A LOCAL LIMIT THEOREM FOR CONVERGENT RANDOM WALKS ON RELATIVELY HYPERBOLIC GROUPS

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Abstract. We study random walks on relatively hyperbolic groups whose law is *convergent*, in the sense that the derivative of its Green function is finite at the spectral radius. When parabolic subgroups are virtually abelian, we prove that such a random walk satisfies a local limit theorem of the form $p_n(e,e) \sim CR^{-n}n^{-d/2}$, where $p_n(e,e)$ is the probability of going back to the origin at time n, R is the inverse of the spectral radius of the random walk and d is the minimal rank of a parabolic subgroup along which the random walk is spectrally degenerate.

1. Introduction

1.1. **General setting.** Consider a finitely generated group Γ and a probability measure μ on Γ. The μ-random walk on Γ starting at $x \in \Gamma$ is defined as

$$X_n^x = xg_1...g_n,$$

where (g_k) are independent random variables in Γ distributed as μ. The law of X_n^x is denoted by $p_n(x,\cdot)$. For $x = e$, it is given by the convolution powers μ^{*n} of the measure μ. The *Local Limit problem* consists in finding the asymptotic behavior of $p_n(x,y)$ when n goes to infinity.

The action by isometries of a discrete group on a Gromov-hyperbolic space (X,d) is said to be *geometrically finite* if for any $o \in X$, the accumulation points of Γo on the Gromov boundary ∂X are either *conical limit points* or *bounded parabolic limit points*. We refer to Section 2.1 below for a definition of these notions. A finitely generated group Γ is *relatively hyperbolic* with respect to a collection of subgroups $\Omega = \{\mathcal{H}_1, ..., \mathcal{H}_p\}$ if it acts via a *geometrically finite action* on a proper geodesic Gromov hyperbolic space X, such that, up to conjugacy, Ω is exactly the set of stabilizers of parabolic limit points for this action. The conjugates of the elements of Ω are called (maximal) *parabolic subgroups*. We will often assume that parabolic subgroups are virtually abelian.

In this paper, we prove a local limit theorem for a special class of random walks on relatively hyperbolic groups. We always assume in the sequel that μ is *admissible*, i.e. its support generates Γ as a semigroup, *symmetric*, i.e. $\mu(g) = \mu(g^{-1})$ for every g, and *aperiodic* i.e. $p_n(e,e) > 0$ for large enough n.

On the one hand, it is known that aperiodic random walks with exponential moments on *virtually abelian groups* of rank d satisfy the following local limit theorem,

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see [46, Theorem 13.12] and references therein :

$$p_n(e,e) \sim CR^{-n}n^{-d/2}, \tag{1.1}$$

where C is a positive constant and $R \geq 1$ is the inverse of the spectral radius of the random walk.

On the other hand, Gouëzel [22] proved that for finitely supported, aperiodic and symmetric random walks on *non-elementary hyperbolic groups*, the local limit theorem always has the following form :

$$p_n(e,e) \sim CR^{-n}n^{-3/2}, \tag{1.2}$$

where, again, C is a positive constant and R the inverse of the spectral radius of the random walk. Notice that $R > 1$ since non-elementary hyperbolic groups are non-amenable, see [28].

On *relatively hyperbolic groups*, the first author proved in [16] that the local limit theorem (1.2) still holds provided the random walk is *spectrally nondegenerate*. This notion is precisely introduced in Definition 3.1 below, see also [17, Definition 2.3]. Roughly speaking, the random walk is *spectrally degenerate along a parabolic subgroup* $\mathcal{H}$ if a suited induced random walk on $\mathcal{H}$ (defined in Section 3) reaches its spectral radius when the initial random walk on Γ reaches its own spectral radius.

In such case, the singularities of the Green function, hence the asymptotics of the convolution powers of μ, are impacted by the singularities of the induced Green functions on this parabolic subgroup. We make further devoted comments in Section 3.2.

On the contrary, a random walk is said to be *spectrally nondegenerate* when it is not spectrally degenerate along any parabolic subgroups in Ω. The spirit of the result in [16] is that a spectrally nondegenerate random walk mainly sees the underlying hyperbolic structure of the group. In contrast, for spectrally degenerate random walks, one would expect to see in the local limit theorem the appearance of a competition between the exponents $d/2$ and $3/2$, related to the competition between parabolic subgroups and the underlying hyperbolic structure.

