The survival probability of a weakly subcritical multitype branching process in iid random environment.

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Abstract. We study the asymptotic behavior of the probability of non extinction of a weakly subcritical multitype branching process in iid random environment. Under suitable assumptions, the survival probability is of order of $\rho^n n^{-3/2}$ for some $\rho \in (0,1)$ to specify.

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1 Introduction and main results

1.1 Historical context

Branching processes in random environment (BPRE) is an important subject of the theory of branching processes. This model was introduced at the beginning of the 1960's; it takes into account the random fluctuations of the reproduction laws over time. Various properties of this and more general models of BPRE have been analyzed these last decades, through a large number of varied articles. There exists a relatively complete description of the basic properties of many models of BPRE either under the annealed approach or the quenched one. In particular, the behavior of the single type BPRE is mainly determined be the properties of the so-called "associated random walk" constructed by the logarithms of the expected population sizes of particles of the respective generations; this random walk divides in a natural way the set of all single type BPRE into the classes of *supercritical*, *critical* and *subcritical* processes (see [6] for more detail). Notice that a precise estimate for single type branching process in finite state space markovian environment does exist [12]; the fact that the underlying Markov chain is finite is essential in this study, this hypothesis is not relevant in the case of product of random matrices.

Analogue statements known for the single type case do exist for the multitype BPRE with finitely types $p \geq 2$. The role of the associated random walk is played in this case by the logarithms of the norms of products of the mean random matrices associated to the environment. Recent useful results on fluctuations of ordinary random walks on the real line have been extended in terms of $p \times p$ random matrices. In particular, the description of the asymptotic behavior of the survival probability for the critical multitype BPRE under general conditions is done in [16] and [5] where it is proved, under natural and quite general assumptions, that the

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probability of survival up to time n is of order $n^{1/2}$ for a multitype branching process evolving in iid random environment. The case of supercritical multitype branching processes is studied in [9], where a Kesten-Stigum type theorem is established. At last, as in the single type case, subcritical BPRE are divided in three categories: *strongly*, *intermediately* and *weakly* subcritical. The first two subcases are the subject of recent work: in the strongly subcritical subcase, the annealed survival probability at time n is equivalent to ρ^n [21] while in the intermediately subcritical subcase it is of order $\rho^n n^{-1/2}$ [6], where $\rho \in (0,1)$ is a constant defined in terms of the Lyapunov exponent for products of the mean value matrices of the laws of reproduction of particles.

In the present paper, we establish a rough asymptotic estimate of the annealed survival probability at time n in the weakly subcritical case. This case was resistant since there was no local limit theorem for the norm of products of random matrices conditioned to remain greater than 1 until time n. Such a local limit theorem is obtained in [18], its proof can be adapted to the present context to show that the probability of survival at time n is of order $\rho^n n^{-3/2}$ where $\rho \in (0,1)$.

1.2 Notations and assumptions

We fix an integer $p \geq 2$ and denote $\mathbb{R}^p$ (resp., $\mathbb{N}^p$) the set of p-dimensional column vectors with real (resp., non negative integers) coordinates; for any column vector $x = (x_i)_{1\leq i\leq p} \in \mathbb{R}^p$, we denote $\tilde{x}$ the row vector $\tilde{x} := (x_1, \ldots, x_p)$. Let $\mathbf{1}$ (resp., $\mathbf{0}$) be the column vector of $\mathbb{R}^p$ whose all coordinates equal 1 (resp., 0). We fix a basis $\{e_i, 1 \leq i \leq p\}$ in $\mathbb{R}^p$ and denote $|\cdot|$ the corresponding $\mathbb{L}^1$ norm.

The multitype Galton-Watson process is a temporally homogeneous vector Markov process $(Z_n)_{n\geq 0}$ whose states are column vectors in $\mathbb{N}^p$. For any $1 \leq i \leq p$, the i-th component $Z_n(i)$ of Z_n may be interpreted as the number of objects of type i in the n-th generation.

A multivariate probability generating function $(f^{(i)})_{1\leq i\leq p}$, denote by f, is a function from $(\mathbb{R}^+)^p$ to $\mathbb{R}^+$ defined by

$$f^{(i)}(s) = \sum_{\alpha\in\mathbb{N}^p} p^{(i)}(\alpha) s^\alpha,$$

for any $s = (s_i)_{1\leq i\leq p} \in (\mathbb{R}^+)^p$, where

(i) $\alpha = (\alpha_i)_i \in \mathbb{N}^p$ and $s^\alpha = s_1^{\alpha_1} \ldots s_p^{\alpha_p}$;

(ii) $p^{(i)}(\alpha) = p^{(i)}(\alpha_1, \ldots, \alpha_p)$ is the probability that an object of type i has α_1 children of type $1, \ldots, \alpha_p$ children of type p.

From now on, the multivariate generating function f is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; let $\mathbf{f}= (f_n)_{n\geq 0}$ be a sequence of iid random copies of f, called the *(random) environment*. The Galton-Watson process with p types of particles in the random environment $\mathbf{f}$ describes the evolution of a particle population $Z_n = (Z_n(i))_{1\leq i\leq p}$ for $n \geq 0$.

For any $s = (s_i)_{1\leq i\leq p}, 0 \leq s_i \leq 1$,

$$\mathbb{E}\left(s^{Z_n} | Z_0, \ldots, Z_{n-1}, f_0, \ldots, f_{n-1}\right) = f_{n-1}(s)^{Z_{n-1}}$$

which yields

$$\mathbb{E}\left(s^{Z_n} | Z_0 = \tilde{e}_i, f_0, \ldots, f_{n-1}\right) = f_0^{(i)}(f_1(\ldots f_{n-1}(s) \ldots)).$$

The probability of non extinction $q_n^{(i)}$ at generation n given the environment $\mathbf{f}$ when the ancestor is of type i is

$$\begin{aligned} q_n^{(i)} &:= \mathbb{P}(|Z_n| > 0 \mid f_0^{(i)}, f_1, \ldots, f_{n-1}) \\ &= 1 - f_0^{(i)}(f_1(\ldots f_{n-1}(\mathbf{0})\ldots)) = \tilde{e}_i(\mathbf{1} - f_0(f_1(\ldots f_{n-1}(\mathbf{0})\ldots))), \end{aligned}$$

so that

$$\mathbb{E}[q_n^{(i)}] = \mathbb{P}(Z_n \neq \tilde{\mathbf{0}} | Z_0 = \tilde{e}_i) = \mathbb{E}[\tilde{e}_i(\mathbf{1} - f_0(f_1(\ldots f_{n-1}(\mathbf{0})\ldots)))].$$

The asymptotic behavior of the quantity above is controlled by the mean of the offspring distributions. We assume that the offspring distributions have finite first and second moments; the generating function $f = (f^{(i)})_{1 \leq i \leq p}$, is thus C^2-functions on $[0,1]^p$ and we introduce:

(i) the mean matrix $M = (M(i,j))_{i,j} = \left(\dfrac{\partial f^{(i)}}{\partial s_j}(\mathbf{1})\right)_{i,j}$;

(ii) the Hessian matrices $B^{(i)} = \left(\dfrac{\partial^2 f^{(i)}}{\partial s_k \partial s_\ell}(\mathbf{1})\right)_{k,\ell}$, for $i = 1, \ldots, p$.

We denote by $(M_n)_{n \geq 0}$ (resp. $(B_n^{(i)})_{n \geq 0}, i = 1, \ldots, p$) the sequence of iid random mean matrices (resp. Hessian matrices) associated with the sequence $(f_n)_{n \geq 0}$. These matrices belong to the semigroup $\mathcal{S}$ of $p \times p$ matrices with positive entries; we endow $\mathcal{S}$ with the $\mathbb{L}^1$-norm.

We assume that the distribution μ of M satisfies the following assumptions.

P1 *Moment assumption:* $\mathbb{E}(|M|) < +\infty$.

P2 *Irreducibility assumption: The support of μ acts strongly irreducibly on the semigroup of matrices with non negative entries, i.e. there exists no affine subspaces A of $\mathbb{R}^p$ such that $A \cap (\mathbb{R}^+)^p$ is non empty, bounded and invariant under the action of all elements of the support of μ.*

P3 *There exists $B > 1$ such that for μ-almost all M and any $1 \leq i, j, k, \ell \leq p$,*

$$\frac{1}{B} M(k,\ell) \leq M(i,j) \leq B\ M(k,\ell). \tag{1.1}$$

From now on, we denote by $\mathcal{S}_B$ the subset of $\mathcal{S}$ of $p \times p$-matrices M satisfying condition (1.1).

P4 *Subcriticality: The upper Lyapunov exponent* $\gamma_\mu := \lim_{n \to +\infty} \dfrac{\mathbb{E}[\ln |M_0 \cdots M_{n-1}|]}{n}$ *is negative.*

P5 *There exists $\varepsilon > 0$ such that* $\mu\{M \in \mathcal{S} : \forall x \in (\mathbb{R}^+)^p \text{ suchthat } |x| = 1, \ln |\tilde{x}M| \geq \varepsilon\} > 0$ *and* $\mu\{M \in \mathcal{S} : \forall x \in (\mathbb{R}^+)^p \text{ suchthat } |x| = 1, \ln |Mx| \geq \varepsilon\} > 0$.

