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Introduction to branching processes in fixed and random environment

Marc PEIGNÉ [1] [2]

[1]**Laboratoire de Mathématiques et Physique Théorique** (LMPT), Parc de Grandmont, Université Fr. Rabelais Tours,37200 France.
email : peigne@lmpt.univ-yours.fr

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Contents

1 Branching processes: the Galton-Watson model

In this course, we intend to give a short introduction to an elementary model for the genealogy of a population. The original branching process was considered by Galton and Watson in the 1870's while seeking a quantitative explanation for the disappearing family names phenomenon, even in a growing population. In general it describes the evolution of successive generations of a "population" under the following assumptions:

1. *the initial generation numbered* 0 *has one member (the ancestor),*
2. *the number of children of any member in any generation of the population is random and follows a offspring distribution called* μ*,*
3. *the* μ*-distributed random variables that represent the number of children of each member in all generations of the population are independent.*

Let us consider a family $(N_{n,i})_{n\geq 0, i\geq 1}$ of $\mathbb{N}$-valued, independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any $n \geq 0$ and $i \geq 1$, the random variable $N_{n,i}$ equals the number of "children" of the i^{th} individual of the generation n.

We write Z_n the number members in the n-th generation; we set $Z_0 = 1$ and

$$Z_n = \begin{cases} N_{n-1,1} + N_{n-1,2} + \ldots + N_{n-1,Z_{n-1}} & \text{if } Z_{n-1} \neq 0 \\ 0 & \text{if } Z_{n-1} = 0. \end{cases} \tag{1.1}$$

The sequence $(Z_n)_{n\geq 0}$ is the *Galton-Watson process*, it is a Markov chain on $\mathbb{N}$ with initial distribution δ_0.

We denote $\mu = (\mu(k))_{k\geq 0}$ the distribution of the random variables $N_{n,i}$ and assume that μ is *non-degenerate*, i.e. $\mu(k) \neq 1$ for any $k \geq 0$ and *aperiodic* i.e. $\gcd\{k \in \mathbb{N}^* \mid \mu(k) > 0\} = 1$.

1.1 Basic properties of the Galton-Watson process

We consider the generating function $f_\mu$of μ defined by: for any $x \geq 0, n \geq 0$ and $i \geq 1$,

$$f(x) = f_\mu(x) := \sum_{k=0}^{+\infty} \mu(k) x^k = \mathbb{E}(x^{N_{n,i}}).$$

When there is no risk of confusion, we will simplify the notation and set $f = f_\mu$.

The radius of convergence of this series is denoted ρ_μ; it is obvious that ρ_μ is greater than or equal to 1 since μ is a probability measure.

Throughout this course, we assume that μ is not degenerate.

Here, we state the first result.

Proposition 1.1. *Assume that* $\mathbb{E}(N_{n,i}^2) < +\infty$.

1. *The function* f *is* C^2 *and convex on* $[0,1]$*; furthermore, if* $\mathbb{P}(N_{n,i} \geq 2) > 0$*, then* f *is strictly convex.*
2. *The expectation* m *and the variance* σ^2 *of* μ *satisfy the equalities*

$$m := \mathbb{E}(N_{n,i}) = f'(1) \quad \text{and} \quad \sigma^2 := V(N_{n,i}) = f''(1) + f'(1) - f'(1)^2.$$

3. *The generating function F_n of Z_n is given by* $F_0 = Id$ and $F_n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$.

4. *The expectation of Z_n is equal to* $= m^n$.

5. *The variance of Z_n is given by*

$$V(Z_n) = \begin{cases} m^{n-1}\dfrac{m^n - 1}{m-1}\sigma^2 & \textit{when} \quad m \neq 1 \\ n\sigma^2 & \textit{otherwise.} \end{cases}$$

1.2 On the extinction probability of the GW process

Let q_n be the extinction probability at the n^{th} generation : $q_n = \mathbb{P}(Z_n = 0)$. The sequence of events $((Z_n = 0))_{n \geq 0}$ is increasing; so is the sequence $(q_n)_{n\geq 0}$. The extinction event $\mathcal{E}$ can be expressed by

$$\mathcal{E} = \{\exists n \in \mathbb{N} \text{ s.t. } Z_n = 0\} = \bigcup_{n \geq 0} (Z_n = 0).$$

Here are some trivial cases.

1. If $\mu(0) = 0$ then $\mathbb{P}(\mathcal{E}) = 0$; furthermore, if $\mu(1) = 1$, then $Z_n = 1$ $\mathbb{P}$-a.s. for any $n \geq 0$.

2. If $\mu(0) = 1$ then $Z_n = 0$ $\mathbb{P}$-a.s. for any $n \geq 1$ and $\mathbb{P}(\mathcal{E}) = 1$.

3. If $0 < \mu(0) < 1$ and $\mu(0) + \mu(1) = 1$, then $\mathbb{P}(Z_n \neq 0, Z_{n+1} = 0) = \mu(1)^n \mu(0)$, which yields

$$\mathbb{P}(\mathcal{E}) = \sum_{n=0}^{+\infty} \mathbb{P}(Z_n \neq 0, Z_{n+1} = 0) = \frac{\mu(0)}{1 - \mu(1)} = 1.$$

From now on, we omit these 3 particular cases and assume that μ satisfies the following assumption

$$0 < \mu(0) < 1 \quad \text{and} \quad \mu(0) + \mu(1) < 1. \tag{1.2}$$

Under condition (1.2), the function f is strictly convex.

We obtain the following proposition.

Proposition 1.2. *1.* $q_n = F_n(0)$.

2. *The sequence $(q_n)_n$ is increasing and its limit q is a fixed point of f (i.e. $f(q) = q$).*

3. *q is the smallest positive fixed point of f.*

4. *$q = 1$ if and only if $m \leq 1$.*

The quantity q is called the *extinction probability* of the Galton-Watson process $(Z_n)_{n\geq 0}$.

Proof. The convexity property of the function f on $[0, 1]$ yields

1. if $f'(1) > 1$, then the equation $f(x) = x$ has two distinct solutions in $[0, 1]$, so $q < 1$;

2. if $f'(1) \leq 1$ then the equation $f(x) = x$ has a unique solution in $[0, 1]$ and so $q = 1$.

□

The Galton-Watson process $(Z_n)_{n\geq 0}$ is called **sub-critical** when $m < 1$, **critical** when $m = 1$ and **super-critical** when $m > 1$.

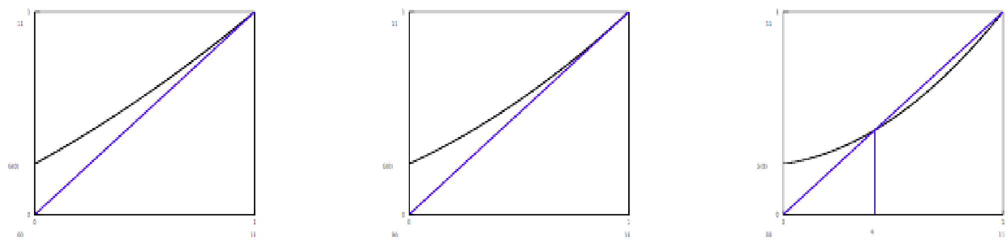

Figure 1: Generating function in the *sub-critical* (left), *critical* (middle) and *super-critical* (right) cases

1.3 Speed of convergence to extinction

Let τ be the moment at which the population dies out:

$$\tau := \inf\{n \geq 1 \mid Z_n = 0\}$$

(with the convention $\inf \emptyset = +\infty$). The quantity $\mathbb{P}(\tau > n)$ is the *survival probability up to time n* $\mathbb{P}(Z_n \neq 0)$ of the GW process; it tends to 0 as $n \to +\infty$ when the GW process is sub-critical or critical.

To study the evolution of a GW process, the following fact is of major importance: the process $(Z_n/m^n)_{n\geq 0}$ is a positive martingale ! Indeed, noticing $\mathcal{T}_n$ the σ-algebra generated by the random variables $N_{k,i}, 0 \leq k \leq n-1, i \geq 0$, we may write

$$\mathbb{E}\left(\frac{Z_{n+1}}{m^{n+1}} \mid \mathcal{T}_n\right) = \frac{1}{m^{n+1}}\mathbb{E}\left(\sum_{i=1}^{Z_n} N_{n,i} \mid \mathcal{T}_n\right) = \frac{1}{m^{n+1}} m Z_n = \frac{Z_n}{m^n}$$

where the last equality relies on the fact that Z_n is $\mathcal{T}_n$-measurable and the $N_{n,i}, i \geq 1$, are independent of $\mathcal{T}_n$. Thus, by the Martingale Convergence Theorem, the sequence $(Z_n/m^n)_{n\geq 0}$ converges $\mathbb{P}$-a.s. to some limit W with value in $[0, \infty]$.

Now, we state a precise results about the extinction of the GW process. The first two statements are due to Kolmogorov [14] and the third to Kesten and Stigum [12].

Theorem 1.3. *Let $(Z_n)_{n\geq 0}$ be a GW process whose offspring distribution μ satisfies (1.2) and has finite mean.*

1. *In the sub-critical case, the process $(Z_n)_{n\geq 0}$ becomes extinct $\mathbb{P}$-a.s. and there exists a constant $c > 0$ such that*
$$\mathbb{P}(\tau > n) = \mathbb{P}(Z_n \neq 0) \sim cm^n \quad \text{as} \quad n \to +\infty. \tag{1.3}$$

2. *In the critical case, $(Z_n)_{n\geq 0}$ becomes extinct $\mathbb{P}$-a.s. and*
$$\mathbb{P}(\tau > n) = \mathbb{P}(Z_n \neq 0) \sim \frac{2}{\sigma^2 n} \quad \text{as} \quad n \to +\infty. \tag{1.4}$$

3. *In the super-critical case, the martingale $(Z_n/m^n)_{n\geq 0}$ converges $\mathbb{P}$-a.s. to some random variable W which satisfies the two properties:*
$$\mathbb{E}(W) \leq 1 \qquad \text{and} \qquad \mathbb{P}(W = 0) \in \{q, 1\}, \tag{1.5}$$
where $q \in]0, 1[$ is the probability of extinction of $(Z_n)_{n\geq 0}$. Furthemore,
$$\mathbb{P}(W = 0) = q \iff \mathbb{E}(N_{n,i} \log^+ N_{n,i}) < +\infty.$$

The last statement is a deep and difficult result, the reader may find the proof in the course by R. Abraham and J.F. Delmas [1], based on the notion of GW trees (see Section 2). We only detail the proof of assertions (1) and (2).
Proof. (1) In the sub-critical case, we may write

$$\mathbb{P}(\tau > n) = \mathbb{P}(Z_n \neq 0) = \mathbb{P}(Z_n \geq 1) \leq \mathbb{E}(Z_n) = m^n.$$

so that $(\mathbb{P}(\tau > n))_{n\geq 0}$ tends exponentially fast to 0 as $n \to +\infty$. To prove (1.3), we have to be more precise. First we notice that the sequence $(\mathbb{P}(Z_n > 0)/m^n)_{n\geq 0}$ is decreasing. Indeed

$$\begin{aligned}\mathbb{P}(Z_{n+1} > 0) = 1 - F_{n+1}(0) &= 1 - f(F_n(0)) \\ &= \frac{1 - f(F_n(0))}{1 - F_n(0)} \times \mathbb{P}(Z_n > 0) \\ &\leq f'(1)\ \mathbb{P}(Z_n > 0) = m\ \mathbb{P}(Z_n > 0).\end{aligned}$$

Thus, the sequence $(\mathbb{P}(Z_n > 0)/m^n)_{n\geq 0}$ converges to some non-negative limit c. To check $c > 0$, we use the fact that $Z_n \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(Z_n > 0) > 0$; consequently [3]

$$\begin{aligned}\mathbb{P}(Z_n > 0) &\geq \frac{\mathbb{E}(Z_n)^2}{\mathbb{E}(Z_n^2)} \\ &= \frac{\mathbb{E}(Z_n)^2}{V(Z_n) + \mathbb{E}(Z_n)^2} \\ &= \frac{m^{n+1}(1-m)}{(1-m^n)\sigma^2 + (1-m)m^{n+1}}\end{aligned}$$

so that $\liminf_{n\to+\infty} \frac{\mathbb{P}(Z_n > 0)}{m^n} \geq \frac{m(1-m)}{\sigma^2} > 0$ since $0 < m < 1$.

(2) Assume that $(Z_n)_{n\geq 0}$ is critical. Set $v_n := \mathbb{P}(\tau > n)$ and notice that $\lim_{n\to+\infty} v_n = 0$ and $v_{n+1} = 1 - f(1 - v_n)$ for any $n \geq 0$. This yields, using the fact that $f(1-x) = 1 - x + \frac{\sigma^2}{2}x^2(1 + \epsilon(x))$ with $\epsilon(x) \to 0$ as $x \to 0^+$,

$$\begin{aligned}\frac{1}{v_{n+1}} - \frac{1}{v_n} &= \frac{1}{1 - f(1 - v_n)} - \frac{1}{v_n} \\ &= \frac{1}{v_n\left(1 - \frac{\sigma^2}{2}v_n(1 + o(n))\right)} - \frac{1}{v_n} \\ &\qquad \text{with } o(n) \to 0 \text{ as } n \to +\infty \\ &= \frac{1}{v_n}\left(1 + \frac{\sigma^2}{2}v_n(1 + o(n))\right) - \frac{1}{v_n} \\ &\qquad \text{with } o(n) \to 0 \text{ as } n \to +\infty \\ &\longrightarrow \frac{\sigma^2}{2} \quad \text{as} \quad n \to +\infty.\end{aligned}$$

Consequently $\frac{1}{v_n} = \frac{1}{v_0} + \sum_{k=0}^{n-1}\Big(\frac{1}{v_{k+1}} - \frac{1}{v_k}\Big) \sim \frac{\sigma^2}{2}n$ as $n \to +\infty$.

