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Return Probabilities for the Reflected Random Walk on $\mathbb{N}_0$

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Abstract Let (Y_n) be a sequence of i.i.d. $\mathbb{Z}$-valued random variables with law μ. The reflected random walk (X_n) is defined recursively by $X_0 = x \in \mathbb{N}_0$, $X_{n+1} = |X_n + Y_{n+1}|$. Under mild hypotheses on the law μ, it is proved that, for any $y \in \mathbb{N}_0$, as $n \to +\infty$, one gets $\mathbb{P}_x[X_n = y] \sim C_{x,y} R^{-n} n^{-3/2}$ when $\sum_{k\in\mathbb{Z}} k\mu(k) > 0$ and $\mathbb{P}_x[X_n = y] \sim C_y n^{-1/2}$ when $\sum_{k\in\mathbb{Z}} k\mu(k) = 0$, for some constants $R, C_{x,y}$ and $C_y > 0$.

Keywords Random walks · Local limit theorem · Generating function · Wiener-Hopf factorization

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1 Introduction

We consider a sequence $(Y_n)_{n\geq 1}$ of $\mathbb{Z}$-valued independent and identically distributed random variables, with commun law μ, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We denote by $(S_n)_{n\geq 0}$ the classical random walk with law μ on $\mathbb{Z}$, defined by $S_0 = 0$ and $S_n = Y_1 + \cdots + Y_n$; the canonical filtration associated with the sequence $(Y_n)_{n\geq 1}$ is denoted $(\mathcal{T}_n)_{n\geq 1}$. The **reflected random walk** on $\mathbb{N}_0$ is defined by

$$\forall n \geq 0 \quad X_{n+1} = |X_n + Y_{n+1}|,$$

where X_0 is a $\mathbb{N}_0$-valued random variable.

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The process $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{N}_0$ with initial law $\mathcal{L}(X_0)$ and transition matrix $Q = (q(x, y))_{x,y\in\mathbb{N}_0}$ given by

$$\forall x, y \geq 0 \quad q(x, y) = \begin{cases} \mu(y-x) + \mu(y+x) & \text{if} \quad y \neq 0 \\ \mu(-x) & \text{if} \quad y = 0 \end{cases}.$$

When $X_0 = x$ $\mathbb{P}-$a.s., with $x \in \mathbb{N}_0$ fixed, the random walk $(X_n)_{n\geq 0}$ is denoted $(X_n^x)_{n\geq 0}$; the probability measure on $(\Omega, \mathcal{T})$ conditioned to the event $[X_0 = x]$ will be denoted $\mathbb{P}_x$ and the corresponding expectation $\mathbb{E}_x$.

We are interested with the behavior as $n \to +\infty$ of the probabilities $\mathbb{P}_x[X_n = y], x, y \in \mathbb{N}_0$; it is thus natural to consider the following generating function $\mathbf{G}$ associated with $(X_n)_{n\geq 0}$ and defined formally as follows:

$$\forall x, y \in \mathbb{N}_0, \ \forall s \in \mathbb{C} \quad \mathbf{G}(s|x, y) := \sum_{n\geq 0} \mathbb{P}_x[X_n = y]s^n.$$

The radius of convergence R of this series is ≥ 1.

The reflected random walk is positive recurrent when $\mathbb{E}[|Y_n|] < +\infty$ and $\mathbb{E}[Y_n] < 0$ (see [8] and [9] for instance and references therein) and consequently $R = 1$; it is also the case when the Y_n are centered, under the stronger assumption $\mathbb{E}[|Y_n|^{3/2}] < +\infty$. On the other hand, when $\mathbb{E}[|Y_n|] < +\infty$ and $\mathbb{E}[Y_n] > 0$, as in the case of the classical random walk on $\mathbb{Z}$, it is natural to assume that μ has exponential moments.

We will extract information about the asymptotic behavior of coefficients of a generating function using the following theorem of Darboux.

Theorem 1.1 *Let* $\mathbf{G}(s) = \sum_{n=0}^{+\infty} g_n s^n$ *be a power series with nonnegative coefficients* g_n *and radius of convergence* $R > 0$. *We assume that* $\mathbf{G}$ *has no singularities in the closed disk* $\left\{s \in \mathbb{C}/|s| \leq R\right\}$ *except* $s = R$ *(in other words,* $\mathbf{G}$ *has an analytic continuation to an open neighborhood of the set* $\left\{s \in \mathbb{C}/|s| \leq R\right\} \setminus \{R\}$*) and that in a neighborhood of* $s = R$

$$\mathbf{G}(s) = \mathbf{A}(s)(R-s)^\alpha + \mathbf{B}(s) \tag{1}$$

where $\mathbf{A}$ *and* $\mathbf{B}$ *are analytic functions.*[1] *Then*

$$g_n \sim \frac{\mathbf{A}(R)R^{1-n}}{\Gamma(-\alpha)n^{1+\alpha}} \quad \text{as} \quad n \to +\infty. \tag{2}$$

This approach has been yet developed by Lalley [5] in the general context of *random walk with a finite reflecting zone*. The transitions $q(x, \cdot)$ of Markov chains of this class are the ones of a classical random walk on $\mathbb{N}_0$ whenever $x \geq K$ for some $K \geq 0$. In our context of the reflected random walk on $\mathbb{N}_0$, it means that the support of μ is

[1] In Eq. 1, this is the positive branch s^α which is meant, which implies that the branch cut is along the negative axis; so the branch cut for the function $\mathbf{G}(s)$ is along the halfline $[R, +\infty]$.

bounded from below (namely by $-K$); we will not assume this in the sequel and will thus not follow the same strategy than S. Lalley. The methods required for the analysis of random walks with non-localized reflections are more delicate, and this is the aim of the present work. We also refer to [6] for a generalization of the main theorem in [5] in another direction

The reflected random walk on $\mathbb{N}_0$ is characterized by the existence of reflection times. We have to consider the sequence $(\mathbf{r}_k)_{k\geq 0}$ of waiting times with respect to the filtration $(\mathcal{T}_n)_{n\geq 0}$ defined by

$$\mathbf{r}_0 = 0 \quad \text{and} \quad \mathbf{r}_{k+1} := \inf\{n > \mathbf{r}_k : X_{\mathbf{r}_k} + Y_{\mathbf{r}_k+1} + \cdots + Y_n < 0\} \quad \text{for all} \quad k \geq 0.$$

In the sequel, we will often omit the index for $\mathbf{r}_1$ and denote this first reflection time $\mathbf{r}$. If one assumes $\mathbb{E}[|Y_n|] < +\infty$ and $\mathbb{E}[Y_n] \leq 0$, one gets $\mathbb{P}_x[\mathbf{r}_k < +\infty] = 1$ for all $x \in \mathbb{N}_0$ and $k \geq 0$; on the contrary, when $\mathbb{E}[|Y_n|] < +\infty$ and $\mathbb{E}[Y_n] > 0$, one gets $\mathbb{P}_x[\mathbf{r}_k < +\infty] < 1$ and in order to have $\mathbb{P}_x[\mathbf{r}_k < +\infty] > 0$ it is necessary to assume that $\mu(\mathbb{Z}^{*-}) > 0$.

The following identity will be essential in this work:

Proposition 1.2 *For all $x, y \in \mathbb{N}_0$, and $s \in \mathbb{C}$, one gets*

$$\mathbf{G}(s|x, y) = \mathbf{E}(s|x, y) + \sum_{w\in\mathbb{N}^*} \mathbf{R}(s|x, w)\mathbf{G}(s|w, y), \tag{3}$$

with

- *for all $x, y \geq 0$*

$$\mathbf{E}(s|x, y) := \sum_{n=0}^{+\infty} s^n \mathbb{P}_x[X_n = y, \mathbf{r} > n]$$

- *for all $x \geq 0$ and $w \geq 1$*

$$\begin{aligned}\mathbf{R}(s|x, w) &:= \mathbb{E}_x[1_{[\mathbf{r}<+\infty, X_\mathbf{r}=w]} s^\mathbf{r}] \\ &= \sum_{n\geq 0} s^n \mathbb{P}[x + S_1 \geq 0, \ldots, x + S_{n-1} \geq 0, x + S_n = -w].\end{aligned}$$

The generating function $\mathbf{E}$ concerns the excursion of the Markov chain $(X_n)_{n\geq 0}$ before its first reflection and $\mathbf{R}$ is related to the process of reflection $(X_{\mathbf{r}_k})_{k\geq 0}$.

By (3), one easily sees that, to make precise the asymptotic behavior of the $\mathbb{P}_x[X_n = y]$, it is necessary to control the excursions of the walk between two successive reflection times. Note that this interrelationship among the Green's functions $\mathbf{G}$, $\mathbf{E}$ and $\mathbf{H}$ may be written as a single matrix equation involving matrix-valued generating functions. For $s \in \mathbb{C}$, let us denote $\mathcal{G}_s$, $\mathcal{E}_s$ and $\mathcal{R}_s$ the following infinite matrices

- $\mathcal{G}_s = (\mathcal{G}_s(x, y))_{x,y\in\mathbb{N}_0}$ with $\mathcal{G}_s(x, y) = \mathbf{G}(s|x, y)$ for all $x, y \in \mathbb{N}_0$,
- $\mathcal{E}_s = (\mathcal{E}_s(x, y))_{x,y\in\mathbb{N}_0}$ with $\mathcal{E}_s(x, y) = \mathbf{E}(s|x, y)$ for all $x, y \in \mathbb{N}_0$,
- $\mathcal{R}_s = (\mathcal{R}_s(x, y))_{x\in\mathbb{N}_0, y\in\mathbb{N}^*}$ with $\mathcal{R}_s(x, y) = \mathbf{R}(s|x, y)$.

Thus, for all $x, y \in \mathbb{N}_0$ and $s \in \mathbb{C}$, one gets

$$\mathcal{G}_s = \mathcal{E}_s + \mathcal{R}_s \mathcal{G}_s. \tag{4}$$

The Green functions $\mathbf{G}(\cdot|x, y)$ may thus be computed when $I - \mathcal{R}_s$ is invertible, in which case one may write $\mathcal{G}_s = (I - \mathcal{R}_s)^{-1}\mathcal{E}_s$.

Let us now introduce some general assumptions:

H1: *the measure μ is* **adapted** *on $\mathbb{Z}$ (i-e the group generated by its support S_μ is equal to $\mathbb{Z}$) and* **aperiodic** *(i-e the group generated by $S_\mu - S_\mu$ is equal to $\mathbb{Z}$)*
H2: *the measure μ has exponential moments of any order (i.e. $\sum_{n\in\mathbb{Z}} r^n \mu(n) < +\infty$ for any $r \in]0, +\infty[$) and $\sum_{n\in\mathbb{Z}} n\mu(n) \geq 0$.*[2]

We now state the main result of this paper, which extends [5] in our situation:

Theorem 1.3 *Let $(Y_n)_{n\geq 1}$ be a sequence of $\mathbb{Z}$-valued independent and identically distributed random variables with law μ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that μ satisfies Hypotheses* **H** *and let $(X_n)_{n\geq 0}$ be the reflected random walk defined inductively by*

$$X_{n+1} = |X_n + Y_{n+1}| \quad \text{for} \quad n \geq 0.$$

- *If $\mathbb{E}[Y_n] = \sum_{k\in\mathbb{Z}} k\mu(k) = 0$, then for any $y \in \mathbb{N}_0$, there exists a constant $C_y \in \mathbb{R}^{*+}$ such that, for any $x \in \mathbb{N}_0$*

$$\mathbb{P}_x[X_n = y] \sim \frac{C_y}{\sqrt{n}} \quad \text{as} \quad n \to +\infty.$$

- *If $\mathbb{E}[Y_n] = \sum_{k\in\mathbb{Z}} k\mu(k) > 0$ then, for any $x, y \in \mathbb{N}_0$, there exists a constant $C_{x,y} \in \mathbb{R}^{*+}$ such that*

$$\mathbb{P}_x[X_n = y] \sim C_{x,y} \frac{\rho^n}{n^{3/2}}$$

  for some $\rho = \rho(\mu) \in [0, 1]$.

The constant $\rho(\mu)$ which appears in this statement is the infimum over $\mathbb{R}$ of the generating function of μ. We also know the exact value of the constants C_y and $C_{x,y}$, $x, y \in \mathbb{N}_0$, which appear in the previous statement: See formulas (36) and (40).

[2] We can in fact consider weaker assumptions: There exists $0 < r_- < 1 < r^+$ such that $\hat{\mu}(r) := \sum_{n\in\mathbb{Z}} r^n \mu(n) < +\infty$ for any $r \in [r_-, r_+]$ and μ_r reaches its minimum on this interval at a (unique) $r_0 \in [r_-, 1]$. We thus need more notations at the beginning, and this complicates in fact the understanding of the proof and is not really of interest.

2 Decomposition of the Trajectories and Factorizations

In this section, we will consider the subprocess of reflections $(X_{\mathbf{r}_k})_{k\geq 0}$ in order to decompose the trajectories of the reflected random walk in several parts which can be analyzed.

We first introduce some notations which appear classically in the fluctuation theory of 1-dimensional random walks.

2.1 On the Fluctuations of a Classical Random Walk on $\mathbb{Z}$

Let τ^{*-} the first strict descending time of the random walk $(S_n)_{n\geq 0}$:

$$\tau^{*-} := \inf\{n \geq 1/S_n < 0\}$$

(with the convention $\inf \emptyset = +\infty$). The variable τ^{*-} is a stopping time with respect to the filtration $(\mathcal{T}_n)_{n\geq 0}$.