As it is usual in studying local probabilities, one has to distinguish between "lattice" and "non lattice" distributions. The "non lattice" assumption ensures that the process $(\ln |\tilde{x} M_{0,n-1}|)_{n \geq 1}$ does not live in the translation of a proper subgroup of $\mathbb{R}$; in the contrary case, when μ is lattice, a phenomenon of cyclic classes appears which involves some complications which are not interesting in our context. In section 2, we give a precise definition of this notion in the context of products of random matrices.

P6 *Non lattice assumption: The measure μ is non lattice.*

We also introduce the following hypotheses concerning the environment $\mathbf{f}$.

P7 *There exist $\varepsilon_0, K_0 > 0$ such that for any $i, j \in \{1, \ldots, p\}$ and any $n \geq 0$,*

$$
\begin{aligned}
&(a)\ \mathbb{P}(Z_{n+1}(i) \geq 2 \mid Z_n = \tilde{e}_j) \geq \varepsilon_0, \\
&(b)\ \mathbb{P}(|Z_{n+1}| = 0 \mid Z_n = \tilde{e}_j) \geq \varepsilon_0, \\
&(c)\ \mathbb{E}[|Z_{n+1}|^2 \mid Z_n = \tilde{e}_j] \leq K_0.
\end{aligned}
$$

Let us first explain some consequences of hypotheses **P1** and **P4** and the way three different subcases appear in the subcritical case. Using the standard subadditivity argument, one infers that, under **P1**, for any $\theta \in [0, 1]$ the limit

$$\lambda(\theta) := \lim_{n \to +\infty} \mathbb{E}\left[|M_{n-1} \cdots M_0|^\theta\right]^{1/n} < +\infty$$

is well defined. The function Λ is the analogue of the moment generating function for the associated random walk in the case of branching processes in random environment with single type of particles. The function $\theta \mapsto \mathbb{E}[|M|^\theta]$ is at least twice continuously differentiable on $[0, 1[$. We set: for $\theta \in [0, 1[$,

$$\Lambda(\theta) := \ln \lambda(\theta).$$

This function is also twice continuously differentiable and strictly convex on $[0, 1[$ (see [2], Proposition 3.1). There are three cases to consider.

1. $\underline{\Lambda'(1) < 0}$ (hence $\Lambda(1) < 0$, by convexity). This case corresponds to the *strongly subcritical* case for single type branching processes, it is studied in [21] when $p \geq 2$. Under suitable conditions, it holds

$$\mathbb{E}[q_n^{(i)}] = \mathbb{P}(|Z_n| > 0 \mid Z_0 = \tilde{e}_i) \sim c^i\ \lambda(1)^n, \quad i = 1, \ldots, p$$

with $c^i > 0$ and $\lambda(1) \in]0, 1[$.

2. $\underline{\Lambda'(1) = 0}$ (hence $\Lambda(1) < 0$). It corresponds to the *intermediately subcritical* case for single type branching processes, it is studied in [6] when $p \geq 2$. Under suitable conditions, it holds the following rough estimate

$$\mathbb{E}[q_n^{(i)}] = \mathbb{P}(|Z_n| > 0 \mid Z_0 = \tilde{e}_i) \asymp c^i\ \frac{\lambda(1)^n}{\sqrt{n}}, \quad i = 1, \ldots, p$$

with $c^i > 0$ and $\lambda(1) \in]0, 1[$.

3. $\underline{\Lambda'(1) > 0}$. This is the *weakly subcritical* case when $p = 1$ and this is the aim of the present article in the case when $p \geq 2$, we only obtain a rough estimate.

We denote by $\theta_\star$ the unique value of θ in $]0, 1[$ such that $\Lambda'(\theta_\star) = 0$. We set $\rho_\star = \lambda(\theta_\star)$; it is obvious that $\rho_\star \in]0, 1[$.

Notation. *Let $c > 0$ and ϕ, ψ be two functions of some variable u; we shall write $\phi \overset{c}{\preceq} \psi$ (or simply $\phi \preceq \psi$) when $\phi(u) \leq c\psi(u)$ for any value of u. The notation $\phi \overset{c}{\asymp} \psi$ (or simply $\phi \asymp \psi$) means $\phi \overset{c}{\preceq} \psi \overset{c}{\preceq} \phi$.*

1.3 Main statement

Now, we may state the main result of this article which concerns a rough estimate of the probability of extinction in the weakly subcritical case.

Theorem 1.1 *Assume that conditions* **P1**-**P7**. *Assume that $\Lambda'(1) > 0$ and let $\theta_\star$ be the unique value of $\theta \in]0,1[$ such that $\Lambda'(\theta_\star) = 0$.*

Then, there exists positive constants $\mathbf{c}$ *and* $\mathbf{C}$ *such that for $1 \le i \le p$,*

$$\mathbf{c}\ \frac{\rho_\star^n}{n^{3/2}} \le \mathbb{P}(|Z_n| > 0 \mid Z_0 = \tilde{e}_i) \le \mathbf{C}\ \frac{\rho_\star^n}{n^{3/2}}$$

with $\rho_\star = \lambda(\theta_\star) \in]0,1[$.

2 On products of positive random matrices

The random variables M_n and $B_n^{(i)}$ are iid with values in the semigroup $\mathcal{S}$ of $p \times p$ matrices with positive coefficients. We set, for any $0 \le k \le n$,

$$M_{k,n} = M_k \cdots M_n \quad \text{and} \quad M_{n,k} = M_n \cdots M_k.$$

In order to control the asymptotic behavior of the matrices $M_{n,0}$ (resp. $M_{0,n}$), we study their action on the cone of column (resp. row) vectors with positive entries.

Let $\mathcal{C}$ (resp. $\tilde{\mathcal{C}}$) be the set of column vector (resp. row vectors) in $\mathbb{R}^p$ with positive entries. For any $x \in \mathcal{C}$, we denote by $\tilde{x}$ the corresponding row vector. We also set $\mathbb{X} := \{x \in \mathcal{C}, |x| = 1\}$ and $\tilde{\mathbb{X}} = \{\tilde{x} \mid x \in \mathbb{X}\}$. We consider the following (left and right) actions of $\mathcal{S}$:

- the linear action of $\mathcal{S}$ on $\mathcal{C}$ (resp. $\tilde{\mathcal{C}}$) defined by $(M, x) \mapsto Mx$ (resp. $(M, \tilde{x}) \mapsto \tilde{x}M$) for any $M \in \mathcal{S}$ and $x \in \mathcal{C}$,
- the projective action of $\mathcal{S}$ on $\mathbb{X}$ (resp. $\tilde{\mathbb{X}}$) defined by $(M, x) \mapsto M \cdot x := \dfrac{Mx}{|Mx|}$ $\left(\text{resp. } (M, \tilde{x}) \mapsto \tilde{x} \cdot M = \dfrac{\tilde{x}M}{|\tilde{x}M|}\right)$ for any $M \in \mathcal{S}$ and $x \in \mathbb{X}$.

For any $M \in \mathcal{S}$, set $v(M) := \min_{1 \le j \le d} \Big(\sum_{i=1}^p M(i,j)\Big)$. Then, for any $x \in \mathcal{C}$,

$$0 < v(M)\ |x| \le |Mx| \le |M|\ |x|.$$

Similarly, noticing ${}^tM$ the transpose of the matrix M, it holds

$$0 < v({}^tM)\ |x| \le |\tilde{x}M| \le |M|\ |x|.$$

Consequently, hypothesis **P5** holds when $\mu\{M \mid v(M) > 1\} > 0$ and $\mu\{M \mid v({}^tM) > 1\} > 0$.

In the context of branching processes, we focus on the right action of $\mathcal{S}$ on $\tilde{\mathbb{X}}$. Therefore, we naturally endow $\tilde{\mathbb{X}}$ with a distance d which is a variant of the Hilbert metric: this distance is bounded on $\tilde{\mathbb{X}}$ and any element M in $\mathcal{S}$ acts on the metric space $(\tilde{\mathbb{X}}, d)$ as a contraction. More precisely: for any $\tilde{x} = (x_i)_{1\leq i\leq p}, \tilde{y} = (y_i)_{1\leq i\leq p} \in \tilde{\mathbb{X}}$, we write

$$m(\tilde{x}, \tilde{y}) = \min\Big\{ \frac{x_i}{y_i} \Big| i = 1, \dots, p \text{ such that } y_i > 0 \Big\}$$

and we set $d(\tilde{x}, \tilde{y}) := \varphi\Big(m(\tilde{x}, \tilde{y})m(\tilde{y}, \tilde{x})\Big)$, where $\varphi(s) := \dfrac{1-s}{1+s}$ for $0 \leq s \leq 1$. For $M \in \mathcal{S}$, we set

$$c(M) := \sup\{d(\tilde{x} \cdot M, \tilde{y} \cdot M) \mid \tilde{x}, \tilde{y} \in \tilde{\mathbb{X}}\}.$$

By [14], the function d is a distance on $\tilde{\mathbb{X}}$ which satisfies the following properties.

Properties 2.1 *1.* $\sup\{d(\tilde{x}, \tilde{y}) \mid \tilde{x}, \tilde{y} \in \tilde{\mathbb{X}}\} = 1$.