The simplest examples of relatively hyperbolic groups are free products. Candellero and Gilch [8] gave an almost complete classification of local limit theorems that can occur for nearest neighbor random walks on free products of finitely many abelian groups. In this context, the free factors play the role of parabolic subgroups. They indeed proved that whenever the random walk gives enough weight to the free factors, the local limit theorem is given by (1.1) as in the abelian case, whereas it is of the form (1.2) in most remaining cases, see in particular the many examples given in [8, Section 7].

Our paper is devoted to the general study of local limit theorems for the so called *convergent* random walks on a relatively hyperbolic group. In this case, the parabolic subgroups have the maximal possible influence on the random walk and the singularities of the Green function on Γ are actually governed by the singularities of the induced Green functions on parabolic subgroups. We give more details in Section 3.2.

Our main results are proved when parabolic subgroups are abelian. Nevertheless, let us emphasize that several intermediary results remain valid for any convergent random walk and we expect that our work can be extended to more general classes of parabolic subgroups. We discuss this problem in Section 8.

1.2. **Main results.** Let μ be an admissible probability measure on a relatively hyperbolic group Γ. Denote by R_μ the inverse of its spectral radius, which is the radius of convergence of the Green function $G(x,y|r)$, defined as

$$G(x,y|r) = \sum_{n\geq 0} p_n(x,y)r^n.$$

This radius of convergence is independent of x and y.

Definition 1.1. *Let Γ be a relatively hyperbolic group and let μ be a probability measure on Γ. We say that μ, or equivalently the random walk driven by μ, is* convergent *if*

$$\frac{d}{dr}_{|r=R_\mu} G(e,e|r) < +\infty.$$

Otherwise, μ is said to be divergent*.*

All non-elementary cases presented above which show a local limit theorem of abelian type (1.1) come from a convergent random walk.

This terminology was introduced in [15]. It comes from the strong analogy between *random walks on relatively hyperbolic groups* on the one hand and the *geodesic flow on geometrically finite negatively curved manifolds* on the other hand. We discuss this analogy in Section 1.4. Spectrally nondegenerate random walks on relatively hyperbolic groups are always divergent as shown in [15, Proposition 5.8].

In particular, if μ is convergent, then it is necessarily spectrally degenerate along some parabolic subgroup. Moreover, whenever parabolic subgroups are virtually abelian, each of them has a well-defined rank.

Definition 1.2. *Let Γ be a relatively hyperbolic group with respect to virtually abelian subgroups and let μ be a convergent probability measure on Γ. The* rank of spectral degeneracy *of μ is the minimal rank of a parabolic subgroup along which μ is spectrally degenerate.*

By the results of Candellero and Gilch [8, Section 7], convergent measures do exist. We do not attempt in this paper to systematically construct such a measure on any relatively hyperbolic group with virtually abelian parabolic subgroups, but we give more details on Candellero and Gilch's constructions in Section 3.2. The central result of our paper is the following local limit theorem.

Theorem 1.3. *Let Γ be a finitely generated relatively hyperbolic group with respect to virtually abelian subgroups. Let μ be a finitely supported, admissible, symmetric and convergent probability measure on Γ. Assume that the corresponding random walk is aperiodic. Let d be the rank of spectral degeneracy of μ. Then for every $x,y\in\Gamma$ there exists $C_{x,y}>0$ such that*

$$p_n(x,y) \sim C_{x,y}R_\mu^{-n}n^{-d/2}.$$

If the μ-random walk is not aperiodic, similar asymptotics hold for $p_{2n}(x,y)$ if the distance between x and x' is even and for $p_{2n+1}(x,y)$ if this distance is odd.

Note that by [15, Proposition 6.1], the rank of any virtually abelian parabolic subgroup along which μ is spectrally degenerate is at least 5. Therefore this local limit theorem cannot coincide with the one given by (1.2) when μ is spectrally nondegenerate. We also get the following corollary.