[3] if Z is a positive r.v. s.t. $\mathbb{P}(Z > 0) > 0$ then $\mathbb{P}(Z > 0) \geq \mathbb{E}(Z)^2/\mathbb{E}(Z^2)$; this is a direct consequence of the Cauchy-Schwartz inequality, since $\mathbb{E}(Z) = \mathbb{E}(Z 1_{Z>0})$.

□

In the previous section, we have seen that the random variable τ is $\mathbb{P}$-a.s. finite if and only if the process $(Z_n)_{n\geq 0}$ is sub-critical or critical. As a direct consequence of (1.3), (1.4) and (1.5) we see that $\mathbb{E}(\tau) < +\infty$ if and only if the process $(Z_n)_{n\geq 0}$ is sub-critical.

2 Branching processes: an approach via trees

In this section we propose another approach to describe GW trees. It is important to obtain strong results in an elementary way on the asymptotic behavior of these processes; it is also of interest to study the limit of GW trees [1]. We present in this section some results concerning super-critical GW processes, conditioned on survival or extinction.

2.1 The set of discrete trees

We recall Neveu's formalism for ordered rooted trees. We denote $\mathcal{U}$ the set of finite sequences of positive integers

$$\mathcal{U} = \bigcup_{n\geq 0} (\mathbb{N}^*)^n$$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $(\mathbb{N}^*)^0 = \{\emptyset\}$. For any $n \geq 1$ and $u = (u_1, \cdots, u_n) \in \mathcal{U}$, the integer denoted by $|u|$ is the length n of u, with the convention $|\emptyset| = 0$.

The concatenation of two sequences $u, v \in \mathcal{U}$ is denoted uv, with the convention that $uv = u$ if $v = \emptyset$ and $uv = v$ if $u = \emptyset$.

A sequence $u \in \mathcal{U}$ is an *ancestor* of $v \in \mathcal{U}$ (we write $u \prec v$) if there exists $w \in \mathcal{U}, w \neq \emptyset$, such that $v = uw$; the set of ancestors of v is the set

$$A_v := \{u \in \mathcal{U} \mid u \prec v\}.$$

Definition 2.1. *A tree* $\mathbf{t}$ *is a subset of* $\mathcal{U}$ *which satisfies the following conditions.*

(i) $\emptyset \in \mathbf{t}$.

(ii) If $u \in \mathbf{t}$ *then* $A_u \subset \mathbf{t}$.

(iii) For any $u \in \mathbf{t}$ *there exists* $k_u(\mathbf{t}) \in \mathbb{N} \cup \{+\infty\}$ *such that, for any* $i \in \mathbb{N}^*$,

$$ui \in \mathbf{t} \Longleftrightarrow 1 \leq i \leq k_u(\mathbf{t}).$$

The integer $k_u(\mathbf{t})$ is the number of offsprings of the vertex $u \in \mathbf{t}$. The vertex $\emptyset$ is called the root of $\mathbf{t}$. For instance, $k_\emptyset(\mathbf{t}) = 3, k_{(1)}(\mathbf{t}) = 3$ and $k_{(3,1)}(\mathbf{t}) = 2$ in the Figure above.

We denote by $\mathbb{T}_\infty$ the set of all trees and introduce some definitions.

Definition 2.2. *For any* $\mathbf{t} \in \mathbb{T}_\infty$,

(i) the **height** *of* $\mathbf{t}$ *is defined by* $H(\mathbf{t}) := \sup\{|\mathbf{u}| \mid \mathbf{u} \in \mathbf{t}\}$;

(ii) the **width at level** $h \geq 0$ *of* $\mathbf{t} \in \mathbb{T}_\infty$ *is the integer* $z_h(\mathbf{t})$ *defined by* $z_h(\mathbf{t}) = \sharp\{u \in \mathbf{t} \mid |u| = h\}$.

For any sequence $v = (v_n)_{n\geq 1} \in (\mathbb{N}^*)^{\mathbb{N}}$, *set* $\bar{v}_0 = \emptyset$ *and* $\bar{v}_n = (v_1, \cdots, v_n), n \geq 1$. *If* $\bar{v}_n \in \mathbf{t}$ *for any* $n \geq 0$, *the sequence* $\bar{\mathbf{v}} = \{\bar{v}_n, n \in \mathbb{N}\}$ *is an* **infinite branch** *of* $\mathbf{t}$.

Now, we fix some notations.

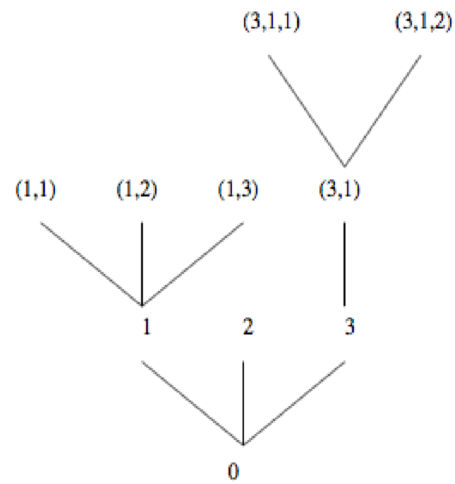

Figure 2: *Example of a finite tree* $\mathbf{t}$

1. The subset of trees with no infinite vertex is denoted
$$\mathbb{T} := \{\mathbf{t} \in \mathbb{T}_\infty \mid \forall u \in \mathbf{t}\ k_u(\mathbf{t}) < +\infty\}.$$
2. The set of finite trees is denoted
$$\mathbb{T}_0 := \{\mathbf{t} \in \mathbb{T} \mid |\mathbf{t}| < +\infty\}.$$
where $|\mathbf{t}|$ denotes the cardinality of $\mathbf{t}$. Notice that $\displaystyle\sum_{u\in\mathbf{t}} k_u(\mathbf{t}) = |\mathbf{t}| - 1$.
3. For any $h \geq 0$, the subset of trees with height less than h is denoted
$$\mathbb{T}^{(h)} = \{\mathbf{t} \in \mathbb{T} \mid H(\mathbf{t}) \leq h\}.$$
4. The restriction function $r_h : \mathbb{T} \to \mathbb{T}^{(h)}$ is defined by: for any tree $\mathbf{t} \in \mathbb{T}$,
$$r_h(\mathbf{t}) = \{u \in \mathbf{T} \mid |u| \leq h\}.$$
5. For any $\mathbf{t} \in \mathbb{T}$ and $u \in \mathbf{t}$, we denote $\mathcal{S}_u(\mathbf{t})$ *the sub-tree of* $\mathbf{t}$ *"above"* u defined by
$$\mathcal{S}_u(\mathbf{t}) = \{v \in \mathcal{U} \mid uv \in \mathbf{t}\}.$$

2.2 Galton-Watson trees

Let $\mu = (\mu(k))_{k\geq 0}$ be a probability distribution on the set of non-negative integers and N be a random variable with distribution μ.

Definition 2.3. *A $\mathbb{T}$-valued random variable τ has the* **branching property** *if for any $n \geq 1$, conditioned on $k_\emptyset(\tau) = n$, the sub-tree $(\mathcal{S}_1(\tau), \cdots, \mathcal{S}_n(\tau))$ are independent and distributed as τ.*

The $\mathbb{T}$-valued random variable τ is a Galton-Watson tree (GW tree) with offspring distribution μ if it has the branching property and if the distribution of $k_\emptyset(\tau)$ is μ.

It is easy to check that a $\mathbb{T}$-valued random variable τ is a Galton-Watson tree with offspring distribution μ if and only if the restriction of the distribution of τ on the set $\mathbb{T}_0$ is given by

$$\forall \mathbf{t} \in \mathbb{T}_0 \qquad \mathbb{P}(\tau = \mathbf{t}) = \prod_{u \in \mathbf{t}} p(k_u(\mathbf{t})) \tag{2.1}$$

and for any $h \geq 1$ and $\mathbf{t} \in \mathbb{T}^{(h)}$ we have $\mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{\substack{u \in \mathbf{t} \\ |u| < h}} p(k_u(\mathbf{t}))$. In this case, the process $(z_h(\tau))_{h \geq 0}$ has the same distribution as the Galton-Watson process $(Z_h)_{h \geq 0}$ defined in (1.1).

The extinction event $\mathcal{E}$ of the GW tree τ is

$$\mathcal{E} = \mathcal{E}(\tau) = (\tau \in \mathbb{T}_0) = \{\omega \in \Omega \mid \tau(\omega) \in \mathbb{T}_0\}.$$

Using the branching property, we obtain

$$\begin{aligned} \mathbb{P}(\mathcal{E}) &= \mathbb{P}(\mathcal{E}(\tau)) \\ &= \sum_{k \geq 0} \mathbb{P}\Big(\mathcal{E}(\mathcal{S}_1(\tau)) \cap \cdots \cap \mathcal{E}(\mathcal{S}_k(\tau)) \mid k_\emptyset(\tau) = k\Big) \mu(k) \\ &= \sum_{k \geq 0} \mathbb{P}(\mathcal{E})^k \mu(k) = f(\mathbb{P}(\mathcal{E})). \end{aligned}$$

In the two next subsections, we apply this formalism by trees to study more deeply the super-critical case.

2.3 On the super-critical GW process conditioned on extinction

We assume in this subsection that assumption (1.2) holds and $m > 1$ while in this case $\mathbb{P}(\mathcal{E}) = q \in]0, 1[$. We introduce the new distribution $\tilde{\mu}$ on $\mathbb{N}$ defined by

$$\forall k \geq 0, \quad \tilde{\mu}(k) = q^{k-1} \mu(k). \tag{2.2}$$

Notice $\sum_{k=0}^{+\infty} \tilde{\mu}(k) = \frac{1}{q} f(q) = 1$ which proves that $\tilde{\mu}$ is a probability measure on $\mathbb{N}$ satisfying assumption (1.2). Furthermore, its generating function $f_{\tilde{\mu}}$ satisfies the equality $f_{\tilde{\mu}}(x) = \frac{1}{q} f(qx)$ so that

$$f'_{\tilde{\mu}}(1) = f'_{\mu}(q) < 1.$$

Thus, the offspring distribution $\tilde{\mu}$ is sub-critical and the associated GW process becomes extinct $\mathbb{P}$-a.s.. In other words, if $\tilde{\tau}$ is a GW tree with offspring distribution $\tilde{\mu}$, then $\mathbb{P}(\mathcal{E}(\tilde{\tau})) = 1$. More precisely, we obtain the following proposition.

Proposition 2.4. *Let τ be a super-critical GW tree with offspring distribution μ satisfying (1.2) and extinction probability $q = \mathbb{P}(\mathcal{E}) \in]0, 1[$. Then, conditioned on the event $\mathcal{E}$, the tree τ is distributed as a sub-critical GW tree $\tilde{\tau}$ with offspring distribution $\tilde{\mu}$ given by (2.2).*

Proof. Since $\tilde{\mu}$ is sub-critical, the random tree $\tilde{\tau}$ belongs a.s. to $\mathbb{T}_0$. For $\mathbf{t} \in \mathbb{T}_0$, we obtain

$$\mathbb{P}(\tilde{\tau} = \mathbf{t}) = \prod_{u \in \mathbf{t}} \mu(k_u(\mathbf{t})) q^{k_u(\mathbf{t}) - 1} = q^{\sum_{u \in \mathbf{t}} (k_u(\mathbf{t}) - 1)} \times \prod_{u \in \mathbf{t}} \mu(k_u(\mathbf{t})) = \frac{1}{q} \mathbb{P}(\tau = \mathbf{t})$$

since $\sum_{u\in\mathbf{t}} k_u(\mathbf{t}) = |\mathbf{t}| - 1$. Taking into account $\mathbb{P}(\tilde\tau \in \mathbb{T}_0) = \mathbb{P}(\mathcal{E}(\tilde\tau)) = 1$ and $\mathbb{P}(\tau \in \mathbb{T}_0) = \mathbb{P}(\mathcal{E}(\tau)) = q$, the last inequality may be written as

$$\mathbb{P}(\tilde\tau = \mathbf{t}) = \mathbb{P}((\tilde\tau = \mathbf{t}) \cap (\tilde\tau \in \mathbb{T}_0)) = \frac{1}{q}\mathbb{P}((\tau = \mathbf{t}) \cap (\tau \in \mathbb{T}_0)) = \mathbb{P}(\tau = \mathbf{t} \mid \mathcal{E}).$$

□

2.4 On the super-critical GW process conditioned on non-extinction

We study here the super-critical GW tree conditioned on non-extinction event $\mathcal{E}^c$. Remind $\mathbb{P}(\mathcal{E}^c) = 1 - q \in]0,1[$.