2. For any $M \in \mathcal{S}$,

$$c(M) = \max_{i,j,k,l \in \{1,\dots,p\}} \frac{|M(i,j)M(k,l) - M(i,l)M(k,j)|}{M(i,j)M(k,l) + M(i,l)M(k,j)}.$$

In particular, there exists $c_B \in [0, 1)$ *such that* $c(M) \leq c_B < 1$ *for any* $M \in \mathcal{S}_B$.

3. $d(\tilde{x} \cdot M, \tilde{y} \cdot M) \leq c(M)d(\tilde{x}, \tilde{y}) \leq c(M)$ *for any* $\tilde{x}, \tilde{y} \in \tilde{\mathbb{X}}$ *and* $M \in \mathcal{S}_B$.

4. $c(MM') \leq c(M)c(M')$ *for any* $M, M' \in \mathcal{S}_B$.

The following lemma plays a crucial role in controlling the behavior of the norm of the product of random matrices $M_0, M_1, \dots$ (see Lemma 2.2 in [1]). We denote by T_B the closed semigroup generated by $\mathcal{S}_B$.

Lemma 2.2 *Under hypothesis* **P3**, *for any* $M \in T_B$ *and* $1 \leq i, j, k, \ell \leq p$,

$$M(i,j) \overset{B^2}{\asymp} M(k, \ell).$$

In particular, there exists $\delta > 1$ *such that for any* $M, N \in T_B$ *and* $x, y \in \mathbb{X}$,

1. $|Mx| \overset{\delta}{\asymp} |M|$ *and* $|\tilde{y}M| \overset{\delta}{\asymp} |M|$,

2. $|\tilde{y}Mx| \overset{\delta}{\asymp} |M|$,

3. $|M||N| \overset{\delta}{\preceq} |MN| \leq |M||N|$.

On the product space $\tilde{\mathbb{X}} \times \mathcal{S}$, we define the function ρ by setting $\rho(\tilde{x}, M) := \log |\tilde{x}M|$ for $(\tilde{x}, M) \in \tilde{\mathbb{X}} \times \mathcal{S}$. This function satisfies the following cocycle property:

$$\rho(\tilde{x}, MN) = \rho(\tilde{x}, M) + \rho(\tilde{x} \cdot M, N)$$

for any $M, N \in \mathcal{S}$ and $\tilde{x} \in \tilde{\mathbb{X}}$.

We achieve this paragraph with the definition of a "lattice" distribution μ (see hypothesis **P6**) in the context of products of random matrices. It may be stated as follows: the measure μ is *lattice* if there exist $t > 0, \epsilon \in [0, 2\pi[$ and a function $\psi : \mathbb{X} \to \mathbb{R}$ such that

$$\forall g \in T_\mu, \forall x \in \mathbb{X}, \quad \exp\{it\rho(g, x) - i\epsilon + i(\psi(g \cdot x) - \psi(x))\} = 1, \tag{2.1}$$

where T_μ is the closed sub-semigroup generated by the support of μ.

2.1 Exponential change of measures

Under hypothesis **P1**, for any $\theta \in [0,1]$, we introduce the operator P_θ defined by: for any bounded Borel function $\varphi : \tilde{\mathbb{X}} \to \mathbb{C}$ and $\tilde{x} \in \tilde{\mathbb{X}}$,

$$P_\theta \varphi(\tilde{x}) := \mathbb{E}[|\tilde{x}M|^\theta \varphi(\tilde{x} \cdot M)].$$

For any $n \geq 1$, it holds

$$P_\theta^n \varphi(\tilde{x}) := \mathbb{E}[|\tilde{x}M_{0,n-1}|^\theta \varphi(\tilde{x} \cdot M_{0,n-1})].$$

The operators P_θ are positive, they act continuously on the space $C(\tilde{\mathbb{X}})$ of $\mathbb{C}$-valued continuous functions on $\tilde{\mathbb{X}}$ endowed with the norm of uniform convergence $|\cdot|_\infty$. Under condition **P3**, their spectral radius on $C(\tilde{\mathbb{X}})$ equals $\lambda(\theta)$.

We denote by $\mathcal{B}_\theta$ the space of θ-Hölder continuous functions $\varphi : (\tilde{\mathbb{X}}, d) \to \mathbb{C}$ such that $m_\theta(\varphi) := \sup_{\substack{\tilde{x},\tilde{y} \in \tilde{\mathbb{X}} \\ \tilde{x} \neq \tilde{y}}} \frac{|\varphi(\tilde{x}) - \varphi(\tilde{y})|}{d(\tilde{x},\tilde{y})^\theta} < +\infty$. Endowed with the norm $|\cdot|_\theta := |\cdot|_\infty + m_\theta(\cdot)$, the space $(\mathcal{B}_\theta, |\cdot|_\theta)$ is a $\mathbb{C}$-Banach space. Furthermore, for any $\varphi \in \mathcal{B}_\theta$ and $n \geq 1$,

$$\begin{aligned} m_\theta(P_\theta^n \varphi) &\leq \mathbb{E}[c(M_{0,n-1})^\theta |M_{0,n-1}|^\theta] \ m_\theta(\varphi) + 2^\theta \ \mathbb{E}[|M_{0,n-1}|^\theta] \ |\varphi|_\infty \\ &\leq r^n \mathbb{E}[|M_{0,n-1}|^\theta] \ m_\theta(\varphi) + 2^\theta \ \mathbb{E}[|M_{0,n-1}|^\theta] \ |\varphi|_\infty \end{aligned}$$

with $r = (c_B)^\theta$ where c_B is introduced in Properties 2.1. By hypothesis **P3**, it holds $c_B \in [0,1[$, hence $r < 1$. In other words, the operator P_θ satisfies the *Doeblin-Fortet* inequality on $\mathcal{B}_\theta$: for any $\varphi \in \mathcal{B}_\theta$ and $n \geq 1$,

$$|P_\theta^n \varphi|_\theta \leq r^n \mathbb{E}[|M_{0,n-1}|^\theta] |\varphi|_\theta + C \times \mathbb{E}[|M_{0,n-1}|^\theta] |\varphi|_\infty$$

for some constants $C > 0$ and $r \in]0,1[$. By [13], the operator P_θ is quasicompact on $\mathcal{B}_\theta$, with spectral radius $\lambda(\theta)$. By [8] and [2], the real positive number $\lambda(\theta)$ is the unique eigenvalue of P_θ with modulus $\lambda(\theta)$, it is simple and there exist a unique strictly positive function $v_\theta \in \mathcal{B}_\theta$ and a unique probability measure ν_θ on $\tilde{\mathbb{X}}$ such that $\nu_\theta(v_\theta) = 1$ satisfying

$$\nu_\theta P_\theta = \lambda(\theta) \nu_\theta \quad \text{and} \quad P_\theta v_\theta = \lambda(\theta) v_\theta.$$

For any $\theta \in [0,1]$, we thus introduce the new transition probability kernel $\bar{P}_\theta$ on $\tilde{\mathbb{X}}$ defined by: for any bounded Borel function $\psi : \tilde{\mathbb{X}} \to \mathbb{C}$ and any $\tilde{x} \in \tilde{\mathbb{X}}$,

$$\bar{P}_\theta \psi(\tilde{x}) := \frac{1}{\lambda(\theta) v_\theta(\tilde{x})} \mathbb{E}[|\tilde{x}M|^\theta v_\theta(\tilde{x} \cdot M) \psi(\tilde{x} \cdot M)].$$

By the above, this Markovian operator is quasicompact on $\tilde{\mathbb{X}}$ with 1 as a simple and isolated eigenvalue and without any other eigenvalue of modulus 1. The powers of $\bar{P}_\theta$ are given by: for any $n \geq 1$,

$$\bar{P}_\theta^n \psi(\tilde{x}) := \frac{1}{\lambda(\theta)^n v_\theta(\tilde{x})} \mathbb{E}[|\tilde{x}M_{0,n-1}|^\theta v_\theta(\tilde{x} \cdot M_{0,n-1}) \psi(\tilde{x} \cdot M_{0,n-1})].$$

Let $(X_n^\theta)_{n \geq 0}$ be the Markov chain on $\tilde{\mathbb{X}}$ with transition operator $\bar{P}_\theta$. For $a \in \mathbb{R}^+$ fixed, any $\tilde{x} \in \tilde{\mathbb{X}}$ and any $n \geq 1$, we set

$$S_0 = a, \quad S_n = S_n(\tilde{x}, a) = a + \ln |\tilde{x}M_{0,n-1}|.$$

We associate to $\bar{P}_\theta$ the Markov walk $(X_n^\theta, S_n)_{n\geq 0}$ on $\tilde{\mathbb{X}} \times \mathbb{R}$ with transition operator $\tilde{P}_\theta$ defined by: for any bounded Borel function $\Psi : \tilde{\mathbb{X}} \times \mathbb{R} \to \mathbb{C}$ and any $(\tilde{x}, a) \in \tilde{\mathbb{X}} \times \mathbb{R}$,

$$\tilde{P}_\theta \Psi(\tilde{x}, a) := \frac{1}{\lambda(\theta) v_\theta(\tilde{x})} \mathbb{E}[|\tilde{x}M|^\theta v_\theta(\tilde{x} \cdot M) \Psi(\tilde{x} \cdot M, a + \ln |\tilde{x}M|)].$$