Corollary 1.4. *Let Γ be a finitely generated relatively hyperbolic group with respect to virtually abelian subgroups. Let μ be a finitely supported, admissible, symmetric and convergent probability measure on Γ such that the corresponding random walk is aperiodic. Let d be the rank of spectral degeneracy of μ. Denote by $q_n(x,y)$ the probability that the first visit to y in positive time starting at x is at time n. Then for every $x,y \in \Gamma$ there exists $C'_{x,y} > 0$ such that*

$$q_n(x,y) \sim C'_{x,y} R_\mu^{-n} n^{-d/2}.$$

Basically, the proof of Theorem 1.3 relies on two main steps. First, we give a precise expansion of the Green functions of induced transition kernels on parabolic subgroups. Second, we compare the singularities of these induced Green functions with the singularities of the Green function associated with the initial random walk. It follows from the proof that the main result of this second step, namely Theorem 5.4, can be shown without assuming that parabolic subgroups are virtually abelian, but as soon as (Γ, μ) satisfy two conditions :

- The Martin boundary of (Γ, μ) is *stable* in the sense of Definition 2.6;
- The Martin boundary of the first return kernel to any dominant parabolic subgroup is reduced to a point at the spectral radius.

An important part in our work, which is of independent interest, is thus the following fact.

Theorem 1.5 (Theorem 4.1)**.** *Let Γ be a finitely generated relatively hyperbolic group with respect to virtually abelian subgroups. Let μ be a finitely supported, admissible and symmetric probability measure on Γ. Then the Martin boundary of (Γ, μ) is stable.*

This complements the results of [17]. Along Section 4, we prove precise results on the asymptotics of the Green function at the spectral radius in a finite extension of a virtually abelian finitely generated group, which show this stability. See precisely Proposition 4.2.

1.3. **Current classification of local limit theorems.** We discuss here some partial results that we get for non-convergent random walks.

For all $k \in \mathbb{N}$, we write

$$I^{(k)}(r) = \sum_{x_1,...,x_k \in \Gamma} G(e,x_1|r)G(x_1,x_2|r)...G(x_{k-1},x_k|r)G(x_k,e|r).$$

It follows from Lemma 2.7 that (Γ, μ) is convergent if and only if $I^{(1)}(R_\mu) < +\infty$. For all parabolic subgroup $\mathcal{H} < \Gamma$, we write

$$I^{(k)}_{\mathcal{H}}(r) = \sum_{x_1,...,x_k \in \mathcal{H}} G(e,x_1|r)G(x_1,x_2|r)...G(x_{k-1},x_k|r)G(x_k,e|r).$$

The following terminology was introduced in [15].

Definition 1.6. *A symmetric admissible and finitely supported random walk μ on a relatively hyperbolic group Γ is said to be* spectrally positive recurrent *if*

(1) μ is divergent, i.e. $I^{(1)}(R_\mu) = +\infty$;
(2) for all parabolic subgroup $\mathcal{H} < \Gamma$,

$$I^{(2)}_{\mathcal{H}}(R_\mu) < +\infty.$$

Any random walk which is spectrally nondegenerate is spectrally positive recurrent, see [15, Proposition 3.7]. Once again, in our setting, this terminology was inspired by the close analogy with the study of the geodesic flow on negatively curved manifolds (see Section 1.4 below and [15, Section 3.3] for more details). In fact, the terminology *positive recurrent* is classical for the study of countable Markov shift, see for instance to [43], [27] and [37]. Note that the analogous notion of spectral nondegeneracy is given for countable Markov shifts by the notion of *strong positive recurrence*, also called *stable positive recurrence*, see [27] or [38].

We discuss in Section 7 the relationship between divergence and spectral positive recurrence of the random walk. As a matter of fact, when parabolic subgroups are virtually abelian, both notions are equivalent unless the random walk is spectrally degenerate along some parabolic subgroup of rank 5 or 6, see Proposition 7.1. This allows us to classify almost all possible behaviors for $p_n(e,e)$ on a relatively hyperbolic group whose parabolic subgroups are virtually abelian, as illustrated by the following corollary, see also the table in Section 1.4.

Notation. For two functions f and g, we write $f \lesssim g$ if there exists a constant C such that $f \leq Cg$. Also, we write $f \asymp g$ if both $f \lesssim g$ and $g \lesssim f$. If the implicit constants depend on a parameter and the dependency is not clear from the context, we avoid this notation.