For any $\mathbf{t} \in \mathbb{T}$, we say that $u \in \mathbb{T}$ is a *survivor in* $\mathbf{t}$ (or u *has type* s) if the set $\{v \in \mathbf{t} \mid u \prec v\}$ is infinite; otherwise, u becomes *extinct* (or u *has type* e).

The *survival process* $(z_h^s(\mathbf{t}))_{h\geq 0}$ is defined by:

$$z_h^s(\mathbf{t}) = \sharp\{u \in \mathbf{t} \mid |u| = h \text{ and } u \text{ is a survivor}\}.$$

For instance, the root $\emptyset$ is a survivor of τ with probability $1-q$. We denote S and E the number of children of the root $\emptyset$ which are survivors and which become extinct, respectively; obviously,

$$\mathcal{E} = (S = 0) \quad \text{and} \quad \mathcal{E}^c = (S \geq 1).$$

We consider the "double" generating function G defined by: for $x, y \geq 0$,

$$G(x,y) = \mathbb{E}(x^S y^E \mid \mathcal{E}^c).$$

Lemma 2.5. *Let τ be a super-critical GW tree with offspring distribution μ satisfying (1.2) and extinction probability $q = \mathbb{P}(\mathcal{E}) \in]0,1[$. Set $\bar q := 1 - q$. Then, for any $x, y \geq 0$*

$$G(x,y) = \frac{f(\bar q x + q y) - f(qy)}{\bar q}. \tag{2.3}$$

Proof. Using the branching property, we obtain

$$\begin{aligned}
\mathbb{E}(x^S y^E \mid \mathcal{E}^c) &= \frac{\mathbb{E}(x^S y^E; S \geq 1)}{\mathbb{P}(S \geq 1)} \\
&= \frac{1}{\bar q} \sum_{n\geq 1} \mu(n) \left(\sum_{k=1}^{n} \binom{n}{k} \bar q^k x^k q^{n-k} y^{n-k} \right) \\
&= \frac{1}{\bar q} \sum_{n\geq 1} \mu(n) \left((\bar q x + qy)^n - (qy)^n\right) \\
&= \frac{f(\bar q x + qy) - f(qy)}{\bar q}.
\end{aligned}$$

□

Conditioned on the event $\mathcal{E}^c$, the GW tree τ is distributed as the following random tree τ^s:

(i) individuals of τ^s have type s or e;

(ii) the root of τ^s has type s;

(iii) an individual of type e produces only individuals of type e according to the sub-critical offspring distribution $\tilde{\mu}$ given by (2.2);

(iv) an individual of type s produces S individuals of type s, with $S \geq 1$, and E individuals of type e, the generating function $G(x, y)$ of (S, E) being given by (2.3); the order of the S and E individuals of type s and e respectively is uniform among the $\binom{E+S}{S}$ possible configurations.

In particular, conditioned on $\mathcal{E}^c$, the survival process $(z_h^s(\tau))_{h \geq 0}$ is a GW process whose offspring distribution $\hat{\mu}$ has generating function

$$f_{\hat{\mu}}(x) = \mathbb{E}(x^S \mid \mathcal{E}^c) = G(x, 1) = \frac{f(\bar{q}x + q) - q}{\bar{q}}.$$

The mean of $\hat{\mu}$ is $f'_{\hat{\mu}}(1) = f'(1) = m$. Since $\hat{\mu}(0) = f_{\hat{\mu}}(0) = 0$, the GW process with offspring distribution $\hat{\mu}$ is super-critical and does not become extinct $\mathbb{P}$-a.s.

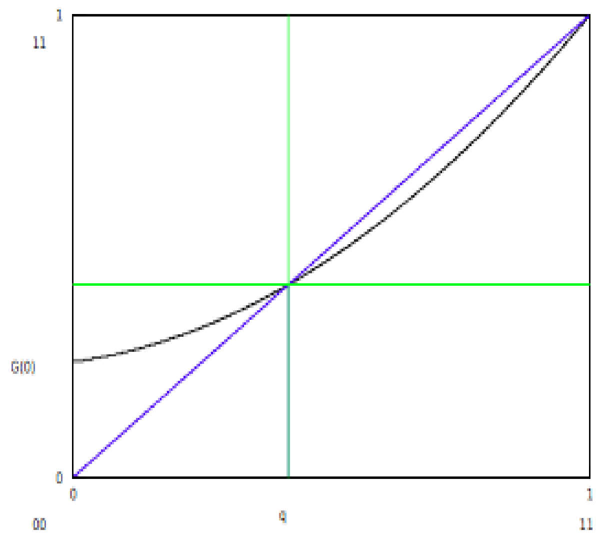

Figure 3: The generating function $f = f_\mu$: up to a scaling factor, $f_{\tilde{\mu}}$ is the "south-west" little square and $f_{\hat{\mu}}$ is the "north-east" little square.

3 Branching processes in random environment

In the previous section, we present a model for the size over time of a population which evolves in a fix environment. Indeed, the offspring distribution μ is always the same: the members of the population produce and die according to the same laws of chance. Furthermore, they do not interfere with one another. Unfortunately, natural processes of multiplication are often affected by many factors which introduce variations over time and also dependencies.

In this section, we take into account some variations of the offspring distributions over time. The first natural idea is to replace the distribution μ by a sequence $(\mu_n)_{n \geq 0}$ of fixed distributions; the probability measures $\mu_n, n \geq 0$, are the offspring distribution of members of generation n (or living at time n). Nevertheless, it is not reasonable to fix once and for all the reproduction rules; for instance, the development of an animal population is often affected by environmental factors such as weather conditions, food supply, an so forth... which cannot be predicted with certainty over a long period.

In this section, we consider branching processes $(Z_n)_{n\geq 0}$ developing in an environment which change stochastically in time and affect the reproductive behaviour of the population. The offspring distributions $\mu_0, \mu_1, \ldots$ are random variables, with values in the set of probability measures on $\mathbb{N}$. In some sense, there are two levels of randomness!

Several types of assumptions may be introduce about the distribution of the sequence $(\mu_n)_{n\geq 0}$. We consider here the case when they are i.i.d. random variables. It is a strong assumption... but the results are hard enough to prove!

3.1 Definitions

From now on, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote $\mathcal{G}$ the set of generating functions of probability measures on $\mathbb{N}$ and $\mathcal{B}(\mathcal{G})$ the σ-algebra of Borel sets in $\mathcal{G}$.

We consider a sequence $\bar{f} = (f_n)_{n\geq 0}$ of i.i.d. $\mathcal{G}$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that for any $n \geq 0$ and $\mathbb{P}$-almost all $\omega \in \Omega$, the probability measure μ_n^ω associated to f_n^ω is not a Dirac mass; it implies $f_n^\omega(x) = 1 \Leftrightarrow x = 1$.

The Galton-Watson process in the random environment $\bar{f}$ describes the evolution of the size Z_n of a particle population at time $n \geq 0$, which the following rules:

1. $Z_0 = 1$
2. each of the Z_n particles existing at time n produces offspring in accordance with the generating function f_n independently of the reproduction of other particles.

In particular, the random variable Z_1 has the generating function: $f_0(x) = \sum_{k\in\mathbb{N}} \mu_0(k)x^n$ (where μ_0 denotes the (random) probability measure with (random) generating function f_0); if $Z_n = k$, then Z_{n+1} is the sum of k independent random variables with (random) generating function f_n.

By the above descriptions, for any $x \geq 0$

$$\mathbb{E}\left(x_n^Z | Z_0, \ldots, Z_{n-1}, f_0, \ldots, f_{n-1}\right) = f_{n-1}(x)^{Z_{n-1}}$$

which yields

$$\mathbb{E}\left(x_n^Z | f_0, \ldots, f_{n-1}\right) = f_0(f_1(\ldots f_{n-1}(x)\ldots)).$$

We assume the two following conditions, which extend (1.2) in the random environment case:

$$\mathbb{P}(\mu_0(0) < 1) = 1 \quad \text{and} \quad \mathbb{P}(\mu_0(0) + \mu_0(1) < 1) > 0. \tag{3.1}$$

3.2 On the extinction probability

Given the environment $\bar{f} = (f_0, f_1, \cdots)$, the probability of extinction q_n at generation $n \geq 1$ is

$$q_n(\bar{f}) := \mathbb{P}(Z_n = 0 | f_0, \ldots, f_{n-1}) = f_0(f_1(\ldots f_{n-1}(0)\ldots)). \tag{3.2}$$

Notice that the sequence $(q_n(\bar{f}))_{n\geq 0}$ is non decreasing $\mathbb{P}$-a.s.; thus it converges $\mathbb{P}$-a.s. to some limit $q(\bar{f})$ which is called the *extinction probability conditioned on the environment* $\bar{f}$.
The probability of extinction at time n is

$$Q_n := \mathbb{P}(Z_n = 0) = \mathbb{E}(q_n(\bar{f})) = \mathbb{E}\Big(f_0(f_1(\ldots f_{n-1}(0)\ldots))\Big). \tag{3.3}$$

As above, the sequence $(Q_n)_{n\geq 0}$ converges to $Q := \mathbb{E}(q(\bar f))$.

Now we study the properties of these two quantities, by trying to "mimic" what exists in the fixed environment case.

Lemma 3.1. *The probability $q(\bar f)$ of extinction conditioned on the environment $\bar f$ satisfies*

$$\mathbb{P}\Big(q(\bar f)=1\Big) = 0 \text{ or } 1.$$

Proof. Letting $n\to+\infty$ in (3.2) yields, for $\mathbb{P}$-almost all $\bar f$

$$q(\bar f) = f_0\Big(q(\theta \bar f)\Big),$$

where θ is the shift operator on $\mathcal{G}^{\mathbb{N}}$ defined by $\theta\Big((f_i)_{i\geq 0}\Big) = (f_{i+1})_{i\geq 0}$. By iterating, we obtain, for any $n\geq 1$,

$$q(\bar f) = f_0\cdots f_{n-1}\Big(q(\theta^n \bar f)\Big) \quad \mathbb{P}\text{-}a.s.. \tag{3.4}$$

So $q(\bar f)=1$ is equivalent to $q(\theta^n\bar f)=1$ $\mathbb{P}$-a.s. for any $n\geq 1$, which implies that the event $(q(\bar f)=1)$ belongs to the asymptotic σ-algebra of the sequence $(f_n)_{n\geq 0}$. By the 0-1 law of Kolmogorov, this event $(q(\bar f)=1)$ as full or zero measure [(4)].

□

Here comes the first main result of this section

Theorem 3.2. *Assume that $\mathbb{E}\big(\log f_0'(1)\big)^+ < +\infty$. If $\mathbb{E}\Big(\log f_0'(1)\Big)\leq 0$ then $\mathbb{P}\Big(q(\bar f)=1\Big)=1$.*

In other words, conditioned on almost all environment $\bar f$, the process $(Z_n)_{n\geq 0}$ becomes extinct a.s.

Proof of Theorem 3.2. We assume $\mathbb{P}\Big(q(\bar f)<1\Big)>0$ and prove that $\mathbb{E}\Big(\log f_0'(1)\Big)>0$.

By Lemma 3.1, we have $\mathbb{P}\Big(q(\bar f)<1\Big)=1$. Thus, the quantity $h(\bar f) := -\log(1-q(\bar f))$ is defined $\mathbb{P}$-a.s and non negative; let us set

$$g(\bar f) := h(\bar f) - h(\theta\bar f) = -\log\Big(\frac{1-q(\bar f)}{1-q(\theta\bar f)}\Big).$$

Notice that, by Lemma 3.1 and convexity of f_0

$$g(\bar f) = -\log\Big(\frac{1-f_0(\theta q(\bar f))}{1-q(\theta\bar f)}\Big) \geq -\log f_0'(1). \tag{3.5}$$

Let us decompose g as $g=g_+-g_-$ with $g_+=\max(g,0)$ and $g_-=\max(-g,0)$.

(i) First, let us check that $0\leq \mathbb{E}(g_-(\bar f))\leq \mathbb{E}(\log f_0'(1))^+<+\infty$. Inequality (3.5) yields

$$\begin{aligned}
0\leq \mathbb{E}(g_-(\bar f)) &= \mathbb{E}(-g(\bar f); g(\bar f)\leq 0)\\
&= \mathbb{E}\left(\log\frac{1-q(\bar f)}{1-q(\theta\bar f)}; \frac{1-q(\bar f)}{1-q(\theta\bar f)}\geq 1\right)\\
&= \mathbb{E}\left(\log\frac{1-f_0(q(\theta\bar f))}{1-q(\theta\bar f)}; \frac{1-f_0(q(\theta\bar f))}{1-q(\theta\bar f)}\geq 1\right)\\
&\leq \mathbb{E}\left(\log f_0'(1); f_0'(1)\geq 1\right)\\
&= \mathbb{E}(\log f_0'(1))^+ < +\infty.
\end{aligned}$$

[4]The asymptotic σ-algebra of the sequence of random variables $(f_n)_{n\geq 0}$ is $\mathcal{F}_\infty := \cap_{n\geq 0}\sigma\{f_k \mid k\geq n\}$. The 0-1 law of Kolmogorov states that, when the $f_k, k\geq 0$, are independant, then $\mathcal{F}_\infty = \{\emptyset,\Omega\}$ $\mathbb{P}$-a.s.