We denote by $(T_n^{*-})_{n\geq 0}$ the sequence of successive strict ladder descending epochs of the random walk $(S_n)_{n\geq 0}$ defined by $T_0^{*-} = 0$ and $T_{n+1}^{*-} = \inf\{k > T_n^{*-}/S_k < S_{T_n^{*-}}\}$ for $n \geq 0$. One gets in particular $T_1^{*-} = \tau^{*-}$; furthermore, setting $\tau_n^{*-} := T_n^{*-} - T_{n-1}^{*-}$ for any $n \geq 1$, one may write $T_n^{*-} = \tau_1^{*-} + \cdots + \tau_n^{*-}$ where $(\tau_n^{*-})_{n\geq 1}$ is a sequence of independent and identically random variables with the same law as τ^{*-}. The sequence $(S_{T_n^{*-}})_{n\geq 0}$ of successive strict ladder descending positions of $(S_n)_{n\geq 0}$ is also a random walk on $\mathbb{Z}$ with independent and identically distributed increments of law $\mu^{*-} := \mathcal{L}(S_{\tau^{*-}})$. The potential associated with μ^{*-} is denoted by U^{*-}; one gets

$$U^{*-}(\cdot) := \sum_{n=0}^{+\infty} \left(\mu^{*-}\right)^{\star n}(\cdot) = \sum_{n=0}^{+\infty} \mathbb{E}\left[\delta_{S_{T_n^{*-}}}(\cdot)\right].$$

Similarly, we can introduce the first ascending time $\tau^{+} := \inf\{n \geq 1/S_n \geq 0\}$ of the random walk $(S_n)_{n\geq 0}$ and the sequence $(T_n^{+})_{n\geq 0}$ of successive large ladder ascending epochs of $(S_n)_{n\geq 0}$ defined by $T_0^{+} = 0$ and $T_{n+1}^{+} = \inf\{k > T_n^{+}/S_k \geq S_{T_n^{+}}\}$ for $n \geq 0$; as above, one may write $T_n^{+} = \tau_1^{+} + \cdots + \tau_n^{+}$ where $(\tau_n^{+})_{n\geq 1}$ is a sequence of i.i.d. random variables. The sequence $(S_{T_n^{+}})_{n\geq 0}$ of successive large ladder ascending positions of $(S_n)_{n\geq 0}$ is also a random walk on $\mathbb{Z}$ with independent and identically distributed increments of law $\mu^{+} := \mathcal{L}(S_{\tau^{+}})$. The potential associated with μ^{+} is denoted by U^{+}; one gets

$$U^{+}(\cdot) := \sum_{n=0}^{+\infty} \left(\mu^{+}\right)^{\star n}(\cdot) = \sum_{n=0}^{+\infty} \mathbb{E}\left[\delta_{S_{T_n^{+}}}(\cdot)\right].$$

We need to control the law of the couple $(\tau^{*-} S_{\tau^{*-}})$ and thus introduce the characteristic function φ^{*-} defined formally by $\varphi^{*-} : (s, z) \mapsto \sum_{n\geq 1} s^n \mathbb{E}\left[1_{[\tau^{*-}=n]} z^{S_n}\right]$ for

$s, z \in \mathbb{C}$. We also introduce the characteristic function associated with the potential of $(\tau^{*-}, S_{\tau^{*-}})$ defined by $\Phi^{*-}(s,z) = \sum_{k\geq 0} \mathbb{E}\left[s^{T_k^{*-}} z^{S_{T_k^{*-}}}\right] = \sum_{k\geq 0} \varphi^{*-}(s,z)^k = \frac{1}{1-\varphi^{*-}(s,z)}$. Similarly, we consider the function $\varphi^+(s,z) := \mathbb{E}[s^{\tau^+} z^{S_{\tau^+}}]$ and the corresponding potential $\Phi^+(s,z) := \sum_{k\geq 0} \mathbb{E}\left[s^{T_k^+} z^{S_{T_k^+}}\right] = \sum_{k\geq 0} \varphi^+(s,z)^k = \frac{1}{1-\varphi^+(s,z)}$. By a straightforward argument, called the *duality lemma* in Feller's book [4], one also gets

$$\Phi^{*-}(s,z) = \sum_{n\geq 0} s^n \mathbb{E}\left[\tau^+ > n, z^{S_n}\right] \quad \text{and} \quad \Phi^+(s,z) = \sum_{n\geq 0} s^n \mathbb{E}\left[\tau^{*-} > n, z^{S_n}\right]. \tag{5}$$

We now introduce the corresponding generating functions $\mathbf{T}^{*-}$, $\mathbf{U}^{*-}$ and $\mathbf{U}^+$ defined by, for $s \in \mathbb{C}$ and $x \in \mathbb{Z}$

$$\begin{aligned}
\mathbf{T}^{*-}(s|x) &= \mathbb{E}\left[s^{\tau^{*-}} 1_{\{x\}}(S_{\tau^{*-}})\right] = \sum_{n\geq 1} s^n \mathbb{P}\left[\tau^{*-} = n, S_n = x\right],\\
\mathbf{T}^{+}(s|x) &= \mathbb{E}\left[s^{\tau^{+}} 1_{\{x\}}(S_{\tau^{+}})\right] = \sum_{n\geq 1} s^n \mathbb{P}\left[\tau^{+} = n, S_n = x\right],\\
\mathbf{U}^{*-}(s|x) &= \sum_{k\geq 0} \mathbb{E}\left[s^{T_k^{*-}} 1_{\{x\}}(S_{T_k^{*-}})\right] = \sum_{n\geq 0} s^n \mathbb{P}\left[\tau^{+} > n, S_n = x\right],\\
\mathbf{U}^{+}(s|x) &= \sum_{k\geq 0} \mathbb{E}\left[s^{T_k^{+}} 1_{\{x\}}(S_{T_k^{+}})\right] = \sum_{n\geq 0} s^n \mathbb{P}\left[\tau^{*-} > n, S_n = x\right].
\end{aligned}$$

Note that $\mathbf{U}^{*-}(s|x) = 0$ when $x \geq 1$ and $\mathbf{U}^+(s|x) = 0$ when $x \leq -1$.

We will first study the regularity of the Fourier transforms φ^{*-} and φ^+ to describe the one of the functions $\mathbf{T}^{*-}(\cdot|x)$ and $\mathbf{T}^+(\cdot|x)$; To do this we will use the Wiener–Hopf factorization theory, in a quite strong version, in order to obtain some uniformity in the estimations we will need. We could adapt the same approach for the functions $\mathbf{U}^{*-}(\cdot|x)$ and $\mathbf{U}^+(\cdot|x)$, but it is more difficult to control the behavior near $s = 1$ of their respective Fourier transforms Φ^{*-} and Φ^+. We will thus prefer to note that, for any $x \in \mathbb{Z}^{*-}$, the function $\mathbf{U}^{*-}(\cdot|x)$ is equal to the finite sum $\sum_{k=0}^{|x|} \mathbb{E}\left[s^{T_k^{*-}} 1_{\{x\}}(S_{T_k^{*-}})\right]$, since $T_k^{*-} \geq k$ a.s; the same remark does not hold for $\mathbf{U}^+(\cdot|x)$ since $\mathbb{P}[S_{\tau^+} = 0] > 0$, but we will see that the series $\sum_{k=0}^{+\infty} \mathbb{E}\left[s^{T_k^+} 1_{\{x\}}(S_{T_k^+})\right]$ converges exponentially fast and a similar approach will be developed.

It will be of interest to consider the following square infinite matrices

- $\mathcal{T}_s^{*-} = \left(\mathcal{T}_s^{*-}(x,y)\right)_{x,y\in\mathbb{Z}^-}$ with $\mathcal{T}_s^{*-}(x,y) := \mathbf{T}^{*-}(s|y-x)$ for any $x, y \in \mathbb{Z}^-$,
- $\mathcal{U}_s^{*-} = \left(\mathcal{U}_s^{*-}(x,y)\right)_{x,y\in\mathbb{Z}^-}$ with $\mathcal{U}_s^{*-}(x,y) := \mathbf{U}^{*-}(s|y-x)$ for any $x, y \in \mathbb{Z}^-$.

The elements of $\mathbb{Z}^-$ are labeled here in the decreasing order. Note that the matrix $\mathcal{T}_s^{*-}$ is strictly upper triangular; so for any $x, y \in \mathbb{Z}^-$ one gets $\mathcal{U}_s^{*-}(x,y) = \sum_{k=0}^{|x-y|} (\mathcal{T}_s^{*-})^k(x,y)$.

- $\mathcal{T}_s^+ = \Big(\mathcal{T}_s^+(x,y)\Big)_{x,y\in\mathbb{N}_0}$ with $\mathcal{T}_s^+(x,y) := \mathbf{T}^+(s|y-x)$ for any $x, y \in \mathbb{N}_0$,
- $\mathcal{U}_s^+ = \Big(\mathcal{U}_s^+(x,y)\Big)_{x,y\in\mathbb{N}_0}$ with $\mathcal{U}_s^+(x,y) := \mathbf{U}^+(s|y-x)$ for any $x, y \in \mathbb{N}_0$.

We will also have $\mathcal{U}_s^+(x,y) = \sum_{k\geq 0}(\mathcal{T}_s^+)^k(x,y)$ for any $x, y \in \mathbb{N}_0$, and the number of terms in the sum will not be finite in this case, but it will not be difficult to derive the regularity of the function $s \mapsto \mathcal{U}_s^+(x,y)$ from the one of each term $s \mapsto \mathcal{T}_s^+(x,y)$.

In the sequel, we will consider the matrices $\mathcal{T}_s^{*-}$ and $\mathcal{T}_s^+$ as operators acting on some Banach space of sequences of complex numbers; we will first consider the case of bounded sequences, that is the set $\mathbb{C}_\infty^{\mathbb{N}_0}$ of sequences $\mathbf{a} = (a_x)_{x\geq 0}$ of complex numbers such that $|\mathbf{a}|_\infty := \sup_{x\geq 0}|a_x| < +\infty$; unfortunately, it will not be possible to give sense to the above inversion formula on the Banach space of linear continuous operators acting on $\Big(\mathbb{C}_\infty^{\mathbb{N}_0}, |\cdot|_\infty\Big)$ and we will have to consider the action of these matrices on a larger space of $\mathbb{C}$-valued sequences introduced in Sect. 2.4.

In the following subsections, we decompose both the excursion of $(X_n)_{n\geq 0}$ before the first reflection and the process of reflections $(X_{\mathbf{r_k}})_{k\geq 0}$.

2.2 The Approach Process and the Matrices $\mathcal{T}_s$

The trajectories of the reflected random walk are governed by the strict descending ladder epochs of the corresponding classical random walk on $\mathbb{Z}$, and the generating function $\mathbf{T}^{*-}$ introduced in the previous section will be essential in the sequel. Since the starting point may be any $x \in \mathbb{N}_0$, we have to consider the first time at which the random walk $(X_n)_{n\geq 0}$ goes on the "left" of the starting point (with eventually a reflection at this time, in which case the arrival point may be $> x$), that is the strict descending ladder epoch τ^{*-} of the random walk $(S_n)_{n\geq 0}$. We thus introduce the matrices $\mathcal{T}_s$ defined by $\mathcal{T}_s = \Big(\mathcal{T}_s(x,y)\Big)_{x,y\in\mathbb{N}_0}$ with

$$\forall x, y \in \mathbb{N}_0 \quad \mathcal{T}_s(x,y) := \mathbf{T}^{*-}(s|y-x). \tag{6}$$

Observe that the $\mathcal{T}_s$ are strictly lower triangular.

2.3 The Excursion Before the First Reflection

We have the following identity: for all $s \in \mathbb{C}$ and $x, y \in \mathbb{N}_0$

$$\mathbf{E}(s|x,y) = \mathbf{U}^+(s|y-x) + \sum_{w=0}^{x-1}\mathbf{T}^{*-}(s|w-x)\mathbf{E}(s|w,y).$$

As above, we introduce the infinite matrices $\mathcal{E}_s = \Big(\mathcal{E}_s(x,y)\Big)_{x,y\in\mathbb{N}_0}$ with $\mathcal{E}_s(x,y) := \mathbf{E}(s|x,y)$ for any $x, y \in \mathbb{N}_0$, and rewrite this identity as follows

$$\mathcal{E}_s = \mathcal{U}_s^+ + \mathcal{T}_s\mathcal{E}_s.$$

Since $\mathcal{T}_s$ is strictly lower triangular, the matrix $I - \mathcal{T}_s$ will be invertible (in a suitable space to made precise) and one will get

$$\mathcal{E}_s = \Big(I - \mathcal{T}_s\Big)^{-1} \mathcal{U}_s^+. \tag{7}$$

In the following sections, we will give sense to this inversion formula and describe the regularity in s of the matrix-valued function $s \mapsto \mathcal{E}_s$.

2.4 The Process of Reflections

Under the hypothesis $\mathbb{P}[\tau^{*-} < +\infty] = 1$,[3] the distribution law of the variable $S_{\tau^{*-}}$ is denoted μ^{*-} and its potential $U^{*-} := \sum_{n\geq 0}(\mu^{*-})^{\star n}$; all the waiting times T_n^{*-} are thus a.s. finite and one gets $(\mu^{*-})^{\star n} = \mathcal{L}(S_{T_n^{*-}})$; furthermore, for any $x \in \mathbb{N}_0$, the successive reflection times $\mathbf{r}_k$, $k \geq 0$, are also a.s. finite. The process $(X_{\mathbf{r}_k})_{k\geq 0}$ appears in a crucial way in [8] to study the recurrence/transience properties of the reflected walk.

