 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6), \omega_6 = \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5$
E_7	ω_7	8	$\omega_7 = \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7)$.

Here, the four infinite families A_n, B_n, C_n and D_n correspond, respectively, to the classical Lie algebras $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$. When δ is minuscule, one may check that each nonzero multiplicity $m^{\lambda}_{\mu,\delta}$ in (2.3) is equal to 1.

REMARK 1. We will also need the following classical properties of minuscule representations.

(i) If δ is minuscule, then for any $\lambda, \mu \in P_+$, we have $K_{\delta,\lambda-\mu} = m^{\lambda}_{\mu,\delta} \in \{0, 1\}$.

(ii) If δ is not minuscule, then there exists a dominant weight $\kappa \neq \delta$ such that κ is a weight for $V(\delta)$, that is $K_{\delta,\kappa} \neq 0$. This follows from the fact that $V(\delta)$ contains a weight β which does not belong to $W \cdot \delta$. Then we can take $\kappa = P_+ \cap W \cdot \beta$.

2.4. *Paths in a weight lattice*

Consider a dominant weight δ associated to the Lie algebra $\mathfrak{g}$. We denote by $Z_\gamma(\delta, \ell)$ the set of paths of length ℓ in the weight lattice P starting at $\gamma \in P$ with steps in the set of weights of $V(\delta)$, that is, the weights β such that $V(\delta)_\beta \neq \{0\}$. When $\gamma = 0$, we write for short $Z(\delta, \ell) = Z_0(\delta, \ell)$. A path in $Z_\gamma(\delta, \ell)$ can be identified with a sequence $(\mu^{(0)}, \mu^{(1)} \ldots, \mu^{(\ell)})$ of weights of $V(\delta)$ such that $\mu^{(0)} = \gamma$. The position at the kth step corresponds to the weight $\mu^{(k)}$. When δ is a minuscule weight, the paths of $Z_\gamma(\delta, \ell)$ will be called minuscule. We will

see in § 5.1 that the paths in $Z(\delta,\ell)$ can then be identified with the vertices of the ℓth tensor product of the Kashiwara crystal graph associated to the representation $V(\delta)$.

Assume $\gamma \in P_+$. We denote by $Z_\gamma^+(\delta,\ell)$ the set of paths of length ℓ which never exit the closed Weyl chamber $\overline{C}$. Since $\overline{C}$ is convex, this is equivalent to saying that $\mu^{(k)} \in P_+$ for any $k = 0,\dots,\ell$.

Example 2. (i) For $\mathfrak{g} = \mathfrak{gl}_n$, the representation $V(\omega_1)$ is minuscule and has dimension n. The weights of $V(\omega_1)$ are the vectors $\varepsilon_1,\dots,\varepsilon_n$ of the standard basis B. The paths $Z(\omega_1,\ell)$ are those appearing in the classical ballot problem.

(ii) For $\mathfrak{g} = \mathfrak{sp}_{2n}$, the representation $V(\omega_1)$ is minuscule and has dimension $2n$. The weights of $V(\omega_1)$ are the vectors $\pm\varepsilon_1,\dots,\pm\varepsilon_n$. In particular, a path of $Z_\gamma(\omega_1,\ell)$ can return to its starting point γ.

(iii) For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the representation $V(\omega_1)$ is not minuscule and has dimension $2n+1$. The weights of $V(\omega_1)$ are 0 and the vectors $\pm\varepsilon_1,\dots,\pm\varepsilon_n$.

(iv) For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the representation $V(\omega_n)$ is minuscule and has dimension 2^n. The coordinates of the weights appearing in $V(\omega_n)$ on the standard basis are of the form $(\beta_1,\dots,\beta_n)$ where for any $i = 1,\dots,n$, $\beta_i = \pm\frac{1}{2}$.

Remark 2. In type A, the coordinates on the standard basis of the weights appearing in any representation are always nonnegative. This notably implies that a path in $Z_\gamma(\delta,\ell)$ can attain a fixed point of $\mathbb{R}^N$ at most one time. This special property does not hold in general for the paths of $Z_\gamma(\delta,\ell)$ in types other than type A. Indeed, the sign of the weight coordinates can be modified under the action of the Weyl group. In particular $Z_\gamma(\delta,\ell)$ contains paths of length $\ell > 0$ which return to γ. This phenomenon introduces some complications in the probabilistic estimations which is described in the following sections.

3. *Background on Markov chains*

3.1. *Markov chains and conditioning*

Consider a probability space $(\Omega,\mathcal{T},\mathbb{P})$ and a countable set M. Let $Y = (Y_\ell)_{\ell\geqslant 1}$ be a sequence of random variables defined on Ω with values in M. The sequence Y is a Markov chain when

$$\mathbb{P}(Y_{\ell+1} = y_{\ell+1} \mid Y_\ell = y_\ell,\dots,Y_1 = y_1) = \mathbb{P}(Y_{\ell+1} = y_{\ell+1} \mid Y_\ell = y_\ell)$$

for any $\ell \geqslant 1$ and any $y_1,\dots,y_\ell,y_{\ell+1} \in M$. The Markov chains considered in the sequel will also be assumed time homogeneous, that is, $\mathbb{P}(Y_{\ell+1} = y_{\ell+1} \mid Y_\ell = y_\ell) = \mathbb{P}(Y_\ell = y_{\ell+1} \mid Y_{\ell-1} = y_\ell)$ for any $\ell \geqslant 2$. For all x and y in M, the transition probability from x to y is then defined by

$$\Pi(x,y) = \mathbb{P}(Y_{\ell+1} = y \mid Y_\ell = x)$$

and we refer to Π as the transition matrix of the Markov chain Y. The distribution of Y_1 is called the initial distribution of the chain Y. It is well known that the initial distribution and the transition probability determine the law of the Markov chain. Conversely, given a probability distribution ν_0 on M and a (possibly infinite) stochastic matrix P (that is, $P = (p_{ij})_{i,j\in M}$ with $p_{ij} \geqslant 0$ and $\sum_j p_{ij} = 1$), there exists a Markov chain on M with initial distribution ν_0 and transition matrix P.

A class of examples of Markov chains is given by random walks on groups. Suppose that M has a group structure and that ν is a probability measure on M; the random walk of law ν is the Markov chain with transition probabilities $\Pi(x,y) = \nu(x^{-1}y)$; this Markov chain starting at the neutral element of M can be realized has $(X_1X_2\dots X_\ell)_{\ell\geqslant 1}$ where $(X_\ell)_{\ell\geqslant 1}$ is a sequence of independent and identically distributed random variables, with law ν.

Let Y be a Markov chain on $(\Omega, \mathcal{T}, \mathbb{P})$, whose initial distribution has full support, that is, $\mathbb{P}(Y_1 = x) > 0$ for any $x \in M$. Let $\mathcal{C}$ be a nonempty subset of M and consider the event $S = (Y_\ell \in \mathcal{C}$ for any $\ell \geqslant 1)$. Assume that $\mathbb{P}(S \mid Y_1 = \lambda) > 0$ for all $\lambda \in \mathcal{C}$. This implies that $\mathbb{P}[S] > 0$, and we can consider the conditional probability $\mathbb{Q}$ relative to this event: $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot|S]$.

It is easy to verify that, under this new probability $\mathbb{Q}$, the sequence (Y_ℓ) is still a Markov chain, with values in $\mathcal{C}$, and with transitions probabilities given by

$$\mathbb{Q}[Y_{\ell+1} = \lambda \mid Y_\ell = \mu] = \mathbb{P}[Y_{\ell+1} = \lambda \mid Y_\ell = \mu] \frac{\mathbb{P}[S \mid Y_1 = \lambda]}{\mathbb{P}[S \mid Y_1 = \mu]}. \tag{3.1}$$

We will denote by $Y^{\mathcal{C}}$ this Markov chain and by $\Pi^{\mathcal{C}}$ the restriction of the transition matrix Π to the entries which belong to $\mathcal{C}$ (in other words, $\Pi^{\mathcal{C}} = (\Pi(\lambda, \mu))_{\lambda,\mu\in\mathcal{C}}$).

3.2. *Doob h-transforms*

A *substochastic matrix* on the countable set M is a map $\Pi : M \times M \to [0, 1]$ such that $\sum_{y\in M} \Pi(x, y) \leqslant 1$ for any $x \in M$. If Π and Π' are substochastic matrices on M, we define their product $\Pi \times \Pi'$ as the substochastic matrix given by the ordinary product of matrices:

$$\Pi \times \Pi'(x, y) = \sum_{z\in M} \Pi(x, z)\Pi'(z, y).$$

The matrix $\Pi^{\mathcal{C}}$ defined in the previous subsection is an example of a substochastic matrix.

A function $h : M \to \mathbb{R}$ is *harmonic* for the substochastic transition matrix Π when it satisfies $\sum_{y\in M} \Pi(x, y)h(y) = h(x)$ for any $x \in M$. Consider a strictly positive harmonic function h. The Doob transform of Π by h (also called the h-transform of Π) is defined by

$$\Pi_h(x, y) = \frac{h(y)}{h(x)}\Pi(x, y).$$

We have $\sum_{y\in M} \Pi_h(x, y) = 1$ for any $x \in M$. Thus, Π_h can be interpreted as the transition matrix for a certain Markov chain.

An example is given in the second part of the previous subsection (see formula (3.1)): the state space is now $\mathcal{C}$, the substochastic matrix is $\Pi^{\mathcal{C}}$ and the harmonic function is $h_{\mathcal{C}}(\lambda) := \mathbb{P}[S \mid Y_1 = \lambda]$; the transition matrix $\Pi^{\mathcal{C}}_{h_{\mathcal{C}}}$ is the transition matrix of the Markov chain $Y^{\mathcal{C}}$.

3.3. *Green function and Martin kernel*

Let Π be a substochastic matrix on the set M. Its Green function is defined as the series

$$\Gamma(x, y) = \sum_{\ell\geqslant 0} \Pi^\ell(x, y).$$

(If Π is the transition matrix of a Markov chain, then $\Gamma(x, y)$ is the expected value of the number of passage at y of the Markov chain starting at x.)

Assume that there exists x^* in M such that $0 < \Gamma(x^*, y) < \infty$ for any $y \in M$. Fix such a point x^*. The Martin kernel associated to Π (with reference point x^*) is then defined by

$$K(x, y) = \frac{\Gamma(x, y)}{\Gamma(x^*, y)}.$$

Consider a positive harmonic function h and denote by Π_h the h-transform of Π. Consider the Markov chain $Y^h = (Y^h_\ell)_{\ell\geqslant 1}$ starting at x^* and whose transition matrix is Π_h.

Theorem 3.1 (Doob). *Assume that there exists a function $f : M \to \mathbb{R}$ such that for all $x \in M$, we have $\lim_{\ell\to+\infty} K(x, Y^h_\ell) = f(x)$ almost surely (a.s.). Then there exists a positive real constant c such that $f = ch$.*

For the sake of completeness, we detail a proof of this theorem in the appendix.

4. *Quotient renewal theorem for a random walk in a cone*

The purpose of this section is to establish a renewal theorem for a random walk forced to stay in a cone. We state this theorem in the weak form of a quotient theorem: see Theorem 4.6. This result is a key ingredient in our proof of Theorem 7.6; it is a purely probabilistic result whose proof can be read independently of the reminder of the article.

We begin by the statement and proof of a quotient local limit theorem for a random walk forced to stay in a cone (Theorem 4.3). This local limit theorem is easier to establish than the renewal one and the ideas of its proof will be reinvested in the proof of Theorem 4.6.

4.1. *Probability to stay in a cone*

Let $(X_\ell)_{\ell\geqslant 1}$ be a sequence of random variables in an Euclidean space $\mathbb{R}^n$, independent and identically distributed, defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$. We assume that these variables have moment of order 1 and denote by m their common mean. Let us denote by $(S_\ell)_{\ell\geqslant 0}$ the associated random walk defined by $S_0 = 0$ and $S_\ell := X_1 + \ldots + X_\ell$.

We consider a cone $\mathcal{C}$ in $\mathbb{R}^n$. In order to see enough paths of the random walk staying in the cone we need some assumptions. We assume that the cone $\mathcal{C}$ contains un open convex subcone $\mathcal{C}_0$ such that:

(h1) the drift of the random walk belongs to $\mathcal{C}_0$: $m \in \mathcal{C}_0$,
(h2) $\exists \ell_0 > 0, \mathbb{P}[S_1 \in \mathcal{C} \cap \overline{\mathcal{C}_0}, \ldots, S_{\ell_0-1} \in \mathcal{C} \cap \overline{\mathcal{C}_0}, S_{\ell_0} \in \mathcal{C}_0] > 0$.

(In our applications, the cone $\mathcal{C}$ will itself be convex and $\mathcal{C}_0$ will be its interior.)

First, one obtains:

Lemma 4.1.

$$\mathbb{P}[\forall \ell \geqslant 1,\ S_\ell \in \mathcal{C}] > 0.$$

Proof. Fix a in $\mathcal{C}_0$ such that, for all $\varepsilon > 0$,

$$\mathbb{P}[S_1 \in \mathcal{C} \cap \overline{\mathcal{C}_0}, \ldots, S_{\ell_0-1} \in \mathcal{C} \cap \overline{\mathcal{C}_0}, S_{\ell_0} \in B(a, \varepsilon)] > 0.$$

(Such a point a exists, by hypothesis (h2).)

By the strong law of large numbers, the sequence $((1/\ell)S_\ell)$ converges a.s. to m. Therefore, by hypothesis (h1), a.s., one obtains $S_\ell \in \mathcal{C}_0$ for any large enough ℓ, that is

$$\mathbb{P}[\exists L, \forall \ell \geqslant L, S_\ell \in \mathcal{C}_0] = \lim_{L\to+\infty} \mathbb{P}[\forall \ell \geqslant L,\ S_\ell \in \mathcal{C}_0] = 1.$$

For any $x \in \mathbb{R}^n$, there exists $k \in \mathbb{N}$ such that $x + ka \in \mathcal{C}_0$. Thus,

$$(\forall \ell \geqslant L,\ S_\ell \in \mathcal{C}_0) = \bigcup_{k\geqslant 0} ((\forall \ell \geqslant L,\ S_\ell \in \mathcal{C}_0) \quad \text{and} \quad (\forall \ell < L,\ S_\ell + ka \in \mathcal{C}_0))$$

and therefore $(\forall \ell \geqslant L,\ S_\ell \in \mathcal{C}_0) \subset \bigcup_{k\geqslant 0}(\forall \ell,\ S_\ell + ka \in \mathcal{C}_0)$ since the cone $\mathcal{C}_0$ is stable under addition. Hence, there exists $k \geqslant 0$ such that $\mathbb{P}[\forall \ell,\ S_\ell + ka \in \mathcal{C}_0] > 0$. Fix such a k and let $\delta > 0$ be such that $B(a, \delta) \subset \mathcal{C}_0$.

We now consider k independent repetitions of the event

$$A_1 := \left(S_1 \in \mathcal{C} \cap \overline{\mathcal{C}_0}, \ldots, S_{\ell_0-1} \in \mathcal{C} \cap \overline{\mathcal{C}_0}, S_{\ell_0} \in B\left(a, \frac{\delta}{k+1}\right)\right),$$

that is, we consider the events A_j, $1 \leqslant j \leqslant k+1$, defined by

$$A_j := \Big(S_{(j-1)\ell_0+1} - S_{(j-1)\ell_0} \in \mathcal{C} \cap \overline{\mathcal{C}_0},\ S_{(j-1)\ell_0+2} - S_{(j-1)\ell_0} \in \mathcal{C} \cap \overline{\mathcal{C}_0}, \ldots,$$
$$S_{(j-1)\ell_0+\ell_0-1} - S_{(j-1)\ell_0} \in \mathcal{C} \cap \overline{\mathcal{C}_0}, S_{j\ell_0} - S_{(j-1)\ell_0} \in B\left(a, \frac{\delta}{k+1}\right)\Big).$$

Note that the sum of an element of the open cone $\mathcal{C}_0$ and an element of its closure $\overline{\mathcal{C}_0}$ is always in $\mathcal{C}_0$. This fact is used in the following claim.

If the events $A_1, \ldots, A_{k+1}$ are fulfilled, then

$$S_1 \in \mathcal{C}, \ldots, S_{(k+1)\ell_0} \in \mathcal{C} \quad \text{and} \quad \|S_{(k+1)\ell_0} - ka - a\| \leqslant \sum_{j=1}^{k+1} \|S_{j\ell_0} - S_{(j-1)\ell_0} - a\| \leqslant \delta.$$

We have

$$\begin{aligned}\mathbb{P}[\forall \ell \geqslant 1,\ S_\ell \in \mathcal{C}] &\geqslant \mathbb{P}\left[\left(\bigcap_{j=1}^{k+1} A_j\right) \text{ and } (\forall \ell > (k+1)\ell_0,\ S_\ell - S_{(k+1)\ell_0} + ka \in \mathcal{C}_0)\right] \\ &= \prod_{j=1}^{k+1} \mathbb{P}(A_j) \times \mathbb{P}[\forall \ell,\ S_\ell + ka \in \mathcal{C}_0] \\ &= (\mathbb{P}(A_1))^{k+1} \times \mathbb{P}[\forall \ell,\ S_\ell + ka \in \mathcal{C}_0] > 0. \end{aligned}$$

□

4.2. *The local limit theorem*

We now assume that the support S_μ of the law μ of the random variables X_l is a subset of $\mathbb{Z}^n$. We suppose that μ is *adapted* on $\mathbb{Z}^n$, which means that the group $\langle S_\mu \rangle$ generated by the elements of S_μ is equal to $\mathbb{Z}^n$.

The local limit theorem specifies the behaviour as $\ell \to +\infty$ of the probability $\mathbb{P}[S_\ell = g]$ for $g \in \mathbb{Z}^n$. It thus appears a phenomenon of 'periodicity'. For instance, for the classical random walk to the closest neighbour on $\mathbb{Z}$, the walk $(S_\ell)_{\ell \geqslant 1}$ may visit an odd site only at an odd time; this is due to the fact that in this case the support of μ is included in $2\mathbb{Z}+1$, that is a coset of a proper subgroup of $\mathbb{Z}$.