Notice that the function h decomposes as $h(\bar f) = g(\bar f) + h(\theta \bar f)$; by iteration, we may write

$$h(\bar f) = S_n g(\bar f) + h(\theta^n \bar f)$$

with $S_n g(\bar f) := g(\bar f) + g(\theta \bar f) + \cdots + g(\theta^{n-1} \bar f)$ for any $n \geq 1$. We may write, for any $\mathbb{P}$-almost all $\bar f$

$$\frac{S_n g_+(\bar f)}{n} - \frac{S_n g_-(\bar f)}{n} = \frac{1}{n} h(\bar f) - \frac{1}{n} h(\theta^n \bar f). \tag{3.6}$$

(ii) We deduce $0 \leq \mathbb{E}(g_+(\bar f)) < +\infty$ and $\mathbb{E} g(\bar f) = 0$.

Namely, since the function h is non negative, identity (3.6) implies

$$\frac{S_n g_+(\bar f)}{n} \leq \frac{S_n g_-(\bar f)}{n} = \frac{1}{n} h(\bar f). \tag{3.7}$$

The ergodic theorem, applied first in inequality (3.7) and secondly in (3.6), yields

$$\mathbb{E}(g_+(\bar f)) \leq \mathbb{E}(g_-(\bar f)) \leq \mathbb{E}(\log f_0'(1))^+ < +\infty$$

and

$$\mathbb{E} g(\bar f) = \mathbb{E} g_+(\bar f) - \mathbb{E} g_-(\bar f) = \lim_{n \to +\infty} \frac{1}{n} h(\bar f) - \frac{1}{n} h(\theta^n \bar f).$$

The sequence $(h(\bar f)/n)_{n \geq 1}$ converges $\mathbb{P}$-a.s. to 0. It implies that $(h(\theta^n \bar f)/n)_{n \geq 1}$ also converges $\mathbb{P}$-a.s. to a constant. This constant is 0 since $h(\theta^n \bar f)$ and $h(\bar f)$ have the same distribution.

Finally $\mathbb{E}|g(\bar f)| < +\infty$ and $\mathbb{E} g(\bar f) = 0$.

(iii) Now, we check that $\mathbb{E}(\log f_0'(1)) \geq 0$.

As above for g_-, we may write

$$\begin{aligned}
0 \leq \mathbb{E}(\log f_0'(1))^- &= \mathbb{E}(-\log f_0'(1)); f_0'(1) \leq 1) \\
&\leq \mathbb{E}\left(-\log \frac{1 - f_0(q(\theta \bar f))}{1 - q(\theta \bar f)}; f_0'(1) \leq 1\right) \\
&= \mathbb{E}\left(-\log \frac{1 - q(\bar f))}{1 - q(\theta \bar f)}; f_0'(1) \leq 1\right) \\
&= \mathbb{E}\left(g(\bar f); f_0'(1) \leq 1\right) \\
. &\leq \mathbb{E}\left(g(\bar f); g(\bar f) \geq 0\right) \\
&= \mathbb{E}(g_+(\bar f)) \\
&\leq \mathbb{E}(\log f_0'(1))^+
\end{aligned}$$

(iv) We conclude, proving that $\mathbb{E}(\log f_0'(1)) > 0$.

Indeed, the equality $\mathbb{E}(\log f_0'(1)) = 0$ implies $\mathbb{E}\Big(g(\bar f) + \log f_0'(1)\Big) = 0$; consequently, by (3.5), the function $g(\bar f) + \log f_0'(1)$ equals 0 $\mathbb{P}$-a.s., which occurs if and only if $\mu_0\{0,1\} = 1$ $\mathbb{P}$-a.s. Contradiction with (3.1).

□

Theorem 3.3. *Assume* $\mathbb{E}\big(\log f_0'(1)\big)^+ < +\infty$ *and* $\mathbb{E}(-\log(1 - f_0(0)) < +\infty$.

If $\mathbb{E}\Big(\log f_0'(1)\Big) > 0$ *then* $\mathbb{P}\Big(q(\bar f) < 1\Big) = 1$. *In other words, conditioned on almost all random environment* $\bar f$, *the process* $(Z_n)_{n \geq 0}$ *survives with some strictly positive probability.*

Proof. Notice that $\mathbb{E}(\log f_0'(1))^- = \mathbb{E}(-\log f_0'(1); f_0'(1) \leq 1) \leq \mathbb{E}(-\log(1-f_0(0))) < +\infty$ and $q_n(\bar{f}) = f_0(\cdots f_{n-1}(0)) = f_0((q_{n-1}(\theta\bar{f})) < 1$ $\mathbb{P}$-a.s., since the f_i are non trivial (see (3.1)). Consequently

$$0 \leq -\log(1-q_n(\bar{f})) < +\infty \quad \mathbb{P}-\text{a.s.}.$$

We claim that in fact $0 \leq \mathbb{E}(-\log(1-q_n(\bar{f}))) < +\infty$. For $n \geq 0$ set $\nu_n := \mathbb{E}(-\log(1-q_n(\bar{f})))$ and notice that $0 \leq \nu_0 < +\infty$. We may write

$$\begin{aligned} -\log(1-q_n(\bar{f})) &= -\log 1 - f_0(q_{n-1}(\theta\bar{f})) \\ &= -\log \frac{1-f_0(q_{n-1}(\theta\bar{f}))}{1-q_{n-1}(\theta\bar{f}))} - \log(1-q_{n-1}(\theta\bar{f})) \\ &\leq \left(-\log \frac{1-f_0(q_{n-1}(\theta\bar{f}))}{1-q_{n-1}(\theta\bar{f}))}\right)^+ - \log(1-q_{n-1}(\theta\bar{f})) \\ &\leq -\log(1-f_0(0)) - \log(1-q_{n-1}(\theta\bar{f})) \end{aligned}$$

which yields $0 \leq \nu_n \leq \mathbb{E}(-\log(1-f_0(0))) + \nu_{n-1}$. Consequently $0 \leq \nu_n < +\infty$ by induction over n. It implies also that the non negative random variable $-\log \dfrac{1-f_0(q_{n-1}(\theta\bar{f}))}{1-q_{n-1}(\theta\bar{f}))}$ has finite expectation, denoted η_n, and that, for any $n \geq 0$,

$$\nu_n = \nu_{n-1} + \eta_n = \nu_0 + \sum_{i=1}^{n} \eta_i.$$

Assume that $\mathbb{P}\Big(q(\bar{f}) < 1\Big) < 1$; Lemma 3.1 implies $\mathbb{P}\Big(q(\bar{f}) < 1\Big) = 0$. Consequently, the sequence $(q_n(\bar{f}))_{n\geq 0}$ increases $\mathbb{P}$-a.s. to 1, and $\nu_n \uparrow +\infty$ by monotone convergence.

Since $q_{n-1}(\theta\bar{f}) \uparrow 1 \quad \mathbb{P}$-a.s., we also deduce that $-\log \dfrac{1-f_0(q_{n-1}(\theta\bar{f}))}{1-q_{n-1}(\theta\bar{f}))} \downarrow -\log f_0'(1)$. Consequently $(\eta_n)_n$ decreases to $\mathbb{E}(-\log f_0'(1))$. Since $\mathbb{E}(-\log f_0'(1))$ is negative, the sum $\sum_{i=1}^{n} \eta_i$ converges to $-\infty$ and so does $(\nu_n)_{n\geq 0}$. Contradiction.

□

By Theorems 3.2 and 3.3, following the classification of GW processes in the deterministic case, we classify the Galton-Watson process in random environment by the value of $\mathbb{E}\log f_0'(1)$ (when it exists, i.e. when $\mathbb{E}\big|\log f_0'(1)\big| < +\infty$). The GW process is

(i) *sub-critical* if $\mathbb{E}\Big(\log f_0'(1)\Big) < 0$;

(ii) *critical* if $\mathbb{E}\Big(\log f_0'(1)\Big) = 0$;

(iii) *super-critical* if $\mathbb{E}\Big(\log f_0'(1)\Big) > 0$ (and $\mathbb{E}(-\log 1 - f_0'(0)) < +\infty$.)

4 Probability of survival a up to time n for BPRE

In this chapter we state the main theorems concerning the behaviour of the sequence $(\mathbb{E}(q_n(\bar{f})))_{n\geq 0}$ as $n \to +\infty$.

Theorem 4.1. *(Critical case) [8] Suppose*

$$(i)\ 0 < \mathbb{E}(\log f'(1))^2 < +\infty, \quad (ii)\ \mathbb{E}\left(\frac{f''(1)}{f'(1)^2}(1+\log_+ f'(1))\right) < +\infty \quad (iii)\ \mathbb{E}\log f_0'(1) = 0.$$

Then, for some $0 < \beta < +\infty$,

$$\mathbb{P}(Z_n > 0) = \mathbb{E}(1 - q_n(\bar{f})) \sim \frac{\beta}{\sqrt{n}} \quad as \quad n \to +\infty.$$

This theorem was first proved by Kozlov [13] for i.i.d. random environment with linear fractional generating functions. Let us emphasize that $\mathbb{P}(Z_n > 0)$ converges to 0 slower than in the fixed environment.

The following statement illustrates the fact that random environment hypothesis may have deep influence on the behaviour of the survival probability up to time n. We omit its proof.

Theorem 4.2. *(sub-critical case) [9]*

1. **Strongly sub-critical case.**

   Suppose

   $$(i)\ \mathbb{E}(f_0'(1)\log_+ f_0'(1)) < +\infty \quad and \quad (ii)\ \mathbb{E}(f_0'(1)\log f_0'(1)) < 0.$$

   Then, there exists $0 < \beta_1 \leq 1$ *such that*

   $$\mathbb{P}(Z_n > 0) \sim \beta_1 \Big(\mathbb{E}(f_0'(1)\Big)^n \quad as \quad n \to +\infty.$$

2. **Intermediate sub-critical case.**

   Suppose

   $$(i)\ \mathbb{E}(f_0'(1)\log^2 f_0'(1)) < +\infty \quad (ii)\ \mathbb{E}((1+\log f_0'(1)|)f''(1)) < +\infty \quad (iii)\ \mathbb{E}(\log f_0'(1)) < 0$$

   and (iv) $\mathbb{E}(f_0'(1)\log f_0'(1)) = 0.$

   Then, there exists $0 < \beta_2 < +\infty$ *such that*

   $$\mathbb{P}(Z_n > 0) \sim \frac{\beta_2}{\sqrt{n}}\Big(\mathbb{E}(f_0'(1)\Big)^n \quad as \quad n \to +\infty.$$

3. **Weakly sub-critical case.**

   Suppose

   $$(i)\ \mathbb{E}\left(\frac{f_0''(1)}{f_0'(1)^{1-\epsilon}}\right) < +\infty, \quad (ii)\ \mathbb{E}\left(\frac{f_0''(1)}{f_0'(1)^{2-\epsilon}}\right) < +\infty, \quad (iii)\ \mathbb{E}(\log f_0'(1)) < 0$$

   and (iv) $0 < \mathbb{E}(f_0'(1)\log f_0'(1)) < +\infty.$

   Then, there exists $0 < \beta_3 < +\infty$ *and* $\gamma \in]0,1[$ *such that*

   $$\mathbb{P}(Z_n > 0) \sim \frac{\beta_2}{n^{3/2}}\gamma^n \quad as \quad n \to +\infty.$$

In the strongly sub-critical case, the condition $\mathbb{E}(f_0'(1)\log f_0'(1)) < 0$ implies $\mathbb{E}(\log f_0'(1)) < 0$ and also $\mathbb{E}f_0'(1) < 1$. Similarly, $\mathbb{E}f_0'(1) < 1$ in the intermediate sub-critical case.
Indeed, in this case, the function $\phi : t \mapsto \mathbb{E}(e^{t\log f_0'(1))})$ is well defined and convex on $[0,1]$ and satisfies $\phi(0) = 1, \phi(1) = \mathbb{E}f_0'(1), \phi'(0) = \mathbb{E}\ \log f_0'(1)$ and $\phi'(1) = \mathbb{E}\ f_0'(1)\log f_0'(1)$. Consequently,

- convexity implies $\phi'(0) \leq \phi'(1) < 0$, so that $\mathbb{E}\ \log f_0'(1) < 0$.
- $\phi(1) < \phi(0)$, so that $\mathbb{E}f_0'(1) < 1$.

Nevertheless, this quantity may be greater than 1 in the weakly sub-critical case.

For super-critical BPRE, there exist results similar to Kesten-Stigum' theorem (see Theorem **??**); they are based on a martingale argument. The reader is referred for instance to [15] and references therein.

5 BPRE: the critical case

In this chapter we present the main steps of the proof of Theorem 4.1, following [8].

5.1 The context: linear fractional generating functions

We focus our attention on the case when the generating functions defining the environment are linear fractional.