Corollary 1.7. *Let Γ be a relatively hyperbolic group whose parabolic subgroups are virtually abelian. Let μ be a finitely supported admissible symmetric probability on Γ. Assume that μ is not spectrally degenerate along any parabolic subgroup of rank* 5 *or* 6*. Then one of the following possibilities occurs.*

[15]**, Theorem 1.4:** *If μ is* spectrally positive recurrent*, then as $n \to +\infty$,*

$$p_n(e,e) \asymp CR^{-n}n^{-3/2}.$$

[16]**, Theorem 1.1:** *Furthermore, if μ is* spectrally nondegenerate*, then as $n \to +\infty$,*

$$p_n(e,e) \sim CR^{-n}n^{-3/2}.$$

Theorem 1.3: *If μ is* convergent*, then as $n \to +\infty$,*

$$p_n(e,e) \sim CR^{-n}n^{-d/2},$$

where d is the rank of spectral degeneracy of μ.

We conjecture that for any spectrally positive recurrent walk (even spectrally degenerate), there should be a local limit theorem $p_n(e,e) \sim CR^{-n}n^{-3/2}$. If Γ has parabolic subgroups which are virtually abelian of rank 5 or 6, it is possible to construct examples of random walks which are divergent but not spectrally positive recurrent, whose local limit theorems are not classified in this corollary. Examples of such groups with their corresponding local limit theorems are studied in a separate paper [19], see also Remark 7.1 at the end of Section 7.

1.4. **Geodesic flow on negatively curved manifolds and random walks.** For convenience of the reader, we present now some results on the ergodic properties of the geodesic flow on geometrically finite manifolds with negative curvature, which greatly influenced this work.

Let $(M,g) = (\tilde{M},g)/\Gamma$ be a complete Riemannian manifold, where $\Gamma = \pi_1(M)$ acts discretely by isometries on the universal cover $(\tilde{M},g)$. Assume that M has pinched negative curvature, i.e. its sectional curvatures κ_g satisfy the inequalities

$-b^2 \leq \kappa_g \leq -a^2 < 0$ for some constants $b > a > 0$. Also assume that the action of Γ on $(\tilde{M}, d_g)$ is *geometrically finite* (see Section 2.1 below).

By definition Γ is then relatively hyperbolic with respect to a finite family of parabolic subgroups $\mathcal{H}_1, ..., \mathcal{H}_p$. Moreover, the pinched curvature hypothesis implies that the $\mathcal{H}_k$ are virtually nilpotent. The interested reader can find in [6] several other equivalent definitions of geometrical finiteness in the context of smooth negatively curved manifolds.

In this context, the Poincaré series associated with the group action is defined for $x, y \in \tilde{M}$ as $P_\Gamma(x, y|s) = \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd_g(x,\gamma y)}$. More generally, for each subgroup H of Γ, the *Poincaré series* of H at s is

$$P_H(x, y|s) = \sum_{\gamma\in H} \mathrm{e}^{-sd_g(x,\gamma y)} \in (0, +\infty].$$

There exists $\delta_H \geq 0$ independent of x, y such that this series converges if $s > \delta_H$ and diverges if $s < \delta_H$. This quantity is called the *critical exponent* of H. The action of H on $(\tilde{M}, g)$ is called *convergent* if $P_H(x, y|\delta_H) < +\infty$, and *divergent* otherwise.

Let μ be a random walk on Γ. We define for $x, y \in \Gamma$ the *symmetrized r-Green distance* by

$$d_r(x, y) = \log\left(\frac{G_\mu(x, y|r)}{G_\mu(e, e|r)}\right) + \log\left(\frac{G_\mu(y, x|r)}{G_\mu(e, e|r)}\right). \tag{1.3}$$

This (signed) distance was introduced in [15] and it is an elaborated version of the classical Green distance defined by Blachère and Brofferio [5].

By Lemma 2.7 below, for all $x, y \in \Gamma$,

$$P_\mu(x, y|r) = \sum_{\gamma\in\Gamma} e^{d_r(x,\gamma y)} = \frac{1}{G(e, e|r)^2}\sum_{z\in\Gamma} G(x, z|r)G(z, y|r) \asymp \frac{d}{dr}_{|r=R_\mu} G(x, y|r).$$

As emphasized by our notation, the classical Poincaré series $P_\Gamma(x, y|s)$ of Γ is analogous in the context of group action on a metric space to the series $P_\mu(x, y|r)$ associated with the random walk on Γ, which is of same order as the r-derivative of the Green function.

The Riemannian metric g plays the role of the law μ and the critical exponent δ_Γ of the Poincaré series plays the role of the logarithm of the radius of convergence of the Green function.