In order to study the behavior of the $(X_n^\theta, S_n)_{n\geq 0}$, we introduce in a natural way the family of "Fourier operators" $(\bar{P}_{\theta,t})_{t\in\mathbb{R}}$ defined by: for any bounded Borel function $\psi : \tilde{\mathbb{X}} \to \mathbb{C}$, for any $\theta \in [0, 1]$ and $t \in \mathbb{R}$,

$$\begin{aligned}\bar{P}_{\theta,t}\psi(\tilde{x}) &:= \frac{1}{\lambda(\theta) v_\theta(\tilde{x})} \mathbb{E}[|\tilde{x}M|^{\theta+it} v_\theta(\tilde{x} \cdot M) \psi(\tilde{x} \cdot M)]. \\ &= \frac{1}{\lambda(\theta) v_\theta(\tilde{x})} \mathbb{E}[e^{(\theta+it)\ln |\tilde{x}M|} v_\theta(\tilde{x} \cdot M) \psi(\tilde{x} \cdot M)]\end{aligned}$$

In the sequel, it is thus natural to consider, for any $\tilde{x}$ in $\tilde{\mathbb{X}}$ and $a \in \mathbb{R}$, the probability measures $\mathbb{P}^\theta_{\tilde{x},a}$ (with associated expectation $\mathbb{E}^\theta_{\tilde{x},a}$) on $(\Omega, \mathcal{F})$ whose restriction to the σ-algebras $\sigma(M_0, \ldots, M_{k-1}), k \geq 1$ is given by: for any positive Borel function $\Phi : \mathcal{S}^k \to \mathbb{C}$,

$$\mathbb{E}^\theta_{\tilde{x},a}(\Phi(M_0, \ldots, M_{k-1})) := \frac{1}{\lambda^k(\theta) v_\theta(\tilde{x})} \mathbb{E}[\Phi(M_0, \ldots, M_{k-1}) e^{\theta(a+\ln |\tilde{x}M_{0,k-1})|} v_\theta(\tilde{x} \cdot M_{0,k-1})] \quad (2.2)$$

and with the same conditional probability measure $\mathbb{P}_{M_0=m_0,\ldots,M_{k-1}=m_{k-1}}$ as $\mathbb{P}$, for any $m_0, \ldots,$ m_{k-1} in $\mathcal{S}$.

Now, we fix $\theta = \theta_\star$. The matrices M_n are iid with respect to the measure $\mathbb{P}$ but this property is no longer relevant with respect to $\mathbb{P}^{\theta_\star}_{\tilde{x},a}$. Neverthelesss, due to the quasicompactness of the operators $\bar{P}_{\theta_\star}$ established above, the fluctuations of the process $(\tilde{x}M_{0,n-1})$ with respect to $\mathbb{P}^{\theta_\star}_{\tilde{x},a}$ may be controlled as in [19]; the Markov chain $(M_n, \tilde{x}\cdot M_{0,n-1})_{n\geq 1}$ driven by the Markov operator $P^{\theta_\star}$ falls within the scope of the article [11] and leads to the following statement.

We denote by

$$\tau_{\tilde{x},a} := \min\{n \geq 1 \mid S_n(\tilde{x}, a) \leq 0\}$$

the first moment when the sequence $(S_n(\tilde{x}, a))_{n\geq 0}$ enters the set $]-\infty, 0]$. Modifying in a natural way the arguments used in [19] one can conclude that under the conditions **P1**–**P6** the function $h^{\theta_\star} : \tilde{\mathbb{X}} \times [0, +\infty[\to [0, +\infty[$ specified by the equality

$$h^{\theta_\star}(\tilde{x}, a) = \lim_{n\to+\infty} \mathbb{E}^{\theta_\star}(S_n(\tilde{x}, a), \tau_{\tilde{x},a} > n)$$

satisfies the property $\mathbb{E}^{\theta_\star}(h^{\theta_\star}(\tilde{x} \cdot M_0, S_1(\tilde{x}, a)); \tau_{\tilde{x},a} > 1) = h^{\theta_\star}(\tilde{x}, a)$. This function $h^{\theta_\star}$ satisfies the following properties (see Theorems 2.2 and 2.3 in [11]).

Proposition 2.3 *There exist constants $C > 1$ and $A > 0$ such that for any $x \in \mathbb{X}$ and $a > 0$,*

$$\mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a} > n) \sim \frac{h^{\theta_\star}(\tilde{x}, a)}{\sqrt{2\pi n}} \quad as \quad n \to +\infty$$

with

$$\mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a} > n) \leq C\ \frac{h^{\theta_\star}(\tilde{x}, a)}{\sqrt{n}} \quad and \quad C^{-1} \max(1, a - A) \leq h^{\theta_\star}(\tilde{x}, a) \leq C(1 + a).$$

In the sequel, we also need a control of the quantities $\mathbb{P}^{\theta_\star}(S_n(\tilde{x},a)\in[b,b+\ell],\tau_{\tilde{x},a}>n)$ for any interval $[b,b+\ell]$ included in $\mathbb{R}^+$. The following statement, in the spirit of Stone's theorem for classical random walks, is stated in [10] for Markov walks over a finite state space and is meaningful when the parameter b is of order $\sqrt{n}$. As announced in [10], this statement is valid in fact in a more general setting, namely when the underlying Markov chain is irreducible and its transition operator satisfies a spectral gap property. When μ is non lattice the spectral radius in $\mathcal{B}_{\theta_\star}$ of the "Fourier operators" $\bar{P}_{\theta_\star,t}, t\in\mathbb{R}^*$, is strictly less than 1; this is crucial in the proof of the following statement.

Proposition 2.4 *Under hypotheses* **P1**–**P6***, for any fixed* $(\tilde{x},a)\in\tilde{\mathbb{X}}\times\mathbb{R},\ell>0$ *and uniformly in* $b\in\mathbb{R}^+$*,*

$$\lim_{n\to+\infty}\left(n\mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a}>n,S_n(\tilde{x},a)\in[b,b+\ell])-\frac{2\ \ell\ h^{\theta_\star}(\tilde{x},a)}{\sqrt{2\pi\sigma^2}}\varphi_+\left(\frac{b}{\sigma\sqrt{n}}\right)\right)=0,$$

where $\varphi_+(t)=te^{-t^2/2}\mathbf{1}_{\mathbb{R}^+}(t)$ *is the Rayleigh density.*

This is now of interest to introduce the following notations. For any $y\in\mathbb{X}$ and $b>0$, we set

$$\tilde{S}_0(y,b)=b\quad\text{and}\quad\tilde{S}_n(y,b):=b-\ln|M_{n-1,0}y|\quad\text{for}\quad n\geq 1.$$

We denote by

$$\tilde{\tau}_{y,b}:=\min\{n\geq 1\mid\tilde{S}_n(y,b)\leq 0\}$$

the first moment when the sequence $(\tilde{S}_n(y,b))_{n\geq 0}$ enters the set $]-\infty,0]$. Propositions 2.3 and 2.4 also hold for the process $(M_{n-1,0}\cdot y,\tilde{S}_n(y,b))_{n\geq 1}$ and the stopping time $\tilde{\tau}_{y,b}$, we denote by $\tilde{h}^{\theta_\star}$ the function analogous to $h^{\theta_\star}$ which appears in the corresponding statements for the process $(M_{n-1,0}\cdot y,\tilde{S}_n(y,b))_{n\geq 1}$.

In the sequel, we set $\Delta:=\ln\delta$ where δ is the constant which appears in Lemma 2.2; the proof is detailed in [18].

Propositions 2.3 and 2.4 yield to following statement.

Corollary 2.5 *Assume hypotheses* **P1**–**P6** *hold. Then, there exists a positive constant* $C>0$ *such that for any* $\tilde{x}\in\tilde{\mathbb{X}},a,b\geq 0$ *and any* $\ell>0$*,*

$$\mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a}>n,S_n(\tilde{x},a)\in[b,b+\ell])\leq\frac{C}{n^{3/2}}\ h^{\theta_\star}(\tilde{x},a)\ \tilde{h}^{\theta_\star}(\tilde{x},b)\ \ell.\tag{2.3}$$

Furthermore, there exist constants $c>0$ *and* ℓ_0 *such that for* $\ell>\ell_0$ *and* $b\geq B$*,*

$$\liminf_{n\to+\infty}n^{3/2}\mathbb{P}^{\theta_\star}_{\tilde{x},a}(\tau_{\tilde{x},a}>n,S_n(\tilde{x},a)\in[b,b+\ell])\geq c\ h^{\theta_\star}(\tilde{x},a)\ \tilde{h}^{\theta_\star}(\tilde{x},b)\ \ell.\tag{2.4}$$

Proof of corollary 2.5. When studying fluctuations of random walks $(S_n)_{n\geq 1}$ with iid increments Y_k on $\mathbb{R}^p,p\geq 1$, we often reverse time as follows. For any $1\leq k\leq n$, the random variables $S_n-S_k=Y_{k+1}+\ldots+Y_n$ and $S_{n-k}=Y_1+\ldots+Y_{n-k}$ have the same distribution. In the case of products of random matrices, even when the M_i are iid (which is not the case under $\mathbb{P}^{\theta_\star}$), the cocycle property $S_n(\tilde{x})=S_n(\tilde{x},0)=\ln|\tilde{x}M_{0,n-1}|=S_k(\tilde{x})+S_{n-k}(\tilde{X}_k)$, with $\tilde{X}_k=\tilde{x}\cdot M_{0,k-1}$, decomposes the sum $S_n(\tilde{x})$ into two terms which are not independent. Hence, the same argument as in the iid case cannot be applied directly. The fact that the matrices $M_0,M_1,\ldots$ belong to

$\mathcal{S}_B$ helps here: for any $x, y \in \mathbb{X}$, we can compare the distribution of $S_n(\tilde{x}) - S_k(\tilde{x})$ to the one of $\ln |M_{n-k-1,0}y|$ (notice here that the non commutativity of the product of matrices forces us to consider in this last quantity the right linear action of the matrices $M_{0,n-k-1}$). It is the strategy that we apply to obtain the following result (see lemma 2.3 in [18]) which is central in the sequel.