Definition 5.1. *The generating function f of a probability measure on $\mathbb{N}$ is linear fractional when there exists constants $0 \leq \alpha \leq 1, 0 \leq \beta < 1$ with $0 \leq \alpha + \beta \leq 1$ such that for all $x \in [0,1]$*

$$f(x) := 1 - \frac{\alpha}{1-\beta} + \frac{\alpha x}{1-\beta x}. \tag{5.1}$$

In other words, f is the generating function of the probability measure μ which is the convex combination $a\delta_0 + (1-a)\mathcal{G}_p$ of the Dirac mass δ_0 at 0 and the geometric distribution $\mathcal{G}_p$ of parameter $0 < p \leq 1$, with $0 \leq a \leq 1$ [5] (in this case, we set $\alpha = (1-a)p$ and $\beta = 1-p$. In particular $f'(1) = \dfrac{\alpha}{(1-\beta)^2}$ and $f''(1) = \dfrac{2\alpha\beta}{(1-\beta)^3}$. Then formula (5.1) may be written as

$$\forall x \in [0,1] \qquad 1 - f(x) = \left(\frac{1}{f'(1)(1-x)} + \frac{f''(1)}{2f'(1)^2})\right)^{-1}. \tag{5.2}$$

Thus, we consider here a sequence $\bar{f} = (f_n)_{n\geq 0}$ of i.i.d. random variables with values in the set of linear fractional generating functions; in other words, we consider a sequence $((\alpha_n, \beta_n))_{n\geq 0}$ of i.i.d. random variables with values in $[0,1] \times [0,1[$ and set

$$\forall x \in [0,1[\quad f_n(x) := 1 - \frac{\alpha_n}{1-\beta_n} + \frac{\alpha_n x}{1-\beta_n x}.$$

The starting point of the proof of KOZLOV is the following formula for the superposition of linear-fractional generating functions, based on (5.2): for any $0 \leq x \leq 1$, setting $\prod_n := \displaystyle\prod_{i=0}^{n-1} f_i'(1)$

$$1 - f_0(\cdots f_{n-1}(x)\cdots) = \left(\frac{1}{(1-x)\prod_n} + \sum_{j=0}^{n-1} \frac{1}{\prod_j}\frac{f_j''(1)}{2f_j'(1)^2}\right)^{-1}. \tag{5.3}$$

[5] the geometric distribution $\mathcal{G}_p$ is the probability mesure $\nu = (\nu_k)_{k\geq 1}$ on $\mathbb{N}^*$ such that $\nu_k = (1-p)^{k-1}p$ for any $k \geq 1$.

Let us set $X_j := \log f'_{j-1}(1)$, $\eta_j := f''_{j-1}(1)/2f'_{j-1}(1)^2$ for $j \geq 1$ and $S_0 = 0, S_n = X_1 + \cdots + X_n$. With these notations, the hypotheses of Theorem 4.1 become

$$(i)\ 0 < \mathbb{E}(X_1^2) < +\infty, \quad (ii)\ \mathbb{E}X_1 = 0 \quad (iii)\ \mathbb{E}\ \eta_1(1 + X_1^+) < +\infty.$$

Formula (5.3) at $x = 0$ implies

$$\begin{aligned} \bar{q}_n(\bar{f}) &:= 1 - q_n(\bar{f}) && (5.4)\\ &= \mathbb{P}(Z_n > 0 \mid f_0, \cdots, f_{n-1}) \\ &= 1 - q_n(\bar{f}) && (5.5)\\ &= 1 - f_0(\cdots f_{n-1}(0) \cdots) \\ &= \left(e^{-S_n} + \sum_{i=0}^{n-1} \eta_{i+1} e^{-S_i}. \right)^{-1} && (5.6) \end{aligned}$$

This yields

$$\begin{aligned} \bar{Q}_n &:= \mathbb{P}(Z_n > 0) \\ &= \mathbb{E}(\bar{q}_n(\bar{f})) \\ &= \mathbb{E}\left(\left(e^{-S_n} + \sum_{i=0}^{n-1} \eta_{i+1} e^{-S_i} \right)^{-1} \right). && (5.7) \end{aligned}$$

The expression (5.7) of $\bar{q}_n(\bar{f})$ allows us to introduce the following notation

$$\bar{q}_\infty(\bar{f}) := \left(\sum_{i=0}^{+\infty} \eta_{i+1} e^{-S_i} \right)^{-1}. \tag{5.8}$$

First, notice that $\bar{q}_\infty(\bar{f})$ exists $\mathbb{P}$-a.s. in $[0, +\infty]$. It is natural to expect that the sequence $(\bar{q}_n(\bar{f}))_{n\geq 0}$ converges to $\bar{q}_\infty(\bar{f})$; unfortunately since the X_i have 0-mean, the terms $e^{-S_i}, 1 \leq i \leq n$, oscillate between 0 and $+\infty$ [6] In fact, the sequence $(\bar{q}_n(\bar{f}))_{n\geq 0}$ does converge with respect to another probability measure $\widehat{\mathbb{P}}_x$ under which the terms e^{-S_i} are all bounded from above. The probability measures $\widehat{\mathbb{P}}_x$ are defined in the next section.

To study the behaviour of $\mathbb{P}(Z_n > 0)$ as $n \to +\infty$, we use results from random walk theory. We just state it here, we present the tools and elements of the proofs in the next section.

5.2 Centered random walks on $\mathbb{R}$: fluctuations and asymptotic behaviour of the minimum

We consider in this section a sequence $(X_i)_{i\geq 1}$ of identically distributed and independent (i.i.d.) real valued random variables, with distribution μ, such that

$$\mathbb{E}(X_i^2) < +\infty \quad \text{and} \quad \mathbb{E}(X_i) = 0. \tag{5.9}$$

[6] Indeed, $\liminf_{n\to+\infty} \frac{S_n}{\sqrt{n}} = -\infty$ and $\limsup_{n\to+\infty} \frac{S_n}{\sqrt{n}} = +\infty$ $\mathbb{P}$-a.s.,. The main steps of the arguments are the following ones: by the central limit theorem, $\lim_{n\to+\infty} \mathbb{P}\Big(\frac{S_n}{\sqrt{n}} > c\Big) = \frac{1}{\sqrt{2\pi}} \int_{\frac{c}{\sigma}}^{+\infty} e^{-u^2/2} du$, for any $c > 0$,; consequently $\mathbb{P}\Big(\limsup_{n\to+\infty}\Big(\frac{S_n}{\sqrt{n}} > c\Big)\Big) > 0$. Notice that the event $\limsup_{n\to+\infty}\Big(\frac{S_n}{\sqrt{n}} > c\Big)$ belongs to the asymptotic σ-algebra associated with the sequence $(X_n)_{n\geq 1}$. The 0-1 law of Kolmogorov yields $\mathbb{P}\Big(\limsup_{n\to+\infty}\Big(\frac{S_n}{\sqrt{n}} > c\Big)\Big) = 1$ and $\mathbb{P}\Big(\limsup_{n\to+\infty} \frac{S_n}{\sqrt{n}} \geq c\Big) = 1$; letting $c \to +\infty$, one obtains $\mathbb{P}\Big(\limsup_{n\to+\infty} \frac{S_n}{\sqrt{n}} = +\infty\Big) = 1$. Similarly the event $\Big(\liminf_{n\to+\infty} \frac{S_n}{\sqrt{n}} = -\infty\Big)$ has full measure.

We also propose some classical assumption in this theory which ensures that the random walk $(S_n)_{n\geq 0}$ does not "stay" in a proper subgroup of $\mathbb{R}$:

$$\textit{The group } G \textit{ generated by the support of } \mu \textit{ is dense in } \mathbb{R}. \tag{5.10}$$

For any $x \in \mathbb{R}$, the process $(S_n)_{n\geq 0}$ defined by $S_0 = x$ and $S_n = x + X_1 + \cdots + X_n$ for $n \geq 1$ is the random walk on $\mathbb{R}$ starting from x and with law μ. It is a Markov chain on $\mathbb{R}$ whose transition kernel P is defined by: for any Borel set $B \subset \mathbb{R}$ and any $x \in \mathbb{R}$,

$$P(x, B) = \int_{\mathbb{R}} 1_B(x + y)\mu(dy).$$

We denote $\mathcal{B}(\mathbb{R})$ the σ-algebra of Borel sets on $\mathbb{R}$ and $((\mathbb{R})^{\otimes \mathbb{N}}, \mathcal{B}(\mathbb{R})^{\otimes \mathbb{N}}, (\mathcal{F}_n)_{n\geq 0}, \theta, (\mathbb{P}_x)_{x\in\mathbb{R}})$ the canonical space associated to the random walk $(S_n)_{n\geq 0}$. To simplify the notations, we set $\mathbb{P} = \mathbb{P}_0$.

Under the hypotheses (5.9), the random walk $(S_n)_{n\geq 0}$ oscillates $\mathbb{P}$-almost surely, for any $x \in \mathbb{R}$; namely

$$\liminf_{n\to+\infty} S_n = -\infty \quad \text{and} \quad \limsup_{n\to+\infty} S_n = +\infty \qquad \mathbb{P}_x\text{-a.s.} \tag{5.11}$$

Let $(\mathbf{l}_k)_{k\geq 0}$ be the sequence of strictly descending ladder epochs of $(S_n)_{n\geq 0}$ defined by: $\mathbf{l_0} = 0$ and for any $k \geq 0$,

$$\mathbf{l}_{k+1} := \inf\{n > \mathbf{l}_k \mid S_n < S_{\mathbf{l}_k}\}.$$

By (5.11), these random variables are $\mathbb{P}$-a.s. finite; furthermore, for any $n \geq 0$

$$\mathbf{l}_{n+1} = \mathbf{l}_n + \mathbf{l}_1 \circ \theta^{\mathbf{l}_n}.$$

It is a classical result on random walks [6] that under hypotheses (5.9),

1. $(\mathbf{l}_{k+1} - \mathbf{l}_k)_{k\geq 0}$ is a sequence of i.i.d. random $\mathbb{N}$-valued random variables with the same distribution as $\mathbf{l}_1$,

2. the sequence $(S_{\mathbf{l}_n})_{n\geq 0}$ is a random walk on $\mathbb{R}^{*-}$, with i.i.d. increments $S_{\mathbf{l}_{n+1}} - S_{\mathbf{l}_n}$ whose distribution is the one of $S_{\mathbf{l_1}}$,

3. $\mathbb{E}(\mathbf{l}_1) = +\infty$ and $-\infty < \mathbb{E}(S_{\mathbf{l_1}}) < 0$.

For any $x \geq 0$, let τ_x be the first time at which the process $(S_n)_{n\geq 0}$, starting from 0, reaches the half-line $]-\infty, -x[$: more precisely,

$$\tau_x := \inf\{n \geq 1 \mid S_n < -x\}.$$

The random variables $\tau_x, x \geq 0$, are stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Notice that $\tau_0 = \mathbf{l}_1$ $\mathbb{P}_0$-a.s. and, more generally, any stopping time $\tau_x, x \geq 0$, is $\mathbb{P}$-a.s. a strictly descending ladder epoch of the random walk $(S_n)_{n\geq 0}$.

It is of interest to consider the restriction P^+ to $\mathbb{R}^+$ of the transition kernel P. It is defined by: for any $x \in \mathbb{R}^+$ and any Borel set $B \subset \mathbb{R}^+$,

$$P^+(x, B) = \int_{-x}^{+\infty} 1_B(x + y)\mu(y) = \mathbb{P}(x + X_1 \in B, \tau_x > 1).$$

This is a sub-markovian kernel , i.e. $P^+1 \leq 1$, whose iterates $(P^+)^n, n \geq 1$, are given by:

$$(P^+)^n(x, B) = \mathbb{P}(x + S_n \in B, \tau_x > n) = \mathbb{P}(x + S_n \in B; m_n \geq -x).$$

A function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is P^+-harmonic when it satisfies the equality: for any $x \geq 0$

$$h(x) = P^+h(x) = \mathbb{E}(h(x + S_1); \tau_x > 1).$$

Such a P^+-harmonic exists and satisfies some important properties.

Proposition 5.2. *Assume conditions (5.9) and (5.10) hold and let h be the function from $\mathbb{R}$ to $\mathbb{R}^+$ defined by: for any $x \in \mathbb{R}$*

$$h(x) = \begin{cases} -\mathbb{E}(S_{\tau_x}) & \text{when} \quad x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

1. *The function h is P_+-harmonic on $\mathbb{R}$.*
2. $h(x) = \lim_{n\to+\infty} \mathbb{E}(S_n; \tau_x > n) \lim_{n\to+\infty} \mathbb{E}(S_n; m_n \geq -x)$.
3. *The function h is increasing on $\mathbb{R}$ and*
$$\lim_{x\to+\infty} \frac{h(x)}{x} = 1. \tag{5.12}$$
4. *There exists $c \geq 1$ such that for any $x \geq 0$ and $y \in \mathbb{R}$,*
$$h(x + y) \leq c\ h(x)\ (1 + y^+). \tag{5.13}$$

Here comes the main ingredient coming from random walks theory that we use in the sequel. First, we set $m_n := \min(S_0, S_1, \cdots, S_n)$ for any $n \geq 0$ and notice that, for any $x \geq 0$,

$$\mathbb{P}_x(m_n \geq 0) = \mathbb{P}_x(\tau_0 > n) = \mathbb{P}(m_n \geq -x) = \mathbb{P}(\tau_x > n).$$

Theorem 5.3. *Under the conditions (5.9) and (5.10) there exist constants $c_0, c_1 > 0$ such that, for any $x \geq 0$*

$$\forall n \geq 1 \qquad \mathbb{P}(m_n \geq -x) \leq c_0 \frac{h(x)}{\sqrt{n}} \tag{5.14}$$

and

$$\mathbf{m}_n(x) := \mathbb{P}(m_n \geq -x) \quad \overset{n\to+\infty}{\sim} \quad c_1 \frac{h(x)}{\sqrt{n}} \tag{5.15}$$

where h is the function defined in Proposition 5.2.