Fact 2.1 [8] *Under the hypothesis $\mathbb{P}[\tau^{*-} < +\infty] = 1$, the process of reflections $(X_{\mathbf{r}_k})_{k\geq 0}$ is a Markov chain on $\mathbb{N}_0$ with transition probability $\mathcal{R}$ given by*

$$\forall x \in \mathbb{N}_0,\ \forall y \in \mathbb{N}_0 \quad \mathcal{R}(x, y) = \begin{cases} 0 & \text{if } y = 0 \\ \displaystyle\sum_0^x U^{*-}(-w)\mu^{*-}(w - x - y) & \text{if } y \geq 1. \end{cases} \tag{8}$$

Furthermore, the measure $\nu_{\mathbf{r}}$ on $\mathbb{N}^*$ defined by

$$\forall x \in \mathbb{N}^* \quad \nu_{\mathbf{r}}(x) := \sum_{y=1}^{+\infty} \left(\frac{\mu^{*-}(-x)}{2} + \mu^{*-}\Big(]-x-y, -x[\Big) + \frac{\mu^{*-}(-x-y)}{2} \right) \mu^{*-}(-y) \tag{9}$$

is stationary for $(X_{\mathbf{r}_k})_{k\geq 0}$ and is unique up to a multiplicative constant; this measure is finite when $\mathbb{E}[|S_{\tau^{*-}}|] = \sum_{k\geq 1} k\mu^{*-}(-k) < +\infty$.

This statement is a bit different from the one in [8] since we assume here that at the reflection time the process $(X_n)_{n\geq 0}$ belongs to $\mathbb{N}^*$; nevertheless, the proof goes exactly along the same lines. It will be useful in the sequel in order to control the spectrum of the stochastic infinite matrix $\mathcal{R} = \Big(\mathcal{R}(x, y)\Big)_{x,y\in\mathbb{N}_0}$; before stating the following crucial property, we introduce the

Notation 2.2 *Let $K = (K(x))_{x\in\mathbb{N}_0}$ be a sequence of nonnegative real numbers which tends to $+\infty$; the set of complex-valued sequences $\mathbf{a} = (a_x)_{x\in\mathbb{N}_0}$ such that $|\mathbf{a}|_K := \sup_{x\in\mathbb{N}_0} \frac{|a_x|}{K(x)} < +\infty$ is denoted $\mathbb{C}_K^{\mathbb{N}_0}$.*

[3] This condition is satisfied for instance when $\mathbb{E}[|Y_n|] < +\infty$ and $\mathbb{E}[Y_n] \leq 0$.

The space $\mathbb{C}_K^{\mathbb{N}_0}$ endowed with the norm $|\cdot|_K$ is a $\mathbb{C}$-Banach space. In the following statement, $\mathbf{h}$ denotes the sequence whose terms are all equal to 1.

Property 2.3 *There exists a constant* $\kappa \in]0, 1[$ *such that, for any* $x \in \mathbb{N}_0$ *and* $y \in \mathbb{N}^*$ *one gets*

$$\mathcal{R}(x, y) \geq \kappa \mu^{*-}(-y).$$

In particular, the operator $\mathcal{R}$ *acting on* $(\mathbb{C}_\infty^{\mathbb{N}_0}, |\cdot|_\infty)$ *is quasi-compact: The eigenvalue* 1 *is simple, with associated eigenvector* $\mathbf{h}$*, and the rest of the spectrum is included in a disk of radius* $\leq 1 - \kappa$.

Furthermore, for any $\mathbf{K} > 1$*, the operator* $\mathcal{R}$ *acts on the Banach space* $(\mathbb{C}_K^{\mathbb{N}_0}, |\cdot|_K)$*, where* K *is the function defined by* $\forall x \geq 0\ K(x) := \mathbf{K}^x$*, the eigenvalue* 1 *is simple with associated eigenvector* $\mathbf{h}$ *and the rest of the spectrum of* $\mathcal{R}$ *acting on* $(\mathbb{C}_K^{\mathbb{N}_0}, |\cdot|_K)$ *is included in a disk of radius* $\leq 1 - \kappa$.

Proof Let $N_\mu := \inf\{k \leq -1/\mu\{k\} > 0\}$ (with $N = -\infty$ is the support of μ is not bounded from below). Since μ is adapted, one gets $\mu^{*-}(k) > 0$ for any $k \in \{-N_\mu, \cdots, -1\}$ (and any $k \in \mathbb{Z}^{*-}$ when $N_\mu = -\infty$); consequently, $U^{*-}(k) > 0$ for any $k \in \mathbb{Z}^{*-}$. In fact, by the 1-dimensional renewal theorem, one knows that $\lim_{k\to-\infty} U^{*-}(k) = \frac{1}{-\mathbb{E}[S_{\tau^{*-}}]} > 0$ since $\mathbb{E}[S_{\tau^{*-}}] > -\infty$ when μ has exponential moments; consequently $\kappa := \inf_{x\in\mathbb{Z}^-} U^{*-}(x) > 0$. Using (8), one may thus write, for any $x \in \mathbb{N}_0$ and $y \in \mathbb{N}^*$

$$\mathcal{R}(x, y) \geq U^{*-}(x)\mu^{*-}(-y) \geq \kappa\mu^{*-}(-y).$$

The matrix $\Big(\mathcal{R}(x, y)\Big)_{x,y\in\mathbb{N}_0}$ thus satisfies the so-called Doeblin condition and it is quasi-compact on $(\mathbb{C}_\infty^{\mathbb{N}_0}, |\cdot|_\infty)$ (see for instance [1] for a precise statement). The same spectral property holds on $(\mathbb{C}_K^{\mathbb{N}_0}, |\cdot|_K)$ for any $\mathbf{K} > 1$; indeed, since μ^{*-} has exponential moments of any order, we have

$$\sup_{x\in\mathbb{N}_0} \sum_{y\in\mathbb{N}_0} \mathcal{R}(x, y)\mathbf{K}^y < +\infty.$$

□

For technical reasons which will appear in Sect. 4, we will replace the function $K : x \mapsto \mathbf{K}^x$ by a function denoted also K which satisfies the following conditions

$$(i)\quad \forall x \in \mathbb{N}_0 \quad K(x) \geq 1 \quad (ii) \quad \lim_{x\to+\infty} \frac{K(x)}{\mathbf{K}^x} = 1 \quad (iii) \quad \mathcal{R}K(x) \leq 1. \qquad (10)$$

It suffices to consider the function $x \mapsto \Big(1 \vee \frac{\mathbf{K}^x}{M}\Big)$ with $M := \sup_{x\in\mathbb{N}_0} \sum_{y\in\mathbb{N}^*} \mathcal{R}(x, y)\mathbf{K}^y$. **The set of functions which satisfy the conditions** (10) will be denoted $\mathcal{K}(\mathbf{K})$.

We now explicit the connection between $\mathcal{R}_s$ and $\mathcal{T}_s$; namely, there exists a similar factorization identity than (3) for the process of reflection. Using the fact that the first reflection time may appear or not at time τ^{*-}, one may write: for all $s \in \mathbb{C}$ and $x \in \mathbb{N}_0$ and $y \in \mathbb{N}^*$

$$\mathbf{R}(s|x, y) = \mathbf{T}(s| - x - y) + \sum_{w=0}^{x-1} \mathbf{T}(s|w - x)\mathbf{R}(s|w, y), \tag{11}$$

which leads to the following equality:

$$\mathcal{R}_s = \left(I - \mathcal{T}_s\right)^{-1} \mathcal{V}_s \tag{12}$$

where we have set $\mathcal{V}_s = \left(\mathcal{V}_s(x, y)\right)_{x,y\in\mathbb{N}_0}$ with

$$\mathcal{V}_s(x, y) := \begin{cases} 0 & \text{if } y = 0 \\ \mathbf{T}^{*-}(s| - x - y) & \text{if } y \in \mathbb{N}^*. \end{cases} \tag{13}$$

The crucial point in the sequel will be thus to describe the regularity of the maps $s \mapsto \mathcal{T}_s$, $s \mapsto \mathcal{V}_s$ and $s \mapsto \mathcal{U}_s^+$ near the point $s = 1$. We will first detail the centered case; the main ingredient is the classical Wiener–Hopf factorization which permits to control both functions φ^{*-} and φ^+.

Another essential point will be to describe the one of the maps $(I - \mathcal{T}_s)^{-1}$ and $(I - \mathcal{R}_s)^{-1}$ and this question is related to the description of the spectrum of the operators $\mathcal{T}_s$ and $\mathcal{R}_s$ when s is close to 1: This is not difficult for $\mathcal{T}_s$ since it is a strictly lower triangular matrix but more subtle for $\mathcal{R}_s$ in the centered case where $\mathcal{R} = \mathcal{R}_1$ is a Markov operator.

3 On the Wiener–Hopf Factorization in the Space of Analytic Functions

3.1 Preliminaries and Notations

The Wiener–Hopf factorization proposes a decomposition of the space–time characteristic function $(s, z) \mapsto 1 - s\mathbb{E}[z^{Y_n}] = 1 - s\hat{\mu}(z)$ in terms of φ^{*-} and φ^+; namely, for all $s, z \in \mathbb{C}$ with modulus < 1

$$1 - s\hat{\mu}(z) = \left(1 - \varphi^{*-}(s, z)\right)\left(1 - \varphi^{+}(s, z)\right). \tag{14}$$

In [3], we use this factorization to obtain local limit theorems for fluctuations of the random walk $(S_n)_{n\geq 0}$; we first propose another such a decomposition, and, by identification of the corresponding factors, we obtain another expression for each of the functions φ^{*-} and φ^+.This new expression allows us to use elementary arguments coming from entire functions theory in order to describe the asymptotic behavior of the sequences $\Big(\mathbb{P}[S_n = x, \tau^{*-} = n]\Big)_{n\geq 1}$ and $\Big(\mathbb{P}[S_n = y, \tau^{*-} > n]\Big)_{n\geq 1}$ for any $x \in \mathbb{Z}^{*-}$ and $y \in \mathbb{Z}^{+}$.

In the present situation, we need first to obtain similar results than in [3] but in terms of regularity with respect to the variable s of the functions φ^{*-} and φ^{+} around the unit circle, with a precise description of their singularity near the point $s = 1$; by the identity (3), we will show that these properties spread to the function $\mathbf{G}(s|x, y)$, which allows us to conclude, using the classical Darboux's method for entire functions.

We will assume that the law μ has exponential moments of any order, i.e. $\sum_{n\in\mathbb{Z}} r^n \mu(n) < +\infty$ for any $r \in \mathbb{R}^{*+}$. This implies that its generating function $\hat{\mu} : z \mapsto \sum_{n\in\mathbb{Z}} z^n \mu(n)$ is analytic on $\mathbb{C}^*$; furthermore, its restriction to $]0, +\infty[$ is strictly convex and $\lim_{r\to+\infty} \hat{\mu}(r) = \lim_{\substack{r\to 0\\ r>0}} \hat{\mu}(r) = +\infty$ when μ charges $\mathbb{Z}^{*+}$ and $\mathbb{Z}^{*-}$. In particular, under these conditions, there exists a unique $r_0 > 0$ such that $\hat{\mu}(r_0) = \inf_{r>0} \hat{\mu}(r)$; it follows $\hat{\mu}'(r_0) = 0$, $\hat{\mu}''(r_0) > 0$. Set $\rho_0 := \hat{\mu}(r_0)$ and $R_\circ := \frac{1}{\rho_0}$; one has $\rho_0 = 1$ when μ is centered and $\rho_0 \in]0, 1[$ otherwise.

We now fix $0 < r_- < r_0 < r_+ < +\infty$ and will denote by $\mathbf{L} = \mathbf{L}[r_-, r_+]$ the space of functions $F : \mathbb{C}^* \to \mathbb{C}$ of the form $F(z) := \sum_{n\in\mathbb{Z}} a_n z^n$ for some (bilateral)-sequence $(a_n)_{n\in\mathbb{Z}}$ such that $\sum_{n\leq 0} |a_n| r_-^n + \sum_{n\geq 0} |a_n| r_+^n < +\infty$; the elements of $\mathbf{L}$ are called **Laurent functions** on the annulus $\{r_- \leq |z| \leq r_+\}$ and $\mathbf{L}$, endowed with the norm $|\cdot|_\infty$ of uniform convergence on this annulus, is a Banach space containing the function $\hat{\mu}$.

3.2 The Centered Case

Let us first consider the centered case: $\mathbb{E}[Y_n] = \hat{\mu}'(1) = 0$; we thus have $r_0 = 1$ and $\rho_0 = R_\circ = 1$. Under the aperiodicity condition on μ, one gets $|1 - s\hat{\mu}(z)| > 0$ for any $z \in \mathbb{C}^*$, $|z| = 1$, and s such that $|s| \leq 1$, except $s = 1$; it follows that for any $z \in \mathbb{C}^*$, $|z| = 1$, the function $s \mapsto \frac{1}{1-s\hat{\mu}(z)}$ may be analytically extended on the set $\{s \in \mathbb{C}/|s| \leq 1 + \delta\} \setminus [1, 1 + \delta[$ for some $\delta > 0$.