Lemma 2.6 *For any $x, y \in \mathbb{X}, a, b \geq 0$ and $\ell > 0$,*

$$\mathbb{P}(\tau_{\tilde{x},a} > n, S_n(\tilde{x}, a) \in [b, b+\ell]) \leq \mathbb{P}(\tilde{\tau}_{y,\ b+\ell+\Delta} > n, \tilde{S}_n(y,\ b+\ell+\Delta) \in [a, a+\ell+2\Delta]).$$

Similarly, for $a \geq \ell > 2\Delta > 0$ and $b \geq \Delta$,

$$\mathbb{P}(\tau_{\tilde{x},a} > n, S_n(\tilde{x}, a) \in [b, b+\ell]) \geq \mathbb{P}(\tilde{\tau}_{y,\ b-\Delta} > n, \tilde{S}_n(y,\ b-B) \in [a-\ell, a-2\Delta]).$$

This allows us to apply the same strategy as in [18] to prove Corollary 2.5. Let us propose a sketch of the argument.
Proof. Inequality (2.3) is proved in [17] Corollary 3.7. The proof of the lower bound (2.4) is based on Proposition 2.4. We set $m = \lfloor n/2 \rfloor$; by the Markov property and Lemma 2.6,

$$\begin{aligned}
&\mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a} > n,\ S_n(\tilde{x}, a) \in [b, b+\ell]) \\
&\geq \mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a} > n, S_m(\tilde{x}, a) \in [\sqrt{n}, \sqrt{2n}], S_n(\tilde{x}, a) \in [b, b+\ell]) \\
&\geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq\ k\ \leq\sqrt{2n}-1}} \mathbb{P}^{\theta_\star}\Big(\tau_{\tilde{x},a} > n, k \leq S_m(\tilde{x}, a) \leq k+1,\ b \leq S_n(\tilde{x}, a) \leq b+\ell\Big) \\
&\geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq\ k\ \leq\sqrt{2n}-1}} \int_{\tilde{\mathbb{X}}\times[k,k+1]} \mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a} > m, (\tilde{x}\cdot M_{0,m-1}, S_m(\tilde{x}, a)) \in \mathrm{d}\tilde{x}'\mathrm{d}a') \\
&\qquad\qquad \mathbb{P}^{\theta_\star}(\tau_{\tilde{x}',a'} > n-m, b \leq S_{n-m}(\tilde{x}', a') \leq b+\ell) \\
&\geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq\ k\ \leq\sqrt{2n}-1}} \int_{\tilde{\mathbb{X}}\times[k,k+1]} \mathbb{P}^{\theta_\star}\Big(\tau_{\tilde{x},a} > m, (\tilde{x}\cdot M_{0,m-1}, S_m(\tilde{x}, a)) \in \mathrm{d}\tilde{x}'\mathrm{d}a') \\
&\qquad\qquad \mathbb{P}^{\theta_\star}(\tilde{\tau}_{x,b-\Delta} > n-m, a'-\ell \leq \tilde{S}_{n-m}(x, b-\Delta) \leq a'-2\Delta) \\
&\geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq\ k\leq\sqrt{2n}-1}} \mathbb{P}^{\theta_\star}\Big(\tau_{\tilde{x},a} > m, k \leq S_m(\tilde{x}, a) \leq k+1\Big) \\
&\qquad\qquad \mathbb{P}^{\theta_\star}\Big(\tilde{\tau}_{x,b-\Delta} > n-m, k+1-\ell \leq \tilde{S}_{n-m}(x, b-\Delta) \leq k-2\Delta\Big)
\end{aligned} \tag{2.5}$$

By Proposition 2.4, there exists a constant $C_0 > 0$ such that when $\sqrt{n} \leq\ k\ \leq \sqrt{2n}-1$,

$$\liminf_{n\to+\infty} n\mathbb{P}^{\theta_\star}\Big(\tau_{\tilde{x},a} > m, k \leq S_m(\tilde{x}, a) \leq k+1\Big) \geq C_0\ h^{\theta_\star}(\tilde{x}, a)$$

and

$$\liminf_{n\to+\infty} \mathbb{P}^{\theta_\star}\Big(\tilde{\tau}_{x,b-\Delta} > n-m, k+1-\ell \leq \tilde{S}_{n-m}(x, b-\Delta) \leq k-2\Delta\Big) \geq C_0(\ell-2\Delta-1)\ \underbrace{\tilde{h}^{\theta_\star}(\tilde{x}, b-\Delta)}_{\succeq\ \tilde{h}^{\theta_\star}(\tilde{x},b)}.$$

Hence, inequality (2.5) yields, for n large enough,

$$n^2\mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a} > n, S_n(\tilde{x},a) \in [b,b+\ell]) \geq \frac{C_0^2}{2}(\sqrt{2n}-\sqrt{n})(\ell-2\Delta-1)\ h^{\theta_\star}(\tilde{x},a)\ \tilde{h}^{\theta_\star}(\tilde{x},b),$$

which implies, for such n,

$$\mathbb{P}^{\theta_\star}(\tau > n, S_n \in [b,b+\ell]) \succeq \frac{h^{\theta_\star}(\tilde{x},a)\ \tilde{h}^{\theta_\star}(\tilde{x},b)}{n^{3/2}}(\ell-2\Delta-1).$$

This achieves the proof of the lower bound (2.4) taking $\ell_0 := 4\Delta+2$.

□

As a direct consequence, we obtain the following "rough local limit theorem" for the process $(S_n(\tilde{x},a))_{n\geq 0}$ and its minimum under the probability $\mathbb{P}$.

Corollary 2.7 *Under hypotheses* **P1**–**P6**, *there exist positive constants c, C and ℓ_0 such that for any $a \geq \ell \geq \ell_0, b \geq \Delta$ and any $x \in \mathbb{X}$,*

$$\mathbb{P}(\tau_{\tilde{x},a} > n, S_n(\tilde{x},a) \in [b,b+\ell]) \leq C\frac{\rho_\star^n}{n^{3/2}}\ h^{\theta_\star}(\tilde{x},a)\ \tilde{h}^{\theta_\star}(\tilde{x},b)\ e^{\theta_\star(a-b)}\quad \ell \tag{2.6}$$

and

$$\liminf_{n\to+\infty}\frac{n^{3/2}}{\rho_\star^n}\mathbb{P}(\tau_{\tilde{x},a} > n, S_n(\tilde{x},a) \in [b,b+\ell]) \geq c\ h^{\theta_\star}(\tilde{x},a)\ \tilde{h}^{\theta_\star}(\tilde{x},b)\ e^{\theta_\star(a-b-\ell)}\ell. \tag{2.7}$$

Similarly,

$$\mathbb{P}(\tau_{\tilde{x},a} > n) \leq C\frac{\rho_\star^n}{n^{3/2}}\ e^{\theta_\star a}\ h^{\theta_\star}(\tilde{x},a) \quad \textit{and} \quad \liminf_{n\to+\infty}\frac{n^{3/2}}{\rho_\star^n}\mathbb{P}(\tau_{\tilde{x},a} > n) \geq c\ e^{\theta_\star a}\ h^{\theta_\star}(\tilde{x},a). \tag{2.8}$$

Proof. By using definition of $\mathbb{E}^{\theta_\star}$ (see (2.2)) and the fact that the function $v_{\theta_\star}$ is non negative and continuous on $\tilde{\mathbb{X}}$, we may write

$$\mathbb{P}(\tau_{\tilde{x},a} > n, S_n(\tilde{x},a) \in [b,b+\ell]) \asymp \rho_\star^n\ e^{\theta_\star a}\ \mathbb{E}^{\theta_\star}[\tau_{\tilde{x},a} > n, e^{-\theta_\star S_n(\tilde{x},a)}\mathbf{1}_{[b,b+\ell]}(S_n(\tilde{x},a))]$$

with $e^{-\theta_\star(b+\ell)}\ \mathbf{1}_{[b,b+\ell]}(S_n(\tilde{x},a)) \leq e^{-\theta_\star S_n(\tilde{x},a)}\ \mathbf{1}_{[b,b+\ell]}(S_n(\tilde{x},a)) \leq e^{-\theta_\star b}\ \mathbf{1}_{[b,b+\ell]}(S_n(\tilde{x},a))$. Similarly $\mathbb{P}(\tau_{\tilde{x},a} > n) \asymp \rho_\star^n\ e^{\theta_\star a}\ \mathbb{E}^{\theta_\star}[\tau_{\tilde{x},a} > n, e^{-\theta_\star S_n(\tilde{x},a)}]$ with

$$\mathbb{E}^{\theta_\star}[\tau_{\tilde{x},a} > n, e^{-\theta_\star S_n(\tilde{x},a)}] \asymp \sum_{k\geq 0} e^{-\theta_\star k}\mathbb{P}^{\theta_\star}(\tau_{\tilde{x},a} > n, S_n(\tilde{x},a) \in [k,k+1[).$$

Inequalities (2.6), (2.7) and (2.8) follow, by applying Corollary 2.5.