Here is a classical definition: we say that the law μ is aperiodic if there exists no proper subgroup G of $\langle S_\mu \rangle$ and no vector $b \in \langle S_\mu \rangle$ such that $S_\mu \subset b + G$. One may easily check that, if the law μ is not aperiodic, then the law μ', defined by $\mu'(\cdot) := \mu(\cdot - b)$ for some $b \in S_\mu$, is aperiodic and the group $G = \langle S_{\mu'} \rangle$ is a proper subgroup of $\langle S_\mu \rangle$.

Let us give some various examples:

Example 3. (i) Denote by $(e_i)_{1 \leqslant i \leqslant n}$, the canonical basis of $\mathbb{R}^n$. If the variables X_ℓ take their values in the set $\{e_i \mid 1 \leqslant i \leqslant n\} \cup \{0\}$, if $\mathbb{P}[X_\ell = e_i] > 0$ for any i and if $\mathbb{P}[X_\ell = 0] > 0$, then $\langle S_\mu \rangle = \mathbb{Z}^n$ and the law is aperiodic.

(ii) If the variables X_ℓ take their values in $\{\pm e_i \mid 1 \leqslant i \leqslant n\}$, if $\mathbb{P}[X_\ell = e_i] > 0$ for any i and if $\mathbb{P}[X_\ell = -e_i] > 0$ for at least one i, then we can take $b = e_1$ and $G = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid x_1 + x_2 + \ldots + x_n \text{ is even}\}$ is the subgroup of $\mathbb{Z}^n$ generated by the $e_i + e_j, 1 \leqslant i, j \leqslant n$.

(iii) If the variables X_ℓ take their values in $\{e_i \mid 1 \leqslant i \leqslant n\}$ and if $\mathbb{P}[X_\ell = e_i] > 0$ for any i, then we can take $b = e_1$ and G is the $n-1$-dimensional subgroup generated by the vectors $e_i - e_1$, $2 \leqslant i \leqslant n$.

The translation by $-b$ allows us to consider an aperiodic random walk in a group of suitable dimension d (with possibly $d < n$). In other words, by replacing the random walk (S_ℓ) by

$(S_\ell - \ell b)$, we can restrict ourselves to an adapted and aperiodic random walk in a discrete group G.

We denote by d the dimension of the group G once this translation is performed. We also assume that the random walk admits a moment of order 2. We write m for the mean vector of X and Γ for its covariance matrix. A classical form of the local limit theorem is the following:

$$\lim_{\ell\to+\infty} \sup_{g\in G} \left| \frac{\ell^{d/2}}{v(G)} \mathbb{P}[S_\ell = g] - \mathcal{N}_\Gamma\left(\frac{1}{\sqrt{\ell}}(g-\ell m)\right)\right| = 0,$$

where $\mathcal{N}_\Gamma$ is the Gaussian density

$$\mathcal{N}_\Gamma(x) = (2\pi)^{-d/2}(\det\Gamma)^{-1/2}\exp(-\tfrac{1}{2}\, x\cdot\Gamma^{-1}\cdot {}^t x), \quad x\in\mathbb{R}^d,$$

and $v(G)$ is the volume of an elementary cell of G. This result gives an equivalent of $\mathbb{P}[S_\ell = g_\ell]$ providing that $g_\ell - \ell m = O(\sqrt{\ell})$. The local limit theorem for large deviations (see the original article [**21**, **22**] or the classical book [**8**]) yields and equivalent when $g_\ell - \ell m = o(\ell)$. It takes a particularly simple form when $g_\ell - \ell m = o(\ell^{2/3})$.

THEOREM 4.2. *Assume that the random variables X_ℓ have an exponential moment, that is, there exists $t > 0$ such that $\mathbb{E}[\exp(t|X_\ell|)] < +\infty$. Let (a_ℓ) be a sequence of real numbers such that $\lim a_\ell \ell^{-2/3} = 0$. Then, when ℓ tends to infinity, we have*

$$\mathbb{P}[S_\ell = g] \sim v(G)\ell^{-d/2}\mathcal{N}_\Gamma\left(\frac{1}{\sqrt{\ell}}(g-\ell m)\right)$$

uniformly in $g\in G$ such that $\|g-\ell m\| \leqslant a_\ell$.

Under the hypotheses of Theorem 4.2, we derive the following equivalent. Given (g_ℓ) and (h_ℓ) two sequences in G such that $\lim \ell^{-2/3}\|g_\ell - \ell m\| = 0$ and $\lim \ell^{-1/2}\|h_\ell\| = 0$, we have

$$\mathbb{P}[S_\ell = g_\ell + h_\ell] \sim \mathbb{P}[S_\ell = g_\ell]. \tag{4.1}$$

4.3. *A quotient LLT for the random walk restricted to a cone*

We assume that the hypotheses of the previous paragraph are satisfied.

THEOREM 4.3. *Assume that the random variables X_ℓ are a.s. bounded. Let (g_ℓ) and (h_ℓ) be two sequences in G and $\alpha < \frac{2}{3}$ such that $\lim \ell^{-\alpha}\|g_\ell - \ell m\| = 0$ and $\lim \ell^{-1/2}\|h_\ell\| = 0$. Then, when ℓ tends to infinity,*

$$\mathbb{P}[S_1\in\mathcal{C},\dots,S_\ell\in\mathcal{C}, S_\ell = g_\ell + h_\ell] \sim \mathbb{P}[S_1\in\mathcal{C},\dots,S_\ell\in\mathcal{C}, S_\ell = g_\ell].$$

The following lemma will play a crucial role; in order to keep a direct way to our principal results, we postpone its proof in the appendix.

LEMMA 4.4. *Assume that the random variables X_ℓ are a.s. bounded. Let $\alpha \in]\frac{1}{2}, \frac{2}{3}[$. If the sequence $(\ell^{-\alpha}\|g_\ell - \ell m\|)_\ell$ is bounded, then there exists $c > 0$ such that, for all large enough ℓ,*

$$\mathbb{P}[S_1\in\mathcal{C},\dots,S_\ell\in\mathcal{C}, S_\ell = g_\ell] \geqslant \exp(-c\ell^\alpha).$$

In the sequel, we will use the following notation: if (u_ℓ) and (v_ℓ) are two real sequences, then we write $u_\ell \preceq v_\ell$ when there exists a constant $\kappa > 0$ such that $u_\ell \leqslant \kappa\, v_\ell$ for all large enough ℓ.

Proof of Theorem 4.3. Fix a real number β such that $\frac{1}{2} < \alpha < \beta < \frac{2}{3}$ and set $b_\ell = [\ell^\beta]$. Let $\delta > 0$ be such that $B(m, \delta) \subset \mathcal{C}$. Set $B_\ell = B(\ell m, \ell\delta)$.

For any $\ell \geqslant 1$, we have $B_\ell \subset \mathcal{C}$. We are going to establish that

$$Q_\ell := \frac{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_\ell = g_\ell]}{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell]} \to 1. \tag{4.2}$$

By the Cramer–Chernoff large deviations inequality, there exists $c = c(\delta) > 0$ such that

$$\mathbb{P}[S_{b_\ell} \notin B_{b_\ell}] = \mathbb{P}[\|S_{b_\ell} - b_\ell m\| \geqslant b_\ell \delta] \leqslant \exp(-cb_\ell) \preceq \exp(-cl^\beta).$$

By Lemma 4.4, there exists $c' > 0$ such that

$$\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_\ell = g_\ell] \geqslant \exp(-c'\ell^\alpha).$$

Since $\alpha < \beta$, we thus have

$$\mathbb{P}[S_{b_\ell} \notin B_{b_\ell}] = o(\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_\ell = g_\ell]).$$

This gives

$$\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell] \sim \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_\ell = g_\ell], \tag{4.3}$$

and, in particular, for any large enough ℓ

$$\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell] \geqslant \tfrac{1}{2}\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_\ell = g_\ell]. \tag{4.4}$$

Now set

$$U_\ell := \frac{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell]}{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell]}.$$

Write $\varepsilon = \operatorname{dist}(m, \mathcal{C}^c)$, and recall that $\varepsilon > 0$. So, when the walk goes out of the cone, its distance to the point km is at least $k\varepsilon$.

We have

$$0 \leqslant 1 - U_\ell \leqslant \frac{\sum_{k \geqslant 1} \mathbb{P}[S_{b_\ell + k} \notin \mathcal{C}]}{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell]}.$$

Thus by (4.4) and our choice of ε, we have for any sufficiently large ℓ,

$$|1 - U_\ell| \leqslant 2\,\frac{\sum_{k \geqslant b_\ell} \mathbb{P}[\|S_k - km\| \geqslant k\varepsilon]}{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_\ell = g_\ell]}. \tag{4.5}$$

The large deviations inequality implies that there exists a constant $c'' = c''(\varepsilon) > 0$ such that for any k,

$$\mathbb{P}[\|S_k - km\| \geqslant k\varepsilon] \leqslant \exp(-c''k).$$

Therefore,

$$\sum_{k \geqslant b_\ell} \mathbb{P}[\|S_k - km\| \geqslant k\varepsilon] \preceq \exp(-c''b_\ell) \preceq \exp(-c''\ell^\beta).$$

Since $\alpha < \beta$, Lemma 4.4 and (4.5) imply $\lim U_\ell = 1$; combined with (4.3), this proves (4.2). Moreover, the same result holds if one replaces g_ℓ by $g_\ell + h_\ell$ for $\lim \ell^{-\alpha}\|g_\ell + h_\ell - \ell m\| = 0$. To achieve the proof of Theorem 4.3, it now suffices to establish that

$$\frac{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell + h_\ell]}{\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell]} \to 1.$$

Since the increments of the random walk are independent and stationary, we have

$$\begin{aligned}
&\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_\ell = g_\ell] \\
&\quad = \sum_{x \in B_{b_\ell} \cap G} \mathbb{P}[S_{\ell - b_\ell} = g_\ell - x] \times \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} = x].
\end{aligned}$$

This leads to the theorem since by (4.1) we have

$$\mathbb{P}[S_{\ell-b_\ell} = g_\ell - x] \sim \mathbb{P}[S_{\ell-b_\ell} = g_\ell + h_\ell - x],$$

uniformly in $x \in B_{b_\ell}$. (Indeed $\|g_\ell - x - (\ell - b_\ell)m\| \preceq \ell^\beta \delta$, uniformly in $x \in B_{b_\ell}$.) □

4.4. *The renewal theorem*

Assume now that the random variables X_ℓ take values in a discrete subgroup of $\mathbb{R}^n$, and denote by G the group generated by the support of their common law μ. We assume that G linearly generates the whole space $\mathbb{R}^n$. Denote by U the associated renewal measure defined by $U := \sum_{j\geqslant 0} \mu^{*j}$. Equivalently, we have for any $g \in G$,

$$U(g) = \sum_{j=0}^{+\infty} \mathbb{P}[S_j = g].$$

Observe that there is no hypothesis of aperiodicity in the following statement; this is due to the fact that the quantity $U(g)$ represents the expected number of visits of g by the whole random walk and not only at a precise time, like the local limit theorem does. In particular, the law of the variables X_l may be supported on a proper coset of G with no consequences on the behaviour of $U(g)$ as $g \to \infty$.

We assume that $m := \mathbb{E}[X_\ell]$ is non-zero. The renewal theorem tells us that, when g tends to infinity in the direction m, we have

$$U(g) \sim \frac{v(G)}{\sigma}(2\pi)^{-(n-1)/2}\|m\|^{(n-3)/2}\|g\|^{-(n-1)/2}$$

where σ^2 is the determinant of the covariance matrix of the orthogonal projection of X_ℓ on the hyperplan orthogonal to m. More precisely, we have the following theorem.

Let $(e_1, e_2, \ldots, e_{n-1})$ be an orthonormal basis of the hyperplan $m^\perp$. If $x \in \mathbb{R}^n$, then denote by x' its orthogonal projection on $m^\perp$ expressed in this basis (here x' is regarded as a row vector). Finally, let B be the covariance matrix of the random vector X'_ℓ.

Recall that $\mathcal{N}_B$ is the $(n-1)$-dimensional Gaussian density given by

$$\mathcal{N}_B(x') = (2\pi)^{-(n-1)/2}(\det B)^{-1/2}\exp(-\tfrac{1}{2}\, x' \cdot B^{-1} \cdot {}^t x').$$

THEOREM 4.5. *We assume that the random variables X_ℓ have an exponential moment. Let (a_ℓ) be a sequence of real numbers such that $\lim a_\ell \ell^{-2/3} = 0$. Then, when ℓ goes to infinity, we have*

$$U(g) \sim \frac{v(G)}{\|m\|}\ell^{-(n-1)/2}\,\mathcal{N}_B\left(\frac{1}{\sqrt{\ell}}g'\right)$$

uniformly in $g \in G$ such that $\|g - \ell m\| \leqslant a_\ell$.

This theorem has been proved by Carlsson and Wainger in the case of absolutely continuous distribution [**4**]. We did not find in the literature the lattice distribution version we state here. A detailed proof of this version is given in [**17**].

Under the hypotheses of Theorem 4.5, we see, in particular, that if (g_ℓ) and (h_ℓ) are two sequences in G such that $\lim \ell^{-2/3}\|g_\ell - \ell m\| = 0$ and $\lim \ell^{-1/2}\|h_\ell\| = 0$, then

$$U(S_\ell = g_\ell + h_\ell) \sim U(S_\ell = g_\ell). \tag{4.6}$$

4.5. *A quotient renewal theorem for the random walk restricted to a cone*

THEOREM 4.6. *Assume the random variables X_ℓ are a.s. bounded. Let (g_ℓ) and (h_ℓ) two sequences in G such that there exists $\alpha < \frac{2}{3}$ with $\lim \ell^{-\alpha}\|g_\ell - \ell m\| = 0$ and $\lim \ell^{-1/2}\|h_\ell\| = 0$.*

Then, when ℓ tends to infinity, we have

$$\sum_{j\geqslant 1}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_j=g_\ell+h_\ell]\sim\sum_{j\geqslant 1}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_j=g_\ell].$$

Proof. Fix a real number β such that $\frac{1}{2}<\alpha<\beta<\frac{2}{3}$ and set $b_\ell=[\ell^\beta]$. Let $\delta>0$ be such that $B(m,\delta)\subset\mathcal{C}$ and set $B_\ell=B(\ell m,\ell\delta)$. For any $\ell\geqslant 1$, we have $B_\ell\subset\mathcal{C}$.

We are going to prove that

$$\frac{\sum_{j\geqslant 1}\mathbb{P}[S_1\in\mathcal{C},\dots,S_{b_\ell}\in\mathcal{C},S_{b_\ell}\in B_{b_\ell},S_j=g_\ell]}{\sum_{j\geqslant 1}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_j=g_\ell]}\to 1\quad\text{when }\ell\to+\infty. \tag{4.7}$$

Observe first that, since the support of μ is bounded, there exists $c_1\in(0,1)$ such that $\mathbb{P}[S_j=g_\ell]=0$ if $j<c_1\ell$.

We prove first that

$$T_\ell:=\frac{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_{b_\ell}\in B_{b_\ell},S_j=g_\ell]}{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_j=g_\ell]}\to 1\quad\text{when }\ell\to+\infty. \tag{4.8}$$

We have

$$0\leqslant 1-T_\ell\leqslant\frac{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_{b_\ell}\notin B_{b_\ell},S_j=g_\ell]}{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_j=g_\ell]}.$$

Write N_ℓ and D_ℓ for the numerator and the denominator of the previous fraction. We have

$$D_\ell\geqslant\mathbb{P}[S_1\in\mathcal{C},\dots,S_\ell\in\mathcal{C},S_\ell=g_\ell].$$

By Lemma 4.4, there exists $c'>0$ such that $D_\ell\geqslant\exp(-c'\ell^\alpha)$. Since the support of μ is bounded, there exists $c_2>0$ such that $|S_{b_\ell}|\leqslant c_2\ell^\beta$. This gives

$$\begin{aligned}N_\ell&\leqslant\sum_{j\geqslant c_1\ell}\mathbb{P}[S_{b_\ell}\notin B_{b_\ell},S_j=g_\ell]\leqslant\sum_{j\geqslant c_1\ell}\mathbb{P}[S_{b_\ell}\notin B_{b_\ell}]\max_{|x|\leqslant c_2\ell^\beta}\mathbb{P}[x+S_{j-b_\ell}=g_\ell]\\&\leqslant\mathbb{P}[S_{b_\ell}\notin B_{b_\ell}]\sum_{j\geqslant c_1\ell}\sum_{|x|\leqslant c_2\ell^\beta}\mathbb{P}[x+S_{j-b_\ell}=g_\ell]\leqslant\mathbb{P}[S_{b_\ell}\notin B_{b_\ell}]\sum_{|x|\leqslant c_2\ell^\beta}U(g_\ell-x).\end{aligned}$$

The cardinality of $B(0,c_2\ell^\beta)\cap G$ is $O(\ell^{n\beta})$ and the function U is uniformly bounded on G. This gives

$$N_\ell\preceq\mathbb{P}[S_{b_\ell}\notin B_{b_\ell}]\times\ell^{n\beta}.$$

By the large deviations inequality, there exists $c_3>0$ such that $\mathbb{P}[S_{b_\ell}\notin B_{b_\ell}]\leqslant\exp(-c_3b_\ell)$. Finally, the polynomial term is absorbed by the exponential term and we obtain

$$N_\ell\preceq\exp(-c_4\ell^\beta).$$

The estimates we have obtained on D_ℓ et N_ℓ clearly yields (4.8).