The following argument is classical, we refer to [10] for the description we present here. One gets $\hat{\mu}'(1) = 0$ and $\hat{\mu}''(1) = \sigma^2 := \mathbb{E}[Y_n^2] > 0$; setting $H(s, z) := 1 - s\hat{\mu}(z)$, it thus follows

$$\frac{\partial H}{\partial z}(1, 1) = 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial z^2}(1, 1) = \sigma^2 > 0.$$

The Weierstrass preparation theorem implies that, on a neighborhood of $(1, 1)$

$$H(s, z) = 1 - s\hat{\mu}(z) = \Big((z - 1)^2 + b(s)(z - 1) + c(s)\Big)\mathcal{H}(s, z)$$

with $\mathcal{H}$ analytic on $\mathbb{C}\times\mathbb{C}^*$ and $\mathcal{H} \neq 0$ on this neighborhood, and $b(s)$ and $c(s)$ analytic on the open ball $B(1, \delta)$ for δ small enough. We compute $\mathcal{H}(1, 1) = -\frac{\sigma^2}{2}$, $b(1) = c(1) = 0$ and $c'(1) = \frac{-1}{\mathcal{H}(1,1)} = \frac{2}{\sigma^2}$. The roots $z_-(s)$ and $z_+(s)$ (with $z_-(s) < 1 < z_+(s)$ when $s \in]0, 1[$ and $z_-(1) = z_+(1) = 1$) of the equation $H(s, z) = 0$ are thus the ones of the quadratic equation $(z - 1)^2 + b(s)(z - 1) + c(s) = 0$; solving, we find

$$z_{\pm}(s) = \mathcal{B}(s) \pm \mathcal{C}(s)\sqrt{1-s}$$

where $\mathcal{B}(s)$ and $\mathcal{C}(s)$ are analytic in $B(1,\delta)$ with $\mathcal{B}(1) = 1$ and $\mathcal{C}(1) = \sqrt{c'(1)} = \frac{\sqrt{2}}{\sigma}$.

Consequently, for δ small enough, the functions $z_{\pm}$ admit the following analytic expansion on $\mathcal{O}_\delta(1) := B(1,\delta) \setminus [1, 1+\delta[$:

$$z_{\pm}(s) = 1 + \sum_{n\geq 1} (\pm 1)^n \alpha_n (1-s)^{n/2} \quad \text{with} \quad \alpha_1 = \frac{\sqrt{2}}{\sigma}.$$

This type of singularity of the functions $z_{\pm}$ near $s = 1$ is essential in the sequel because it contains the one of the functions $\varphi^{*-}(s,z)$ and $\varphi^{+}(s,z)$ near $(1,1)$. The Wiener–Hopf factorization has several versions in the literature; we emphasize here that we need some kind of uniformity with respect to the parameter z in the local expansion of the function φ^{*-} near $s = 1$, this is why we consider the map $s \mapsto \varphi^{*-}(s,\cdot)$ with values in $\mathbf{L}$. It is proved in particular in [1] (see also [7] for a more precise statement, in the context of Markov walks) that there exists $\delta > 0$ such that the function $s \mapsto \left(z \mapsto \phi^{*-}(s,z) := \frac{1-\varphi^{*-}(s,z)}{z-z_-(s)}\right)$ is analytic on the open ball $B(1,\delta) \subset \mathbb{C}$, with values in $\mathbf{L}$. Setting $\phi^{*-}(s,\cdot) := \sum_{k\geq 0}(1-s)^k \phi^{*-}_{(k)}(\cdot)$ for $|1-s| < \delta$ and $\phi^{*-}_{(k)} \in \mathbf{L}$ and using the local expansion $z_-(s) = 1 - \frac{\sqrt{2}}{\sigma}\sqrt{1-s} + \cdots$, one thus gets for δ small enough and $s \in \mathcal{O}_\delta(1)$

$$\varphi^{*-}(s,\cdot) = \varphi^{*-}(1,\cdot) + \sum_{k\geq 1}(1-s)^{k/2}\varphi^{*-}_{(k)}(\cdot)$$

with $\sum_{k\geq 0} |\varphi^{*-}_{(k)}|_\infty \delta^k < +\infty$ and $\varphi^{*-}_{(1)} : z \mapsto \frac{\sqrt{2}}{\sigma} \times \frac{1-\mathbb{E}[z^{S_{\tau^{*-}}}]}{1-z}$.

We summarize the information we will need in the following

Proposition 3.1 *For any $r_- < 1 < r_+$, the function $s \mapsto \varphi^{*-}(s,\cdot)$ has an analytic continuation to an open neighborhood of $\overline{B(0,1)} \setminus \{1\}$ with values in $\mathbf{L}$; furthermore, for $\delta > 0$, this function is analytic in the variable $\sqrt{1-s}$ on the set $\mathcal{O}_\delta(1)$ and its local expansion of order 1 in $\mathbf{L}$ is*

$$\varphi^{*-}(s,\cdot) = \varphi^{*-}(1,\cdot) + \sqrt{1-s}\;\varphi^{*-}_{(1)}(\cdot) + \mathbf{O}(s,\cdot) \tag{15}$$

with $\varphi^{*-}_{(1)} : z \mapsto \frac{\sqrt{2}}{\sigma} \times \frac{1-\mathbb{E}[z^{S_{\tau^{*-}}}]}{1-z}$ and $\mathbf{O}(s,\cdot)$ uniformly bounded in $\mathbf{L}$.

A similar statement holds for the function φ^+; in particular, the local expansion near $s = 1$ follows from the one of the root $z_+(s)$, namely $z_+(s) = 1 + \frac{\sqrt{2}}{\sigma}\sqrt{1-s} + \ldots$. We may thus state the

Proposition 3.2 *The function $s \mapsto \varphi^{+}(s,\cdot)$ has an analytic continuation to an open neighborhood of $\overline{B(0,1)} \setminus \{1\}$ with values in $\mathbf{L}$; furthermore, for $\delta > 0$ small enough, this function is analytic in the variable $\sqrt{1-s}$ on the set $\mathcal{O}_\delta(1)$ and one gets*

$$\varphi^{+}(s,\cdot) = \varphi^{+}(1,\cdot) + \sqrt{1-s}\;\varphi^{+}_{(1)}(\cdot) + \mathbf{O}(s,\cdot) \tag{16}$$

with $\varphi^+_{(1)} : z \mapsto -\frac{\sqrt{2}}{\sigma} \times \frac{1-\mathbb{E}[z^{S_{\tau^+}}]}{1-z}$ *and* $\mathbf{O}(s,\cdot)$ *uniformly bounded in* $\mathbf{L}$.

3.3 The Maps $s \mapsto \mathbf{T}^{*-}(s|x)$ and $s \mapsto \mathbf{T}^{+}(s|x)$ for $x \in \mathbb{Z}$

We use here the inverse Fourier's formula: for any $x \in \mathbb{Z}^{*-}$ and $s \in \mathbb{C}$, $|s| < 1$, one gets

$$\mathbf{T}^{*-}(s|x) = \frac{1}{2i\pi} \int_{\mathbb{T}} z^{-x-1} \varphi^{*-}(s,z) \mathrm{d}z.$$

Similarly $\mathbf{T}^{+}(s|x) = \frac{1}{2i\pi} \int_{\mathbb{T}} z^{-x-1} \varphi^{+}(s,z) \mathrm{d}z$ for any $x \in \mathbb{N}_0$. We will apply Propositions 3.1 and 3.2 and first identify the coefficients which appear in the local expansion as Fourier transforms of some known measures; Let us denote

- δ_x the Dirac mass at $x \in \mathbb{Z}$,
- $\lambda^{*-} = \sum_{x \le -1} \delta_x$ the counting measures on $\mathbb{Z}^{*-}$,
- $\lambda^{+} = \sum_{n \ge 0} \delta_x$ the counting measures on $\mathbb{N}_0$.

One easily checks that $z \mapsto \frac{1-\mathbb{E}[z^{S_{\tau^{*-}}}]}{z-1}$ and $z \mapsto \frac{1-\mathbb{E}[z^{S_{\tau^{+}}}]}{1-z}$ are the generating functions associated, respectively, with the measures $(\delta_0 - \mu^{*-}) \star \lambda^{*-}$ and $(\delta_0 - \mu^{+}) \star \lambda^{+}$.

Proposition 3.3 *There exists an open neighborhood* Ω *of* $\overline{B(0,1)} \setminus \{1\}$ *such that, for any* $x \in \mathbb{Z}$, *the functions* $s \mapsto \mathbf{T}^{*-}(s|x) := \mathbb{E}[s^{\tau^{*-}} 1_{\{x\}}(S_{\tau^{*-}})]$ *and* $s \mapsto \mathbf{T}^{+}(s|x) := \mathbb{E}[s^{\tau^{+}} 1_{\{x\}}(S_{\tau^{+}})]$ *have an analytic continuation to* Ω; *furthermore, for* $\delta > 0$ *small enough, these functions are analytic in the variable* $\sqrt{1-s}$ *on the set* $\mathcal{O}_\delta(1)$ *and their local expansions of order 1 are*

$$\mathbf{T}^{*-}(s|x) = \mu^{*-}(x) - \sqrt{1-s}\frac{\sqrt{2}}{\sigma}\mu^{*-}\Big(]-\infty, x]\Big) + (1-s)\,\mathbf{O}(s|x) \tag{17}$$

and

$$\mathbf{T}^{+}(s|x) = \mu^{+}(x) - \sqrt{1-s}\frac{\sqrt{2}}{\sigma}\mu^{+}\Big(]x, +\infty[\Big) + (1-s)\,\mathbf{O}(s|x) \tag{18}$$

with $\mathbf{O}(s|x)$ *analytic in the variable* $\sqrt{1-s}$ *and uniformly bounded in* $s \in \mathcal{O}_\delta(1)$ *and* $x \in \mathbb{Z}$.

Furthermore, for any $K > 1$, *there exists a constant* $\mathbf{O} > 0$ *such that*

$$K^{|x|}\Big|\mathbf{T}^{*-}(s|x)\Big| \le \mathbf{O}, \quad K^{|x|}\Big|\mathbf{T}^{+}(s|x)\Big| \le \mathbf{O} \quad \text{and} \quad K^{|x|}\Big|\mathbf{O}(s|x)\Big| \le \mathbf{O}. \tag{19}$$

for any $s \in \Omega \cup \mathcal{O}_\delta(1)$ *and* $x \in \mathbb{Z}$.

Proof The analyticity property and the local expansions (17) and (18) are direct consequences of Propositions 3.1 and 3.2. To establish for instance the first inequality in (19), we use the fact that for $s \in \Omega \cup \mathcal{O}_\delta(1)$, the function $z \mapsto \varphi^{*-}(s, z)$ is analytic on any annulus $\{z \in \mathbb{C}/r_- < |z| < r_+\}$; for any $K > 1$ and $x \in \mathbb{Z}^{*-}$, one thus gets

$$\mathbf{T}^{*-}(s|x) = \frac{1}{2i\pi}\int_{\mathbb{T}} z^{-x-1}\varphi^{*-}(s,z)\mathrm{d}z = \frac{1}{2i\pi}\int_{\{z/|z|=1/K\}} z^{-x-1}\varphi^{*-}(s,z)\mathrm{d}z.$$

So $\left|\mathbf{T}^{*-}(s|x)\right| \leq \frac{K^{-|x|}}{2\pi} \times \sup_{\substack{s\in\Omega\cup\mathcal{O}_\delta(1)\\ |z|=1/K}} |\varphi*-(s,z)|$. The same argument holds for the quantities $\mathbf{T}^{+}(s|x)$ and $\mathbf{O}(s|x)$. □

3.4 The Coefficient Maps $s \mapsto \mathcal{T}_s^{*-}(x, y)$ and $s \mapsto \mathcal{T}_s^{+}(x, y)$ for $x, y \in \mathbb{Z}$

We first present some consequences of the previous statement for the matrix coefficients $\mathcal{T}_s^{*-}(x, y)$ and $\mathcal{T}_s^{+}(x, y)$.