□

3 Proof of the main theorem

3.1 Expression of non extinction probability

For every realization of the environmental sequence $\mathbf{f} = (f_0, f_1, \ldots)$ and $0 \leq k \leq n-1$, we define

$$F_{k,n-1} = f_k \circ \ldots \circ f_{n-1} \quad \text{and} \quad F_{n,n-1} = \mathrm{Id}.$$

For $1 \le i \le p$, we set $F^{(i)}_{k,n-1} = f^{(i)}_k \circ \ldots \circ f_{n-1}$. From the definition of the process $(Z_n)_{n\ge 0}$, it holds

$$\mathbb{E}[s^{Z_n} \mid Z_0 = \tilde{e}_i, f_0, \ldots f_{n-1}] = f^{(i)}_0 \circ \ldots \circ f_{n-1}(s) = F^{(i)}_{0,n-1}(s)$$

so that

$$\mathbb{E}[q^{(i)}_n] = \mathbb{P}(|Z_n| > 0 \mid Z_0 = \tilde{e}_i) = \mathbb{E}[1 - F^{(i)}_{0,n-1}(\mathbf{0})].$$

For a generating function f with corresponding mean matrix M and a matrix $\mathbf{a} = (\mathbf{a}(k,\ell))_{1\le k,\ell\le p}$ with positive entries, we set, for $s \in [0,1]^p$,

$$\psi_{f,\mathbf{a}}(s) := \frac{|\mathbf{a}|}{|\mathbf{a}(\mathbf{1} - f(s))|} - \frac{|\mathbf{a}|}{|\mathbf{a}M(\mathbf{1} - s)|}.$$

We fix $i \in \{1,\ldots,p\}$. Let $\mathbf{a}^{(i)}$ be the matrix with $\mathbf{a}^{(i)}(i,i) = 1$ and $\mathbf{a}^{(i)}(k,\ell) = 0$ for all $(k,\ell) \neq (i,i)$. Using the definition of the functions $\psi_{f,\mathbf{a}}$, we write for $n \ge 1$,

$$\frac{1}{1 - F^{(i)}_{0,n-1}(s)} = \frac{1}{|\mathbf{a}^{(i)}M_{0,n-1}(\mathbf{1} - s)|} + \sum_{k=0}^{n-2} \frac{1}{|\mathbf{a}^{(i)}M_{0,k}|}\psi_{f_k,\mathbf{a}^{(i)}M_{0,k-1}}(F_{k+1,n-1}(s)),$$

where, as previously $M_{0,k} = M_0 \ldots M_k$ for any $k \ge 0$. Consequently,

$$\begin{aligned}
\mathbb{E}[q^{(i)}_n] &= \mathbb{E}[1 - F^{(i)}_{0,n-1}(\mathbf{0})] \\
&= \mathbb{E}\left[\left(\frac{1}{|\mathbf{a}^{(i)}M_{0,n-1}|} + \sum_{k=0}^{n-2} \frac{1}{|\mathbf{a}^{(i)}M_{0,k}|}\underbrace{\psi_{f_k,\mathbf{a}^{(i)}M_{0,k-1}}(F_{k+1,n-1}(\mathbf{0}))}_{\eta_{k,n-1}}\right)^{-1}\right] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{n-1} \frac{1}{|\mathbf{a}^{(i)}M_{0,k}|}\underbrace{\psi_{f_k,\mathbf{a}^{(i)}M_{0,k-1}}(F_{k+1,n-1}(\mathbf{0}))}_{\eta_{k,n-1}}\right)^{-1}\right]
\end{aligned}$$

where the random variable $\eta_{k,n-1}$ is defined as above with the convention $\eta_{0,n-1} = 1$. By Lemma 3 in [5], it holds $0 \le \eta_{k,n-1} \le \text{Cste } p^2 \sum_{i=1}^p |B^{(i)}_k|/|M_k|^2$.

In the sequel, we also need a control of the lower bound of the random variables $\eta_{k,n-1}$. This is why we introduce the restrictive assumption **P7**, which readily implies, by a straightforward computation, that the random variables are bounded from below and above by non negative constants (see Proposition 2.1 and Lemma 3.1 in [3]). Therefore, under this additive assumption, it holds

$$\mathbb{E}[q^{(i)}_n] \asymp \mathbb{E}\left[\frac{1}{|M_{0,0}|^{-1} + \ldots + |M_{0,n-1}|^{-1}}\right]. \tag{3.1}$$

This is not an exact formula but it is sufficient in our context since we only have rough estimates in Corollary 2.7. We cannot improve our result as long as we do not have a precise estimate in the local theorem for the norms of products of random matrices conditioned to remain strictly greater than 1 until time n.

3.2 The change of measure $\widehat{\mathbb{P}}_{\tilde{x},a}$

In the case when the Lyapunov exponent of the sequence $(M_n)_{n\geq 0}$ equals 0, it is natural to introduce the following new probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ on the canonical path space $((\mathbb{X}\times\mathbb{R})^{\otimes\mathbb{N}}, \sigma(X_n, S_n : n\geq 0), \theta)$ of the Markov chain $(X_n, S_n)_{n\geq 0}$ [4], whose associated expectation $\widehat{\mathbb{E}}_{\tilde{x},a}$ is characterized by the following property: for any bounded and non negative Borel function φ on $(\tilde{\mathbb{X}}\times\mathbb{R})^{k+1}$ with compact support,

$$\widehat{\mathbb{E}}_{\tilde{x},a}[\varphi(X_0,S_0,\dots,X_k,S_k)] := \lim_{n\to+\infty}\mathbb{E}_{\tilde{x},a}[\varphi(X_0,\dots,S_k)\mid \tau>n],$$

when the limit exists. The fact that the Lyapunov exponent of the sequence $(M_n)_{n\geq 0}$ equals 0 implies that the probability $\mathbb{P}_{\tilde{x},a}(\tau>n)$ decreases towards 0 as $1/\sqrt{n}$ (see [19]) and the expectation $\widehat{\mathbb{E}}_{\tilde{x},a}$ is given explicitely as follows

$$\widehat{\mathbb{E}}_{\tilde{x},a}[\varphi(X_0,S_0,\dots,X_k,S_k)] = \frac{1}{h(\tilde{x},a)}\mathbb{E}_{\tilde{x},a}[\varphi(X_0,\dots,S_k)h(X_k,S_k),\ \tau>k],$$

for some non negative function h (see [16] for the details). Furthermore, if $(Y_k)_{k\geq 0}$ is a sequence of bounded real-valued random variables, adapted to the filtration $(\mathcal{F}_k)_{k\geq 0}$ and which converges in $\mathbb{L}^1(\widehat{\mathbb{P}}_{\tilde{x},a})$ to some random variable Y_∞, then

$$\lim_{n\to+\infty}\mathbb{E}_{\tilde{x},a}\big[Y_n\mid\tau>n\big] = \widehat{\mathbb{E}}_{\tilde{x},a}[Y_\infty]. \tag{3.2}$$

This last property yields to the speed of convergence to 0 of the extinction probability in a rather luminous way [16], [17].

In the present case, a similar construction does exist under the probability $\mathbb{P}^{\theta_\star}$; in this case, the sequence $(M_n)_{n\geq 0}$ is not iid but the choice of the parameter $\theta_\star$ is done in such a way the corresponding Lyapunov exponent $\Lambda'(\theta_\star)$ equals 0. Therefore, we introduce the following family of probability measures $\widehat{\mathbb{P}}^{\theta_\star}_{\tilde{x},a}$ (with corresponding expectation $\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{x},a}$) whose restriction to the σ-algebras $\sigma(X_0,S_0,\dots,X_k,S_k), k\geq 0$ are defined as follows: for any bounded and non negative Borel function φ on $(\mathbb{X}\times\mathbb{R})^{k+1}$ with compact support

$$\begin{aligned}\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{x},a}&[\varphi(X_0,S_0,\dots,X_k,S_k)]\\ &:= \frac{1}{h^{\theta_\star}(\tilde{x},a)}\mathbb{E}^{\theta_\star}_{\tilde{x},a}[\varphi(X_0,\dots,S_k)h^{\theta_\star}(X_k,S_k),\ \tau>k]\\ &= \frac{\mathbb{E}_{\tilde{x},a}\left[\varphi(X_0,\dots,S_k)h^{\theta_\star}(X_k,S_k)e^{\theta_\star S_k}v_{\theta_\star}(X_k),\ \tau>k\right]}{\rho_\star^k\ h^{\theta_\star}(\tilde{x},a)\ v_{\theta_\star}(\tilde{x})}\end{aligned} \tag{3.3}$$

The rough estimate of the quantity $\mathbb{P}_{\tilde{x},a}(\tau>n)$ given by (2.8) immediately yields

$$\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{x},a}[\varphi(X_0,S_0,\dots,X_k,S_k)] \asymp \frac{\mathbb{E}_{\tilde{x},a}\left[\varphi(X_0,\dots,S_k)h^{\theta_\star}(X_k,S_k)e^{\theta_\star S_k},\ \tau>k\right]}{\rho_\star^k\ h^{\theta_\star}(\tilde{x},a)}. \tag{3.4}$$

Let us conclude this paragraph noticing that the property (3.2) does not hold anymore, even in a weaker form, as the reader can see by following the demonstration of lemma 4.1 in [17]. The strategy to get the speed of convergence towards 0 of the survival probability is thus different and uses a clever decomposition of the denominator of the right hand side in (3.1), inspired by [15] and [7].