Next, we prove that

$$T'_\ell:=\frac{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_j\in\mathcal{C},S_{b_\ell}\in B_{b_\ell},S_j=g_\ell]}{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_{b_\ell}\in\mathcal{C},S_{b_\ell}\in B_{b_\ell},S_j=g_\ell]}\to 1\quad\text{when }\ell\to+\infty. \tag{4.9}$$

We have

$$0\leqslant 1-T'_\ell\leqslant\frac{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_{b_\ell}\in\mathcal{C},S_{b_\ell}\in B_{b_\ell},\exists k\in\{1,\dots,j-b_\ell\}\,S_{b_\ell+k}\notin\mathcal{C},S_j=g_\ell]}{\sum_{j\geqslant c_1\ell}\mathbb{P}[S_1\in\mathcal{C},\dots,S_{b_\ell}\in\mathcal{C},S_{b_\ell}\in B_{b_\ell},S_j=g_\ell]}.$$

Write N'_ℓ and D'_ℓ for the numerator and the denominator of the previous fraction. Let $\varepsilon > 0$ be as in the proof of Theorem 4.3. Since $[S_k \notin \mathcal{C}] \subset [\|S_k - km\| \geqslant k\varepsilon]$, one obtains

$$\begin{aligned}
N'_\ell &\leqslant \sum_{j \geqslant c_1 \ell} \mathbb{P}[\exists k \in \{1, \dots, j - b_\ell\}\, S_{b_\ell + k} \notin \mathcal{C}, S_j = g_\ell] \\
&\leqslant \sum_{k \geqslant 1} \sum_{j \geqslant \max(c_1 \ell, k + b_\ell)} \mathbb{P}[S_{b_\ell + k} \notin \mathcal{C}, S_j = g_\ell] \\
&\leqslant \sum_{k \geqslant 1} \sum_{j \geqslant k + b_\ell} \sum_{y \notin \mathcal{C}} \mathbb{P}[S_{b_\ell + k} = y] \mathbb{P}[S_{j - b_\ell - k} = g_\ell - y] \\
&\leqslant \sum_{k \geqslant 1} \sum_{y \notin \mathcal{C}} \mathbb{P}[S_{b_\ell + k} = y]\ U(g_\ell - y) \\
&\leqslant \sum_{k \geqslant 1} \sum_{y \notin \mathcal{C}} \mathbb{P}[S_{b_\ell + k} = y]\ \max_{g \in G} U(g) \\
&\leqslant \sum_{k \geqslant b_\ell} \mathbb{P}[S_k \notin \mathcal{C}]\ \max_{g \in G} U(g) \\
&\leqslant \sum_{k \geqslant b_\ell} \mathbb{P}[\|S_k - km\| \geqslant k\varepsilon]\ \max_{g \in G} U(g).
\end{aligned}$$

Using the large deviations estimate, we obtain as in the proof of Theorem 4.3

$$N'_\ell \preceq \exp(-c\ell^\beta).$$

Let us now look at the denominator D'_ℓ; we have

$$D'_\ell \geqslant \sum_{j \geqslant c_1 \ell} \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_j \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_j = g_\ell].$$

Since $T_\ell \to 1$, we obtain for large enough ℓ,

$$D'_\ell \geqslant \frac{1}{2} \sum_{j \geqslant c_1 \ell} \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_j \in \mathcal{C}, S_j = g_\ell] = D_\ell.$$

So $D'_\ell \geqslant \frac{1}{2} \exp(-c'\ell^\alpha)$ which leads to (4.9); the convergence (4.7) follows, combining (4.8) and (4.9).

The limit (4.7) also holds when we replace g_ℓ by $g_\ell + h_\ell$. Therefore, to prove the theorem, it suffices to check that

$$\frac{\sum_{j \geqslant c_1 \ell} \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_j = g_\ell + h_\ell]}{\sum_{j \geqslant c_1 \ell} \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} \in B_{b_\ell}, S_j = g_\ell]} \to 1.$$

We write the denominator of this fraction as

$$\sum_{j \geqslant c_1 \ell} \sum_{y \in B_{b_\ell}} \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} = y] \times \mathbb{P}[S_{j - b_\ell} = g_\ell - y],$$

which is equal to

$$\sum_{y \in B_{b_\ell}} \mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{b_\ell} \in \mathcal{C}, S_{b_\ell} = y] \times U(g_\ell - y).$$

This is sufficient to conclude since, by (4.6), we have $U(g_\ell + h_\ell - y) \sim U(g_\ell - y)$, uniformly in $y \in B_{b_\ell}$. □

5. *Crystals and random walks*

5.1. *Brief review on crystals*

We now recall some basics on quantum groups and Kashiwara crystals. For a complete review, we refer to [**10**, **12**]. The quantum group $U_q(\mathfrak{g})$ is a q-deformation of the enveloping Lie algebra $U(\mathfrak{g})$. To each dominant weight $\lambda \in P_+$ corresponds a unique (up to isomorphism) irreducible $U_q(\mathfrak{g})$-module $V_q(\lambda)$. The representation theory of the finite-dimensional $U_q(\mathfrak{g})$-modules is essentially parallel to that of $U(\mathfrak{g})$. In particular, any tensor product $V_q(\lambda) \otimes V_q(\mu)$ decomposes into irreducible components. The outer multiplicities so obtained are the same as those appearing in the decomposition of $V(\lambda) \otimes V(\mu)$. Similarly, there exists a relevant notion of weight subspaces in $V_q(\lambda)$ and for any $\beta \in P$, we have $\dim V_q(\lambda)_\beta = \dim V(\lambda)_\beta = K_{\lambda,\beta}$.

To each irreducible module $V_q(\lambda)$ is associated its Kashiwara crystal $B(\lambda)$. Formally, $B(\lambda) := L(\lambda)/qL(\lambda)$ where $L(\lambda)$ is a particular lattice in $V_q(\lambda)$ over the ring

$$A(q) := \{F \in \mathbb{C}(q) \text{ without pole at } q = 0\}.$$

It was proved by Kashiwara that $B(\lambda)$ has the structure of a coloured and oriented graph. This graph encodes many informations on the representation $V_q(\lambda)$ (and thus also on $V(\lambda)$). In particular, the crystal $B(\lambda)$ contains $\dim V_q(\lambda) = \dim V(\lambda)$ vertices. Its arrows are labelled by the simple roots $\alpha_i \in \Delta_+$. The graph structure is obtained from the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$, $i \in I$, which are renormalizations of the action of the Chevalley generators e_i and f_i of $U_q(\mathfrak{g})$. More precisely, we have an arrow $a \xrightarrow{i} b$ when $b = \tilde{f}_i(a)$ or equivalently $a = \tilde{e}_i(b)$. When there is no arrow $\xrightarrow{i}$ starting from a (respectively ending at b), we write $\tilde{f}_i(a) = 0$ (respectively $\tilde{e}_i(b) = 0$).

The notion of a crystal can be extended to a category $\mathcal{O}_{\text{int}}$ of $U_q(\mathfrak{g})$ modules containing the irreducible modules $V_q(\lambda)$ and stable by tensorization. The crystal associated to the module $M \in \mathcal{O}_{\text{int}}$ is unique up to isomorphism: given two crystals B and B' associated to M, there exists a bijection $\psi : B \to B'$ which commutes with the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$.

Given any $b \in B$ and $i \in I$, we set $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k(b) \neq 0\}$ and $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k(b) \neq 0\}$. The weight of the vertex $b \in B$ is then defined by

$$\text{wt}(b) = \sum_{i \in I} \text{wt}(b)_i\, \omega_i \in P \quad \text{where } \text{wt}(b)_i = \varphi_i(b) - \varepsilon_i(b). \tag{5.1}$$

One can then prove that $\text{wt}(\tilde{f}_i(b)) = \text{wt}(b) - \alpha_i$ for any $i \in I$ and any $b \in B$ such that $\tilde{f}_i(b) \neq 0$. For any $\beta \in P$, the dimension $\dim M_\beta$ of the weight space β in M is the cardinality of the set B_β of vertices of weight β in the crystal B associated to M. A vertex $b \in B$ is said to be of highest weight when $\varepsilon_i(b) = 0$ for any $i \in I$. In that case, we immediately obtain $\text{wt}(b) \in P_+$. Denote by $\text{HW}(B)$ the set of highest weight vertices in B; the elements of $\text{HW}(B)$ are in one-to-one correspondence with the connected components of the crystal B. In particular, the crystal $B(\lambda)$ is connected with a unique highest weight vertex of weight λ. For any $b \in B$, we denote by $B(b)$ the connected component of B containing b and by $\text{hw}(b)$ the highest weight vertex of $B(b)$.

The following two properties of crystals will be essential for our purpose.

THEOREM 5.1. *Consider $M \in \mathcal{O}_{\text{int}}$ and B its crystal graph.*

(i) *The decomposition of the $U_q(\mathfrak{g})$-module M in irreducible components is given by the decomposition of B in connected components. More precisely, we have*

$$M \simeq \bigoplus_{b \in \text{HW}(B)} V_q(\text{wt}(b)).$$

(ii) *Consider $\lambda, \mu \in P_+$ and $B(\lambda), B(\mu)$ the crystals associated to $V_q(\lambda)$ and $V_q(\mu)$. The crystal associated to $V_q(\lambda) \otimes V_q(\mu)$ is the crystal $B(\lambda) \otimes B(\mu)$ whose set of vertices is the direct product of the sets of vertices of $B(\lambda)$ and $B(\mu)$ and whose crystal structure is given by the following rules:*

$$\tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i(v) \text{ if } \varepsilon_i(v) > \varphi_i(u), \\ \tilde{e}_i(u) \otimes v \text{ if } \varepsilon_i(v) \leqslant \varphi_i(u), \end{cases} \tag{5.2}$$

and

$$\tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i(u) \otimes v \text{ if } \varphi_i(u) > \varepsilon_i(v), \\ u \otimes \tilde{f}_i(v) \text{ if } \varphi_i(u) \leqslant \varepsilon_i(v). \end{cases}$$

We thus have

$$\varphi_i(u \otimes v) = \varphi_i(v) + \max\{\varphi_i(u) - \varepsilon_i(v), 0\},$$
$$\varepsilon_i(u \otimes v) = \varepsilon_i(u) + \max\{\varepsilon_i(v) - \varphi_i(u), 0\}.$$

In particular, $u \otimes v \in \mathrm{HW}(B(\lambda) \otimes B(\mu))$ if and only if $u \in \mathrm{HW}(B(\lambda))$ and $\varepsilon_i(v) \leqslant \varphi_i(v)$ for any $i \in I$.

Example 4. In type C_3, the crystal corresponding to the minuscule weight ω_1 is

$$B(\omega_1) : 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \bar{3} \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}.$$

The tensor power $B(\omega_1)^{\otimes 2}$ is the crystal

$$\begin{array}{ccccccccc}
1\otimes 1 & \xrightarrow{1} & 2\otimes 1 & \xrightarrow{2} & 3\otimes 1 & \xrightarrow{3} & \bar{3}\otimes 1 & \xrightarrow{2} & \bar{2}\otimes 1 & \xrightarrow{1} & \bar{1}\otimes 1 \\
 & & 1\downarrow & & 1\downarrow & & 1\downarrow & & & & 1\downarrow \\
1\otimes 2 & & 2\otimes 2 & \xrightarrow{2} & 3\otimes 2 & \xrightarrow{3} & \bar{3}\otimes 2 & \xrightarrow{2} & \bar{2}\otimes 2 & & \bar{1}\otimes 2 \\
2\downarrow & & & & 2\downarrow & & & & 2\downarrow & & 2\downarrow \\
1\otimes 3 & \xrightarrow{1} & 2\otimes 3 & & 3\otimes 3 & \xrightarrow{3} & \bar{3}\otimes 3 & & \bar{2}\otimes 3 & \xrightarrow{1} & \bar{1}\otimes 3 \\
3\downarrow & & 3\downarrow & & & & 3\downarrow & & 3\downarrow & & 3\downarrow \\
1\otimes \bar{3} & \xrightarrow{1} & 2\otimes \bar{3} & \xrightarrow{2} & 3\otimes \bar{3} & & \bar{3}\otimes \bar{3} & \xrightarrow{2} & \bar{2}\otimes \bar{3} & \xrightarrow{1} & \bar{1}\otimes \bar{3} \\
2\downarrow & & & & 2\downarrow & & & & 2\downarrow & & 2\downarrow \\
1\otimes \bar{2} & \xrightarrow{1} & 2\otimes \bar{2} & & 3\otimes \bar{2} & \xrightarrow{3} & \bar{3}\otimes \bar{2} & & \bar{2}\otimes \bar{2} & \xrightarrow{1} & \bar{1}\otimes \bar{2} \\
 & & 1\downarrow & & 1\downarrow & & 1\downarrow & & & & 1\downarrow \\
1\otimes \bar{1} & & 2\otimes \bar{1} & \xrightarrow{2} & 3\otimes \bar{1} & \xrightarrow{3} & \bar{3}\otimes \bar{1} & \xrightarrow{2} & \bar{2}\otimes \bar{1} & & \bar{1}\otimes \bar{1}
\end{array}$$

and it admits three connected components with highest weight vertices $1\otimes 1$, $1 \otimes 2$ and $1\otimes \bar{1}$ of weights $2\omega_1, \omega_2$ and 0. This gives the decomposition

$$V_q(\omega_1)^{\otimes 2} \simeq V_q(2\omega_1) \oplus V_q(\omega_2) \oplus V_q(0).$$

Remark 3. We have seen that $\dim M_\beta$ is equal to the number of vertices of weight β in the crystal B of M. Similarly, the number of highest weight vertices in B with weight λ gives the multiplicity of $V(\lambda)$ in the decomposition of M into its irreducible components.

Consider C and C' two connected components of the crystal B. The components C and C' are isomorphic when there exists a bijection $\phi_{C,C'}$ from C to C' which commutes with the action of the Kashiwara operators, that is $b_1 \xrightarrow{i} b_2$ in C if and only if $\phi_{C,C'}(b_1) \xrightarrow{i} \phi_{C,C'}(b_2)$ in C'. In that case, the isomorphism $\phi_{C,C'}$ is unique since it must send the highest weight vertex of C on the highest weight vertex of C'.

The following lemma is a straightforward consequence of Theorem 5.1.

LEMMA 5.2. *Assume $b^0 = a_1^0 \otimes \ldots \otimes a_\ell^0$ is a highest weight vertex of $B(\delta)^{\otimes \ell}$.*

(i) *For any $k = 1, \ldots, \ell$, the vertex $b^{(k),0} := a_1^0 \otimes \ldots \otimes a_k^0$ of $B(\delta)^{\otimes \ell}$ is a highest weight vertex of $B(\delta)^{\otimes k}$ and $\varepsilon_i(a_{k+1}^0 \otimes \ldots \otimes a_\ell^0) \leqslant \varphi_i(b^{(k),0})$ for any $i \in I$.*

(ii) *Consider $b = a_1 \otimes \ldots \otimes a_\ell$ a vertex of $B(b^0)$. Then, for any $k = 1, \ldots, \ell$, the vertex $b^{(k)} := a_1 \otimes \ldots \otimes a_k$ belongs to $B(b^{(k),0})$.*

We will need the following proposition in § 7.

PROPOSITION 5.3. *Consider $\lambda, \mu \in P_+$. With the notation of § 2.2, the following properties hold.*

(i) *$f^\ell_{\lambda/\mu} = \sum_{\kappa \in P_+} f^\ell_\kappa m^\lambda_{\kappa,\mu}$ for any $\ell \geqslant 1$.*

(ii) *Assume $(\lambda^{(a)})_a$ is a sequence of weights of the form $\lambda^{(a)} = am + o(a)$ with $m \in C$ and consider $\kappa \in P_+$. Then, for a sufficiently large, the weight $\lambda^{(a)}$ belongs to P_+ and $m^{\lambda^{(a)}}_{\kappa,\mu} = K_{\mu,\lambda^{(a)}-\kappa}$. Therefore*

$$f^\ell_{\lambda^{(a)}/\mu} = \sum_{\kappa \in P_+} f^\ell_\kappa K_{\mu,\lambda^{(a)}-\kappa} = \sum_{\gamma \in P} f^\ell_{\lambda^{(a)}-\gamma} K_{\mu,\gamma} \tag{5.3}$$

for any $\ell \geqslant 1$.

Proof. To prove (i), we write

$$s_\mu \cdot (s_\delta)^{\otimes \ell} = \sum_\kappa f^\ell_\kappa s_\kappa s_\mu = \sum_\kappa \sum_\lambda f^\ell_\kappa m^\lambda_{\kappa,\mu} s_\lambda = \sum_\lambda f^\ell_{\lambda/\mu} s_\lambda,$$

where all the sums run over P_+. The assertion immediately follows, comparing the two last expressions.

For a sufficiently large, we must have $\lambda^{(a)} \in P_+$ for $\lambda^{(a)} \sim am$ and $m \in C$. By Lemma 5.2, $m^{\lambda^{(a)}}_{\kappa,\mu}$ is equal to the number of vertices of weight $\lambda^{(a)}$ of the form $b_\kappa \otimes b$ where b_κ is the highest weight vertex of $B(\kappa)$ and $b \in B(\mu)$ satisfies $\varepsilon_i(b) \leqslant \varphi_i(b_\kappa)$ for any $i \in I$. In particular, b has weight $\lambda^{(a)} - \kappa$. Thus, $m^{\lambda^{(a)}}_{\kappa,\mu} \leqslant K_{\mu,\lambda^{(a)}-\kappa}$.

Now assume $b \in B(\mu)$ has weight $K_{\mu,\lambda^{(a)}-\kappa}$ so that the vertex $b_\kappa \otimes b \in B(\kappa) \otimes B(\mu)$ has dominant weight $\lambda^{(a)}$. We have, by assertion (ii) of Theorem 5.1,

$$\mathrm{wt}(b_\kappa \otimes b)_i = \varphi_i(b_\kappa \otimes b) - \varepsilon_i(b_\kappa \otimes b) \leqslant \varphi_i(b_\kappa \otimes b).$$

Since $\lambda^{(a)} = am + o(a)$, the weight $\mathrm{wt}(b_\kappa \otimes b)_i$ tends to infinity with a, for any $i \in I$. Since $b \in B(\mu)$ and μ is fixed,

$$\varphi_i(b_k \otimes b) = \varphi_i(b) + \max\{\varphi_i(b_\kappa) - \varepsilon_i(b)\} > \varphi_i(b)$$

for a sufficiently large. This means that $\varepsilon_i(b) \leqslant \varphi_i(b_\kappa)$ for such an a. So $b_\kappa \otimes b$ is a highest weight vertex with dominant weight $\lambda^{(a)}$. Therefore, $m^{\lambda^{(a)}}_{\kappa,\mu} \geqslant K_{\mu,\lambda^{(a)}-\kappa}$. □

5.2. *Paths in weight lattices and crystals*

Let δ be a dominant weight and $B(\delta)$ the crystal of $V_q(\delta)$.[†] By Theorem 5.1 and (2.2), we derive for any $\ell \geqslant 1$ the decomposition of $B(\delta)^{\otimes \ell}$ in its irreducible components.

$$B(\delta)^{\otimes \ell} \simeq \bigsqcup_{\lambda \in P_+} B(\lambda)^{\oplus f^\ell_{\lambda,\delta}}.$$

[†]For the sake of simplicity, we only consider in this section paths obtained from tensor powers of irreducible modules. This hypothesis will be relaxed in Section 8.