Proposition 3.4 *There exists an open neighborhood Ω of $\overline{B(0,1)} \setminus \{1\}$ such that for any $x, y \in \mathbb{Z}$, the functions $s \mapsto \mathcal{T}_s^{*-}(x, y)$ and $s \mapsto \mathcal{T}_s^{+}(x, y)$ have an analytic continuation to Ω; furthermore, for $\delta > 0$ small enough, these functions are analytic in the variable $\sqrt{1-s}$ on the set $\mathcal{O}_\delta(1)$ and their local expansions of order 1 are*

$$\mathcal{T}_s^{*-}(x, y) = \mathcal{T}^{*-}(x, y) + \sqrt{1-s}\,\widetilde{\mathcal{T}}^{*-}(x, y) + (1-s)\,\mathbf{O}_s(x, y) \tag{20}$$

and

$$\mathcal{T}_s^{+}(x, y) = \mathcal{T}^{+}(x, y) + \sqrt{1-s}\,\widetilde{\mathcal{T}}^{+}(x, y) + (1-s)\,\mathbf{O}_s(x, y) \tag{21}$$

where

- $\mathcal{T}^{*-}(x, y) = \mu^{*-}(y - x)$,
- $\widetilde{\mathcal{T}}^{*-}(x, y) = -\frac{\sqrt{2}}{\sigma}\mu^{*-}\Big(]-\infty, y-x]\Big)$,
- $\mathcal{T}^{+}(x, y) = \mu^{+}(y - x)$,
- $\widetilde{\mathcal{T}}^{+}(x, y) = -\frac{\sqrt{2}}{\sigma}\mu^{+}\Big(]y-x, +\infty[\Big)$,
- $\mathbf{O}_s(x, y)$ *is analytic in the variable $\sqrt{1-s}$ for $s \in \mathcal{O}_\delta(1)$.*

Proof We give the details for the maps $s \mapsto \mathcal{T}_s^{*-}(x, y)$. Let Ω be the open neighborhood of $\overline{B(0,1)}\setminus\{1\}$ given by Proposition 3.3 and fix $\delta > 0$ such that (17), (18) and (19) hold. In particular, for any $x, y \in \mathbb{Z}^{-}$, the function $s \mapsto \mathcal{T}_s^{*-}(x, y) = \mathbf{T}^{*-}(s|y - x)$ is analytic on Ω and has the local expansion, for $s \in \mathcal{O}_\delta(1)$

$$\mathcal{T}_s^{*-}(x, y) = \mathcal{T}^{*-}(x, y) + \sqrt{1-s}\,\widetilde{\mathcal{T}}^{*-}(x, y) + (1-s)\,\mathbf{O}_s(x, y)$$

whose coefficients are given in the statement of the proposition and $s \mapsto \mathbf{O}(x, y)$ is analytic in the variable $\sqrt{1-s}$; furthermore, the quantities $K^{|y-x|}\left|\mathcal{T}_s^{+}(x, y)\right|$ and $K^{|y-x|}\left|\mathbf{O}_s(x, y)\right|$ are bounded, uniformly in $x, y \in \mathbb{Z}^-$ and $s \in \Omega \cup \mathcal{O}_\delta(1)$. □

3.5 The Coefficient Maps $s \mapsto \mathcal{U}_s^{*-}(x, y)$ and $s \mapsto \mathcal{U}_s^{+}(x, y)$ for $x, y \in \mathbb{Z}$

We consider here the maps $s \mapsto \mathcal{U}_s^{*-}(x, y)$ and $s \mapsto \mathcal{U}_s^{+}(x, y)$. The matrix $\mathcal{U}_s^{*-} = \left(\mathcal{U}_s^{*-}(x, y)\right)_{x,y\in\mathbb{Z}}$ is the potential of $\mathcal{T}_s^{*-} = \left(\mathcal{T}_s^{*-}(x, y)\right)_{x,y\in\mathbb{Z}}$; since $\mathcal{T}_s^{*-}$ is strictly upper triangular, each $\mathcal{U}_s^{*-}(x, y)$ will be the combination by summations and products of finitely many coefficients $\mathcal{T}_s^{*-}(i, j), i, j \in \mathbb{Z}$, and their regularity will thus be a direct consequence of the previous statement.

Proposition 3.5 *There exists an open neighborhood Ω of $\overline{B(0,1)} \setminus \{1\}$ such that, for any x, y in $\mathbb{Z}^-$, the functions $s \mapsto \mathcal{U}_s^{*-}(x, y)$ have an analytic continuation to Ω; furthermore, for $\delta > 0$ small enough, these functions are analytic in the variable $\sqrt{1-s}$ on the set $\mathcal{O}_\delta(1)$ and their local expansions of order 1 are*

$$\mathcal{U}_s^{*-}(x, y) = \mathcal{U}^{*-}(x, y) + \sqrt{1-s}\,\widetilde{\mathcal{U}}^{*-}(x, y) + (1-s)\,\mathbf{O}_s(x, y) \tag{22}$$

where

- $\mathcal{U}^{*-}(x, y) = U^{*-}(y-x)$,
- $\widetilde{\mathcal{U}}^{*-}(x, y) = -\frac{\sqrt{2}}{\sigma} U^{*-}\Big(]y-x, 0]\Big)$,
- $\mathbf{O}_s(x, y)$ *is analytic in the variable $\sqrt{1-s}$ and bounded for $s \in \mathcal{O}_\delta(1)$.*

Similarly, for any $x, y \in \mathbb{N}_0$, the functions $s \mapsto \mathcal{U}_s^{+}(x, y)$ have an analytic continuation to Ω, and these functions are analytic in the variable $\sqrt{1-s}$ on the set $\mathcal{O}_\delta(1)$ with the following local expansions of order 1

$$\mathcal{U}_s^{+}(x, y) = \mathcal{U}^{+}(x, y) + \sqrt{1-s}\,\widetilde{\mathcal{U}}^{+}(x, y) + (1-s)\,\mathbf{O}_s(x, y) \tag{23}$$

where

- $\mathcal{U}^{+}(x, y) = U^{+}(y-x)$,
- $\widetilde{\mathcal{U}}^{+}(x, y) = -\frac{\sqrt{2}}{\sigma} U^{+}\Big([0, y-x]\Big)$,
- $\mathbf{O}_s(x, y)$ *is analytic in the variable $\sqrt{1-s}$ and bounded for $s \in \mathcal{O}_\delta(1)$.*

Proof The matrix $\mathcal{T}_s^{*-}$ being strictly upper triangular, for any $x, y \in \mathbb{Z}^-$, one gets $\left(\mathcal{T}_s^{*-}\right)^n(x, y) = 0$ when $n > |x-y|$, so

$$\mathcal{U}_s^{*-}(x, y) = \sum_{n=0}^{|x-y|} \left(\mathcal{T}_s^{*-}\right)^n(x, y). \tag{24}$$

The analyticity of the coefficients $\mathcal{U}_s^{*-}(x, y)$ with respect to $s \in \Omega$ and $\sqrt{1-s}$ when $s \in \mathcal{O}_\delta(1)$ follows from the previous Proposition.

Let us now establish the local expansion (22); for any fixed $x, y \in \mathbb{Z}^-$, one gets

$$\mathcal{U}_s^{*-}(x, y) = \sum_{n=0}^{|x-y|} \left(\mathcal{T}^{*-} + \sqrt{1-s}\, \widetilde{\mathcal{T}}^{*-} + (1-s)\, \mathbf{O}_s\right)^n (x, y).$$

The constant term $\mathcal{U}^{*-}(x, y)$ is thus equal to $\sum_{n=0}^{|x-y|} \left(\mathcal{T}^{*-}\right)^n (x, y) = \sum_{n=0}^{+\infty} \left(\mathcal{T}^{*-}\right)^n (x, y)$; on the other hand, the coefficient corresponding to $\sqrt{1-s}$ in this expansion is equal to

$$\widetilde{\mathcal{U}}^{*-}(x, y) = \sum_{n=0}^{|x-y|} \sum_{k=0}^{n-1} \left(\mathcal{T}^{*-}\right)^k \widetilde{\mathcal{T}}^{*-} \left(\mathcal{T}^{*-}\right)^{n-k-1} (x, y).$$

Inverting the order of summations and using the expression of $\widetilde{\mathcal{T}}^{*-}$ in Proposition 3.4, one gets

$$\begin{aligned}
\widetilde{\mathcal{U}}^{*-}(x, y) &= \mathcal{U}^{*-}\widetilde{\mathcal{T}}^{*-}\mathcal{U}^{*-}(x, y) \\
&= -\frac{\sqrt{2}}{\sigma} \left(U^{*-} \star \left(\sum_{k \le -1} \mu^{*-}\left(]-\infty, k]\right)\, \delta_k \right) \star U^{*-} \right) (y - x) \\
&= -\frac{\sqrt{2}}{\sigma} U^{*-}\left(]y - x, 0]\right).
\end{aligned}$$

To obtain the last equality, observe that the measures $U^{*-} \star \left(\sum_{k\le -1} \mu^{*-}(]-\infty, k])\ \delta_k\right) \star U^{*-}$ and $U^{*-} \star \lambda^{*-} = \sum_{k\le -1}\ U^{*-}(]k, 0])\ \delta_k$ have the same generating function.

The proof goes along the same lines for $\mathcal{U}_s^+(x, y) = \sum_{n=0}^{+\infty} \left(\mathcal{T}_s^+\right)^n (x, y)$, but there are infinitely many terms in the sum since $\mu^+(0) > 0$; for $s \in \Omega \cup \mathcal{O}_\delta(1)$, one thus first sets $\mathcal{T}_s^+ = \varepsilon_s I + T_s$ with $\varepsilon_s := \mathbb{E}\left[s^{\tau^+} 1_{\{0\}}(S_{\tau^+})\right]$. One gets $\delta_1 = \mu^+(0) \in]0, 1[$, so $|\varepsilon_s| < 1$ for Ω and δ small enough. Since I and T_s commute and T_s is strictly upper triangular, one may write, for any $x, y \in \mathbb{N}_0$, and $n \ge |x - y|$,

$$\begin{aligned}
\left(\mathcal{T}_s^+\right)^n (x, y) &= \sum_{k=0}^{n} \binom{n}{k} \varepsilon_s^{n-k} T_s^k (x, y) \\
&= \sum_{k=0}^{|x-y|} \binom{n}{k} \varepsilon_s^{n-k} T_s^k (x, y)
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{U}_s^+(x,y) &= \sum_{n\geq 0}\left(\mathcal{T}_s^+\right)^n(x,y)\\
&= \sum_{n=0}^{|x-y|}\left(\mathcal{T}_s^+\right)^n(x,y) + \sum_{n>|x-y|}\sum_{k=0}^{|x-y|}\binom{n}{k}\varepsilon_s^{n-k}T_s^k(x,y)\\
&= \sum_{n=0}^{|x-y|}\left(\mathcal{T}_s^+\right)^n(x,y) + \sum_{k=0}^{|x-y|}\frac{1}{k!}\left(\sum_{n>|x-y|} n\ldots(n-k+1)\varepsilon_s^{n-k}\right)T_s^k(x,y)
\end{aligned}$$

with $s\mapsto\left(\sum_{n>|x-y|} n\ldots(n-k+1)\varepsilon_s^{n-k}\right)$ analytic on Ω and analytic in $\sqrt{1-s}$ on $\mathcal{O}_\delta(1)$. The analyticity of $s\mapsto\mathcal{U}_s^+(x,y)$ follows, and the computation of the coefficients in (23) goes along the same line than in (22). □

4 The Centered Reflected Random Walk

Throughout this section, we will assume that hypotheses **H** hold and that μ is centered. In this case, the radius of convergence of the generating functions $\mathbf{G}(\cdot|x,y)$, $x,y\in\mathbb{N}_0$, is equal to 1 and we study the type of their singularity near $s=1$.

We denote by $\mathcal{M}_\infty$ the space of infinite matrices $M=(M(x,y))_{x,y\in\mathbb{N}_0}$ such that

$$\|M\|_\infty := \sup_{x\in\mathbb{N}_0}\sum_{y\in\mathbb{N}_0}|M(x,y)| < +\infty.$$

The quantity $\|M\|_\infty$ is the norm of the matrix M considered as an operator acting continuously on the Banach space $(\mathbb{C}_\infty^{\mathbb{N}_0},|\cdot|_\infty)$. As we have already seen, we also work on the space of infinite sequences $\mathcal{C}_K^{\mathbb{N}_0}$ for some $K\in\mathcal{K}(1+\eta)$ where $\eta>0$; consequently, we will consider the space $\mathcal{M}_K$ of infinite matrices $M=(M(x,y))_{x,y\in\mathbb{N}_0}$ such that

$$\|M\|_K := \sup_{x\in\mathbb{N}_0}\sum_{y\in\mathbb{N}_0}\frac{K(y)}{K(x)}|M(x,y)| < +\infty.$$

The quantity $\|M\|_K$ is the norm of M considered as an operator acting continuously on $(\mathbb{C}_K^{\mathbb{N}_0},|\cdot|_K)$.

4.1 The Map $s\mapsto\mathcal{T}_s$ and Its Potential $\mathcal{U}_s$

Recall that the matrix $\mathcal{T}_s$ is the lower triangular with coefficients $\mathcal{T}_s(x,y)$, $x,y\in\mathbb{N}_0$, given by

$$\mathcal{T}_s(x,y)=\mathbf{T}_s(y-x)=\mathbb{E}\left[s^{\tau^{*-}}1_{\{y-x\}}(S_{\tau^{*-}})\right].$$

Proposition 4.1 *There exists an open neighborhood* Ω *of* $\overline{B(0,1)} \setminus \{1\}$ *such that the* $\mathcal{M}_\infty$*-valued function* $s \mapsto \mathcal{T}_s$ *has an analytic continuation to* Ω*; furthermore, for* $\delta > 0$ *small enough, this function is analytic in the variable* $\sqrt{1-s}$ *on the set* $\mathcal{O}_\delta(1)$ *and its local expansions of order 1 in* $(\mathcal{M}_\infty, \|\cdot\|_\infty)$ *is*

$$\mathcal{T}_s = \mathcal{T} + \sqrt{1-s}\,\widetilde{\mathcal{T}} + (1-s)\,\mathbf{O}_s \tag{25}$$

where

- $\mathcal{T} = \Big(\mathcal{T}(x,y)\Big)_{x,y\in\mathbb{N}_0}$ *with* $\mathcal{T}(x,y) = \begin{cases} \mu^{*-}(y-x) & \text{if } 0 \le y \le x-1 \\ 0 & \text{if } y \ge x \end{cases},$
- $\widetilde{\mathcal{T}} = \Big(\widetilde{\mathcal{T}}(x,y)\Big)_{x,y\in\mathbb{N}_0}$ *with* $\widetilde{\mathcal{T}}(x,y) = \begin{cases} -\frac{\sqrt{2}}{\sigma}\mu^{*-}\Big(]-\infty, y-x]\Big) & \text{if } 0 \le y \le x-1 \\ 0 & \text{if } y \ge x \end{cases},$
- $\mathbf{O}_s$ *is analytic in the variable* $\sqrt{1-s}$ *and uniformly bounded in* $(\mathcal{M}_\infty, \|\cdot\|_\infty)$ *for* $s \in \mathcal{O}_\delta(1)$.