[4] θ denotes the shift operator on $(\mathbb{X}\times\mathbb{R})^{\otimes\mathbb{N}}$ defined by $\theta\Big((x_k,s_k)_{k\geq 0}\Big) = (x_{k+1},s_{k+1})_{k\geq 0}$ for any $(x_k,s_k)_{k\geq 0}$ in $(\mathbb{X}\times\mathbb{R})^{\otimes\mathbb{N}}$

3.3 Proof of Theorem 1.1

We set here $m_k := \min(|M_{0,0}|, \ldots, |M_{0,k-1}|) = \min(|\tilde{\mathbf{1}}M_{0,0}|, \ldots, |\tilde{\mathbf{1}}M_{0,k-1}|)$ for any $k \geq 1$. Hence $(m_n \geq e^{-a}) = (\tau_{\tilde{\mathbf{1}},a} > n)$ for any $a \geq 0$ and (2.7) and (2.8) may be restated in particular as follows: for $\ell > \ell_0$,

$$\liminf_{n\to+\infty} \frac{n^{3/2}}{\rho_\star^n} \mathbb{P}(\tau_{\tilde{x},a} > n, S_n(\tilde{x},a) \in [b, b+\ell]) \geq c h^{\theta_\star}(\tilde{x},a)\ \tilde{h}^{\theta_\star}(\tilde{x},b)\ e^{\theta_\star(a-b-\ell)}\ \ell \tag{3.5}$$

and

$$\mathbb{P}(m_n \geq e^{-a}) \leq\ C\ \frac{\rho_\star^n}{n^{3/2}}\ e^{\theta_\star a} h^{\theta_\star}(\tilde{\mathbf{1}}, a), \tag{3.6}$$

where c and C are two strictly positive constants.

3.3.1 Proof of the upper bound

It holds

$$\begin{aligned}
\mathbb{P}(|Z_n| > 0 \mid Z_0 = \tilde{e_i}) &= \mathbb{E}[q_n^{(i)}] \\
&\leq \mathbb{P}(m_n \geq 1) + \mathbb{E}[q_n^{(i)}; m_n < 1] \\
&\preceq \mathbb{P}(m_n \geq 1) + \mathbb{E}[m_n; m_n < 1] \\
&\leq \mathbb{P}(m_n \geq 1) + \sum_{j=1}^{+\infty} e^{-j+1}\mathbb{P}(e^{-j} \leq m_n < e^{-j+1}) \\
&\leq \mathbb{P}(m_n \geq 1) + \sum_{j=1}^{+\infty} e^{-j+1}\mathbb{P}(m_n \geq e^{-j}) \\
&\preceq \frac{\rho_\star^n}{n^{3/2}} + \frac{\rho_\star^n}{n^{3/2}} \sum_{j=1}^{+\infty} h^{\theta_\star}(\tilde{\mathbf{1}}, j)\ e^{(\theta_\star-1)j} \quad \text{by (3.6)}, \\
&\preceq \quad \frac{\rho_\star^n}{n^{3/2}} \quad \text{since} \quad h^{\theta_\star}(\tilde{\mathbf{1}}, j) \preceq 1 + j \quad \text{and} \quad \theta_\star < 1.
\end{aligned}$$

The upper bound is established.

3.3.2 Proof of the lower bound

We fix $0 < k < n/2$ and decompose $\displaystyle\sum_{\ell=0}^{n-1} \frac{1}{|M_{0,\ell}\ |}$ as $\mathbf{A} + \mathbf{B} + \mathbf{C}$ where

$$\mathbf{A} := \sum_{\ell=0}^{k-1} |M_{0,\ell}|^{-1}, \quad \mathbf{B} := \sum_{\ell=k}^{n-k-1} |M_{0,\ell}|^{-1} \quad \text{and} \quad \mathbf{C} := \sum_{\ell=n-k}^{n-1} |M_{0,\ell}|^{-1}.$$

It holds

$$\begin{aligned}
\mathbb{P}(|Z_n| > 0 \mid Z_0 = \tilde{e_i}) &= \mathbb{E}[q_n^{(i)}] \\
&\geq \mathbb{E}\left[\frac{1}{1 + \mathbf{A} + \mathbf{B} + \mathbf{C}}; m_n \geq 1\right] \\
&\geq \mathbb{E}\left[\frac{1}{1 + \mathbf{A} + \mathbf{C}}; m_n \geq 1\right] - \mathbb{E}\left[\mathbf{B}; m_n \geq 1\right].
\end{aligned}$$

First step: control of the term $\mathbb{E}\left[\frac{1}{1+\mathbf{A}+\mathbf{C}}; m_n \geq 1\right]$

It holds

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{\mathbf{A}+\mathbf{C}}; m_n \geq 1\right] &\geq \mathbb{E}[\frac{1}{1+\mathbf{A}+\sum_{\ell=n-k}^{n-1}|M_{0,\ell}|^{-1}};\\
&\qquad m_{n-k} \geq 1, |M_{n-k,n-1}| \leq 1, \ldots, |M_{n-1,n-1}| \leq 1]\\
&\quad \text{since } [m_{n-k} \geq 1, |M_{n-k,n-1}| \leq 1, \ldots, |M_{n-1,n-1}| \leq 1] \subset [m_n \geq 1]\\
&\geq \mathbb{E}[\frac{1}{2+\mathbf{A}+\sum_{\ell=n-k}^{n-1}|M_{\ell,n-1}|};\\
&\qquad m_{n-k} \geq 1, |M_{n-k,n-1}| \leq 1, \ldots, |M_{n-1,n-1}| \leq 1]\\
&\quad \text{since } |M_{\ell+1,n-1}| \geq |M_{0,\ell}|^{-1}, n-k \leq \ell \leq n-2 \text{ and } |M_{0,n-1}|^{-1} \leq 1\\
&\quad \text{on the event } [|M_{0,n-1}| \geq 1]\\
&\geq \mathbb{E}\left[\frac{1}{2+\mathbf{A}+\sum_{\ell=n-k}^{n-1}|M_{\ell,n-1}|};\right.\\
&\qquad \left. m_{n-k} \geq 1, 1 \leq |M_{0,n-k-1}| \leq K, \frac{1}{K} \leq |M_{n-k,n-1}| \leq 1, \ldots, |M_{n-1,n-1}| \leq 1\right]\\
&\quad \text{for any constant } K > 1,\\
&\geq \mathbb{E}\left[\frac{1}{2+\mathbf{A}+\sum_{\ell=1}^{k}|M'_{\ell,1}|};\right.\\
&\qquad \left. m_{n-k} \geq 1, 1 \leq |M_{0,n-k-1}| \leq K, \frac{1}{K} \leq |M'_{k,1}| \leq 1, \ldots, |M'_{1,1}| \leq 1\right]\\
&\quad \text{by setting } (M_{n-k}, \ldots, M_{n-1}) = (M'_k, \ldots, M'_1) \text{ and } \quad M'_{\ell,1} = M'_\ell \ldots M'_1\\
&= \mathbb{E}\left[\frac{1}{2+\mathbf{A}+\mathbf{C}'}; m_{n-k} \geq 1, 1 \leq |M_{0,n-k-1}| \leq K, \frac{1}{K} \leq |M'_{k,1}| \leq 1, \mathbf{M}'_k \leq 1\right]\\
&\quad \text{with } \mathbf{C}' := \sum_{\ell=1}^{k}|M'_{\ell,1}| \text{ and } \mathbf{M}'_k := \max(|M'_{1,1}|, \ldots, |M'_{k,1}|)\\
&\geq \frac{1}{2}\mathbb{E}\left[\frac{1}{1+\mathbf{A}} \times \frac{1}{1+\mathbf{C}'}; m_{n-k} \geq 1, 1 \leq |M_{0,n-k-1}| \leq K,\right.\\
&\qquad\qquad \left.\frac{1}{K} \leq |M'_{k,1}| \leq 1, \mathbf{M}'_k \leq 1\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\mathbb{E}\left[\frac{1}{1+\mathbf{A}}; m_{n-k} \geq 1, 1 \leq |M_{0,n-k-1}| \leq K\right]\\
&\qquad\qquad \times \mathbb{E}\left[\frac{1}{1+\mathbf{C}'}; \frac{1}{K} \leq |M'_{k,1}| \leq 1, \mathbf{M}'_k \leq 1\right]
\end{aligned}
$$

by independence of the random variables $\frac{1}{1+\mathbf{A}}\mathbf{1}_{[m_{n-k}\geq 1, 1 \leq |M_{0,n-k-1}| \leq K]}$ and $\frac{1}{1+\mathbf{C}'}\mathbf{1}_{[\frac{1}{K}|M'_{k,1}| \leq 1, \mathbf{M}'_k \leq 1]}$.