We refer to Section 6 for comments and elements of the proofs of these two statements.
To prove Theorem 4.1, we follow the approach of J. Geiger and G. Kersting [8]. In the next section, we construct a family of new probability measures $\widehat{\mathbb{P}}_x$ on $(\Omega, \mathcal{T})$. For any $x \geq 0$, the probability measure $\widehat{\mathbb{P}}_x$ takes into account the event $(m_n \geq -x)$; it allows to control a.s. each term of the quantity $e^{-S_n} + \sum_{i=0}^{n-1} \eta_{i+1} e^{-S_i}$ which appears in the expression (5.4) of $q_n(\bar{f})$.

5.3 The measure probability $\widehat{\mathbb{P}}_x$ on $(\Omega, \mathcal{T})$

Since the function h is P_+-harmonic on $\mathbb{R}^+$, it gives rise to a Markov kernel P_+^h on $\mathbb{R}^+$ defined by

$$P_+^h \phi = \frac{1}{h} P_+(h\phi)$$

for any bounded measurable function ϕ on $\mathbb{R}^+$. The kernels P_+ and P_+^h are related to the stopping times τ by the following identity: for any $x \geq 0$ and $n \geq 1$,

$$\begin{aligned}
(P_+^h)^n \phi(x) &= \frac{1}{h(x)} P_+^n(h\phi)(x) \\
&= \frac{1}{h(x)} \mathbb{E}_x \left(h\phi(S_n); \tau > n \right) \\
&= \frac{1}{h(x)} \mathbb{E}_x \left(h\phi(S_n); m_n \geq 0 \right) \\
&= \frac{1}{h(x)} \mathbb{E} \left(h\phi(x + S_n); m_n \geq -x \right).
\end{aligned}$$

This new Markov chain with kernel P_+^h allows us to change the measure on the canonical path space $((\mathbb{R})^{\otimes \mathbb{N}}, \sigma(S_n : n \geq 0), \theta)$ of the random walk $(S_n)_{n\geq 0}$ from $\mathbb{P}$ to the measure $\widehat{\mathbb{P}}_x$ characterized by the property

$$\widehat{\mathbb{E}}_x[\varphi(S_0, \ldots, S_k)] = \frac{1}{h(x)} \mathbb{E}_x[\varphi(S_0, \ldots, S_k) h(S_k);\ m_k \geq 0] \tag{5.16}$$

for any positive Borel function φ on $(\mathbb{X} \times \mathbb{R})^{k+1}$.

By the Markov property, for any $0 \leq k \leq n$, we may write

$$\mathbb{E}_x \left(\varphi(S_0, \ldots, S_k); m_n \geq 0 \right) = \mathbb{E}_x(\varphi(S_0, \ldots, S_k) \mathbf{m}_{n-k}(S_k); m_k \geq 0). \tag{5.17}$$

Consequently, the dominated convergence theorem and (5.17) yield, for any bounded function φ with compact support,

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{E}_x[\varphi(S_0, \ldots, S_k) | m_n \geq 0] &= \frac{1}{h(x)} \mathbb{E}_x[\varphi(S_0, \ldots, S_k) h(S_k); m_k \geq 0] \\
&= \widehat{\mathbb{E}}_x[\varphi(S_0, \ldots, S_k)], \qquad (5.18)
\end{aligned}$$

which clarifies the interpretation of $\widehat{\mathbb{P}}_x$.

Now we formalize in three steps the construction of a new probability measure, denoted again $\widehat{\mathbb{P}}_x$, for each $x \geq 0$, but defined this time on the bigger σ-algebra $\sigma(f_n, Z_n : n \geq 0)$. Retaining the notations from the previous parts, the measure $\widehat{\mathbb{P}}_x$ is characterized by properties (5.16), (5.19) and (5.20).

Step 1. The marginal distribution of $\widehat{\mathbb{P}}_x$ on $\sigma(S_n : n \geq 0)$ is $\widehat{\mathbb{P}}_x$ characterized by the property (5.16).

Step 2. For any $n \geq 0$, the conditional distribution of $\bar{f}$ under $\widehat{\mathbb{P}}_x$ given $S_0 = s_0 = x, \ldots, S_n = s_n$ is given by:

$$\begin{aligned}
&\widehat{\mathbb{P}}_x(f_0 \in A_0, \cdots, f_n \in A_n \mid S_0 = s_0, \cdots, S_n = s_n) \\
&\qquad = \mathbb{P}(f_k \in A_k, 0 \leq k \leq \mid S_i = s_i, 0 \leq i \leq n),
\end{aligned} \tag{5.19}$$

for any Borel subset $A_0, \ldots, A_n$ of generating functions and almost all $(s_i)_{0\leq i\leq n}$ with respect to the law of $(S_0, \ldots, S_n)$ under $\mathbb{P}$ (and also under $\widehat{\mathbb{P}}_x$ since, by formula (5.16), the probability measure $\widehat{\mathbb{P}}_x$ is absolutely continuous with respect to $\mathbb{P}$ on each σ-algebra $\sigma(S_0, \ldots, S_n)$).

Step 3. The conditional distribution of $(Z_n)_{n\geq 0}$ under $\widehat{\mathbb{P}}_x$ given $f_0, \ldots$ is the same as under $\mathbb{P}$, namely

$$\widehat{\mathbb{E}}_x \left(x^{Z_n} | Z_0, \ldots, Z_{n-1}, f_0, \ldots, f_{n-1} \right) = \mathbb{E} \left(x^{Z_n} | Z_0, \ldots, Z_{n-1}, f_0, \ldots, f_{n-1} \right) = f_{n-1}(s)^{Z_{n-1}}. \tag{5.20}$$

5.4 Proof of Theorem 4.1

We follow J. Geiger and G. Kersting's approach. First, notice that the quantity $\mathbb{P}(Z_n \neq 0)$ equals $\mathbb{P}_x(Z_n \neq 0)$ for any $x \geq 0$; thus we fix $x \geq 0, \rho > 1$ and $0 \leq k \leq n$ and decompose $\mathbb{P}(Z_n \neq 0)$ as

$$\mathbb{P}(Z_n \neq 0) = A_n(x) + B_n(x,\rho) + C_n(x,\rho,k) - D_n(x,\rho,k), \tag{5.21}$$

where

- $A_n(x) = \mathbb{P}_x(Z_n \neq 0, m_n < 0)$;
- $B_n(x,\rho) = \mathbb{P}_x(Z_n \neq 0, m_n \geq 0) - \mathbb{P}_x(Z_n \neq 0, m_{\rho n} \geq 0)$;
- $C_n(x,\rho,k) = \mathbb{P}_x(Z_k \neq 0, m_{\rho n} \geq 0)$;
- $D_n(x,\rho,k) = \mathbb{P}_x(Z_k \neq 0, Z_n = 0, m_{\rho n} \geq 0)$.

We separate the proof in 5 steps.

1. The sequence $(\widehat{\mathbb{P}}_x(Z_k \neq 0))_{k\geq 0}$ converges to some limit $v(x) > 0$.
2. The quantity $A(x) := \limsup_{n\to+\infty} \sqrt{n}A_n(x)$. tends to 0 when $x \to +\infty$.
3. For any $x \geq 0$, the quantity $B(x,\rho) := \limsup_{n\to+\infty} \sqrt{n}B_n(x,\rho)$ tends to 0 when $\rho \to +1$.
4. The sequence $\big(\sqrt{n}C_n(x,\rho,k)\big)_{n\geq 0}$ converges to $c_1 \dfrac{h(x)}{\sqrt{\rho}} \widehat{\mathbb{P}}_x(Z_k \neq 0)$ as $n \to +\infty$.
5. For any $x \geq 0$ and $\rho > 1$ the quantity $D(x,\rho,k) := \limsup_{n\to+\infty} \sqrt{n}D_n(x,\rho,k)$ tends to 0 when $k \to +\infty$.

The assertion arrives, letting first $k \to +\infty$, then $\rho \to 1$ and at last $x \to +\infty$.
Step 1. First, we state the following lemmas. Their proofs are postponed in the next subsection.

Lemma 5.4. *For any $x \geq 0$,* $\displaystyle \widehat{\mathbb{E}}_x \sum_{k=0}^{+\infty} e^{-S_k}\eta_{k+1} < +\infty.$

Lemma 5.5. *For any $x \geq 0$, the sequence $(\bar{q}_n)_{n\geq 1}$ converges to $\bar{q}_\infty$ in $\mathbb{L}^1(\widehat{\mathbb{P}}_x)$; in particular*

$$\lim_{k\to+\infty} \widehat{\mathbb{P}}_x(Z_k \neq 0) = \widehat{\mathbb{E}}_x \bar{q}_\infty, \tag{5.22}$$

where $\bar{q}_\infty$ is defined in (5.8).

From Lemma 5.5, it is obvious that $\widehat{\mathbb{E}}_x(\bar{q}_\infty) \leq 1$. On the other hand, the expression of $\bar{q}_\infty$ combined with Lemma 5.4 yields $\widehat{\mathbb{E}}_x(\bar{q}_\infty) > 0$. In other words, for any $x \geq 0$,

$$0 < v(x) := \widehat{\mathbb{E}}_x(\bar{q}_\infty) < +\infty.$$

□

Step 2. We may write $A_n(x) = \mathbb{E}_x\left(\mathbb{E}\left(Z_n \neq 0 | f_0, \dots, f_{n-1}\right); m_n \leq 0\right) = \mathbb{E}_x\left(\bar{q}_n; m_n \leq 0\right)$.
Notice that $\bar{q}_k = \mathbb{P}(Z_k > 0 | f_0, \dots, f_{k-1}) \leq \mathbb{E}(Z_k | f_0, \dots, f_{k-1}) = e^{S_k}$ for any $0 \leq k \leq n-1$, so that $\bar{q}_n = \min_{0 \leq k \leq n-1} \bar{q}_k \leq \exp(m_n)$. This yields

$$
\begin{aligned}
\mathbb{P}_x(Z_n \neq 0, m_n < 0) &= \mathbb{P}(Z_n \neq 0, m_n < -x) \\
&\preceq \mathbb{E}[\exp(m_n); m_n < -x] \\
&\leq \sum_{k=x}^{+\infty} e^{-k} \mathbb{P}(-k \leq m_n < -k+1) \\
&\leq \sum_{k=x}^{+\infty} e^{-k} \mathbb{P}(m_n \geq -k) \\
&\preceq \frac{1}{\sqrt{n}} \sum_{k=x}^{+\infty} (k+1) e^{-k}.
\end{aligned} \tag{5.23}
$$

Consequently $A(x) = \limsup_{n \to +\infty} \sqrt{n} \mathbb{P}_x(Z_n \neq 0, m_n < 0) \longrightarrow 0$ as $x \to +\infty$.

□

Step 3. By (5.15),

$$
\begin{aligned}
0 \leq B_n(x, \rho) &= \mathbb{P}(Z_n \neq 0, m_n \geq -x) - \mathbb{P}(Z_n \neq 0, m_{\rho n} \geq -x) \\
&= \mathbb{P}_x(Z_n \neq 0, n < \tau \leq \rho n) \\
&\leq \mathbb{P}_x(n < \tau \leq \rho n) \\
&= \mathbb{P}_x(\tau > n) - \mathbb{P}_x(\tau > \rho n) \\
&\sim c_1 \frac{h(x)}{\sqrt{n}} \left(1 - \frac{1}{\sqrt{\rho}}\right) \quad \text{as} \quad n \to +\infty
\end{aligned}
$$

so that $B(x, \rho) = \limsup_{n \to +\infty} \sqrt{n} B_n(x, \rho) \leq c_1 h(x) \left(1 - \frac{1}{\sqrt{\rho}}\right) \overset{\rho \to 1}{\longrightarrow} 0$.