Let $\mathfrak{P}$ be the map defined on $B(\delta)^{\otimes\ell}$ which associates to each vertex b the highest weight vertex $\mathfrak{P}(b)$ of $B(b)$. Following ideas essentially analogous to those used in [1], the map $\mathfrak{P}$ can be interpreted as a Pitman transform on paths in the weight lattice.

Remark 4. In [18], the transformation $\mathfrak{P}$ was computed using the Knuth insertion algorithm on semistandard tableaux. One can prove that these semistandard tableaux yield simple parametrizations of the crystals $B(\delta)$. There exist analogous notions of tableaux for types B_n, C_n, D_n and G_2 which similarly give a simple parametrization of $B(\delta)$ for any dominant weight δ. They were introduced by Kashiwara and Nakashima in [13] for the classical types and by Kang and Misra for type G_2 (see [11]). In [14–16] one describes combinatorial procedures on these tableaux generalizing the Knuth insertion algorithm. They also permit us to compute the transformation $\mathfrak{P}$ similarly to the original paper by O'Connell. The computation is then more efficient for it avoids the determination of a path from b to the highest weight vertex $\mathfrak{P}(b)$ of $B(b)$. We do not pursue in this direction and refer to [16] for a simple exposition of these procedures in types B_n, C_n, D_n and G_2.

To each vertex $b = a_1 \otimes \ldots \otimes a_\ell$ in $B(\delta)^{\otimes\ell}$ naturally corresponds a path in $Z(\delta,\ell)$, namely the path $(\mu^{(1)},\ldots,\mu^{(\ell)})$ where for any $k = 1,\ldots,\ell$, we have $\mu^{(k)} = \sum_{j=1}^{k} \mathrm{wt}(a_j)$. We shall consider the map $\mathcal{R}$ defined by

$$\mathcal{R} : \begin{cases} B(\delta)^{\otimes\ell} \to Z(\delta,\ell), \\ b \mapsto (\mu^{(1)},\ldots,\mu^{(\ell)}). \end{cases} \tag{5.4}$$

This map is surjective by definition of $Z(\delta,\ell)$. Write $\mathrm{HW}(B(\delta)^{\otimes\ell})$ for the set of highest weight vertices in $B(\delta)^{\otimes\ell}$. The image of the restriction $\mathcal{R}_{\mathrm{HW}}$ of $\mathcal{R}$ to $\mathrm{HW}(B(\delta)^{\otimes\ell})$ is a subset of $Z^+(\delta,\ell)$. Indeed, by Lemma 5.2, if $b = a_1 \otimes \ldots \otimes a_\ell$ belongs to $\mathrm{HW}(B(\delta)^{\otimes\ell})$, then $a_1 \otimes \ldots \otimes a_k$ belongs to $\mathrm{HW}(B(\delta)^{\otimes k})$ for any $k = 1,\ldots,\ell$.

When each weight space in $V_q(\delta)$ has dimension 1, the map $\mathcal{R}$ is bijective, since the steps in the path of $Z(\delta,\ell)$ are in one-to-one correspondence with the weights of $B(\delta)$. We then identify the paths $Z(\delta,\ell)$ with the vertices of $B(\delta)^{\otimes\ell}$. More precisely, the path $(0,\mu^{(1)},\ldots,\mu^{(\ell)})$ is identified with the vertex $b = a_1 \otimes \ldots \otimes a_\ell \in B(\delta)^{\otimes\ell}$ where, for any $k = 1,\ldots,\ell$, the vertex a_k is the unique one with weight $\mu^{(k)} - \mu^{(k-1)}$ in $B(\delta)$; this situation happens in particular when δ is a minuscule weight. Indeed, since each weight in $V_q(\delta)$ belongs to the orbit of δ, each weight space has dimension 1. We have the following crucial property:

Proposition 5.4. *The restriction $\mathcal{R}_{\mathrm{HW}} : \mathrm{HW}(B(\delta)^{\otimes\ell}) \to Z^+(\delta,\ell)$ is a one-to-one correspondence for any $\ell \geqslant 1$ if and only if δ is minuscule.*

Proof. Assume δ is minuscule. We identify the paths in $Z^+(\delta,\ell)$ with vertices of $B(\delta)^{\otimes\ell}$, as explained above, and $\mathcal{R}_{\mathrm{HW}}$ is injective. We thus have to show that any vertex $b = a_1 \otimes \ldots \otimes a_\ell \in B(\delta)^{\otimes\ell}$ such that $\mathrm{wt}(a_1 \otimes \ldots \otimes a_k)$ is dominant for any $k = 1,\ldots,\ell$, is a highest weight vertex. We proceed by induction on ℓ. For $\ell = 1$, the highest weight vertex of $B(\delta)$ is a_1, for $B(\delta)$ contains a unique vertex of dominant weight (δ being minuscule). Assume $b = a_1 \otimes \ldots \otimes a_\ell \in B(\delta)^{\otimes\ell}$ is such that $\mathrm{wt}(a_1 \otimes \ldots \otimes a_k)$ is dominant for any $k = 1,\ldots,\ell$. By the induction hypothesis, $b^\flat = a_1 \otimes \ldots \otimes a_{\ell-1}$ is of highest weight. According to assertion (ii) of Theorem 5.1, it suffices to show that $\varepsilon_i(a_\ell) \leqslant \varphi_i(b^\flat)$ for any $i \in I$. But $\varepsilon_i(a_\ell) \in \{0,1\}$ since δ is minuscule. One can therefore assume that $\varepsilon_i(a_\ell) = 1$. In this case, we have $\varphi_i(a_\ell) = 0$ because, once again, δ is minuscule. The condition $\mathrm{wt}(b) \in P_+$ implies that $\varphi_i(b) \geqslant \varepsilon_i(b)$ for any $i \in I$. Moreover,

the condition $\mathrm{wt}(b) = \mathrm{wt}(b^\flat) + \mathrm{wt}(a_\ell)$ can be written

$$\varphi_i(b) - \varepsilon_i(b) = \varphi_i(b^\flat) - \varepsilon_i(b^\flat) + \varphi_i(a_\ell) - \varepsilon_i(a_\ell) = \varphi_i(b^\flat) - \varepsilon_i(b^\flat) - 1 \quad \text{for any } i \in I.$$

Thus $\varphi_i(b^\flat) \geqslant \varepsilon_i(b^\flat) + 1 \geqslant \varepsilon_i(a_\ell)$ as expected.

Conversely, assume $\mathcal{R}_{\mathrm{HW}} : \mathrm{HW}(B(\delta)^{\otimes \ell}) \to Z^+(\delta, \ell)$ is a one-to-one correspondence for any $\ell \geqslant 1$. If δ is not minuscule, then there exists a weight μ such that $\dim V_q(\delta)_\mu > 0$ which is not in the orbit of δ under the action of the Weyl group W. The orbit of μ under W intersects the cone of dominant weights P_+. Therefore, we can assume $\mu \in P_+$. The crystal $B(\delta)$ thus contains a vertex b_μ of dominant weight μ which is not a highest weight vertex since $\mu \neq \delta$. Then the path in P from 0 to μ belongs to $Z^+(\delta, 1)$ but does not belong to the image of $\mathrm{HW}(B(\delta))$ by $\mathcal{R}_{\mathrm{HW}}$, and this contradicts our assumption. □

Remark 5. There exist dominant weights δ which are not minuscule but such that each weight space of $V(\delta)$ has dimension 1. This is notably the case for $\delta = k\omega_1, k \in \mathbb{N}$ in type A_{n-1} and $\delta = \omega_1$ in type B_n. In that case, according to the previous proposition, the map $\mathcal{R}_{\mathrm{HW}}$ does not provide a bijection between highest weight vertices and paths in P_+.

Let $Z^+(\delta, \ell, \lambda)$ be the subset of $Z^+(\delta, \ell)$ of paths starting at 0 and ending at $\lambda \in P_+$. Theorem 5.1 and the previous proposition immediately yield the following corollary.

Corollary 5.5. *Assume δ is minuscule. We have* $\mathrm{card}(Z^+(\delta, \ell, \lambda)) = f^\ell_{\lambda,\delta}$ *where the outer multiplicity $f^\ell_{\lambda,\delta}$ is defined by* (2.2).

As far as we are aware, this equality was first established by Grabiber and Mayard in [**7**]; their proof relied on some previous works by Proctor [**20**]. Crystal basis theory permits us to derive a very short proof of this identity.

5.3. *Probability distribution on $B(\delta)$*

The aim of this paragraph is to endow $B(\delta)$ with a probability distribution. We are going to associate to each vertex $a \in B(\delta)$ a probability p_a such that

$$0 < p_a < 1 \quad \text{and} \quad \sum_{a \in B(\delta)} p_a = 1. \tag{I}$$

The probability distributions we consider are compatible with the weight graduation, that is for any $a, a' \in B(\delta)$,

$$\mathrm{wt}(a) = \mathrm{wt}(a') \Longrightarrow p_a = p_{a'}. \tag{II}$$

We proceed as follows. Let $t = (t_1, \ldots, t_n)$ be a n-tuple of positive reals (recall that n is the rank of the root system we consider). Let $a \in B(\delta)$. For any $\kappa \in Q_+$ such that $\kappa = \sum_{i=1}^n m_i \alpha_i$, we set $t^{[\kappa]} = t_1^{m_1} \ldots t_n^{m_n}$. Since $\mathrm{wt}(a)$ is a weight of $V_q(\delta)$, there exist nonnegative integers $m_1, \ldots, m_n$ such that $\mathrm{wt}(a) = \delta - \sum_{i=1}^n m_i \alpha_i$. We can compute the sum $\Sigma_\delta(t) = \sum_{a \in B(\delta)} t^{[\mathrm{wt}(a)]}$. The probability distribution is then defined by

$$p_a = \frac{t^{[\mathrm{wt}(a)]}}{\Sigma_\delta(t)} \quad \text{for any } a \in B(\delta), \tag{5.5}$$

so that we have $\sum_{a \in B(\delta)} p_a = 1$. It is clear that conditions (I) and (II) are satisfied. Note that if we have an arrow $a \xrightarrow{i} a'$ in $B(\delta)$, then $p_{a'} = p_a \times t_i$ since $\mathrm{wt}(a') = \mathrm{wt}(a) - \alpha_i$. Also, the integer m_i can easily be read on the crystal $B(\delta)$. It corresponds to the number of arrows $\xrightarrow{i}$ in

any path connecting the highest weight vertex of $B(\delta)$ to a. Recall that these numbers do not depend on the path since the crystals are endowed with the weight graduation wt. The case when $t_i = 1$ for any $i = 1, \ldots, n$, corresponds to the uniform probability distribution on $B(\delta)$ considered in [1].

Remark 6. Observe that our construction depends only on the fixed n-tuple $t = (t_1, \ldots, t_n)$.

When $t_1 = \ldots = t_n = 1$, one gets $p_a = 1/\dim V(\delta)$ for any $a \in B(\delta)$. We shall say in this case that the distribution is uniform. In the rest of the paper, we assume that the n-tuple $t = (t_1, \ldots, t_n)$ is fixed for each root system corresponding to a simple Lie algebra. We denote by $x = (x_1, \ldots, x_N) \in \mathbb{R}^N_{>0}$ any solution of the algebraic system

$$x^{\alpha_i} = t_i^{-1}, \ i = 1, \ldots, n. \tag{5.6}$$

Let us briefly explain why such a solution necessarily exists. Let $U = (u_{i,j})$ be the $N \times n$ matrix whose entries are determined by the decompositions $\alpha_i = \sum_{j=1}^N u_{i,j}\varepsilon_j, \ i = 1, \ldots, n$. Taking the logarithm of each equation in (5.6), we are led to solve the equation ${}^tUX = Y$ where $X = (\ln x_1, \ldots, \ln x_N)$ and $Y = (-\ln t_1, \ldots, -\ln t_n)$. This linear system necessarily admits solutions because U has rank n; this follows from the fact that the set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ generates a n-dimensional subspace in $\mathbb{R}^N$. Each solution X yields a positive solution $x = (x_1, \ldots, x_N)$ of (5.6), and x is unique when $N = n$. With the convention of [3], this happens for the root systems B_n, C_n, D_n, E_8, F_4 and G_2. Now consider $a \in B(\delta)$. By the previous definition of t_a, we derive the relation $t_a = x^{-\delta}x^{\mathrm{wt}(a)}$. This allows us to write

$$\Sigma_\delta(t) = x^{-\delta} \sum_{a \in B(\delta)} x^{\mathrm{wt}(a)} = x^{-\delta} s_\delta(x) \quad \text{and} \quad p_a = \frac{x^{\mathrm{wt}(a)}}{s_\delta(x)}.$$

Example 5. (i) Assume $\delta = \omega_1$ for type A_{n-1}. Then we have

$$B(\delta) : a_1 \xrightarrow{1} a_2 \xrightarrow{2} \cdots \xrightarrow{n-1} a_n.$$

The simple roots are the $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ i = 1, \ldots, n-1$. We thus have $t_i = x_{i+1}/x_i$ for any $i = 1, \ldots, n-1$. We obtain $x_i = x_1 t_1 \ldots t_{i-1}$ for any $i = 2, \ldots, n$. In that case, we have $N = (n-1) + 1$ where $n-1$ is the rank of the root system considered. We have $p_i = x_i/(x_1 + \ldots + x_n)$ for $i = 1, \ldots, n$. This gives $p_i = x_i$ for $i = 1, \ldots, n$. The corresponding random walk in P corresponds to the ballot problem where each transition in the direction ε_i has probability p_i.

(ii) Assume $\delta = \omega_1$ for type C_n. Then, we have

$$B(\delta) : a_1 \xrightarrow{1} a_2 \xrightarrow{2} \cdots \xrightarrow{n-1} a_n \xrightarrow{1} a_{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} a_{\overline{2}} \xrightarrow{1} a_{\overline{1}}.$$

The simple roots are the $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ i = 1, \ldots, n-1$ and $\alpha_n = 2\varepsilon_n$. We thus have $t_i = x_{i+1}/x_i$ for any $i = 1, \ldots, n-1$ and $t_n = x_n^{-2}$. This gives $x_i = 1/(t_i \ldots t_{n-1}\sqrt{t_n})$ for any $i = 1, \ldots n-1$ and $x_n = 1/\sqrt{t_n}$. So $p_{a_i} = x_i/s_{\omega_1}(x)$ and $p_{a_{\overline{i}}} = x_i^{-1}/s_{\omega_1}(x)$ where $s_{\omega_1}(x) = \sum_{i=1}^n (x_i + x_i^{-1})$. We thus obtain a random walk in $\mathbb{Z}^n$ with transitions to nearest neighbours and such that the quantities $p_{a_i}p_{a_{\overline{i}}}$ do not depend on i.

5.4. *The random walk $\mathcal{W}$ in the weight lattice*

5.4.1. *From a probability measure on the crystal $B(\delta)$ to a random walk on the weight lattice.* A random walk in the weight lattice P isomorphic to $\mathbb{Z}^n$ is characterized by the law of its increments, which is a probability measure on P. As the map $b \mapsto \mathrm{wt}(b)$ sends the crystal $B(\delta)$ into P, any probability measure p on $B(\delta)$ can be pushed forward and defines a probability $\mathrm{wt}_* p$ on P.

The random walk with law $\mathrm{wt}_* p$ can be naturally realized as a random process defined on the infinite product space $\mathcal{B}(\delta)^{\otimes\mathbb{N}}$ equipped with the product measure $p^{\otimes\mathbb{N}}$. Here is a 'concrete' realization of this construction. The infinite product space $\mathcal{B}(\delta)^{\otimes\mathbb{N}}$ is the projective limit of the sequence of tensor products $(B(\delta)^{\otimes\ell})_{\ell\in\mathbb{N}}$ associated to the projections $\pi_{\ell',\ell}(a_1\otimes a_2\otimes\ldots\otimes a_{\ell'}) = a_1\otimes a_2\otimes\ldots\otimes a_\ell$ if $\ell\leqslant\ell'$. We denote by $\pi_{\infty,\ell}$ the canonical projection from $\mathcal{B}(\delta)^{\otimes\mathbb{N}}$ onto $B(\delta)^{\otimes\ell}$. By Kolmogorov's theorem, we know that there exists a unique probability measure $\mathbb{P}$ on the space $\mathcal{B}(\delta)^{\otimes\mathbb{N}}$ whose image by each projection $\pi_{\infty,\ell}$ is the probability $\mathbb{P}^{(\ell)}$ defined by

$$\mathbb{P}^{(\ell)}(a_1\otimes a_2\otimes\ldots\otimes a_\ell) = \prod_{i=1}^{\ell} p_{a_i}. \tag{5.7}$$

On the probability space $(\mathcal{B}(\delta)^{\otimes\mathbb{N}},\mathbb{P})$, the random variables

$$a_1\otimes a_2\otimes\ldots\mapsto a_\ell,\quad \ell\in\mathbb{N}$$

are independent and identically distributed with law p.

Now, on this probability space we define the random variables $\mathcal{W}_\ell$ by

$$\mathcal{W}_\ell = \mathrm{wt}\circ\pi_{\infty,\ell}, \tag{5.8}$$

that is

$$\mathcal{W}_\ell(b) = \mathrm{wt}(b^{(\ell)})$$

if $b = a_1\otimes a_2\otimes\ldots$ and $b^{(\ell)} = a_1\otimes a_2\otimes\ldots\otimes a_\ell$.

The random process $(\mathcal{W}_\ell)_{\ell\in\mathbb{N}}$ is a realization of the random walk on P with law $\mathrm{wt}_* p$. Indeed, for any $\beta,\beta'\in P$, we have

$$\begin{aligned}\mathbb{P}(\mathcal{W}_{\ell+1}=\beta'\mid\mathcal{W}_\ell=\beta) &= \mathbb{P}^{(\ell+1)}(\mathrm{wt}(b^{(\ell)}\otimes a_{\ell+1})=\beta'\mid\mathrm{wt}(b^{(\ell)})=\beta)\\ &= \mathbb{P}^{(\ell+1)}(\mathrm{wt}(b^{(\ell)})+\mathrm{wt}(a_{\ell+1})=\beta'\mid\mathrm{wt}(b^{(\ell)})=\beta)\\ &= \mathbb{P}^{(\ell+1)}(\mathrm{wt}(a_{\ell+1})=\beta'-\beta).\end{aligned}$$

5.4.2. *Application to our particular situation.* Now, we come back to the particular choice of a probability distribution p on the crystal $B(\delta)$ which has been described in the previous section; the probability of a vertex b depends only on its weight, and if there is an arrow $b\xrightarrow{i}b'$ then $p_{b'} = p_b\times t_i$.