Now, we assume $\ln K \geq \ell_0$, where ℓ_0 is defined in Lemma 2.6; it holds, on the one hand,

$$\begin{aligned}
\liminf_{n\to+\infty} \frac{n^{3/2}}{\rho_\star^{n-k}} \mathbb{E}\left[\frac{1}{1+\mathbf{A}}; m_{n-k} \geq 1, 1 \leq |M_{0,n-k-1}| \leq K\right] \\
&\geq \liminf_{n\to+\infty} \frac{n^{3/2}}{\rho_\star^{n-k}} \mathbb{E}\left[\frac{1}{1+\mathbf{A}}; m_k \geq 1, \mathbb{P}_{\tilde{x}_k, a_k}(m_{n-2k}\circ\theta^k \geq 1, 1 \leq |M_{0,n-2k}\circ\theta^k| \leq K)\right] \\
&\text{with } \tilde{x}_k = \tilde{\mathbf{1}}\cdot M_{0,k-1} \text{ and } a_k = \ln|M_{0,k-1}| \\
&\geq \rho_\star^{-k}\mathbb{E}\left[\frac{1}{1+\mathbf{A}}; m_k \geq 1, \right. \\
&\qquad \left. \underbrace{\liminf_{n\to+\infty} \frac{n^{3/2}}{\rho_\star^{n-2k}} \mathbb{P}_{\tilde{x}_k, a_k}(m_{n-2k}\circ\theta^k \geq 1, 1 \leq |M_{0,n-2k}\circ\theta^k| \leq K)}_{\geq c\ h^{\theta_\star}(\tilde{x}_k, a_k)\ \tilde{h}^{\theta_\star}(\tilde{x}_k, 0) e^{\theta_\star(a_k - \ln K)}\ \ \ln K\ >0 \text{ by } (3.5)}\right] \\
&\geq c\ \frac{\ln K}{K^{\theta_\star}} \quad \underbrace{\rho_\star^{-k}\mathbb{E}\left[\frac{1}{1+\mathbf{A}}|M_{0,k-1}|^{\theta_\star}; m_k \geq 1, h^{\theta_\star}(\tilde{\mathbf{1}}\cdot M_{0,k-1}, \ln|M_{0,k-1}|)\right]}_{\asymp\ \ \widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[\frac{1}{1+\mathbf{A}}\right]}
\end{aligned}$$

On the other hand, since the function $\tilde{h}^{\theta_\star}$ associated with the process $(M'_{n,1}\cdot x, b+\ln|M'_{n,1}x|)_{n\geq 1}$ satisfies $\tilde{h}^{\theta_\star}(x,b) \asymp 1$ for b in a compact set, it holds

$$\rho_\star^{-k}\mathbb{E}\left[\frac{1}{1+\mathbf{C}'}; \frac{1}{K} \leq |M'_{k-1,1}| \leq 1, \mathbf{M}'_{k-1} \leq 1\right] \geq c\ (\ln K)\ \widehat{\mathbb{E}}^{\theta_\star}_{\mathbf{1},0}\left[\frac{1}{1+\mathbf{C}'}\right]$$

Finally, by (3.4),

$$\liminf_{n\to+\infty} \frac{n^{3/2}}{\rho_\star^{n}} \mathbb{E}\left[\frac{1}{\mathbf{A}+\mathbf{C}}; m_n \geq 1\right] \succeq (\ln K)\ \widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[\frac{1}{1+\mathbf{A}}\right] \times \widehat{\mathbb{E}}^{\theta_\star}_{\mathbf{1},0}\left[\frac{1}{1+\mathbf{C}'}\right]. \tag{3.7}$$

Second step: control of the term $\mathbb{E}[\mathbf{B}; m_n \geq 1]$

$$\begin{aligned}
\mathbb{E}[\mathbf{B}; m_n \geq 1] &= \sum_{\ell=k}^{n-k-1} \mathbb{E}\left[|M_{0,\ell}|^{-1}; m_n \geq 1\right] \\
&\leq \sum_{j\geq 0}\sum_{\ell=k}^{n-k-1} 2^{-j}\mathbb{P}\left(|M_{0,\ell}|^{-1} \in [2^j, 2^{j+1}[; m_n \geq 1\right) \\
&\leq \sum_{j\geq 0}\sum_{\ell=k}^{n-k-1} 2^{-j}\mathbb{P}(|M_{0,\ell}|^{-1} \in [2^j, 2^{j+1}[; m_\ell \geq 1, \\
&\qquad\qquad |M_{0,\ell+1}| \geq 1, \ldots, |M_{0,n-1}| \geq 1)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\geq 0}\sum_{\ell=k}^{n-k-1} 2^{-j}\mathbb{P}(|M_{0,\ell}|^{-1}\in[2^j,2^{j+1}[; m_\ell\geq 1,\\
&\qquad\qquad\qquad\qquad |M_{\ell+1,\ell+1}|\geq 2^{-j},\ldots,|M_{\ell+1,n-1}|\geq 2^{-j})\\
&= \sum_{j\geq 0}\sum_{\ell=k}^{n-k-1} 2^{-j}\underbrace{\mathbb{P}(|M_{0,\ell}|^{-1}\in[2^j,2^{j+1}[; m_\ell\geq 1)}_{\asymp\frac{\rho_\star^\ell}{\ell^{3/2}}}\underbrace{\mathbb{P}(m_{n-\ell-1}\geq 2^{-j})}_{\asymp j\frac{\rho_\star^{n-\ell}}{(n-\ell)^{3/2}}}\\
&\preceq \frac{\rho_\star^n}{n^{3/2}}\left(\sum_{j\geq 0} j2^{-j}\right)\underbrace{\left(\sum_{\ell=k}^{n-k-1}\frac{n^{3/2}}{\ell^{3/2}(n-\ell)^{3/2}}\right)}_{\preceq\frac{1}{\sqrt{k}}}
\end{aligned}
$$

Last step: choosing k large enough

By the above $\sup_{n\geq}\frac{n^{3/2}}{\rho_\star^n}\mathbb{E}\left[\mathbf{B}; m_n\geq 1\right]\longrightarrow 0$ as $k\to+\infty$. It remains to check that the factors $\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[\frac{1}{1+\mathbf{A}}\right]$ and $\widehat{\mathbb{E}}^{\theta_\star}_{\mathbf{1},0}\left[\frac{1}{1+\mathbf{C}'}\right]$ in the right hand side of (3.7) do not converge to 0 as $k\to+\infty$.

By (3.4) and by convexity of the function $x\mapsto\frac{1}{x}$ on $\mathbb{R}^{*+}$, it holds

$$
\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[\frac{1}{1+\mathbf{A}}\right]\geq\frac{1}{\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}[1+\mathbf{A}]}\geq\frac{1}{\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[1+\sum_{\ell=0}^{+\infty}|\tilde{\mathbf{1}}M_{0,\ell}|^{-1}\right]}.
$$

and it suffices to check that $\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[\sum_{\ell=0}^{+\infty}|\tilde{\mathbf{1}}M_{0,\ell}|^{-1}\right]<+\infty$. Indeed, for any $\ell\geq 0$,

$$
\begin{aligned}
\widehat{\mathbb{E}}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[|\tilde{\mathbf{1}}M_{0,\ell}|^{-1}\right] &\preceq \mathbb{E}^{\theta_\star}_{\tilde{\mathbf{1}},0}\left[|M_{0,\ell}|^{-1}h^{\theta_\star}(X_{\ell+1},S_{\ell+1}),\tau>\ell+1\right]\\
&\preceq \mathbb{E}^{\theta_\star}\left[|M_{0,\ell}|^{-1}(1+\ln|M_{0,\ell}|), m_{\ell+1}\geq 1\right]\\
&\leq \sum_{j=0}^{+\infty}(1+j)e^{-j}\underbrace{\mathbb{P}^{\theta_\star}(e^j\leq|M_{0,\ell}|<e^{j+1}, m_{\ell+1}\geq 1)}_{=\mathbb{P}^{\theta_\star}_{\tilde{\mathbf{1}},0}(j\leq S_{\ell+1}<j+1,\tau>\ell+1)}\\
&\leq \frac{1}{\ell^{3/2}}\sum_{j=0}^{+\infty}(1+j)^2e^{-j}\quad\text{by inequality (2.3),}
\end{aligned}
$$

which yields the expected result. The same argument works for the factor $\widehat{\mathbb{E}}^{\theta_\star}_{\mathbf{1},0}\left[\frac{1}{1+\mathbf{C}'}\right]$ by considering the random variables $\tilde{S}_\ell(\mathbf{1},0)$ and $\tilde{\tau}_{\mathbf{1},0}$ instead of $S_\ell(\tilde{\mathbf{1}},0)$ and $\tau_{\tilde{\mathbf{1}},0}$.

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