□

Step 4. Fix $0 \leq k \leq n$. By (5.17)

$$
\begin{aligned}
\mathbb{P}_x\left(Z_k \neq 0, m_{\rho n} \geq 0\right) &= \mathbb{P}_x\left(Z_k \neq 0, m_k \geq 0, \mathbf{m}_{\rho n - k}(S_k)\right) \\
&= \mathbb{E}_x\left(\mathbb{E}(1_{[Z_k \neq 0]} 1_{[m_k \geq 0]} \mathbf{m}_{\rho n - k}(S_k) \mid f_0 \dots, f_{k-1})\right) \\
&= \mathbb{E}_x\left(\mathbb{P}(Z_k \neq 0 | f_0 \dots, f_{k-1}) 1_{[m_k \geq 0]} \mathbf{m}_{\rho n - k}(S_k)\right) \\
&= \mathbb{E}_x\left(\bar{q}_k, m_{\rho n} \geq 0\right) \\
&= \mathbb{E}_x\left(\bar{q}_k \mid m_{\rho n} \geq 0\right) \mathbb{P}_x(m_{\rho n} \geq 0).
\end{aligned} \tag{5.24}
$$

As a direct consequence, using (5.18),

$$
\lim_{n \to +\infty} \mathbb{P}_x(Z_k \neq 0 | m_{\rho n} \geq 0) = \lim_{n \to +\infty} \mathbb{E}_x\left(\bar{q}_k | m_{\rho n} > 0\right) = \widehat{\mathbb{E}}_x\left(\bar{q}_k\right) = \widehat{\mathbb{P}}_x(Z_k \neq 0).
$$

Thus, equivalence (5.15) yields $\qquad \lim_{n \to +\infty} \sqrt{n} C_n(x, \rho, k) = c_1 \frac{h(x)}{\sqrt{\rho}} \widehat{\mathbb{P}}_x(Z_k \neq 0).$

□

Step 5. We may write, using equality (5.24)

$$\begin{aligned}
D_n(x,\rho,k) &= \mathbb{P}_x(Z_k \neq 0, Z_n = 0, m_{\rho n} \geq 0) \\
&= \mathbb{P}_x(Z_k \neq 0, m_{\rho n} \geq 0) - \mathbb{P}_x(Z_n \neq 0, m_{\rho n} \geq 0) \\
&= \mathbb{E}_x\left(\bar{q}_k, m_{\rho n} \geq 0\right) - \mathbb{E}_x\left(\bar{q}_n, m_{\rho n} \geq 0\right) \\
&= \mathbb{E}_x\Big((\bar{q}_k - \bar{q}_n)\mathbf{m}_{(\rho-1)n}(S_n); m_n \geq 0\Big) \\
&\preceq \frac{1}{\sqrt{(\rho-1)n}}\frac{1}{h(x)}\mathbb{E}_x\left((\bar{q}_k - \bar{q}_n)h(S_n); m_n \geq 0\right).
\end{aligned}$$

Since $1_{[m_n \geq 0]}$ and $h(S_n)$ are $\sigma(S_0,\ldots,S_n)$-measurable, we observe that

$$\begin{aligned}
\sqrt{n}D_n(x,\rho,k) &\preceq \frac{1}{\sqrt{\rho-1}}\widehat{\mathbb{E}}_x\left(\bar{q}_k - \bar{q}_n\right) \\
&= \frac{1}{\sqrt{\rho-1}}\Big(\widehat{\mathbb{P}}_x(Z_k \neq 0) - \widehat{\mathbb{P}}_x(Z_n \neq 0)\Big) \qquad (5.25)
\end{aligned}$$

so that

$$D(x,\rho,k) := \limsup_{n\to+\infty}\sqrt{n}D_n(x,\rho,k) \preceq \frac{1}{\sqrt{\rho-1}}\Big(\widehat{\mathbb{P}}_x(Z_k \neq 0) - v(x)\Big) \stackrel{k\to+\infty}{\longrightarrow} 0.$$

□

Combining expression (5.21) and the 5 steps above, we obtain for any $x \geq 0, \rho > 1$ and $k \geq 1$,

$$\begin{aligned}
c_1 h(x)v(x) - D(x,\rho,k) &\leq \liminf_{n\to+\infty}\sqrt{n}\mathbb{P}(Z_n \neq 0) \\
&\leq \limsup_{n\to+\infty}\sqrt{n}\mathbb{P}(Z_n \neq 0) \\
&\leq c_1 h(x)v(x) + A(x) + B(x,\rho) + D(x,\rho,k).
\end{aligned}$$

By Lemma 5.4, letting first $k \to +\infty$ then $\rho \to 1$, we obtain

$$c_1 h(x)v(x) \leq \liminf_{n\to+\infty}\sqrt{n}\mathbb{P}(Z_n \neq 0) \leq \limsup_{n\to+\infty}\sqrt{n}\mathbb{P}(Z_n \neq 0) \leq c_1 h(x)v(x) + A(x).$$

Since $v(x) > 0, h(x) > 0$ and $A(x) < +\infty$ for any $x \geq 0$, we obtain

$$0 < \liminf_{n\to+\infty}\sqrt{n}\mathbb{P}(Z_n \neq 0) \leq \limsup_{n\to+\infty}\sqrt{n}\mathbb{P}(Z_n \neq 0) < +\infty.$$

Finally, since $\lim_{x\to+\infty} A(x) = 0$, we conclude that both limits

$$\lim_{n\to+\infty}\sqrt{n}\mathbb{P}(Z_n \neq 0) \quad \text{and} \quad \lim_{x\to+\infty} c_1 h(x)v(x)$$

exist, coincide and belong to $]0,+\infty[$.

□

5.5 Proofs of Lemmas

Proof of Lemma 5.4. First, we get rid of the terms $\eta_{k+1}, k \geq 0$; by (5.13), for any $x \geq 0$ and $k \geq 0$, one may write, $\mathbb{P}$-a.s. on the event $(m_k \geq -x)$,

$$h(x + S_{k+1}) = h(x + S_k + X_{k+1}) \leq c\, h(x + S_k)(1 + X_{k+1}^+).$$

Hence

$$\begin{aligned}
\widehat{\mathbb{E}}_x(e^{-S_k}\eta_{k+1}) &= \frac{1}{h(x)}\mathbb{E}(e^{-S_k}\eta_{k+1}h(x+S_{k+1}); m_{k+1}\geq -x) \\
&\leq \frac{1}{h(x)}\mathbb{E}(e^{-S_k}\eta_{k+1}h(x+S_k)(1+cX_{k+1}^+); m_k\geq -x) \\
&= \frac{1}{h(x)}\mathbb{E}(e^{-S_k}h(x+S_k); m_k\geq -x)\times\mathbb{E}(\eta_{k+1}(1+cX_{k+1}^+)) \qquad (5.26)\\
&\preceq \widehat{\mathbb{E}}_x(e^{-S_k}). \qquad (5.27)
\end{aligned}$$

Thus, it suffices to prove that $\widehat{\mathbb{E}}_x\sum_{k=0}^{+\infty}e^{-S_k}<+\infty$ for any $x\geq 0$. By (5.12), there exists $c>0$ such that $h(x)\leq c\,x$; hence, we may write, for some $\lambda\in]0,1[$ and $c(\lambda)>0$,

$$\begin{aligned}
\widehat{\mathbb{E}}_x\left(\sum_{n=0}^{+\infty}e^{-S_n}\right)] &\leq 1+\frac{1}{h(x)}\sum_{n=1}^{+\infty}\mathbb{E}\left(e^{-(x+S_n)}h(S_n); x+S_0\geq 0,\dots,x+S_n\geq 0\right)\\
&\leq 1+\frac{c}{h(x)}\sum_{n=1}^{+\infty}\mathbb{E}\left(S_n e^{-(x+S_n)}; x+S_0\geq 0,\dots,x+S_n\geq 0\right)\\
&\leq 1+\frac{c(\lambda)}{h(x)}\sum_{n=1}^{+\infty}\mathbb{E}\left(e^{-(x+\lambda S_n)}; x+S_0\geq 0,\dots,x+S_n\geq 0\right)\\
&\leq 1+\frac{c(\lambda)}{h(x)}\sum_{n=1}^{+\infty}\mathbb{E}\left(e^{-\lambda(x+S_n)}; x+S_1\geq 0,\dots,x+S_n\geq 0\right). \qquad (5.28)
\end{aligned}$$

From now on, we set, for any $x\geq 0$,

$$\Phi(x):=\sum_{n=1}^{+\infty}\mathbb{E}\left(e^{-\lambda(x+S_n)}; x+S_1\geq 0,\dots,x+S_n\geq 0\right).$$

The function Φ is increasing on $\mathbb{R}^+$ and finite for any $x\geq 0$. Indeed, it suffices to check that

1. $\Phi(0)<+\infty$;
2. there exists $\delta>0$ such that:

$$\Phi(x)<+\infty\Rightarrow\Phi(x+\delta)<+\infty. \qquad (5.29)$$

To prove the first assertion, we use the so called "duality principle": the vectors $(X_1, X_1+X_2,\cdots,X_1+\cdots+X_n)$ and $(X_n, X_n+X_{n-1},\cdots,X_n+\cdots+X_1)$ have the same distribution. It yields

$$\begin{aligned}
\Phi(0) &= \sum_{n=1}^{+\infty}\mathbb{E}\left(e^{-\lambda S_n}; X_n\geq 0, X_n+X_{n-1}\geq 0,\dots,X_n+\cdots+X_1\geq 0\right)\\
&= \sum_{n=1}^{+\infty}\mathbb{E}\left(e^{-\lambda S_n}; S_n\geq S_{n-1}, S_n\geq S_{n-2},\dots,S_n\geq 0\right)\\
&= \sum_{n=1}^{+\infty}\mathbb{E}\left(e^{-\lambda S_n}; \exists k\geq 1\mid n=\ell_k\right)\\
&= \sum_{k\geq 1}\mathbb{E}\left(e^{-\lambda S_{\ell_k}}\right), \qquad (5.30)
\end{aligned}$$

where $\ell_k, k \geq 1$, is the k-th ascending ladder epoch of the random walk $(S_n)_{n\geq 0}$ defined by: $\ell_0 = 0$ and for $k \geq 1$,

$$\ell_k := \inf\{n > \ell_{k-1} \mid S_n \geq S_{\ell_{k-1}}\}.$$

The random process $(S_{\ell_k})_{k\geq 0}$ is a random walk on $\mathbb{R}^{*+}$, which is obviously transient. As a consequence of the renewal theorem, the quantities $\sum_{k=1}^{+\infty} \mathbb{E}\left(1_{[a,a+1[}(S_{\ell_k})\right)$ are uniformly bounded on $\mathbb{R}^+$. By (5.30), it follows that $\Phi(0) < +\infty$.

For the second assertion (5.29), since the distribution μ of the X_i is non degenerated and $\mathbb{E}(X_i) = 0$, there exists $\delta > 0$ such that $\mathbb{P}(X_i \geq \delta) > 0$. For any $x \geq 0$, we obtain

$$\begin{aligned}
\Phi(x) &= \sum_{n\geq 1} \mathbb{E}\left(e^{-\lambda S_n}; S_1 \geq -x, \ldots, S_n \geq -x\right) \\
&= \sum_{n\geq 1} \int_{-x}^{+\infty} e^{-\lambda y} \mathbb{E}\left(e^{-\lambda(X_2+\cdots+X_n)}; y + X_2 \geq -x, \ldots, y + X_2 + \cdots + X_n \geq -x\right) \mu(dy) \\
&\geq \sum_{n\geq 0} \int_{\delta}^{+\infty} e^{-\lambda y} \mathbb{E}\left(e^{-\lambda S_n}; y + S_1 \geq -x, \ldots, y + S_n \geq -x\right) \mu(dy) \\
&\geq \int_{\delta}^{+\infty} e^{-\lambda y} \sum_{n\geq 0} \mathbb{E}\left(e^{-\lambda S_n}; \delta + S_1 \geq -x, \ldots, \delta + S_n \geq -x\right) \mu(dy) \\
&\geq \Phi(x+\delta) \int_{\delta}^{+\infty} e^{-\lambda y} \mu(dy).
\end{aligned}$$

Assertion (5.29) follows immediately since $\int_{\delta}^{+\infty} e^{-\lambda y} \mu(dy) > 0$.

□

Proof of Lemma 5.5. We claim that

$$\lim_{n\to+\infty} \widehat{\mathbb{E}}_x \Big| \frac{1}{\bar{q}_n} - \frac{1}{\bar{q}_\infty} \Big| = 0. \tag{5.31}$$

By definition, the quantities $\bar{q}_n$ are always less than or equal to 1. Therefore, (5.31) implies that the same property holds $\widehat{\mathbb{P}}_x$-almost surely for $\bar{q}_\infty$. Hence

$$\widehat{\mathbb{E}}_x |\bar{q}_n - \bar{q}_\infty| = \widehat{\mathbb{E}}_x \left(\bar{q}_n \bar{q}_\infty \Big| \frac{1}{\bar{q}_n} - \frac{1}{\bar{q}_\infty}\Big|\right) \leq \widehat{\mathbb{E}}_x \Big| \frac{1}{\bar{q}_n} - \frac{1}{\bar{q}_\infty} \Big| \quad \overset{n\to+\infty}{\longrightarrow} 0$$

and $\lim_{n\to+\infty} \widehat{\mathbb{P}}_x(Z_n \neq 0) = \widehat{\mathbb{E}}_x \bar{q}_\infty$.

Finally, it remains to verify (5.31). From (5.4), (5.8) and (5.26)

$$\Big| \frac{1}{\bar{q}_n} - \frac{1}{\bar{q}_\infty} \Big| \leq e^{-S_n} + \sum_{i=n}^{+\infty} \eta_{i+1} e^{-S_i} \preceq \sum_{i=n}^{+\infty} e^{-S_i},$$

with $\widehat{\mathbb{E}}_x(\eta_{i+1} e^{-S_i}) \preceq \mathbb{E}_x(e^{-S_i})$ for any $i \geq 0$ and $\sum_{i=0}^{+\infty} \mathbb{E}_x(e^{-S_i}) < +\infty$.