We extend the notation p_b to the vertices $b\in B(\delta)^{\otimes\ell}$, using the rule (5.7):

$$p_{a_1\otimes a_2\otimes\ldots\otimes a_\ell} = p_{a_1}p_{a_2}\cdots p_{a_\ell}.$$

We recall that if $b\in B(\delta)$, then $p_b = x^{\mathrm{wt}(b)}/s_\delta(x)$. We see that if $b\in B(\delta)^{\otimes\ell}$, then

$$p_b = \frac{x^{\mathrm{wt}(b)}}{s_\delta(x)^\ell}. \tag{5.9}$$

Using the rules of construction of the tensor powers of the crystal graph, it is straightforward to verify that the previous properties extend to tensor powers: in particular, for b and b' in $B(\delta)^{\otimes\ell}$, one obtains the following.

(i) If $\mathrm{wt}(b)=\mathrm{wt}(b')$, then $p_b = p_{b'}$,

(ii) If there exists an arrow $b\xrightarrow{i}b'$ in $B(\delta)^{\otimes\ell}$. Then $p_{b'} = p_b\times t_i$.

Following the construction described in § 5.4.1, we obtain a random walk $(\mathcal{W}_\ell)_{\ell\in\mathbb{N}}$ on the weight lattice P satisfying the properties:

(i) the law of the increments of the random walk $(\mathcal{W}_\ell)$ is given by

$$\mathbb{P}(\mathcal{W}_{\ell+1}=\beta'\mid\mathcal{W}_\ell=\beta) = \frac{K_{\delta,\beta'-\beta}x^{\beta'-\beta}}{s_\delta(x)} \tag{5.10}$$

(ii) the expectation of the increments, also called the drift of the random walk, is

$$m := \frac{1}{s_\delta(x)} \sum_\alpha K_{\delta,\alpha} x^\alpha \alpha. \tag{5.11}$$

(iii) the probability of a finite path $\pi = (\beta_0 = 0, \beta_1, \beta_2, \ldots, \beta_\ell)$ is given by

$$\frac{1}{s_\delta(x)^\ell} K_{\delta,\alpha^{(1)}} \times \ldots \times K_{\delta,\alpha^{(\ell)}} \times x^{\beta_\ell},$$

where $\alpha^{(i)} = \beta_i - \beta_{i-1}$ for any $i = 1, \ldots, \ell$.

Remark 7. (i) Consider a dominant weight λ and B a connected component in $B(\delta)^{\otimes \ell}$ isomorphic to $B(\lambda)$. We set

$$S_{\lambda,\ell}(x) := \sum_{b \in B} p_b = \sum_{\beta \in P} \sum_{b \in B_\beta} \frac{x^\beta}{s_\delta(x)^\ell} = \frac{1}{s_\delta(x)^\ell} \sum_{\beta \in P} K_{\lambda,\beta} x^\beta = \frac{s_\lambda(x)}{s_\delta(x)^\ell}, \tag{5.12}$$

where the second equality is a consequence of (5.9) and the last equality is obtained by definition of the Weyl character. Clearly, $S_{\lambda,\ell}(x)$ represents the probability $\mathbb{P}(B)$ that a random vertex of $B(\delta)^{\otimes \ell}$ belongs to B; observe that this probability does not depend on the connected component B itself but only on λ and ℓ.

(ii) When $\lambda = \delta$ is minuscule, the situation simplifies. Indeed $K_{\delta,\alpha} = 1$ for any weight α of $V(\delta)$, therefore the map wt is one-to-one on $B(\delta)$. The probability of the path $\pi = (\beta_0 = 0, \beta_1, \beta_2, \ldots, \beta_\ell)$ is then equal to $x^{\beta_\ell}/s_\delta(x)^\ell$. In particular, *when δ is minuscule, two paths starting and ending at the same points have the same probability.*

6. *Markov chains in the Weyl chamber*

The purpose of this section is to introduce a Markov chain $(\mathcal{H}_\ell)$ in the Weyl chamber, obtained from $(\mathcal{W}_\ell)_{\ell \geqslant 1}$ by an operation on crystals. This operation consists in the composition of $\mathcal{W}_\ell$ with a transformation of $B(\delta)^{\otimes \mathbb{N}}$ which plays the role of the Pitman transformation. Here and in the sequel, δ is a fixed dominant weight, that is, an element of P_+.

6.1. *The Markov chain $\mathcal{H}$*

The map $\mathfrak{P}$ has been introduced in § 2.4. To any vertex $b(\ell) \in B(\delta)^{\otimes \ell}$, it associates the highest weight vertex $\mathfrak{P}(b(\ell))$ of $B(b(\ell)) \subset B(\delta)^{\otimes \ell}$. By Lemma 5.2, the transformation on $B(\delta)^{\otimes \mathbb{N}}$ (also denoted by $\mathfrak{P}$) such that $\mathfrak{P}(b) = (\mathfrak{P}(b(\ell)))_{\ell \geqslant 1}$ for any $b = (b(\ell))_{\ell \geqslant 1} \in B(\delta)^{\otimes \mathbb{N}}$ is well defined. We consider the random variable $\mathcal{H}_\ell := \mathcal{W}_\ell \circ \mathfrak{P}$ (see (5.8)) defined on the probability space $\Omega(\delta) = (B(\delta)^{\otimes \mathbb{N}}, \mathbb{P})$ with values in P_+. This yields a stochastic process $\mathcal{H} = (\mathcal{H}_\ell)_{\ell \geqslant 0}$.

Proposition 6.1. *Consider $\ell \in \mathbb{Z}$ and $\lambda \in P_+$. Then $\mathbb{P}(\mathcal{H}_\ell = \lambda) = f^\ell_{\lambda,\delta}(s_\lambda(x)/s_\delta(x)^\ell)$.*

Proof. By definition of the random variable $\mathcal{H}_\ell$, we have

$$\mathbb{P}(\mathcal{H}_\ell = \lambda) = \sum_{b_\lambda \in \mathrm{HW}(B(\delta)^{\otimes \ell}), \mathrm{wt}(b_\lambda) = \lambda} \mathbb{P}(B(b_\lambda)).$$

We have seen in (5.12) that $\mathbb{P}(B(b_\lambda)) = s_\lambda(x)/s_\delta(x)^\ell$ does not depend on b_λ but only on λ. By definition of $f^\ell_{\lambda,\delta}$ (see (2.2)) and Theorem 5.1, the number of connected components in $B(\delta)^{\otimes \ell}$ which are isomorphic to $B(\lambda)$ is equal to $f^\ell_{\lambda,\delta}$. This gives $\mathbb{P}(\mathcal{H}_\ell = \lambda) = f^\ell_{\lambda,\delta}(s_\lambda(x)/s_\delta(x)^\ell)$. □

We can now state the main result of this section:

THEOREM 6.2. *The stochastic process $\mathcal{H}$ is a Markov chain with transition probabilities*

$$\Pi_{\mathcal{H}}(\mu,\lambda) = \frac{m^{\lambda}_{\mu,\delta} s_\lambda(x)}{s_\delta(x) s_\mu(x)} \quad \lambda,\mu \in P_+.$$

Proof. Consider a sequence of dominant weights $\lambda^{(1)},\dots,\lambda^{(\ell)},\lambda^{(\ell+1)}$ such that $\lambda^{(1)} = \delta, \lambda^{(\ell)} = \mu$ and $\lambda^{(\ell+1)} = \lambda$. Recall that, for any $k = 1,\dots,\ell$, the integer $m^{\lambda^{(k+1)}}_{\lambda^{(k)},\delta}$ is the multiplicity of $V(\lambda^{(k+1)})$ in $V(\lambda^{(k)}) \otimes V(\delta)$. Let $b \in \Omega(\delta)$ be such that $\mathcal{H}_k(b) = \lambda^{(k)}$ for any $k = 1,\dots,\ell+1$. Write $h^{(\ell+1)} = a^h_1 \otimes \dots \otimes a^h_{\ell+1}$ for the highest weight vertex of $B(b^{(\ell+1)}) \subset B(\delta)^{\otimes \ell+1}$. By Lemma 5.2, for $k = 1,\dots,\ell+1$, the vertex $h^{(k)} = a^h_1 \otimes \dots \otimes a^h_k$ is the highest weight vertex of $b^{(k)}$ and has weight $\lambda^{(k)}$. Let us denote by S the set of highest weight vertices of $B(\delta)^{\otimes \ell+1}$ whose projection on $B(\delta)^k$ has weight $\lambda^{(k)}$ for any $k = 1,\dots,\ell+1$. By Theorem 5.1 and a straightforward induction, we have

$$\mathrm{card}(S) = \prod_{k=1}^{\ell} m^{\lambda^{(k+1)}}_{\lambda^{(k)},\delta}.$$

We can now write

$$\mathbb{P}(\mathcal{H}_{\ell+1} = \lambda, \mathcal{H}_k = \lambda^{(k)} \text{for any } k = 1,\dots,\ell) = \sum_{h\in S} \sum_{b \in B(h)} p_b = \sum_{h \in S} \mathbb{P}(B(h)).$$

Each h in S is a highest weight vertex of weight λ and one obtains $\mathbb{P}(B(h)) = S_{\lambda,\ell+1}(x)$ (see (5.12)). This gives

$$\mathbb{P}(\mathcal{H}_{\ell+1} = \lambda, \mathcal{H}_k = \lambda^{(k)} \text{ for any } k = 1,\dots,\ell) = \prod_{k=1}^{\ell} m^{\lambda^{(k+1)}}_{\lambda^{(k)},\delta} \times S_{\lambda,\ell+1}(x).$$

Similarly, we have

$$\mathbb{P}(\mathcal{H}_k = \lambda^{(k)} \text{ for any } k = 1,\dots,\ell) = \prod_{k=1}^{\ell-1} m^{\lambda^{(k+1)}}_{\lambda^{(k)},\delta} \times S_{\mu,\ell}(x).$$

Hence,

$$\mathbb{P}(\mathcal{H}_{\ell+1} = \lambda \mid \mathcal{H}_k = \lambda^{(k)} \text{ for any } k = 1,\dots,\ell) = \frac{m^{\lambda^{(\ell+1)}}_{\lambda^{(\ell)},\delta} S_{\lambda^{(\ell+1)},\ell+1}(x)}{S_{\lambda^{(\ell)},\ell}(x)} = \frac{m^{\lambda}_{\mu,\delta} s_\lambda(x)}{s_\delta(x) s_\mu(x)}.$$

In particular, $\mathbb{P}(\mathcal{H}_{\ell+1} = \lambda \mid \mathcal{H}_k = \lambda^{(k)}$ for any $k = 1,\dots,\ell)$ depends only on λ and $\mu = \lambda^{(\ell)}$. This shows the Markov property. □

6.2. *Intertwining operators*

For any $\lambda \in P_+$ and $\beta \in P$, the event $(\mathcal{W}_\ell = \beta, \mathcal{H}_\ell = \lambda)$ contains all the elements b in $\Omega(\delta)$ such that $b^{(\ell)}$ has weight β and belongs to a connected component of $B(\delta)^{\otimes \ell}$ with highest weight λ. This gives $\mathbb{P}(\mathcal{W}_\ell = \beta, \mathcal{H}_\ell = \lambda) = (1/s_\delta(x)^\ell) f_{\ell,\lambda} K_{\lambda,\beta} x^\beta$ since there are $f_{\ell,\lambda}$ connected components in $B(\delta)^{\otimes \ell}$ of highest weight λ and each of them contains $K_{\lambda,\beta}$ vertices of weight β whose common probability is $x^\beta / s_\delta(x)^\ell$. Using Proposition 6.1, we obtain

$$\mathbb{P}(\mathcal{W}_\ell = \beta \mid \mathcal{H}_\ell = \lambda) = \frac{K_{\lambda,\beta} x^\beta}{s_\lambda(x)} \tag{6.1}$$

which is independent of ℓ. We set $\mathcal{K}(\lambda,\beta) := \mathbb{P}(\mathcal{W}_\ell = \beta \mid \mathcal{H}_\ell = \lambda) = K_{\lambda,\beta} x^\beta / s_\lambda(x)$.

THEOREM 6.3. *We have the intertwining relation* $\Pi_{\mathcal{H}}\mathcal{K} = \mathcal{K}\Pi_{\mathcal{W}}$.

Proof. For any $\mu \in P_+$ and $\beta \in P$, let $(\Pi_{\mathcal{H}}\mathcal{K})(\mu,\beta)$ (respectively $(\mathcal{K}\Pi_{\mathcal{W}})(\mu,\beta)$) be the coefficient of the matrix $\Pi_{\mathcal{H}}\mathcal{K}$ (respectively $\mathcal{K}\Pi_{\mathcal{W}}$) corresponding to the pair (μ,β). By Theorem 6.2 and (6.1), one obtains

$$(\Pi_{\mathcal{H}}\mathcal{K})(\mu,\beta) = \sum_{\lambda\in P_+} \Pi_{\mathcal{H}}(\mu,\lambda)\mathcal{K}(\lambda,\beta) = \sum_{\lambda\in P_+} \frac{m^{\lambda}_{\mu,\delta}s_\lambda(x)}{s_\delta(x)s_\mu(x)}\frac{K_{\lambda,\beta}x^\beta}{s_\lambda(x)} = \sum_{\lambda\in P_+} \frac{m^{\lambda}_{\mu,\delta}K_{\lambda,\beta}x^\beta}{s_\delta(x)s_\mu(x)}.$$

This gives

$$(\Pi_{\mathcal{H}}\mathcal{K})(\mu,\beta) = \frac{x^\beta}{s_\delta(x)s_\mu(x)}\sum_{\lambda\in P_+} m^{\lambda}_{\mu,\delta}K_{\lambda,\beta}.$$

On the other hand, using (5.10)

$$(\mathcal{K}\Pi_{\mathcal{W}})(\mu,\beta) = \sum_{\gamma\in P}\mathcal{K}(\mu,\gamma)\Pi_{\mathcal{W}}(\gamma,\beta) = \sum_{\gamma\in P}\frac{K_{\mu,\gamma}x^\gamma}{s_\mu(x)}K_{\delta,\beta-\gamma}\frac{x^{\beta-\gamma}}{s_\delta(x)}.$$

This yields $(\mathcal{K}\Pi_{\mathcal{W}})(\mu,\beta) = (x^\beta/s_\delta(x)s_\mu(x))\sum_{\gamma\in P}K_{\mu,\gamma}K_{\delta,\beta-\gamma}$. Therefore, the equality $\Pi_{\mathcal{H}}\mathcal{K} = \mathcal{K}\Pi_{\mathcal{W}}$ reduces to $\sum_{\lambda\in P_+} m^{\lambda}_{\mu,\delta}K_{\lambda,\beta} = \sum_{\gamma\in P}K_{\mu,\gamma}K_{\delta,\beta-\gamma}$ which was established in Lemma 2.1. □

7. *Restriction to the Weyl chamber*

We have explicit formulae for the transition matrices $\Pi_{\mathcal{W}}$ and $\Pi_{\mathcal{H}}$ of the Markov chains $\mathcal{W}$ and $\mathcal{H}$. The matrix $\Pi_{\mathcal{W}}$ has entries in P and the matrix $\Pi_{\mathcal{H}}$ has entries in P_+. We will see that if the representation δ is minuscule then $\Pi_{\mathcal{H}}$ is a Doob transform of the restriction of $\Pi_{\mathcal{W}}$ to P_+.

7.1. *Doob transform of the random walk $\mathcal{W}$ restricted to the Weyl chamber*

Recall that $(\mathcal{W}_\ell)_{\ell\geqslant 1}$ is a Markov chain with transition matrix $\Pi_{\mathcal{W}}$. Since the closed Weyl chamber $\overline{C}$ is a subset of $\mathbb{R}^N$, it makes sense to consider the substochastic matrix $\Pi^{\overline{C}}_{\mathcal{W}}(\mu,\lambda)$, that is, the restriction of the transition matrix of $\mathcal{W}$ to $\overline{C}$. The following proposition answers the question whether the transition matrix $\Pi_{\mathcal{H}}$ of the Markov chain $\mathcal{H}$ (see Theorem 6.2) can be considered as a Doob transform of $\Pi^{\overline{C}}_{\mathcal{W}}$. We denote by ψ the function defined on P_+ by $\psi(\lambda) = x^{-\lambda}s_\lambda(x)$.