The local limit theorem describes the asymptotic behavior as $n \to +\infty$ of the quantity $p^{(n)}(x, y)$ for any $x, y \in \Gamma$; in the geometrical setting, it is replaced by the *orbital counting asymptotic*, that is the asymptotic behavior as $R \to +\infty$ of the orbital function $N_\Gamma(x, y, R)$ defined for all $x, y \in \tilde{M}$ by

$$N_\Gamma(x, y, R) := \#\left\{\gamma \in \Gamma\ ;\ d(x, \gamma y) \leq R\right\}.$$

The following definition, which is similar to Definition 1.6 above, comes from the results of [10] even though the terminology has been fixed in [34] (in the full general setting of negatively curved manifolds, not necessarily geometrically finite).

Definition 1.8. *Let $(M, g) = (\tilde{M}, g)/\Gamma$ be a geometrically finite Riemannian manifold with pinched negative curvature, where $\Gamma = \pi_1(M)$. Let $o \in \tilde{M}$ be fixed. The action of Γ on $(\tilde{M}, g)$ is said to be* positive recurrent *if*

(1) the action of Γ on $(\tilde{M}, g)$ is divergent;

(2) for all parabolic subgroup $\mathcal{H} \subset \Gamma$,

$$\sum_{h \in \mathcal{H}} d(o, h.o) e^{-sd(o,po)} < +\infty.$$

We refer to [34, Definition 1.3] for a definition of positive recurrence beyond geometrically finite manifolds. The action of Γ is said to be *strongly positive recurrent* - in the literature, one also says that Γ has a *critical gap* - if for all parabolic subgroup $\mathcal{H} \subset \Gamma$, we have $\delta_{\mathcal{H}} < \delta_\Gamma$. This is similar to the notion of *spectral nondegeneracy* for random walks on relatively hyperbolic groups.

Theorem A of [10] shows that strongly positive recurrent actions are positive recurrent (only the divergence is non-trivial). This has later been shown for more general negatively curved manifolds in [40, Theorem 1.7] and the analogous result for random walks is given by Proposition 3.7 of [15].

Moreover, Theorem B of [10] shows that the action is positive recurrent if and only if the geodesic flow admits an invariant probability measure of maximal entropy. This has been shown for general negatively curved manifolds in [34, Theorem 1.4]. Combined with Theorem 4.1.1 of [36], it gives the following asymptotic counting.

Theorem 1.9. *Let $(M, g) = (\tilde{M}, g)/\Gamma$ be a negatively curved manifold.*

- *If the action of Γ is positive recurrent, then for all $x, y \in \tilde{M}$, there is $C_{xy} > 0$ such that, as $R \to +\infty$,*
$$N_\Gamma(x, y, R) \sim C_{xy} e^{\delta_\Gamma R}.$$
- *If the action of Γ is not positive recurrent, then for all $x, y \in \tilde{M}$,*
$$N_\Gamma(x, y) = o\left(e^{\delta_\Gamma R}\right).$$

Getting precise asymptotics of $N_\Gamma(x, y, R)$ when the action of Γ is not positive recurrent is in general difficult. To the authors' knowledge, the only known examples are abelian coverings (cf [35]), which are not geometrically finite, and geometrically finite Schottky groups whose parabolic factors have counting functions satisfying some particular tail condition and for which asymptotics have been obtained in [11], [44] and [32]. Recall that Schottky groups are free products of elementary groups whose limit sets are at a positive distance from each other, see for instance Section 2.4 of [32] for a definition.

In view of our analogy with random walks on relatively hyperbolic groups, we recall the following result from [31], which gathers in some particular cases results of [44] and [32]. It can be thought as a Riemannian version of the work of Candellero and Gilch [8] presented above.