Proof The regularity of each coefficient map $s \mapsto \mathcal{T}_s(x,y)$ may be proved as in Proposition 3.4; we thus focus our attention on the analyticity of the $\mathcal{M}_\infty$-valued map $s \mapsto \mathcal{T}_s$. By a classical result in the theory of vector-valued analytic functions of the complex variable (see for instance [2], Theorem 9.13), it suffices to check that this property holds for the functions $s \mapsto \mathcal{T}_s(\mathbf{a})$ for any bounded sequence $\mathbf{a} = (a_i)_{i\ge 0} \in \mathbb{C}^{\mathbb{N}_0}$; to check this, we will use the fact that any uniform limit on some open set of analytic functions is analytic on this set.

Fix $N \ge 1$ and let $\mathcal{T}_{s,N}$ be the "truncated" matrix defined by

$$\mathcal{T}_{s,N}(x,y) = \begin{cases} \mathcal{T}_s(x,y) & \text{if } \max(x-N,0) \le y \le x-1 \\ 0 & \text{otherwise.} \end{cases}$$

One gets $\mathcal{T}_{s,N}(\mathbf{a}) = \sum_1^N \mathbf{T}_s^{*-}(-k)\mathbf{a}^{(k)}$ with $\mathbf{a}^{(k)} := \underbrace{0,\ldots,0}_{k\ times}, a_0, a_1, \ldots,$ which implies that the $\mathcal{M}_\infty$-valued map $s \to \mathcal{T}_{s,N}$ is analytic on Ω and analytic in the variable $\sqrt{1-s}$ on $\mathcal{O}_\delta(1)$. The same property holds for the map $s \to \mathcal{T}_s$ since, by (19), one gets

$$\|\mathcal{T}_s - \mathcal{T}_{s,N}\|_\infty = \sup_{x\in\mathbb{N}_0} \sum_{|y-x|>N} |\mathcal{T}_s(x,y)| \le \sum_{|y-x|>N} \frac{\mathbf{O}}{K^{|x-y|}} = \frac{\mathbf{O}}{(K-1)K^N} \stackrel{N\to+\infty}{\longrightarrow} 0.$$

□

Let us now give sense to the matrix $(I - \mathcal{T}_s)^{-1}$; formally one may write

$$(I - \mathcal{T}_s)^{-1} = \mathcal{U}_s := \sum_{k\ge 0} (\mathcal{T}_s)^k.$$

Since the matrices $\mathcal{T}_s$ are strictly lower triangular, one gets $\mathcal{T}_s^k(x, y) = 0$ for any $x, y \in \mathbb{N}_0$ and $k \geq |x - y| + 1$; it follows that, for any $x, y \in \mathbb{N}_0$

$$(I - \mathcal{T}_s)^{-1}(x, y) = \mathcal{U}_s(x, y) = \sum_{k=0}^{|x-y|} (\mathcal{T}_s)^k(x, y). \tag{26}$$

The analyticity in the variable s (resp. $\sqrt{1-s}$) on Ω (resp. on $\mathcal{O}_\delta(1)$) of each coefficient $\mathcal{U}_s(x, y)$ follows by the previous fact and one may compute its local expansion near $s = 1$. Nevertheless, this property does not hold in the Banach space $(\mathcal{M}_\infty, \|\cdot\|_\infty)$, as it can be seen easily in the following statement (clearly $\mathcal{U}$ and $\widetilde{\mathcal{U}} \notin \mathcal{M}_\infty$), we have in fact to consider the bigger space $\mathcal{M}_K$ to obtain a similar statement.

Proposition 4.2 *Fix $\eta > 0$ and $K \in \mathcal{K}(1+\eta)$. There exists an open neighborhood Ω of $\overline{B(0,1)} \setminus \{1\}$ such that the function $s \mapsto \mathcal{U}_s$ has an analytic continuation to Ω, with values in $\mathcal{M}_K$; furthermore, for $\delta > 0$ small enough, this function is analytic in the variable $\sqrt{1-s}$ on the set $\mathcal{O}_\delta(1)$ and its local expansion of order 1 in $\mathcal{M}_K$ is*

$$\mathcal{U}_s = \mathcal{U} + \sqrt{1-s}\,\widetilde{\mathcal{U}} + (1-s)\,\mathbf{O}_s \tag{27}$$

where

- $\mathcal{U} = (\mathcal{U}(x, y))_{x,y\in\mathbb{N}_0}$ *with* $\mathcal{U}(x, y) = \begin{cases} U^{*-}(y-x) & \text{if } 0 \leq y \leq x \\ 0 & \text{if } y > x \end{cases}$,
- $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}}(x, y))_{x,y\in\mathbb{N}_0}$ *with* $\widetilde{\mathcal{U}}(x, y) = \begin{cases} -\frac{\sqrt{2}}{\sigma} U^{*-}\left(]y-x, 0]\right) & \text{if } 0 \leq y \leq x-1 \\ 0 & \text{if } y \geq x \end{cases}$,
- $\mathbf{O}_s = (\mathbf{O}_s(x, y))_{x,y\in\mathbb{N}_0}$ *is analytic in the variable $\sqrt{1-s}$ for $s \in \mathcal{O}_\delta(1)$ and uniformly bounded in $\mathcal{M}_K$.*

Proof Since $\|\mathcal{T}\|_\infty = 1$, one may choose $\delta > 0$ in such a way $\|\mathcal{T}_s\|_\infty \leq 1 + \frac{\eta}{2}$ for any $s \in \mathcal{O}_\delta(1)$; it thus follows that, for such s, any $x \in \mathbb{N}_0$ and $y \in \{0, \cdots, x-1\}$

$$|\mathcal{U}_s(x, y)| \leq \sum_{n=0}^{|x-y|} \|\mathcal{T}_s\|_\infty^k \leq (1 + 2/\eta)\left(1 + \frac{\eta}{2}\right)^{|x-y|}. \tag{28}$$

So, $\|\mathcal{U}_s\|_K < +\infty$ when $s \in \mathcal{O}_\delta(1)$ and $K \in \mathcal{K}(1+\eta)$. To prove the analyticity of the function $s \mapsto \mathcal{U}_s$, we consider as above the truncated matrix $\mathcal{U}_{s,N}$ and check, first that for any $\mathbf{a} \in \mathbb{C}^{\mathbb{N}_0}$ the maps $s \mapsto \mathcal{U}_{s,N}(\mathbf{a})$ are analytic on Ω and analytic in the variable $\sqrt{1-s}$ on $\mathcal{O}_\delta(1)$, and second that the sequence $(\mathcal{U}_{s,N})_{N\geq 1}$ converges to $\mathcal{U}_s$ in $(\mathcal{M}_K, \|\cdot\|_K)$. The expansion (27) is a straightforward computation. □

From now on, we fix a constant $\eta > 0$ **and a function** $K \in \mathcal{K}(1+\eta)$.

4.2 The Excursions $\mathcal{E}_s(\cdot, y)$ for $y \in \mathbb{N}_0$

The excursion $\mathcal{E}_s$ before the first reflection has been defined formally in (7) as follows

$$\mathcal{E}_s = \Big(I - \mathcal{T}_s\Big)^{-1} \mathcal{U}_s^+ = \mathcal{U}_s\, \mathcal{U}_s^+.$$

The regularity with respect to the parameter s of the matrix coefficients $\mathcal{U}_s^+(x, y)$ and the matrix $\mathcal{U}_s = (I - \mathcal{T}_s)^{-1}$ is well described in Propositions 3.5 and 4.2. Each coefficient of $\mathcal{E}_s$ is a finite sum of products of coefficients of $\mathcal{U}_s$ and $\mathcal{U}_s^+$, so the regularity of the map $s \mapsto \mathcal{E}_s(x, y)$ will follow immediately. The number of terms in this sum is equal to $\min(x, y)$, it thus increases with x and y and it is not easy to obtain some kind of uniformity with respect to these parameters. In fact, it will be sufficient to fix the arrival site y and to describe the regularity of the map $s \mapsto (\mathcal{E}_s(x, y))_{x\in\mathbb{N}_0}$.

Proposition 4.3 *There exists an open neighborhood Ω of $\overline{B(0,1)} \setminus \{1\}$ (depending on the function K) such that, for any $y \in \mathbb{N}_0$, the function $s \mapsto \mathcal{E}_s(\cdot, y)$ has an analytic continuation on Ω with values in the Banach space $\mathbb{C}_K^{\mathbb{N}_0}$; furthermore, for $\delta > 0$ small enough, this function is analytic in the variable $\sqrt{1-s}$ on the set $\mathcal{O}_\delta(1)$ and its local expansion of order 1 in $\mathbb{C}_K^{\mathbb{N}_0}$ is*

$$\mathcal{E}_s(\cdot, y) = \mathcal{E}(\cdot, y) + \sqrt{1-s}\, \widetilde{\mathcal{E}}(\cdot, y) + (1-s)\, \mathbf{O}_s(y) \tag{29}$$

where

- $\mathcal{E}(\cdot, y) = (I - \mathcal{T})^{-1}\mathcal{U}^+(\cdot, y) = \mathcal{U}\mathcal{U}^+(\cdot, y)$,
- $\widetilde{\mathcal{E}}(\cdot, y) = \widetilde{\mathcal{U}}\mathcal{U}^+(\cdot, y) + \mathcal{U}\widetilde{\mathcal{U}}^+(\cdot, y)$,
- $\mathbf{O}_s(y)$ *is analytic in the variable* $\sqrt{1-s}$ *and uniformly bounded in* $\mathbb{C}_K^{\mathbb{N}_0}$ *for* $s \in \mathcal{O}_\delta(1)$.

Proof For any $x, y \in \mathbb{N}_0$, one gets $\mathcal{E}_s(x, y) = \sum_{z=0}^{y} \mathcal{U}_s(x, z)\mathcal{U}_s^+(z, y)$, and the conclusions above follow from Propositions 3.5 and 4.2; in particular, for any fixed $y \in \mathbb{N}_0$ and $N \geq 1$, the $\mathbb{C}_K^{\mathbb{N}_0}$-valued map $s \mapsto (\mathcal{E}_{s,N}(x, y))_x$ defined by $\mathcal{E}_{s,N}(x, y) = \mathcal{E}_s(x, y)$ if $0 \leq x \leq N$ and $\mathcal{E}_{s,N}(x, y)$ otherwise, is analytic in $s \in \Omega$ and $\sqrt{1-s}$ when $s \in \mathcal{O}_\delta(1)$. It is sufficient to check that this sequence of vectors converges to $\mathcal{E}_s(\cdot, y)$ in the sense of the norm $|\cdot|_K$ for some suitable choice of $K > 1$; by (28), one gets

$$\Big|\mathcal{E}_s(x, y)\Big| \leq (y+1)(1+2/\eta)\left(1 + \frac{\eta}{2}\right)^x \times \max_{0\leq z\leq y} |\mathcal{U}_s^+(z, y)|$$

so that $\frac{\left|\mathcal{E}_s(x,y)\right|}{(1+\eta/2)^x} \leq \frac{C_y}{(1+\eta/2)^x}$, for some constant $C_y > 0$ depending only on y. Since $K \in \mathcal{K}(1+\eta)$, one gets $\sup_{x\geq N} \frac{\left|\mathcal{E}_s(x,y)\right|}{K(x)} \to 0$ as $N \to +\infty$; this proves that the sequence $\Big(\mathcal{E}_{s,N}(\cdot, y)\Big)_{N\geq 0}$ converges in $\mathbb{C}_K^{\mathbb{N}_0}$ to $\mathcal{E}(\cdot, y)$ as $N \to +\infty$ and that $s \mapsto \mathcal{E}_s(\cdot, y)$ is analytic. The local expansion (29) follows by a direct computation. □

4.3 On the Map $s \mapsto \mathcal{R}_s$

The matrice $\mathcal{R}_s$ describes the dynamic of the space–time reflected process $(\mathbf{r}_k, X_{\mathbf{r}_k})_{k\geq 0}$ and is defined formally in Sect. 2:

$$\mathcal{R}_s = \Big(I - \mathcal{T}_s\Big)^{-1}\mathcal{V}_s = \mathcal{U}_s\mathcal{V}_s$$

with $\mathcal{V}_s = \Big(\mathcal{V}_s(x,y)\Big)_{x,y\in\mathbb{N}_0}$ and $\mathcal{V}_s(x,y) := \begin{cases} 0 & \text{if } y=0 \\ \mathbf{T}^{*-}(s|-x-y) & \text{if } y\in\mathbb{N}^* \end{cases}$. So, one first needs to control the regularity of the map $s \mapsto \mathcal{V}_s$.

Fact 4.4 *The $\mathcal{M}_K$-valued function $s \mapsto \mathcal{V}_s$ is analytic in s on Ω and in $\sqrt{1-s}$ on $\mathcal{O}_\delta(1)$; furthermore, it has the following local expansion of order 1 near $s=1$*

$$\mathcal{V}_s = \mathcal{V} + \sqrt{1-s}\,\widetilde{\mathcal{V}} + (1-s)\,\mathbf{O}_s \tag{30}$$

where

- $\mathcal{V} = \Big(\mathcal{V}(x,y)\Big)_{x,y\in\mathbb{N}_0}$ *with* $\mathcal{V}(x,y) := \begin{cases} 0 & \text{if } y=0 \\ \mu^{*-}(-x-y) & \text{if } y\in\mathbb{N}^*, \end{cases}$
- $\widetilde{\mathcal{V}} = \Big(\widetilde{\mathcal{V}}(x,y)\Big)_{x,y\in\mathbb{N}_0}$ *with* $\widetilde{\mathcal{V}}(x,y) := \begin{cases} 0 & \text{if } y=0 \\ -\frac{\sqrt{2}}{\sigma}\mu^{*-}\Big(]-\infty,-x-y]\Big) & \text{if } y\in\mathbb{N}^*, \end{cases}$
- $\mathbf{O}_s$ *is analytic in the variable $\sqrt{1-s}$ and uniformly bounded in $\mathcal{M}_K$ for $s\in\mathcal{O}_\delta(1)$.*

We now may describe the regularity of the map $s \mapsto \mathcal{R}_s$.