PROPOSITION 7.1. *If δ is a minuscule representation, then the transition matrix $\Pi_{\mathcal{H}}$ is the Doob ψ-transform of the substochastic matrix $\Pi^{\overline{C}}_{\mathcal{W}}$, in particular ψ is harmonic with respect to this substochastic matrix. If δ is not minuscule, then the transition matrix $\Pi_{\mathcal{H}}$ cannot be realized as a Doob transform of the substochastic matrix $\Pi^{\overline{C}}_{\mathcal{W}}$.*

Proof. Given λ and μ in P_+ we have

$$\Pi^{\overline{C}}_{\mathcal{W}}(\mu,\lambda) = K_{\delta,\lambda-\mu}p_{\lambda-\mu} \quad \text{and} \quad \Pi_{\mathcal{H}}(\mu,\lambda) = \frac{m^{\lambda}_{\mu,\delta}s_\lambda(x)}{s_\delta(x)s_\mu(x)}.$$

By Remark 1, if δ is minuscule, then $K_{\delta,\lambda-\mu} = m^{\lambda}_{\mu,\delta} \in \{0,1\}$ for any $\lambda,\mu \in P_+$. Therefore,

$$\Pi_{\mathcal{H}}(\mu,\lambda) = \frac{\psi(\lambda)}{\psi(\mu)}\Pi^{\overline{C}}_{\mathcal{W}}(\mu,\lambda).$$

Conversely, if $\Pi_{\mathcal{H}}$ can be realized as a h-transform of the substochastic matrix $\Pi_{\mathcal{W}}^{\overline{C}}$, then we must have

$$\frac{h(\lambda)}{h(\mu)} K_{\delta,\lambda-\mu} \frac{x^\lambda}{x^\mu s_\delta(x)} = \frac{m_{\mu,\delta}^\lambda s_\lambda(x)}{s_\delta(x) s_\mu(x)} \quad \text{for any } \lambda, \mu \in P_+.$$

This is equivalent to the equality

$$K_{\delta,\lambda-\mu} \frac{h(\lambda)}{h(\mu)} = m_{\mu,\delta}^\lambda \frac{x^{-\lambda} s_\lambda(x)}{x^{-\mu} s_\mu(x)} \quad \text{for any } \lambda, \mu \in P_+. \tag{7.1}$$

Since h and ψ are positive on P^+, the function $g := h/\psi$ is well defined and positive on P^+ and one obtains

$$K_{\delta,\lambda-\mu} \frac{g(\lambda)}{g(\mu)} = m_{\mu,\delta}^\lambda \quad \text{for any } \lambda, \mu \in P_+. \tag{7.2}$$

Assume (7.2) holds and δ is not minuscule. By Remark 1, there exists a dominant weight $\kappa \in P_+$, $\kappa \neq \delta$, such that $K_{\delta,\kappa} \neq 0$. For $\mu = 0$ and $\lambda = \kappa$, we obtain

$$K_{\delta,\kappa} \frac{g(\kappa)}{g(0)} = m_{0,\delta}^\kappa.$$

Now, since $\kappa \neq \delta$, we have $m_{0,\delta}^\kappa = 0$, and so $g(\kappa) = 0$ since $K_{\delta,\kappa} \neq 0$. Contradiction. □

7.2. *Limit of ψ along a drift*

As announced in Introduction, we will now describe the connection between the Markov chain $\mathcal{H}$ and the random walk $\mathcal{W}$ conditioned to never exit $\overline{C}$. We have seen in § 5.2 that, for the minuscule representation $V(\delta)$, the vertices of $B(\delta)^{\otimes \ell}$ can be identified with the paths $Z(\delta, \ell)$. Moreover, by Proposition 5.4, the highest weight vertices of $B(\delta)^{\otimes \ell}$ are identified with the paths $Z^+(\delta, \ell)$ which remain in $\overline{C}$. The drift m of $\mathcal{W}$ given by (5.11) belongs to C (the open Weyl chamber) when

$$m = \sum_{i \in I} m_i \omega_i \text{ with } m_i > 0 \quad \text{for any } i \in I. \tag{7.3}$$

Lemma 7.2. (i) *We have $m \in C$ if and only if $0 < t_i < 1$ for any $i \in I$.*

(ii) *For any direction $\vec{d}$ in C, there exists an n-tuple $t = (t_1, \ldots, t_n)$ with $0 < t_i < 1$ such that $\vec{d}$ is the direction of the drift m associated to the random walk $\mathcal{W}$ defined from t as in § 5.2.*

Proof. (i) Recall that $m = \sum_{b \in B(\delta)} p_b \mathrm{wt}(b)$. By (5.1), for any $b \in B(\delta)$, the coordinates of the weight $\mathrm{wt}(b)$ are determined by the i-chains containing b (that is, the sub-crystal containing all the vertices connected to b by arrows i). By (5.1) and (5.9), the contribution to m of any i-chain $a_1 \xrightarrow{i} a_2 \xrightarrow{i} \cdots \xrightarrow{i} a_{k+1}$ of length k is equal to $p_{a_1} \sum_{j=0}^{[k/2]} (k-2j)(t_i^j - t_i^{k-j})\omega_i$. For each fixed $i = 1, \ldots, n$, all these contributions are positive if and only if $0 < t_i < 1$. This proves (i).

(ii) For any $i = 1, \ldots, n$, let S_i be the set of vertices a in $B(\delta)$ such that $\varepsilon_i(a) = 0$. For any $a \in S_i$, write k_a the length of the i-chain in $B(\delta)$ starting at a. We denote by $m(t) = \sum_{i=1}^n m_i(t)\omega_i \in C$ the drift corresponding to the n-tuple $t = (t_1, \ldots, t_n)$ with $0 < t_i < 1$. By the previous arguments

$$m_i(t) = m_i(t_i) = \sum_{a \in S_i} p_a \sum_{j=0}^{[k_a/2]} (k_a - 2j)(t_i^j - t_i^{k_a - j})$$

depends only on t_i. Write $M_i = \max\{m_i(t_i) \mid 0 < t_i < 1\}$. One obtains $M_i > 0$ and for any u_i in $]0, M_i]$, there exists $t_i \in]0, 1[$ such that $m_i(t_i) = u_i$. Now consider a direction $\vec{d}$ in C.

Assume $v = (v_1, \ldots, v_n)$ belongs to $\mathbb{R}\vec{d}$. There exists $c \in \mathbb{R}_{>0}$ such that $u_i = cv_i \in]0, M_i]$ for any $i = 1, \ldots, n$. It then suffices to choose t_i such that $m_i(t_i) = u_i$. □

In the sequel, we *assume that* $m \in C$. Consider a positive root α, that we decompose into $\alpha = \alpha_{i_1} + \ldots + \alpha_{i_r}$ on the basis of simple roots. Then $t^{[\alpha]} = t_{i_1} \ldots t_{i_r}$. According to the fact that $m \in C$, we immediately derive from the previous lemma that $0 < t^{[\alpha]} < 1$. In particular, the product

$$\nabla = \prod_{\alpha \in R_+} \frac{1}{1 - x^{-\alpha}} = \prod_{\alpha \in R_+} \frac{1}{1 - t^{[\alpha]}} \tag{7.4}$$

running on the (finite) set R_+ of positive roots is well defined and finite.

Proposition 7.3. *Assume $m \in C$ and consider a sequence $(\lambda^{(a)})_{a \in \mathbb{N}}$ of dominant weights such that $\lambda^{(a)} = am + o(a)$. Then $\lim_{a \to +\infty} x^{-\lambda^{(a)}} s_{\lambda^{(a)}}(x) = \nabla$.*

Proof. By the Weyl character formula, we have

$$s_{\lambda^{(a)}}(x) = \nabla \sum_{w \in W} \varepsilon(w) x^{w(\lambda^{(a)} + \rho) - \rho}.$$

This gives

$$x^{-\lambda^{(a)}} s_{\lambda^{(a)}}(x) = \nabla \sum_{w \in W} \varepsilon(w) x^{w(\lambda^{(a)} + \rho) - \lambda^{(a)} - \rho} = \nabla \sum_{w \in W} \varepsilon(w) t^{[\lambda^{(a)} + \rho - w(\lambda^{(a)} + \rho)]}.$$

For $w = 1$, one obtains $\varepsilon(w) t^{[\lambda^{(a)} + \rho - w(\lambda^{(a)} + \rho)]} = 1$; it thus suffices to prove that, for any $w \neq 1$, we have $\lim_{a \to +\infty} t^{[\lambda^{(a)} + \rho - w(\lambda^{(a)} + \rho)]} = 0$.

Fix such a $w \neq 1$ and set $u(a) = \lambda^{(a)} + \rho - w(\lambda^{(a)} + \rho) = \lambda^{(a)} - w(\lambda^{(a)}) + \rho - w(\rho)$. Since $m \in C$, the weight $\lambda^{(a)}$ belongs to C for a large enough. In the sequel, we can thus assume that $\lambda^{(a)} + \rho \in C$. Its stabilizer under the action of the Weyl group is then trivial. Now the weights of the finite-dimensional representation $V(\lambda^{(a)} + \rho)$ are stable under the action of W. Thus, $w(\lambda^{(a)} + \rho)$ is a weight of $V(\lambda^{(a)} + \rho)$. This implies that $w(\lambda^{(a)} + \rho) \leqslant \lambda^{(a)} + \rho$, that is $u(a) \in Q_+$ is a linear combination of simple roots with nonnegative coefficients. Since $\lambda^{(a)} = am + o(a)$ and ρ is fixed, we have

$$u(a) = a(m - w(m)) + o(a) \in Q_+.$$

Since $w \neq 1$, one obtains $m \neq w(m)$ and one can set $m - w(m) = \sum_{i=1}^n m_i \alpha_i$ where the m_i belong to $\mathbb{R}_{\geqslant 0}$ for any $i = 1, \ldots, n$ and $m_{i_0} = \max_{i=1,\ldots,n}\{m_i\} > 0$. This gives $u(a) = a \sum_{i=1}^n m_i \alpha_i + o(a)$ and

$$t^{[\lambda^{(a)} + \rho - w(\lambda^{(a)} + \rho)]} = (t^{[m]})^a \times t^{[o(a)]} \leqslant t_{i_0}^{a m_{i_0}} \times t^{[o(a)]}$$

tends to 0 when a tends to infinity for $0 < t_{i_a} < 1$. □

7.3. *Random walks $\mathcal{W}$ with fixed drift*

Let δ be a minuscule representation. There is a bijection between the paths from 0 to $\lambda \in P_+$ and the highest weight vertices of $B(\delta)^{\otimes \ell}$ of weight λ. Assume the probability distribution on $B(\delta)$ is such that $m \in C$. The following proposition shows that the probability distribution on $B(\delta)$ is completely determined by m under the previous hypotheses. This remark will not be needed in the sequel of the article.

PROPOSITION 7.4. *Assume δ is minuscule. If two probability distributions on $B(\delta)$ satisfying the conditions imposed in § 5.3 have the same drift $m \in C$, then they coincide.*

Proof. Assume that we have a probability distribution on $B(\delta)$ as in § 5.3. Consider $\lambda^{(\ell)} = \ell m + O(1)$ a sequence of dominant weights which tends to infinity in the direction of the drift m. Let $\gamma \in P$. Since δ is minuscule, all the paths from 0 to $\lambda^{(\ell)} - \gamma$ (respectively to $\lambda^{(\ell)}$) have the same probability; furthermore, by Proposition 5.4, there is $f^{\ell}_{\lambda^{(\ell)}-\gamma,\delta}$ (respectively $f^{\ell}_{\lambda^{(\ell)},\delta}$) such paths. We thus have

$$\frac{f^{\ell}_{\lambda^{(\ell)}-\gamma,\delta} x^{\lambda^{(\ell)}-\gamma}}{f^{\ell}_{\lambda^{(\ell)},\delta} x^{\lambda^{(\ell)}}} = \frac{\mathbb{P}(\mathcal{W}_1 \in \overline{C}, \ldots, \mathcal{W}_\ell \in \overline{C}, \mathcal{W}_\ell = \lambda^{(\ell)} - \gamma)}{\mathbb{P}(\mathcal{W}_1 \in \overline{C}, \ldots, \mathcal{W}_\ell \in \overline{C}, \mathcal{W}_\ell = \lambda^{(\ell)})}.$$

By Theorem 4.3, we know that this quotient tends to 1 when ℓ tends to infinity. This gives

$$\lim_{\ell \to +\infty} \frac{f^{\ell}_{\lambda^{(\ell)}-\gamma,\delta}}{f^{\ell}_{\lambda^{(\ell)},\delta}} = x^{\gamma}, \tag{7.5}$$

hence x^{γ} is determined by m. The unique probability distribution with drift m defined on $B(\delta)$ thus satisfies $p_a = x^{\mathrm{wt}(a)}/s_\delta(x)$ where $x^{\mathrm{wt}(a)}$ and $s_\delta(x)$ are given by (7.5). □

7.4. *Transition matrix of $\mathcal{W}$ conditioned to never exit the Weyl chamber*

In this section, we *assume δ is minuscule* and $m \in C$. Set $\Pi = \Pi^{\overline{C}}_{\mathcal{W}}$. Let us denote by Γ the Green function associated with the substochastic matrix Π. Consider $\mu \in P_+$. For any $\lambda \in P$, we have $\Gamma(\mu, \lambda) = \sum_{\ell \geqslant 0} \Pi^{\ell}(\mu, \lambda)$ and, clearly, $\Gamma(\mu, \lambda) = 0$ if $\lambda \notin P_+$. The Martin kernel

$$K(\mu, \mathcal{W}_\ell) = \frac{\Gamma(\mu, \mathcal{W}_\ell)}{\Gamma(0, \mathcal{W}_\ell)} \tag{7.6}$$

is defined a.s., for ℓ large enough, along the random walk trajectories.

In order to apply Theorem 3.1, we want to prove that K converges a.s. to the harmonic function ψ of § 7.1.

Write b_μ for the highest weight vertex of $B(\mu)$. For $\lambda \in P_+$, let $B(\mu, \ell, \lambda)$ be the subset of vertices $b = a_1 \otimes \ldots \otimes a_\ell$ in $B(\delta)^{\otimes \ell}$ such that

$$\mathrm{wt}(b_\mu \otimes a_1 \otimes \ldots \otimes a_k) \in P_+ \text{ for any } k = 1, \ldots, \ell \text{ and } \mathrm{wt}(b_\mu \otimes b) = \lambda.$$

By definition of $\Pi = \Pi^{\overline{C}}_{\mathcal{W}}$, and since δ is minuscule, one obtains

$$\Pi^{\ell}(\mu, \lambda) = \mathrm{card}(B(\mu, \ell, \lambda)) \frac{x^{\lambda - \mu}}{s_\delta(x)^{\ell}}.$$

Indeed, all the paths from μ to λ of length ℓ have the same probability $x^{\lambda-\mu}/s_\delta(x)^{\ell}$. By (2.2), Theorem 5.1 and Proposition 5.4, we know that

$$\mathrm{card}(B(\mu, \ell, \lambda)) = f^{\ell}_{\lambda/\mu}$$

and $f^{\ell}_{\lambda/\mu}$ is the number of highest weight vertices of weight λ in $B(\mu) \otimes B(\delta)^{\otimes \ell}$. According to Lemma 5.2, they can be written as $b_\mu \otimes b$ with $\varepsilon_i(b) \leqslant \varphi_i(b_\mu)$ for any $i \in I$.

REMARK 8. When δ is not minuscule, the vertices $b_\mu \otimes b$ with $b \in B(\lambda, \mu, \ell)$ are not necessarily of highest weight and we can have $\mathrm{card}(B(\lambda, \mu, \ell)) > f^{\ell}_{\lambda/\mu}$. A vertex of $B(\delta)^{\ell}$ which is not of highest weight can yield a path in $\overline{C}$.

According to Proposition 5.3, given any sequence $\lambda^{(a)}$ of weights of the form $\lambda^{(a)} = am + o(a)$, one can write, for a large enough

$$\Gamma(\mu, \lambda^{(a)}) = \sum_{\ell \geqslant 0} f^{\ell}_{\lambda^{(a)}/\mu} \frac{x^{\lambda^{(a)}-\mu}}{s_\delta(x)^\ell} = x^{\lambda^{(a)}-\mu} \sum_{\ell \geqslant 0} \frac{1}{s_\delta(x)^\ell} \sum_{\gamma \in P} f^{\ell}_{\lambda^{(a)}-\gamma} K_{\mu,\gamma}$$
$$= x^{\lambda^{(a)}-\mu} \sum_{\gamma \in P} K_{\mu,\gamma} \sum_{\ell \geqslant 0} \frac{f^{\ell}_{\lambda^{(a)}-\gamma}}{s_\delta(x)^\ell}.$$

Since

$$\Gamma(0, \lambda^{(a)}) = x^{\lambda^{(a)}} \sum_{\gamma \in P} K_{0,\gamma} \sum_{\ell \geqslant 0} \frac{f^{\ell}_{\lambda^{(a)}-\gamma}}{s_\delta(x)^\ell} = \sum_{\ell \geqslant 0} f^{\ell}_{\lambda} \frac{x^{\lambda^{(a)}}}{s_\delta(x)^\ell},$$

one obtains, for a large enough,

$$K(\mu, \lambda^{(a)}) = x^{-\mu} \sum_{\gamma \in P} K_{\mu,\gamma} \frac{\sum_{\ell \geqslant 0} (f^{\ell}_{\lambda^{(a)}-\gamma}/s_\delta(x)^\ell)}{\sum_{\ell \geqslant 0} (f^{\ell}_{\lambda^{(a)}}/s_\delta(x)^\ell)}$$
$$= x^{-\mu} \sum_{\gamma \in P} K_{\mu,\gamma} x^\gamma \frac{\sum_{\ell \geqslant 0} f^{\ell}_{\lambda^{(a)}-\gamma} (x^{\lambda^{(a)}-\gamma}/s_\delta(x)^\ell)}{\sum_{\ell \geqslant 0} f^{\ell}_{\lambda^{(a)}} (x^{\lambda^{(a)}}/s_\delta(x)^\ell)},$$

and thus

$$K(\mu, \lambda^{(a)}) = x^{-\mu} \sum_{\gamma \text{ weight of } V(\mu)} K_{\mu,\gamma} x^\gamma \frac{\Gamma(0, \lambda^{(a)} - \gamma)}{\Gamma(0, \lambda^{(a)})}. \tag{7.7}$$

We will use the following:

Proposition 7.5. *Under the previous assumptions on m, for any $\gamma \in P$, we have*

$$\lim_{a \to +\infty} \frac{\Gamma(0, \mathcal{W}_a - \gamma)}{\Gamma(0, \mathcal{W}_a)} = 1 \quad \text{(a.s.)}.$$

Proof. The weights γ run over the set of weights of $V(\mu)$. This set is finite, therefore the statement is a direct consequence of Theorem 4.6. □

The strong law of large numbers states that $\mathcal{W}_a = am + o(a)$ a.s. With (7.7) this implies that, a.s., for a large enough

$$K(\mu, \mathcal{W}_a) = x^{-\mu} \sum_{\gamma \text{ weight of } V(\mu)} K_{\mu,\gamma} x^\gamma \frac{\Gamma(0, \mathcal{W}_a - \gamma)}{\Gamma(0, \mathcal{W}_a)},$$

and, by Proposition 7.5

$$L(\mu) := \lim_{a \to +\infty} K(\mu, \mathcal{W}_a) = x^{-\mu} \sum_{\gamma \text{ weight of } V(\mu)} K_{\mu,\gamma} x^\gamma = x^{-\mu} s_\mu(x) = \psi(\mu) \quad \text{(a.s.)}. \tag{7.8}$$

The function L thus coincides with the harmonic function of Proposition 7.1. By Theorem 3.1, there exists a constant c such that $\psi = c h_{\overline{C}}$ where $h_{\overline{C}}$ is the harmonic function defined in § 3.2 associated to the restriction of $(\mathcal{W}_\ell)_{\ell \geqslant 0}$ to the closed Weyl chamber $\overline{C}$. By Theorems 7.1 and 6.2, we thus derive

$$\Pi_{h_{\overline{C}}}(\mu, \lambda) = \Pi_\psi(\mu, \lambda) = \Pi_{\mathcal{H}}(\mu, \lambda) = \frac{m^{\lambda}_{\mu,\delta} s_\lambda(x)}{s_\delta(x) s_\mu(x)} = \frac{s_\lambda(x)}{s_\delta(x) s_\mu(x)}$$

since δ is a minuscule representation (which implies in particular $m^{\lambda}_{\mu,\delta} = 1$ for any $\lambda, \mu \in P_+$ such that $\lambda - \mu$ is a weight of $B(\delta)$, see § 2.3).