We conclude by using Lemma 5.4.

□

6 Fluctuations of random walks on $\mathbb{R}$

First we prove the Proposition 5.2. and then present the main tools which yield to Theorem 5.3.
Proof of Proposition 5.2. (1) We decompose $h(x)$ as

$$h(x) = -\mathbb{E}(S_{\tau_x}, \tau_x = 1) - \mathbb{E}(S_{\tau_x}, \tau_x > 1).$$

On one hand,

$$-\mathbb{E}(S_{\tau_x}, \tau_x = 1) = -\mathbb{E}(S_1, \tau_x = 1) = -\mathbb{E}(S_1) + \mathbb{E}(S_1, \tau_x > 1) = \mathbb{E}(S_1, \tau_x > 1)$$

and, on the other hand, by the Markov property,

$$\begin{aligned} -\mathbb{E}(S_{\tau_x}, \tau_x > 1) &= -\mathbb{E}(S_{\tau_x}, x + X_1 > 0, \tau_x > 1) \\ &= -\int_{-x}^{+\infty} \mathbb{E}(y + S_{\tau_{x+y}})\mu(dy) \\ &= -\int_{-x}^{+\infty} y\mu(dy) - \int_{-x}^{+\infty} \mathbb{E}(S_{\tau_{x+y}})\mu(y) \\ &= -\mathbb{E}(S_1, \tau_x > 1) + \mathbb{E}(h(x + S_1), \tau_x > 1). \end{aligned}$$

Consequently $h(x) = \mathbb{E}(h(x + S_1), \tau_x > 1) = P^+h(x)$.

(2) Noticing that $(S_n)_{n\geq 0}$ it is a martingale with respect to $(\mathcal{F})_{n\geq 0}$, we may write

$$\begin{aligned} \mathbb{E}(S_n; \tau_x > n) &= \mathbb{E}(S_n) - \mathbb{E}(S_n; \tau_x \leq n) \\ &= -\mathbb{E}(\mathbb{E}(S_n; \mid \mathcal{F}_{\tau_x}); \tau_x \leq n) \\ &= -\mathbb{E}(S_{\tau_x}; \tau_x \leq n) \end{aligned}$$

so that, by Monotone Convergence Theorem,

$$\lim_{n\to+\infty} \mathbb{E}(S_n; \tau_x > n) = -\mathbb{E}(S_{\tau_x}).$$

(3) The inequality $S_{\tau_x} < -x \leq S_{\tau_x - 1}$ $\mathbb{P}-$a.s readily implies $\mathbb{E}(S_{\tau_x}) \leq -x$ and $\liminf_{x\to+\infty} \frac{-\mathbb{E}(S_{\tau_x})}{x} \geq 1$. If the r.v. X_k were bounded from below by $-A$, with $A \geq 0$, we would have $S_{\tau_x} < -x \leq S_{\tau_x - 1} \leq S_{\tau_x} + A$ $\mathbb{P}-$a.s, which yields

$$\limsup_{x\to+\infty} \frac{-\mathbb{E}(S_{\tau_x})}{x} \leq \limsup_{x\to+\infty} \frac{x + A}{x} \leq 1.$$

When the X_k are not bounded from below by $-A$, the same result holds as a consequence of the renewal theory for non negative random variables (see Blackwell's renewal theorem).

(4) When $y < 0$ we have $h(x + y) \leq h(x) \leq ch(x)(1 + y^+)$ since $c \geq 1$. Now, assume $y \geq 0$. When $y \geq 0$, the inequality $S_{\tau_{x+y}} \geq S_{\tau_x} + S_{\tau_y} \circ \theta^{\tau_x}$ holds $\mathbb{P}$-a.s and yields

$$h(x + y) = -\mathbb{E}(S_{\tau_{x+y}}) \leq \mathbb{E}(S_{\tau_x}) + \mathbb{E}(S_{\tau_y} \circ \theta^{\tau_x}) = h(x) + h(y).$$

By (5.12), it follows $h(y) \preceq y$ so that

$$h(x + y) \leq h(x)(1 + h(y)/h(0)) \leq ch(x)(1 + y) \leq h(x)\left(1 + \frac{h(y)}{h(0)}\right) \leq ch(x)(1 + y^+),$$

for some $c \geq 1$.

□

6.1 Ideas of the proof of Theorem 5.3 via the Wiener-Hopf's factorization

Throughout this section, we assume $S_0 = 0$ and set $\mathbb{P}_0 = \mathbb{P}$. Notice that the following identity holds: for any $x \geq 0$ and $n \geq 1$

$$\mathbb{P}(\tau_x > n) = \mathbb{P}(m_n \geq -x).$$

Remember that the stopping time τ_0 coincides with the first descending ladder epoch $\mathbf{l}_1$. The proof of Theorem 5.3 is decomposed in 3 steps.

Step 1: we explicit the tail behaviour of the distribution of $\mathbf{l}_1$.

Step 2 concerns the asymptotic behaviour of of the sequence $\mathbb{E}(e^{aS_n}; \mathbf{l}_1 > n)$

The conclusion arrives in step 3.

The classical approach used to study the behaviour as $n \to +\infty$ of the quantity $\mathbb{P}(\tau_x > n)$ is based on the "Wiener-Hopf's factorization". This famous identity allows to decompose the generating function of the couple $(\mathbf{l_1}, S_{l_1})$ into quantities associated to the r.w. $(S_n)_{n\geq 0}$. In this section, we present the main ingredients of the proof but we do not explicit the constant $-\mathbb{E}(S_{\tau_x})$, the reader can find all the details in [16].

By [4] (chapter IV, section 15, P5) or [6] (chapter XVII, section 2, Lemma 3), the following identities hold: for all $s \in [0, 1[$ and $a > 0$

$$\sum_{n=0}^{+\infty} s^n \mathbb{P}(\mathbf{l_1} > n) = \exp\left(\sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{P}(S_n < 0)\right) \tag{6.1}$$

and

$$\sum_{n=0}^{+\infty} s^n \mathbb{E}(e^{aS_n}; \mathbf{l_1} > n) = \exp\left(\sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{E}(e^{aS_n}; S_n < 0)\right). \tag{6.2}$$

Such identities are quite a miracle ! Their proofs rely on deep properties of functions of the complex variable. They allow to obtain the asymptotic behaviour of the sequence $(\mathbb{P}(m_n \geq -x))_{n\geq 1}$.

Step 1- Asymptotic behaviour of the sequence $(\mathbb{P}(\mathbf{l_1} > n))_{n\geq 1}$.

By Central limit theorem, one knows that the sequence $\left(\mathbb{P}(S_n < 0) - \frac{1}{2}\right)_{n\geq 0}$ converges to 0; more precisely, it is proved in [21] that the series $\sum_{n\geq 1} \frac{1}{n}\left(\mathbb{P}(S_n < 0) - \frac{1}{2}\right)$ converges (and even absolutely) to some limit c; thus, the equality (6.1) implies

$$\sum_{n=0}^{+\infty} s^n \mathbb{P}(\mathbf{l_1} > n) = \frac{e^c}{\sqrt{1-s}}(1 + \varepsilon(s)).$$

Using the classical tauberian theorem [6] (chapter XIII, section 5), one gets, as $n \to +\infty$,

$$\mathbb{P}(\mathbf{l_1} > n) \sim \frac{e^c}{\sqrt{\pi n}}. \tag{6.3}$$

Step 2- We prove that for any $a > 0$, the sequence $\left(n^{3/2}\mathbb{E}(e^{aS_n}; \mathbf{l_1} > n)\right)_{n\geq 1}$ converges to a non zero constant (which can be expressed in terms of the potentials of the descending and ascending ladder random walks).

Derivating the two sides of (6.2) and multiplying by s, we obtain, for any $|s| < 1$ and $a > 0$,

$$\sum_{n=1}^{+\infty} s^n n\mathbb{E}(e^{aS_n}; \mathbf{l_1} > n) = \sum_{n=1}^{+\infty} s^n \mathbb{E}(e^{aS_n}; S_n < 0) \times \sum_{n=0}^{+\infty} s^n \mathbb{E}(e^{aS_n}; \mathbf{l_1} > n).$$

Hence $n\mathbb{E}(e^{aS_n}; \mathbf{l_1} > n)$ may be decomposed as $\sum_{k=0}^{n-1} a_k b_{n-k}$ with, for any $n \geq 0$,

$$a_n = \mathbb{E}(e^{aS_n}; S_n < 0) \quad \text{and} \quad b_n = \mathbb{E}(e^{aS_n}; \mathbf{l_1} > n). \tag{6.4}$$

As a direct consequence of the local limit theorem for random walks on $\mathbb{R}$, we obtain $a_n \sim \frac{1}{a\sigma\sqrt{n}}$ as $n \to +\infty$, where $\sigma^2 = \mathbb{E}(X_1^2)$. We conclude using the following elementary lemma ([11]) with $\alpha_n = a_n$ and $\beta_n = b_n, n \geq 0$.

Lemma 6.1. *Let $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ be two sequences of non negative real numbers such that*

1. $\lim_{n\to+\infty} \sqrt{n}\alpha_n = \alpha > 0$,
2. $\sum_{n=0}^{+\infty} \beta_n = B < +\infty$,
3. *the sequence $(n\beta_n)_n$ is bounded.*

Then

$$\lim_{n\to+\infty} \sqrt{n} \sum_{k=0}^{n} \alpha_k \beta_{n-k} = \alpha B.$$

Notice that hypotheses *2.* and *3.* of this lemma are not direct consequences or foregoing. We need to apply the following lemma, with $\beta_n = b_n, n \geq 0$, and $\gamma_n = \dfrac{\mathbb{E}(e^{aS_n}; S_n < 0)}{n}, n \geq 1$.

Lemma 6.2. *Let $(\beta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ be two sequences of real numbers such that for any s such that $|s| \leq 1$,*

$$\sum_{n=0}^{+\infty} \beta_n s^n = \exp\left(\sum_{n=1}^{+\infty} \gamma_n s^n\right).$$

If the sequence $(n^{3/2}\gamma_n)_{n\geq 1}$ is bounded, then so is the sequence $(n^{3/2}\beta_n)_{n\geq 0}$.

Step 3- Asymptotic behaviour of the sequence $(\mathbb{P}(\tau_x > n))_{n\geq 1}$, for any $x > 0$.

For any $n \geq 1$, set $m_n := \min(0, S_1, \cdots, S_n)$ and let T_n be the $\mathbb{N}$-valued random variable corresponding to the first time $k \in \{0, \cdots, n\}$ such that $m_n = S_k$ (Warning! The random variable T_n is not a stopping time!). One obtains

$$\begin{aligned}
\mathbb{P}(\tau_x > n) &= \mathbb{P}(m_n \geq -x) \\
&= \sum_{k=0}^{n} \mathbb{P}(m_n \geq -x, T_n = k) \\
&= \sum_{k=0}^{n} \mathbb{P}(S_1 > S_k, \cdots, S_{k-1} > S_k, -x \leq S_k < 0, S_{k+1} \geq S_k, \cdots, S_n \geq S_k) \\
&= \sum_{k=0}^{n} \mathbb{P}(S_1 > S_k, \cdots, S_{k-1} > S_k, -x \leq S_k < 0) \\
&\qquad \times \mathbb{P}(X_{k+1} \geq 0, \cdots, X_{k+1} + \cdots + X_n \geq 0).
\end{aligned} \tag{6.5}$$

Since $(X_1, \cdots, X_k) = (X_k, \cdots, X_1)$ have the same distribution, one may write the last expression as

$$\begin{aligned}\mathbb{P}(S_1 > S_k, \cdots, S_{k-1} > S_k, -x \leq S_k < 0) &= \mathbb{P}(X_2 + \cdots + X_k < 0, \cdots, X_k < 0, -x \leq S_k < 0) \\ &= \mathbb{P}(S_1 < 0, S_2 < 0, \cdots, S_{k-1} < 0, -x \leq S_k < 0).\end{aligned}$$

On one hand, by using a similar argument to the one developed in Step 2, one can check that the term $\mathbb{P}(S_1 > S_k, \cdots, S_{k-1} > S_k, -x \leq S_k < 0)$ is asymptotically equivalent to $1/k^{3/2}$, up to a multiplicative constant. On the other hand, using the fact that the vectors $(X_{k+1}, \cdots, X_n)$ and $(X_1, \cdots, X_{n-k})$ have the same distribution, one gets

$$\mathbb{P}(X_{k+1} \geq 0, \cdots, X_{k+1} + \cdots + X_n \geq 0) = \mathbb{P}(S_1 \geq 0, \cdots, S_{n-k} \geq 0).$$

The reader will be easily convinced (see [16] for more details) that the quantity $\mathbb{P}(S_1 \geq 0, \cdots, S_{n-k} \geq 0)$ behaves, as $n \to +\infty$, like

$$\mathbb{P}(S_1 > 0, \cdots, S_{n-k} > 0) = \mathbb{P}(\tau > n - k) \sim \frac{c}{n-k}$$

for some constant $c > 0$. One finishes the proof by applying again Lemma 6.1 to the equality (6.5).

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