Proposition 4.5 *The $\mathcal{M}_K$-valued function $s \mapsto \mathcal{R}_s$ is analytic in s on Ω and in $\sqrt{1-s}$ on $\mathcal{O}_\delta(1)$; furthermore, and it has the following local expansion of order 1 near $s=1$*

$$\mathcal{R}_s = \mathcal{R} + \sqrt{1-s}\,\widetilde{\mathcal{R}} + (1-s)\,\mathbf{O}_s \tag{31}$$

where

- $\widetilde{\mathcal{R}} = \widetilde{\mathcal{U}}\mathcal{V} + \mathcal{U}\widetilde{\mathcal{V}}$.
- $\mathbf{O}_s$ *is analytic in the variable $\sqrt{1-s}$ and uniformly bounded in $\mathcal{M}_K$ for $s\in\mathcal{O}_\delta(1)$.*

Proof The analyticity of this function with respect to the variables s or $\sqrt{1-s}$ is clear by Proposition 4.2 and Fact 4.4 and one may write, for $s\in\mathcal{O}_\delta(1)$,

$$\begin{aligned}
\mathcal{R}_s &= \Big(I - \mathcal{T}_s\Big)^{-1}\mathcal{V}_s \\
&= \mathcal{U}_s\mathcal{V}_s \\
&= \Big(\mathcal{U} + \sqrt{1-s}\,\widetilde{\mathcal{U}} + (1-s)\mathbf{O}_s\Big)\Big(\mathcal{V} + \sqrt{1-s}\,\widetilde{\mathcal{V}} + (1-s)\mathbf{O}_s\Big) \\
&= \mathcal{U}\mathcal{V} + \sqrt{1-s}\Big(\widetilde{\mathcal{U}}\mathcal{V} + \mathcal{U}\widetilde{\mathcal{V}}\Big) + (1-s)\mathbf{O}_s.
\end{aligned}$$

□

A direct computation gives in particular

$$\mathcal{E}(x,y) = \sum_{k=0}^{\min(x,y)} U^{*-}(k-x)U^{+}(y-k) \tag{32}$$

and

$$\widetilde{\mathcal{R}}(x,y) = \mathcal{A}(x,y) + \mathcal{B}(x,y) \tag{33}$$

with

$$\mathcal{A}(x,y) := \begin{cases} 0 & \text{if } x=0 \text{ or } y=0 \\ -\frac{\sqrt{2}}{\sigma}\sum_{k=0}^{x-1} U^{*-}\Big(]k-x,0]\Big)\mu^{*-}(-k-y) & \text{otherwise} \end{cases},$$

and

$$\mathcal{B}(x,y) := \begin{cases} 0 & \text{if } y=0 \\ -\frac{\sqrt{2}}{\sigma}\sum_{k=0}^{x} U^{*-}(k-x)\mu^{*-}\Big(]-\infty,-k-y]\Big) & \text{otherwise} \end{cases}.$$

4.4 On the Spectrum of $\mathcal{R}_s$ and Its resolvent $\Big(I-\mathcal{R}_s\Big)^{-1}$

The question is more delicate in the centered case since the spectral radius of $\mathcal{R}$ is equal to 1 (we will see in the next Section that it is < 1 in the non-centered case, which simplifies this step).

4.4.1 The Spectrum of $\mathcal{R}_s$ for $|s|=1$ and $s \neq 1$

Using Property 2.3, we first control the spectral radius of the $\mathcal{R}_s$ for $s \neq 1$; indeed, we may control the norm of $\mathcal{R}_s^2$:

Fact 4.6 *For $|s| = 1$ and $s \neq 1$, one gets $\|\mathcal{R}_s^2\|_K < 1$; in particular, the spectral radius of $\mathcal{R}_s$ in $\mathcal{M}_K$ is < 1.*

Proof Fix $s \in \mathbb{C}\setminus\{1\}$ of modulus 1; by strict convexity, for any $w \in \mathbb{N}_0$ and $y \in \mathbb{N}^*$, there exists $\rho_{w,y} \in]0,1[$, depending also on s, such that $|\mathcal{R}_s(w,y)| \leq \rho_{w,y}\mathcal{R}(w,y)$; on the other hand, by Property 2.3, we may choose $\epsilon > 0$ and a finite set $F \subset \mathbb{N}_0$ such that, for any $x \in \mathbb{N}_0$,

$$\mathcal{R}(x,F) := \sum_{w\in F}\mathcal{R}(x,w) \geq \epsilon.$$

For any $y \in \mathbb{N}_0$, we set $\rho_y := \max_{w\in F}\rho_{w,y}$; since F is finite, one gets $\rho_y \in]0,1[$. Consequently, for any $x \in \mathbb{N}_0$

$$\left|\mathcal{R}_s^2 K(x)\right| \le \sum_{w\in\mathbb{N}^*}\sum_{y\in\mathbb{N}^*}\mathcal{R}(x,w)\times\left|\mathcal{R}_s(w,y)\right|K(y) \le \mathcal{S}_1(s|x)+\mathcal{S}_2(s|x)$$

with

$$\mathcal{S}_1(s|x) := \sum_{w\in F}\sum_{y\in\mathbb{N}^*}\mathcal{R}(x,w)\times\left|\mathcal{R}_s(w,y)\right|K(y)$$
$$\mathcal{S}_2(s|x) := \sum_{w\notin F}\sum_{y\in\mathbb{N}^*}\mathcal{R}(x,w)\times\left|\mathcal{R}_s(w,y)\right|K(y).$$

One gets

$$\mathcal{S}_1(s|x) \le \sum_{w\in F}\mathcal{R}(x,w)\sum_{y\in\mathbb{N}^*}\rho_y\mathcal{R}(w,y)K(y) \le \rho\mathcal{R}(x,F)$$

with $\rho := \max_{w\in F}\sum_{y\in\mathbb{N}^*}\rho_y\mathcal{R}(w,y)K(y) \in]0,1[$.

On the other hand, $\mathcal{S}_2(s|x) \le \mathcal{R}(x,\mathbb{N}^*\setminus F) = 1-\mathcal{R}(x,F)$. Finally, since $K\ge 1$, one gets

$$\frac{\left|\mathcal{R}_s^2 K(x)\right|}{K(x)} \le \Big(\rho\mathcal{R}(x,F)+1-\mathcal{R}(x,F)\Big) \le 1-(1-\rho)\epsilon < 1,$$

which achieves the proof of the Fact 4.6. □

Since the map $s\mapsto\mathcal{R}_s$ is analytic on the set $\{s\in\mathbb{C}/|s|<1+\delta\}\setminus[1,1+\delta[$, the same property holds for the map $s\mapsto(I-\mathcal{R}_s)^{-1}$ on a neighborhood of $\{s\in\mathbb{C}/|s|\le 1\}\setminus\{1\}$.

4.4.2 Perturbation Theory and Spectrum of $\mathcal{R}_s$ for s close to 1

We now focus our attention on s close to 1.[4] Recall that h denotes the sequence whose terms are equal to 1 and observe that $\nu_r(\mathbf{h})=1$. By Property 2.3, the operator $\mathcal{R}$ may be decomposed as follows on $\mathcal{M}_K$

$$\mathcal{R}=\pi+\mathcal{Q}$$

where

- π is the rank one projector on the space $\mathbb{C}\cdot\mathbf{h}$ defined by

$$\mathbf{a}=(a_k)_{k\ge0}\mapsto\left(\sum_{i\ge1}\nu_{\mathbf{r}}(k)a_k\right)\mathbf{h},$$

[4] Recall that $\mathbf{h}$ denotes the sequence whose terms are all equal to 1 and observe that $\nu_{\mathbf{r}}(\mathbf{h})=1$.

- $\mathcal{Q}$ is a bounded operator on $\mathbb{C}_K^{\mathbb{N}_0}$ with spectral radius < 1,
- $\pi \circ \mathcal{Q} = \mathcal{Q} \circ \pi = 0$.

The map $s \mapsto \dfrac{\mathcal{R}_s - \mathcal{R}}{\sqrt{1-s}}$ is bounded on $\mathcal{O}_\delta(1)$. By perturbation theory, for $s \in \mathcal{O}_\delta(1)$ with δ small enough, the operator $\mathcal{R}_s$ admits a similar spectral decomposition as above; namely, one gets

$$\forall s \in \mathcal{O}_\delta(1) \quad \mathcal{R}_s = \lambda_s \pi_s + \mathcal{Q}_s \tag{34}$$

with

- λ_s is the dominant eigenvalue of $\mathcal{R}_s$, with corresponding eigenvector $\mathbf{h}_s$, normalized in such a way that $\nu_{\mathbf{r}}(\mathbf{h}_s) = 1$,
- π_s is a rank one projector on the space $\mathbb{C} \cdot \mathbf{h}_s$,
- $\mathcal{Q}_s$ is a bounded operator on $\mathbb{C}_K^{\mathbb{N}_0}$ with spectral radius $\leq \rho_\delta$ for some $\rho_\delta < 1$,
- $\pi_s \circ \mathcal{Q}_s = \mathcal{Q}_s \circ \pi_s = 0$.

Furthermore, the maps $s \mapsto \dfrac{\lambda_s - 1}{\sqrt{1-s}}$, $s \mapsto \dfrac{\pi_s - \pi}{\sqrt{1-s}}$, $s \mapsto \dfrac{\mathbf{h}_s - \mathbf{h}}{\sqrt{1-s}}$ and $s \mapsto \dfrac{\mathcal{Q}_s - \mathcal{Q}}{\sqrt{1-s}}$ are bounded on $\mathcal{O}_\delta(1)$. We may in fact make precise the local behavior of the map $s \mapsto \lambda_s$; by the above decomposition and Proposition 4.5, one gets, for $s \in \mathcal{O}_\delta(1)$,

$$\begin{aligned} \lambda_s &= \nu_{\mathbf{r}}(\mathcal{R}_s \mathbf{h}) + \nu_{\mathbf{r}}\Big((\mathcal{R}_s - \mathcal{R})(\mathbf{h}_s - \mathbf{h})\Big) \\ &= 1 + \sqrt{1-s}\ \ \nu_{\mathbf{r}}(\widetilde{\mathcal{R}}\mathbf{h}) + (1-s) O(s) \end{aligned}$$

with $O(s)$ bounded on $\mathcal{O}_\delta(1)$. Since $\nu_{\mathbf{r}}(\widetilde{\mathcal{R}}\mathbf{h}) \neq 0$, the operator $I - \mathcal{R}_s$ is invertible when $s \in \mathcal{O}_\delta(1)$ and δ small enough, with inverse

$$\Big(I - \mathcal{R}_s\Big)^{-1} = \frac{1}{1-\lambda_s} \pi_s + \Big(I - \mathcal{Q}_s\Big)^{-1}.$$

Fact 4.7 *For $\delta > 0$ small enough, the function $s \mapsto \Big(I - \mathcal{R}_s\Big)^{-1}$ admits on $\mathcal{O}_\delta(1)$ the following local expansion of order 1 with values in $\mathcal{M}_K$.*

$$\Big(I - \mathcal{R}_s\Big)^{-1} = -\frac{1}{\sqrt{1-s} \times \nu_{\mathbf{r}}\Big(\widetilde{\mathcal{R}}\mathbf{h}\Big)} \pi + \mathbf{O}_s \tag{35}$$

where $\mathbf{O}_s$ is analytic in the variable $\sqrt{1-s}$ and uniformly bounded in $\mathcal{M}_K$.

4.5 The Return Probabilities in the Centered Case: Proof of the Main Theorem

We use here the identity $\mathcal{G}_s = \Big(I - \mathcal{R}_s\Big)^{-1} \mathcal{E}_s$ given in the introduction. By Proposition 4.5 and Fact 4.6, for any fixed $y \in \mathbb{N}_0$, the function $s \mapsto \mathcal{G}_s(\cdot, y)$ is analytic on a neighborhood of $\overline{B(0,1)} \setminus \{1\}$. Furthermore, for $\delta > 0$ small enough and $s \in \mathcal{O}_\delta(1)$, one may write, using (29) and (38)

$$\mathcal{G}_s(\cdot, y) = -\frac{\nu_{\mathbf{r}}(\mathcal{E}(\cdot, y))}{\nu_{\mathbf{r}}\left(\widetilde{\mathcal{R}}\mathbf{h}\right)} \times \frac{1}{\sqrt{1-s}} + \mathbf{O}_s$$

with $\nu_{\mathbf{r}}$, $\mathcal{E}(\cdot, y)$ and $\widetilde{\mathcal{R}}$ given, respectively, by formulas (9), (32) and (33) and $s \mapsto \mathbf{O}_s$ analytic on $\mathcal{O}_\delta(1)$ in the variable $\sqrt{1-s}$ and uniformly bounded in $\mathcal{M}_K$.