THEOREM 7.6. *Assume δ is a minuscule representation and $m \in C$. The transition matrix $\Pi_{\mathcal{H}}$ of the Markov chain $\mathcal{H}$ is the same as the transition matrix of $(\mathcal{W}^{\overline{C}}_{\ell})_{\ell \geqslant 0}$, the random walk $\mathcal{W}$ conditioned to never exit the cone $\overline{C}$: the coefficients of $\Pi_{\mathcal{H}}$ are given by*

$$\Pi_{\mathcal{H}}(\mu, \lambda) = \frac{s_\lambda(x)}{s_\delta(x) s_\mu(x)} \quad \text{for any } \lambda, \mu \in P_+ \text{ such that } \lambda - \mu \text{ is a weight of } B(\delta).$$

COROLLARY 7.7. *With the above notation and assumptions, for any $\lambda \in P_+$, we have*

$$\mathbb{P}_\lambda(\mathcal{W}_\ell \in \overline{C} \text{ for any } \ell \geqslant 1) = x^{-\lambda} s_\lambda(x) \prod_{\alpha \in R_+} (1 - x^{-\alpha}).$$

Proof. Recall that the function $\lambda \longmapsto \mathbb{P}_\lambda(\mathcal{W}_\ell \in \overline{C}$ for any $\ell \geqslant 1)$ is harmonic. By Theorem 7.6, there is a positive constant c such that $\mathbb{P}_\lambda(\mathcal{W}_\ell \in \overline{C}$ for any $\ell \geqslant 1) = c x^{-\lambda} s_\lambda(x)$. But, for any sequence $\lambda^{(a)} = am + o(a)$ of dominant weights

$$\lim_{a \to +\infty} \mathbb{P}_{\lambda^{(a)}}(\mathcal{W}_\ell \in \overline{C} \text{ for any } \ell \geqslant 1) = \lim_{a \to +\infty} \mathbb{P}_0(\mathcal{W}_\ell + \lambda^{(a)} \in \overline{C} \text{ for any } \ell \geqslant 1) = 1.$$

On the other hand, we know by Proposition 7.3 that $\lim_{a \to +\infty} x^{-\lambda} s_\lambda(x) = \nabla$ (see (7.4)). Therefore, $c = 1/\nabla$. □

EXAMPLE 6. Consider a random walk $\mathcal{W}_\ell$ in $\mathbb{Z}^n$ to the $2n$ nearest neighbours such that the quantities $p_{a_i} p_{a_{\overline{i}}}$ do not depend on i (see Example 5). Let $\nu \in P_+$. Let us compute $\mathbb{P}_\nu(\mathcal{W}_\ell \in \overline{C}$ for any $\ell \geqslant 1)$ for

$$\overline{C} = \{\lambda \mid \lambda_1 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant \lambda_n \geqslant 0, \lambda_i \in \mathbb{Z}\} \quad \text{and for}$$
$$\overline{C} = \{\lambda \mid \lambda_1 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant |\lambda_n| \geqslant 0, \lambda_i \in \mathbb{Z} \cup \tfrac{1}{2}\mathbb{Z}\}$$

that is, for the Weyl chambers of types C_n and D_n.

(i) In type C_n, the process $\mathcal{W}_\ell$ is obtained from $B(\omega_1)$. The simple roots are the $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \ldots, n-1$ and $\alpha_n = 2\varepsilon_n$, we obtain $x_i = 1/(t_i \ldots t_{n-1}\sqrt{t_n})$ for any $i = 1, \ldots n-1$ and $x_n = 1/\sqrt{t_n}$. The positive roots are $\varepsilon_i \pm \varepsilon_i$ with $1 \leqslant i < j \leqslant n$ and $2\varepsilon_i$ with $i = 1, \ldots, n$. Therefore,

$$\mathbb{P}_\nu(\mathcal{W}_\ell \in \overline{C} \text{ for any } \ell \geqslant 1) = x^{-\nu} s^{C_n}_\nu(x) \prod_{1 \leqslant i < j \leqslant n} \left(1 - \frac{x_j}{x_i}\right)\left(1 - \frac{1}{x_i x_j}\right) \prod_{1 \leqslant i \leqslant n} \left(1 - \frac{1}{x_i^2}\right)$$

where $s^{C_n}_\nu(x)$ is the Weyl character of type C_n associated to ν, specialized in $x_1, \ldots, x_n$.

(ii) In type D_n, the process $\mathcal{W}_\ell$ is also obtained from $B(\omega_1)$. The simple roots are the $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \ldots, n-1$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. We obtain $x_i = 1/(t_i \ldots t_{n-2}\sqrt{t_{n-1}t_n})$ for any $i = 1, \ldots, n-2$, $x_{n-1} = 1/\sqrt{t_{n-1}t_n}$ and $x_n = \sqrt{t_{n-1}/t_n}$. The positive roots are $\varepsilon_i \pm \varepsilon_i$ with $1 \leqslant i < j \leqslant n$. Then

$$\mathbb{P}_\nu(\mathcal{W}_\ell \in \overline{C} \text{ for any } \ell \geqslant 1) = x^{-\nu} s^{D_n}_\nu(x) \prod_{1 \leqslant i < j \leqslant n} \left(1 - \frac{x_j}{x_i}\right)\left(1 - \frac{1}{x_i x_j}\right)$$

where $s^{D_n}_\nu(x)$ is the Weyl character of type D_n associated to ν, specialized in $x_1, \ldots, x_n$.

8. *Complementary results*

8.1. *Asymptotic behaviour of the coefficients* $f^{\ell}_{\lambda/\mu,\delta}$

In [23], Stanley studied the asymptotic behaviour of $f^{\ell}_{\lambda/\mu,\delta}$ when $V(\delta)$ is the defining representation of $\mathfrak{gl}_n$ (that is, $\delta = \omega_1$ is associated to the classical ballot problem). More precisely, he established that for any fixed $\mu \in P_+$ and any direction $\vec{d}$ in C

$$\lim_{\ell\to\infty} \frac{f^{\ell}_{\lambda^{(\ell)}/\mu,\omega_1}}{f^{\ell}_{\lambda^{(\ell)},\omega_1}} = s_\mu(m), \tag{8.1}$$

where $m = (m_1, \ldots, m_n) \in d$ is such that $m_i \geqslant 0$ for any $i = 1, \ldots n$, the sum $m_1 + \ldots + m_n$ equals 1 and $\lambda^{(\ell)} = \ell m + o(\ell)$ tends to ∞ in the direction $\vec{d}$.

By the previous theorem, one may extend this result as follows. Assume $\vec{d}$ is a direction in C. By (ii) of Lemma 7.2, there exists a n-tuple $t = (t_1, \ldots, t_n)$ with $0 < t_i < 1$ such that $\vec{d}$ is the direction of the drift $m \in C$ associated to the random walk $\mathcal{W}$ defined from t as in § 5.2.

THEOREM 8.1. *Assume δ is minuscule. If $\lambda^{(\ell)} = \ell m + o(\ell^\alpha)$ with $\alpha < \frac{2}{3}$, then*

$$\lim_{\ell\to\infty} \frac{f^{\ell}_{\lambda^{(\ell)}/\mu,\delta}}{f^{\ell}_{\lambda^{(\ell)},\delta}} = s_\mu(x). \tag{8.2}$$

Proof. Consider $\lambda^{(\ell)} = \ell m + o(\ell)$ a sequence of dominant weights which tends to infinity in the direction $\vec{d}$. By Proposition 5.3

$$\frac{f^{\ell}_{\lambda^{(\ell)}/\mu,\delta}}{f^{\ell}_{\lambda^{(\ell)},\delta}} = \sum_{\gamma\in P} K_{\mu,\gamma} \frac{f^{\ell}_{\lambda^{(\ell)}-\gamma,\delta}}{f^{\ell}_{\lambda^{(\ell)},\delta}} = \sum_{\gamma\in P} K_{\mu,\gamma} \frac{f^{\ell}_{\lambda^{(\ell)}-\gamma,\delta} x^{\lambda^{(\ell)}-\gamma}}{f^{\ell}_{\lambda^{(\ell)},\delta} x^{\lambda^{(\ell)}}} x^\gamma, \tag{8.3}$$

where the sums are finite since the set of weight in $V(\mu)$ is finite. Note that, for any $\gamma \in P$,

$$\frac{f^{\ell}_{\lambda^{(\ell)}-\gamma,\delta} x^{\lambda^{(\ell)}-\gamma}}{f^{\ell}_{\lambda^{(\ell)},\delta} x^{\lambda^{(\ell)}}} = \frac{\mathbb{P}(\mathcal{W}_1 \in \overline{C}, \ldots, \mathcal{W}_\ell \in \overline{C}, \mathcal{W}_\ell = \lambda^{(\ell)} - \gamma)}{\mathbb{P}(\mathcal{W}_1 \in \overline{C}, \ldots, \mathcal{W}_\ell \in \overline{C}, \mathcal{W}_\ell = \lambda^{(\ell)})}.$$

By Theorem 4.3, we know that this quotient tends to 1 when ℓ tends to infinity. This implies

$$\lim_{\ell\to+\infty} \frac{f^{\ell}_{\lambda^{(\ell)}/\mu,\delta}}{f^{\ell}_{\lambda^{(\ell)},\delta}} = \sum_{\gamma\in P} K_{\mu,\gamma} x^\gamma = s_\mu(x)$$

as announced. □

REMARK 9. When $\delta = \omega_1$ in type A_{n-1} (ballot problem), we have $p_i = x_i/s_\delta(x)$. Then $m = (1/s_\delta(x)) \sum_{i=1}^n x_i \varepsilon_i$. We can normalize the law so that $s_\delta(x) = 1$ (it suffices to replace each x_i by $x_i/s_\delta(x)$). Then $p_i = m_i = x_i$. This shows that our theorem is indeed a generalization of (8.1). Finally, note that the proof of (8.1) mainly uses the representation theory of the symmetric group; it seems nevertheless difficult to obtain a purely algebraic proof of the limits (8.2).

8.2. *Random walks defined from non-irreducible representations*

In Section 6, we have defined the random walk $(\mathcal{W}_\ell)_{\ell\geqslant 0}$ starting from the crystal $B(\delta)$ of the irreducible module $V_q(\delta)$. In fact, most of our results can be easily adapted to the case when

$(\mathcal{W}_\ell)_{\ell\geqslant 0}$ is defined from the crystal $B(M)$ of a $U_q(\mathfrak{g})$-module M possibly non-irreducible. These random walks are therefore based on tensor products of non-irreducible representations. We may also associate to such products random walks for which the conditioned law to never exit $\overline{C}$ can be computed explicitly in terms of characters.

To do this, we consider similarly some positive real numbers $t_1, \ldots, t_n$ and $x_1, \ldots, x_N$ such that $x^{\alpha_i} = t_i^{-1}$. We define a probability distribution on $B(M)$, setting $p_a = x^{\mathrm{wt}(a)}/s_M(x)$ where s_M is the character of M (that is the sum of the characters of its irreducible components). The random variable X on $B(M)$ is given by $X(a) = \mathrm{wt}(a)$ for any $a \in B(M)$. The product probability distribution on $B(M)^{\otimes\ell}$ is $p_b = x^{\mathrm{wt}(b)}/s_M(x)^\ell$ for any $b \in B(\delta)^{\otimes\ell}$. We introduce similarly $\Omega(M)$ as the projective limit of the tensor products $B(M)^{\otimes\ell}$ and the Markov chain $\mathcal{W} = (\mathcal{W}_\ell)_{\ell\geqslant 0}$ such that $\mathcal{W}_\ell(b) = \mathrm{wt}(b^{(\ell)})$ for any $\ell \geqslant 0$. Its transition matrix is given by

$$\Pi_{\mathcal{W}}(\beta, \beta') = K_{M,\beta'-\beta}\frac{x^{\beta'-\beta}}{s_M(x)},$$

where $K_{M,\beta'-\beta}$ is the dimension of the weight space $\beta' - \beta$ in M. Write $M = \bigoplus_{\nu\in P_+} V_q^{\oplus m_\nu}(\nu)$ for the decomposition of M into irreducible components. The set of increments for the Markov chain corresponding to M is the union of the sets of transition for the Markov chains corresponding to each $V(\nu)$ with $m_\nu > 0$. Observe that the Markov chain obtained for an isotypical representation $M = V_q^{\oplus m_\nu}(\nu)$ is the same as the Markov chain for $V(\nu)$. Nevertheless, when M admits at least two non isomorphic irreducible components, the matrix $\Pi_{\mathcal{W}}$ depends on the multiplicities m_ν.

One can consider the random variable $\mathcal{H}_\ell$ on $\Omega(M)$ such that $\mathcal{H}_\ell(b) = \mathrm{wt}(\mathfrak{P}(b(\ell)))$ for any $b \in \Omega(M)$, that is $\mathcal{H}_\ell(b)$ is the highest weight of $B(b)$, the connected component of $B(M)^{\otimes\ell}$ containing $b^{(\ell)}$. In the proof of Theorem 6.2, we do not use the fact that $B(\delta)$ is connected (or equivalently that $V_q(\delta)$ is irreducible). So, by the same arguments, one may check that $\mathcal{H} = (\mathcal{H}_\ell)_{\ell\geqslant 0}$ is yet a Markov chain with transition probabilities

$$\Pi_{\mathcal{H}}(\mu, \lambda) = \frac{m^\lambda_{\mu,M} s_\lambda(x)}{s_M(x) s_\mu(x)},$$

where $m^\lambda_{\mu,M}$ is the multiplicity of $V_q(\lambda)$ in $V_q(\mu) \otimes M$.

In order to obtain an analogue of Proposition 7.1, we will say that M is of *minuscule type* if all its irreducible components are minuscule representations and $\dim M_\mu \in \{0, 1\}$ for any weight μ. In particular, the nonzero multiplicities in the decomposition $M = \bigoplus_{\nu\in P_+} V_q(\nu)$ are equal to 1. When M is of minuscule type, we have $K_{M,\lambda-\mu} = m^\lambda_{\mu,M} \in \{0, 1\}$ for any $\lambda, \mu \in P_+$ since $K_{\nu,\lambda-\mu} = m^\lambda_{\mu,\nu}$ for any ν and the irreducible components $V_q(\nu)$ have no common weight. We give below the table of the possible non-irreducible minuscule-type representations for each root system. For a table of the minuscule representations see § 2.3.

Type	Non-irreducible minuscule-type representations
A_n	$\bigoplus_{j\in J} V(\omega_j)$ for $J \subset \{1, \ldots, n-1\}$ and $\lvert J\rvert > 1$
D_n	$\bigoplus_{j\in J} V(\omega_j)$ for $J \subset \{1, n-1, n\}$ and $\lvert J\rvert > 1$
E_6	$V(\omega_1) \oplus V(\omega_6)$

THEOREM 8.2. *Assume M is of* minuscule type. *Then, the transition matrix $\Pi_{\mathcal{H}}$ is the ψ-transform of the substochastic matrix $\Pi^{\overline{C}}_{\mathcal{W}}$ where ψ is the harmonic function defined by $\psi(\lambda) = x^{-\lambda} s_\lambda(x)$ for any $\lambda \in P_+$.*

Proof. Write $M = \bigoplus_{\nu\in P_+} V_q(\nu)$ the decomposition of M into its irreducible components. We have

$$K_{M,\lambda-\mu} = \sum_{\nu} K_{\nu,\lambda-\mu} \in \{0,1\} \quad \text{and} \quad m^{\lambda}_{\mu,M} = \sum_{\nu} m^{\lambda}_{\mu,\nu} \in \{0,1\}. \tag{8.4}$$

As in Proposition 7.1, the matrix $\Pi_{\mathcal{H}}$ is the ψ-transform of the substochastic matrix $\Pi^{\overline{C}}_{\mathcal{W}}$, since we have $K_{M,\lambda-\mu} = m^{\lambda}_{\mu,M}$ for any $\lambda,\mu \in P_+$. □

In the rest of this paragraph, we assume that $m = \mathbb{E}(X)$ belongs to C. As in Lemma 7.2, this is equivalent to the assumption $0 < t_i < 1$ for any $i = 1,\dots,n$. If we denote by $f^{\ell}_{\lambda/\mu,M}$ the multiplicity of $V_q(\lambda)$ in $V_q(\mu)\otimes M^{\otimes\ell}$, then the decomposition of Proposition 5.3 yet holds. Moreover, Proposition 5.4 admits a straightforward analogue which guarantees that each vertex $b_\mu \otimes b$ with $b \in B(M)^{\otimes\ell}$ yielding a path in $\overline{C}$, is of highest weight. If we consider a family $(\lambda^{(a)})_a$ of weights of the form $\lambda^{(a)} = am + o(a)$, then one has, for a large enough, $\lambda^{(a)} \in P_+$ and

$$f^{\ell}_{\lambda^{(a)}/\mu,M} = \sum_{\kappa\in P_+} f^{\ell}_{\kappa,M} K_{\mu,\lambda^{(a)}-\kappa} = \sum_{\gamma\in P} f^{\ell}_{\lambda^{(a)}-\gamma,M} K_{\mu,\gamma}.$$

Next theorem is analogous to Theorem 7.6.