We may thus apply Darboux's theorem 1.1 with $R = 1$, $\alpha = -\frac{1}{2}$ (and so $\Gamma(-\alpha) = \sqrt{\pi}$) and $\mathbf{A}(1) = -\frac{\nu_{\mathbf{r}}(\mathcal{E}(\cdot, y))}{\nu_{\mathbf{r}}\left(\widetilde{\mathcal{R}}\mathbf{h}\right)} > 0$. One gets, for all $x, y \in \mathbb{N}_0$

$$\mathbb{P}_x[X_n = y] \sim \frac{C_y}{\sqrt{n}} \quad \text{with} \quad C_y = -\frac{1}{\sqrt{\pi}} \times \frac{\nu_{\mathbf{r}}(\mathcal{E}(\cdot, y))}{\nu_{\mathbf{r}}\left(\widetilde{\mathcal{R}}\mathbf{h}\right)} > 0. \tag{36}$$

5 The Non-centered Random Walk

We assume here $\mathbb{E}[Y_n] > 0$ and use a standard argument in probability theory to reduce the question to the centered case.

5.1 The Relativisation Principle and Its Consequences

For any $r > 0$, we denote by μ_r the probability measure defined on $\mathbb{Z}$ by

$$\forall n \in \mathbb{Z} \quad \mu_r(n) = \frac{1}{\hat{\mu}(r)} r^n \mu(n).$$

For any $k \geq 0$, one gets $(\mu^{*k})_r = (\mu_r)^{*k}$ and that the generating function $\hat{\mu}_r$ is related to the one of μ by the following identity $\forall z \in \mathbb{C} \quad \hat{\mu}_r(z) := \frac{\hat{\mu}(rz)}{\hat{\mu}(r)}$.

The waiting times τ^{*-} and τ^+ are defined on the space $(\Omega, \mathcal{T})$, with values in $\mathbb{N}_0 \cup \{+\infty\}$; they are both a.s. finite if and only if μ_r is centered, i.e. $r = r_0$ (see Sect. 3.1 for the notations).

Throughout this section, we will denote $\mathbb{P}^\circ$ the probability on $(\Omega, \mathcal{T})$ which ensures that the Y_n are i.i.d. with law μ_{r_0}; the expectation with respect to $\mathbb{P}^\circ$ is denoted $\mathbb{E}^\circ$. We set $\rho_\circ = \hat{\mu}(r_0)$ and $R_\circ = 1/\rho_\circ \in]1, +\infty[$. The variables Y_n have common law μ_{r_0} under $\mathbb{P}^\circ$, and they are in particular centered; we may thus apply the results of the previous section when we refer to this probability measure on $(\Omega, \mathcal{T})$.

Fact 5.1 *Let $n \geq 1$ and $\Phi : \mathbb{R}^{n+1} \to \mathbb{C}$ a bounded Borel function; then, one gets*

$$\mathbb{E}\Big[\Phi(S_0, S_1, \ldots, S_n)\Big] = \rho_\circ^n \times \mathbb{E}^\circ\Big[\Phi(S_0, S_1, \ldots, S_n) r_0^{-S_n}\Big].$$

As a direct consequence, for any $x, y \in \mathbb{N}_0$ and $s \in \mathbb{C}$, one gets, at least formally

$$\mathcal{E}_s(x, y) = r_0^{x-y} \mathcal{E}^\circ_{\rho_\circ s}(x, y) \quad \text{and} \quad \mathcal{R}_s(x, y) = r_0^{x+y} \mathcal{R}^\circ_{\rho_\circ s}(x, y)$$

where we have set $\mathcal{E}^\circ_s(x, y) := \sum_{n\geq 0} s^n \mathbb{E}^\circ_x[\mathbf{r} > n, X_n = y]$ and $\mathcal{R}^\circ_s(x, y) := \sum_{n\geq 0} s^n \mathbb{E}^\circ_x[\mathbf{r} = n, X_n = y]$. We may thus introduce the diagonal matrix $\Delta = (\Delta(x, y))_{x,y\in\mathbb{N}_0}$ defined by $\Delta(x, y) = 0$ when $x \neq y$ and $\Delta(x, x) = r_0^x$ for any $x \geq 0$; by the above, one gets formally

$$\mathcal{E}_s = \Delta \mathcal{E}^\circ_s \Delta^{-1} \quad \text{and} \quad \mathcal{R}_s = \Delta \mathcal{R}^\circ_s \Delta.$$

In the sequel, we will add the exponent $\circ$ to the quantities $\mathcal{U}^+, \mathcal{T}, \mathcal{U}, \mathcal{V}$ defined in the previous section when μ was assumed to be centered and considered here as variables defined on $(\Omega, \mathcal{F}, \mathbb{P}^\circ)$; with these notations, we will have

$$\mathcal{E}^\circ = \mathcal{U}^\circ \mathcal{U}^{\circ+}, \quad \widetilde{\mathcal{E}}^\circ = \widetilde{\mathcal{U}}^\circ \mathcal{U}^{\circ+} + \mathcal{U}^\circ \widetilde{\mathcal{U}}^{\circ+}, \mathcal{R}^\circ = \mathcal{U}^\circ \mathcal{V}^\circ \quad \text{and} \quad \widetilde{\mathcal{R}}^\circ = \widetilde{\mathcal{U}}^\circ \mathcal{V}^\circ + \mathcal{U}^\circ \widetilde{\mathcal{V}}^\circ.$$

Combining Propositions 4.3 and 4.5, we may thus state the

Proposition 5.2 *There exists a function K, an open neighborhood Ω of $\overline{B(0, R_\circ)} \setminus \{R_\circ\}$, and $\delta > 0$ small enough such that, for any $y \in \mathbb{N}_0$, the functions $s \mapsto \Big(\mathcal{E}_s(x, y)\Big)_{x\in\mathbb{N}_0}$ have an analytic continuation on Ω with values in the Banach space $\mathbb{C}_K^{\mathbb{N}_0}$ and are analytic in the variable $\sqrt{R_\circ - s}$ on the set $\mathcal{O}_\delta(R_\circ) := B(R_\circ, \delta) \setminus [R_\circ, R_\circ + \delta)$, with the following local expansion of order 1 in $\mathbb{C}_K^{\mathbb{N}_0}$:*

$$\mathcal{E}_s = \mathcal{E} + \sqrt{R_\circ - s}\, \widetilde{\mathcal{E}} + (R_\circ - s)\, \mathbf{O}_s \tag{37}$$

where

- $\mathcal{E} = \Delta \mathcal{E}^\circ \Delta^{-1}$,
- $\widetilde{\mathcal{E}} = \sqrt{\rho_0}\, \Delta \widetilde{\mathcal{E}}^\circ \Delta^{-1}$,
- $\mathbf{O}_s = (\mathbf{O}_s(x, y))_{x\in\mathbb{N}_0}$ *is analytic in the variable* $\sqrt{R_\circ - s}$ *and uniformly bounded in* $\mathbb{C}_K^{\mathbb{N}_0}$ *for* $s \in \mathcal{O}_\delta(R_\circ)$.

Similarly, the function $s \mapsto \mathcal{R}_s = \Big(\mathcal{R}_s(x, y)\Big)_{x,y\in\mathbb{N}_0}$ has an analytic continuation to Ω, with values in the Banach space $\mathcal{M}_K$, and is analytic in the variable $\sqrt{R_\circ - s}$ on the set $\mathcal{O}_\delta(R_\circ)$ with the following local expansion of order 1 in $\mathcal{M}_K$:

$$\mathcal{R}_s = \mathcal{R} + \sqrt{R_\circ - s}\, \widetilde{\mathcal{R}} + (R_\circ - s)\, \mathbf{O}_s \tag{38}$$

where

- $\mathcal{R} = \Delta \mathcal{R}^\circ \Delta$,
- $\widetilde{\mathcal{R}} = \sqrt{\rho_0}\, \Delta \widetilde{\mathcal{R}}^\circ \Delta$,
- $\mathbf{O}_s = (\mathbf{O}_s(x, y))_{x\in\mathbb{N}_0}$ *is analytic in the variable* $\sqrt{R_\circ - s}$ *and uniformly bounded in* $\mathbb{C}_K^{\mathbb{N}_0}$ *for* $s \in \mathcal{O}_\delta(R_\circ)$.

To prove the main theorem in the non-centered case, we will thus apply the same strategy than in the previous section. The proof simplifies in this case since the operator $I - \mathcal{R}_s$ becomes invertible.

Fact 5.3 *For K suitably choosen, $\delta > 0$ small enough and any $s \in \mathcal{O}_\delta(R_\circ)$, the spectral radius of the operator $\mathcal{R}_s$ on $\mathcal{M}_K$ is < 1.*

Proof It will be a direct consequence of the continuity of the map $s \mapsto \mathcal{R}_s$ on $K_\delta(R_0)$ and the inequality $\|\mathcal{R}_{R_\circ}\|_K < 1$. Indeed, one gets, using the definition of $\mathcal{R}$ and setting $\phi := 1_{[-x,+\infty[}$

$$\begin{aligned}
\|\mathcal{R}_{R_\circ}\|_K &\leq \sup_{x\in\mathbb{N}_0} \sum_{y\geq 1}\Big(\sum_{n\geq 0} R_\circ^n \mathbb{P}\left[\phi(S_1)\ldots\phi(S_{n-1})1_{\{-x-y\}}(S_n)\right]\Big) K(y)\\
&= \sup_{x\in\mathbb{N}_0} \sum_{y\geq 1}\Big(\sum_{n\geq 0} R_\circ^n r_0^{x+y} K(y)\mathbb{P}^\circ\left[\phi(S_1)\ldots\phi(S_{n-1})1_{\{-x-y\}}(S_n)\right]\Big)\\
&\leq r_0 \sup_{x\in\mathbb{N}_0}\Big(\sum_{n\geq 0} R_\circ^n \mathbb{P}^\circ\left[\phi(S_1)\ldots\phi(S_{n-1})(1-\phi)(S_n)\right]\Big)\\
&\quad \text{if} \quad r_0^{y-1}K(y) \leq 1 \quad \text{for all} \quad y \geq 1 \leq r_0
\end{aligned}$$

which achieves the proof, assuming that $y \mapsto r_0^{y-1}K(y)$ is ≤ 1 on $\mathbb{N}_0$. □

As a direct consequence, one may write

$$(I - \mathcal{R}_s)^{-1} = \sum_{n\geq 0} \mathcal{R}_s^n.$$

Furthermore, the map $s \mapsto (I - \mathcal{R}_s)^{-1}$ is analytic in the variable s on Ω and analytic in the variable $\sqrt{R_\circ - s}$ on $O_\delta(R_\circ)$ and the local expansion near $R_\circ$ is

$$\begin{aligned}
(I - \mathcal{R}_s)^{-1} &= (I - \mathcal{R})^{-1} + \sqrt{R_\circ - s}\,(I - \mathcal{R})^{-1}\,\widetilde{\mathcal{R}}\,(I - \mathcal{R})^{-1}\\
&\quad +(R_\circ - s)\ldots
\end{aligned} \tag{39}$$

5.2 The Return Probabilities in the Non-centered Case: Proof of the Main Theorem

We use here the identity $\mathcal{G}_s = \Big(I - \mathcal{R}_s\Big)^{-1}\mathcal{E}_s$ given in the introduction. By Proposition 5.2 and Fact 5.3, for any fixed $y \in \mathbb{N}_0$, the function $s \mapsto \mathcal{G}_s(\cdot, y)$ is analytic on a neighborhood of $\overline{B(0, R_\circ)} \setminus \{R_\circ\}$. Furthermore, for $\delta > 0$ small enough and $s \in \mathcal{O}_\delta(R_\circ)$, one may write, using Proposition 5.2 and the local expansion (39)

$$\begin{aligned}
\mathcal{G}_s(\cdot, y) &= (I - \mathcal{R})^{-1}(\cdot, y) + \sqrt{R_\circ - s}\Big((I - \mathcal{R})^{-1}\\
&\quad\times\widetilde{\mathcal{R}}\,(I - \mathcal{R})^{-1}\,\mathcal{E}(\cdot, y) + (I - \mathcal{R})^{-1}\,\widetilde{\mathcal{E}}(\cdot, y)\Big) + (R_\circ - s)\mathbf{O}_s
\end{aligned}$$

with $s \mapsto \mathbf{O}_s$ analytic on $\mathcal{O}_\delta(R_\circ)$ in the variable $\sqrt{R_\circ - s}$ and uniformly bounded in $\mathcal{M}_K$.

We may thus apply Darboux's theorem 1.1 with $R = R_\circ$, $\alpha = \frac{1}{2}$ (and so $\Gamma(-\alpha) = -2\sqrt{\pi}$) and $\mathbf{A}(R_\circ) = (I - \mathcal{R})^{-1}\, \widetilde{\mathcal{R}}\, (I - \mathcal{R})^{-1}\, \mathcal{E}(\cdot, y) + (I - \mathcal{R})^{-1}\, \widetilde{\mathcal{E}}(\cdot, y) < 0$.

For all $x, y \in \mathbb{N}_0$, one gets

$$\mathbb{P}_x[X_n = y] \sim C_{x,y} \frac{\rho_\circ^n}{n^{3/2}}$$

with $\rho_\circ = \frac{1}{R_\circ} = \hat{\mu}(r_0) \in]0, 1[$ and

$$C_{x,y} = -\frac{1}{2\rho_\circ\sqrt{\pi}} \times \Big((I - \mathcal{R})^{-1}\, \widetilde{\mathcal{R}}\, (I - \mathcal{R})^{-1}\, \mathcal{E}(x, y) + (I - \mathcal{R})^{-1}\, \widetilde{\mathcal{E}}(x, y) \Big) > 0 \quad (40)$$

where the matrices $\mathcal{R}$, $\widetilde{\mathcal{R}}$, $\mathcal{E}$, and $\widetilde{\mathcal{E}}$ are given explicitly in Proposition 5.2.

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