THEOREM 8.3. *Assume M is a minuscule type representation and $m \in C$. Then the Markov chains $\mathcal{H}$ and $(\mathcal{W}^{\overline{C}}_{\ell})_{\ell\geqslant 0}$ have the same the transition matrix Π, namely*

$$\Pi(\mu,\lambda) = \frac{s_\lambda(x)}{s_M(x)s_\mu(x)} \quad \textit{for any } \lambda,\mu \in P_+ \textit{ such that } \lambda-\mu \textit{ is a weight of } B(M).$$

EXAMPLE 7. Consider the minuscule-type representation $M = V(\omega_1)\oplus V(\omega_{n-1})\oplus V(\omega_n)$ in type D_n. The simple roots are the $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1,\dots,n-1$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. We thus have $t_i = x_{i+1}/x_i$ for any $i = 1,\dots,n-1$ and $t_n = 1/x_{n-1}x_n$; this gives $x_i = 1/(t_i\dots t_{n-2}\sqrt{t_{n-1}t_n})$ for any $i = 1,\dots,n-2$, $x_{n-1} = 1/\sqrt{t_{n-1}t_n}$ and $x_n = \sqrt{t_{n-1}/t_n}$. We have $s_M(x) = s_{\omega_1}(x) + s_{\omega_{n-1}}(x) + s_{\omega_n}(x)$. The weights μ of M (and thus the possible transitions for $\mathcal{W}$ in P) are such that

$$\mu \in \{\pm\varepsilon_i \mid i = 1,\dots,n\} \sqcup \{\pm\tfrac{1}{2}\varepsilon_1 \pm \dots \pm \tfrac{1}{2}\varepsilon_n\}.$$

There exist $2^n + 2n$ possible transitions and the probability corresponding to the transition μ is $p_\mu = x^\mu/s_M(x)$.

Appendix A. *Miscellaneous proofs*

A.1. *Proof of Theorem* 3.1

We endow the space $(\Omega,\mathcal{T})$ with the probability $\mathbb{P}_{x^*}$ defined by: $\mathbb{P}_{x^*}(\cdot) = \mathbb{P}[\cdot \mid Y_1 = x^*]$. We denote by K_h the Martin kernel associated to Π_h; for any $x,y \in M$

$$K_h(x,y) = \frac{h(x^*)}{h(x)} K(x,y).$$

The Markov chain (Y^h_ℓ) is transient in M and it converges a.s. in the Martin topology to a random variable Y^h_∞ taking values in the Martin boundary $\mathcal{M}$ (see, for instance, [24, Theorem 7.19]). That means that for $\mathbb{P}_{x^*}$-almost all ω, the sequence of functions $(K_h(\cdot, Y^h_\ell(\omega)))_{\ell\geqslant 1}$ converges pointwise to some (random) function $K_h(\cdot, Y^h_\infty(\omega))$; in this

context, the hypothesis of the theorem may be written

$$\forall x \in M, \quad K_h(x, Y_\infty^h(\omega)) = \frac{h(x^*)}{h(x)} f(x) \quad \text{(a.s.)}.$$

Since the family of functions $\{K_h(\cdot, \xi), \xi \in \mathcal{M}\}$ separates the boundary points, the function $K_h(\cdot, Y_\infty^h(\omega))$ is a.s. constant, that is, $Y_\infty^h(\omega) = \xi_\infty$ a.s. for some fixed boundary point ξ_∞.

It remains to prove that $K_h(x, \xi_\infty)$ does not depend on x. Note that $\mathbb{P}_x \ll \mathbb{P}_{x^\star}$ for all $x \in M$, since $\Gamma(x^*, x) > 0$. Setting $\nu_x(B) := \mathbb{P}_x[Y_\infty^h \in B]$ for any Borel set B in $\mathcal{M}$ and any $x \in M$, one obtains (see, for instance, [**24**, Theorem 7.42])

$$\nu_x(B) = \int_B K_h(x, \xi)\ \nu_{x^*}(d\xi). \tag{A.1}$$

By the above, ν_{x^*} is the Dirac mass at the point ξ_∞; the equality (A.1) with $B = M$ gives $1 = K_h(x, \xi_\infty)$ for any $x \in M$, that is, $f(x) = h(x)/h(x^*)$. The proof is complete.

A.2. *Proof of Lemma* 4.4

We begin with a lemma coming from Garbit's thesis [**5**, **6**]. This lemma gives a uniform lower bound for the probability that a centred random walk goes from the ball of radius $r + 1$ to the ball of radius r without leaving a cone.

Let $(R_\ell)_{\ell \geqslant 1}$ be a centred random walk in $\mathbb{R}^n$, with finite second moment. Let $\mathcal{C}_0$ be an open convex cone in $\mathbb{R}^n$. We fix a unitary vector $\vec{u}$ in $\mathcal{C}_0$. For any $a > 0$, we denote by $\mathcal{C}_a$ the translated cone $\mathcal{C}_a := a\vec{u} + \mathcal{C}_0$, and for any $r > 0$ we denote by $\mathcal{C}_a(r)$ the truncated translated cone $\mathcal{C}_a(r) := \{x \in \mathcal{C}_a \mid \|x\| \leqslant r\}$, which is non-empty if $a < r$.

Lemma A.1 [**5**, **6**]. *There exist a positive integer k and real positive numbers r_0, a and ρ, with $a \leqslant r_0$ such that*

$$\forall r \geqslant r_0, \quad \forall x \in \mathcal{C}_a(r+1), \quad \mathbb{P}_x[R_1 \in \mathcal{C}_0, \ldots, R_k \in \mathcal{C}_0, R_k \in \mathcal{C}_a(r)] \geqslant \rho.$$

Proof of Lemma A.4. For $y \in \mathbb{R}^n$ and $\ell \in \mathbb{N}$, we set

$$p_\ell(y) := \mathbb{P}_0\left[y + \frac{R_1}{\sqrt{\ell}} \in \mathcal{C}_0, \ldots, y + \frac{R_\ell}{\sqrt{\ell}} \in \mathcal{C}_0, y + \frac{R_\ell}{\sqrt{\ell}} \in \mathcal{C}_1(2)\right].$$

Note that we also get

$$p_\ell(y) = \mathbb{P}_{\sqrt{\ell} y}[R_1 \in \mathcal{C}_0, \ldots, R_\ell \in \mathcal{C}_0, R_\ell \in \mathcal{C}_{\sqrt{\ell}}(2\sqrt{\ell})].$$

If a sequence (y_ℓ) in $\mathbb{R}^n$ converges to a point y in $\mathcal{C}_0$, the 'Donsker line' going through the points $(y_\ell + R_i/\sqrt{\ell})_{i=1,\ldots,\ell}$ converges in law to a Brownian path (between times 0 and 1) starting at y; the probability that this Brownian path stays in $\mathcal{C}_0$ and ends in $\mathcal{C}_1(2)$ is positive; from Portemanteau theorem (cf. [**2**]), we deduce that $\liminf p_\ell(y_\ell) > 0$. It follows that, for any compact $K \subset \mathcal{C}_0$, there exists a positive integer ℓ such that $\inf_{y \in K} p_\ell(y) > 0$.

We choose a positive integer k and a positive real ρ such that $p_k(y) \geqslant \rho$ for all $y \in \overline{\mathcal{C}_1(3)}$. We note that $y \in \overline{\mathcal{C}_1(3)}$ as soon as $\sqrt{k} y \in \mathcal{C}_{\sqrt{k}}(2\sqrt{k} + 1)$ and we conclude

$$\forall z \in \mathcal{C}_{\sqrt{k}}(2\sqrt{k} + 1), \quad \mathbb{P}_z[R_1 \in \mathcal{C}_0, \ldots, R_k \in \mathcal{C}_0, R_k \in \mathcal{C}_{\sqrt{k}}(2\sqrt{k})] \geqslant \rho.$$

We define $r_0 := 2\sqrt{k}$ and $a := \sqrt{k}$. The four parameters of the lemma are now fixed, and the result is proved for $r = r_0$.

We consider $r > r_0$ and $z \in \mathcal{C}_a(r+1)$ and we distinguish two cases.

(i) If $z \in \mathcal{C}_a(r_0 + 1)$, then $\mathbb{P}_z[R_1 \in \mathcal{C}_0, \ldots, R_k \in \mathcal{C}_0, R_k \in \mathcal{C}_a(r)] \geqslant \rho$, because $\mathcal{C}_a(r_0) \subset \mathcal{C}_a(r)$ and $\mathbb{P}_z[R_1 \in \mathcal{C}_0, \ldots, R_k \in \mathcal{C}_0, R_k \in \mathcal{C}_a(r_0)] \geqslant \rho$.

(ii) If $z \notin \mathcal{C}_a(r_0+1)$, then we remark that $\|a\vec{u}\| < r_0+1 < \|z\|$ and we denote by z' the point on the segment $[a\vec{u}, z]$ such that $\|z'\| = r_0+1$. We have $z' \in \mathcal{C}_a(r_0+1)$ and we verify, by inclusion of events, that

$$\mathbb{P}_{z'}[R_1 \in \mathcal{C}_0, \dots, R_k \in \mathcal{C}_0, R_k \in \mathcal{C}_a(r_0)] \leqslant \mathbb{P}_z[R_1 \in \mathcal{C}_0, \dots, R_k \in \mathcal{C}_0, R_k \in \mathcal{C}_a(r)].$$

We conclude that the right-hand term is $\geqslant \rho$.
The proof of the lemma is complete. □

Proof of Lemma 4.4. We come back to the study of a sequence of bounded independent identically distributed random variables (X_ℓ) taking values in a discrete subgroup G of $\mathbb{R}^n$. We suppose that the law of these random variables is adapted and aperiodic. We denote by m their common mean and by (S_ℓ) the associated random walk. Moreover, we consider a cone $\mathcal{C}$ and an open convex subcone $\mathcal{C}_0$ satisfying assumptions (h1) and (h2) stated in Section 4.1.

Let $\alpha \in]\frac{1}{2}, \frac{2}{3}[$. We suppose that the sequence $(\ell^{-\alpha}\|g_\ell - \ell m\|)_\ell$ is bounded, and we want to prove that

$$\liminf (\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, S_\ell = g_\ell])^{\ell^{-\alpha}} > 0.$$

It is sufficient to prove this result with $\mathcal{C}$ replaced by $\mathcal{C} \cap \overline{\mathcal{C}_0}$, so we suppose in the sequel that $\mathcal{C} \subset \overline{\mathcal{C}_0}$. With this assumption, we get that the sum of an element of $\mathcal{C}$ and an element of $\mathcal{C}_0$ is always in $\mathcal{C}_0$, hence in $\mathcal{C}$. By Lemma 4.1, we know that $p := \mathbb{P}[\forall \ell \geqslant 1,\ S_\ell \in \mathcal{C}] > 0$. Since $\alpha > \frac{1}{2}$, the sequence $(\ell^{-\alpha}(S_\ell - \ell m))_\ell$ goes to zero in quadratic mean, hence in probability. For r_0 large enough and all ℓ, we have

$$\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, \|S_\ell - \ell m\| \leqslant r_0 + [\ell^\alpha]] \geqslant \frac{p}{2}. \tag{A.2}$$

From Lemma A.4 applied to the centred random walk $R_\ell = S_\ell - \ell m$, we know that there exist an integer $k > 0$, a positive real number r_0 and a positive real number ρ such that, for all $r \geqslant r_0$ and all $x \in \mathcal{C}_0$ with $\|x\| \leqslant r+1$,

$$\mathbb{P}_x[S_1 - m \in \mathcal{C}_0, \dots, S_k - km \in \mathcal{C}_0, \|S_k - km\| \leqslant r] \geqslant \rho.$$

Moreover, for all $r > r_0$ and all $x \in \mathcal{C}_0$ with $\|x\| \leqslant r+1$,

$$\mathbb{P}_x[S_1 \in \mathcal{C}_0, \dots, S_k \in \mathcal{C}_0, \|S_k - km\| \leqslant r] \geqslant \rho. \tag{A.3}$$

We fix k, r_0 and ρ.

For a fixed $\ell \geqslant 1$, we consider now paths of the random walk satisfying the following $(1 + [\ell^\alpha])$ independent properties:

$$S_1 \in \mathcal{C}, \dots, S_\ell \in \mathcal{C}, \|S_\ell - \ell m\| \leqslant r_0 + [\ell^\alpha],$$

$$S_{\ell+1} - S_\ell \in \mathcal{C}_0, \dots, S_{\ell+k} - S_\ell \in \mathcal{C}_0 \text{ and } \|S_{\ell+k} - S_\ell - km\| \leqslant r_0 + [\ell^\alpha] - 1,$$

$$S_{\ell+k+1} - S_{\ell+k} \in \mathcal{C}_0, \dots, S_{\ell+2k} - S_{\ell+k} \in \mathcal{C}_0 \text{ and } \|S_{\ell+2k} - S_{\ell+k} - km\| \leqslant r_0 + [\ell^\alpha] - 2,$$

$$\vdots$$

$$S_{\ell+k([\ell^\alpha]-1)+1} - S_{\ell+k([\ell^\alpha]-1)} \in \mathcal{C}_0, \dots, S_{\ell+k[\ell^\alpha]} - S_{\ell+k([\ell^\alpha]-1)} \in \mathcal{C}_0$$
$$\text{and } \|S_{\ell+k[\ell^\alpha]} - S_{\ell+k([\ell^\alpha]-1)} - km\| \leqslant r_0 + [\ell^\alpha] - [\ell^\alpha].$$

The probabilities that these path properties hold are controlled by (A.2) and (A.3). By inclusion, we conclude that, for all ℓ, we have

$$\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{\ell+k[\ell^\alpha]} \in \mathcal{C}, \|S_{\ell+k[\ell^\alpha]} - (\ell + k[\ell^\alpha])m\| \leqslant r_0] \geqslant \frac{p}{2}\rho^{\ell^\alpha}.$$

The distance between the ball $B(\ell m, r_0)$ and the complementary of the cone $\mathcal{C}$ increases linearly with ℓ. We can fix an integer d such that, for all ℓ, the distance between the ball $B(d\ell m, r_0)$ and the complementary of the cone is greater than 2ℓ times the L^∞ norm of the random variables X_ℓ.

Let us summarize what we know as follows.

(i) For all ℓ,

$$\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{d\ell+k[(d\ell)^\alpha]} \in \mathcal{C}, \|S_{d\ell+k[(d\ell)^\alpha]} - (d\ell + k[(d\ell)^\alpha])m\| \leqslant r_0] \geqslant \frac{p}{2}\rho^{(d\ell)^\alpha}, \tag{A.4}$$

(ii) Starting from a point in the ball $B((d\ell + k[(d\ell)^\alpha])m, r_0)$, the random walk cannot exit the cone $\mathcal{C}$ in less than 2ℓ steps (we use here the fact that the X_ℓ are bounded).

We now want to apply Theorem 4.2; for all integers $\ell \geqslant 0$ and i between ℓ and 2ℓ, we define $g'_{\ell,i} := g_{d\ell+k[(d\ell)^\alpha]+i} - x - (d\ell + k[(d\ell)^\alpha])m$ and we note that, if $\|x\| \leqslant r_0$, then

$$\|g'_{\ell,i} - im\| = O((d\ell + k[(d\ell)^\alpha] + i)^\alpha) = O(i^\alpha).$$

Since $\alpha < \frac{2}{3}$, Theorem 4.2 gives us the existence of positive constants c and C such that, for all ℓ large enough,

$$\inf_{x:\|x\|\leqslant r_0 \text{ and } g'_{\ell,i}\in G} \mathbb{P}_x[S_i = g'_{\ell,i}] \geqslant Ci^{-n/2} \exp\left(-\frac{c}{i}\|g'_{\ell,i} - im\|^2\right).$$

Modifying c and C if necessary, we obtain

$$\inf_{x:\|x\|\leqslant r_0 \text{ and } g'_{\ell,i}\in G} \mathbb{P}_x[S_i = g'_{\ell,i}] \geqslant C\ell^{-n/2} \exp(-c\ell^{2\alpha-1}). \tag{A.5}$$

We apply this estimate to the set of paths of the random walk starting at time $d\ell + k[(d\ell)^\alpha]$ from the point $x + (d\ell + k[(d\ell)^\alpha])m$. Using (A.4) and (A.5), we obtain

$$\begin{aligned}\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{d\ell+k[(d\ell)^\alpha]} \in \mathcal{C}, \|S_{d\ell+k[(d\ell)^\alpha]} - (d\ell + k[(d\ell)^\alpha])m\| \leqslant r_0,\\ S_{d\ell+k[(d\ell)^\alpha]+i} = g_{d\ell+k[(d\ell)^\alpha]+i}] \geqslant \frac{p}{2}\rho^{(d\ell)^\alpha} \times C\ell^{-n/2} \exp(-c\ell^{2\alpha-1}).\end{aligned}$$

Moreover, we remember that, by the choice of d, if $\|S_{d\ell+k[(d\ell)^\alpha]} - (d\ell + k[(d\ell)^\alpha])m\| \leqslant r_0$ then $S_{d\ell+k[(d\ell)^\alpha]+j} \in \mathcal{C}$ for all j between 1 and 2ℓ. We conclude that

$$\mathbb{P}[S_1 \in \mathcal{C}, \dots, S_{d\ell+k[(d\ell)^\alpha]+i} \in \mathcal{C}, S_{d\ell+k[(d\ell)^\alpha]+i} = g_{d\ell+k[(d\ell)^\alpha]+i}] \geqslant \frac{p}{2}\rho^{(d\ell)^\alpha} C\ell^{-n/2} \exp(-c\ell^{2\alpha-1}).$$

Since $\alpha < 1$, the $(d\ell + k[(d\ell)^\alpha] + i)^{-\alpha}$th power of the right-hand side has a positive limit when ℓ goes to infinity. This gives the announced conclusion, modulo the following final easy claim: all large enough integers can be written under the form $d\ell + k[(d\ell)^\alpha] + i$ with $0 < \ell \leqslant i \leqslant 2\ell$. □

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Cédric Lecouvey, Emmanuel Lesigne and Marc Peigné
Laboratoire de Mathématiques et Physique Théorique,
Université François-Rabelais Tours
Fédération Denis Poisson - CNRS
Parc de Grandmont
37200 Tours
France

cedric.lecouvey@lmpt.univ-tours.fr
emmanuel.lesigne@lmpt.univ-tours.fr
marc.peigne@lmpt.univ-tours.fr