Theorem 1.10. *Let $(M, g_H) = \mathbb{H}^2/\Gamma$ be a hyperbolic surface where Γ is a Schottky group with at least one parabolic free factor $\mathcal{H} = \langle h \rangle$. We fix a parameter $b \in (1, 2)$. Then, there exists a family $(g_{a,b})_{a \in (0,+\infty)}$ of negatively curved Riemannian metrics on M obtained by perturbation of the hyperbolic metric g_H in such a way that*

- the metrics $g_{a,b}$ coincides with g_H outside a small neighborhood (controlled by the value a) of the cuspidal end associated with $\mathcal{H}$;

- the distance $d_{a,b}$ induced by $g_{a,b}$ satisfies the following condition: for any fixed point $x \in \mathbb{H}^2$,

$$d_{a,b}(x, h^n x) = 2\big(\ln|n| + b \ln|\ln|n||\big) + O(1).$$

Then, there exists a "critical value" $a^* > 0$ such that :

- *if $a > a^*$ then the action of Γ on $(\mathbb{H}^2, g_a)$ is strongly positive recurrent. In particular for all $x, y \in \mathbb{H}^2$,*
$$N_\Gamma(x, y, R) \sim C_{xy} e^{\delta_\Gamma R};$$
- *if $a = a^*$, then the action of Γ on $(\mathbb{H}^2, g_a)$ is divergent but non positive recurrent. Moreover, for all $x, y \in \mathbb{H}^2$,*
$$N_\Gamma(x, y, R) \sim C_{xy} \frac{e^{\delta_\Gamma R}}{R^{2-b}};$$
- *if $a \in (0, a^*)$, then the action of Γ on $(\mathbb{H}^2, g_a)$ is convergent and for all $x, y \in \mathbb{H}^2$,*
$$N_\Gamma(x, y, R) \sim C_{xy} \frac{e^{\delta_\Gamma R}}{R^b}.$$

We end this paragraph with a table that summarizes the different cases that arise in the study of local limit theorems of relatively hyperbolic groups. We also indicate the corresponding results obtained in the framework of geometrically finite non-compact surfaces endowed with the metric g_a defined in Theorem 1.10.

Local Limit Theorem	**Counting problem**
μ **spectrally nondegenerate** (*so μ spectrally positive recurrent*) $p_n(x,y) \sim C_{x,y} R_\mu^{-n} n^{-3/2}$ see [16]	**critical gap property** $\delta_\Gamma > \delta_{\mathcal{H}}$ (*so Γ positive recurrent*) $N_\Gamma(x, y, R) \sim C_{xy} e^{\delta_\Gamma R}$ see [36], [31]
μ **spectrally degenerate** $+$ **spectrally positive recurrent** Rough estimate: $p_n(x,y) \asymp R_\mu^{-n} n^{-3/2}$ see [15] Conjecture: $p_n(x,y) \sim C_{x,y} R_\mu^{-n} n^{-3/2}$	Γ **exotic** i.e $\delta_\Gamma = \delta_{\mathcal{H}}$ $+$ **positive recurrent** $N_\Gamma(x, y, R) \sim C_{xy} e^{\delta_\Gamma R}$ see [36] [31]
μ **spectrally degenerate** $+$ **divergent** $+$ **not spectrally positive recurrent** rank of spectral degeneracy 5 or 6 possible exotic local limit theorem see [19]	Γ **exotic** i.e $\delta_\Gamma = \delta_{\mathcal{H}}$ $+$ **divergent** $+$ **not positive recurrent** $N_\Gamma(x, y, R) \sim C_{xy} \frac{e^{\delta_\Gamma R}}{R^{2-b}}$ see [44], [31]
μ **convergent** (*so μ spectrally degenerate*) $p_n(x,y) \sim C_{x,y} R_\mu^{-n} n^{-d_\mu/2}$ see Theorem 1.3	Γ **convergent** (*so Γ exotic*) $N_\Gamma(x, y, R) \sim C_{xy} \frac{e^{\delta_\Gamma R}}{R^b}$ see [32]

On the left column, Γ is a relatively hyperbolic group with respect to virtually abelian parabolic subgroups $\mathcal{H}_1, \ldots, \mathcal{H}_p$ (up to conjugacy). We consider a probability measure μ on Γ which is finitely supported, admissible and symmetric and such that the random walk is aperiodic. In the case where μ is convergent, we denote by d_μ its rank of spectral degeneracy.

On the right column, Γ is a geometrically finite Fuchsian group with parabolic subgroups $\mathcal{H}_1, \ldots, \mathcal{H}_p$ (up to conjugacy). We assume that $\mathbb{H}^2/\Gamma$ is endowed with the metric $g_{a,b}$ and set $\delta_{\mathcal{H}} = \max(\delta_{\mathcal{H}_1}, \ldots, \delta_{\mathcal{H}_p})$.

1.5. **Organization of the paper.** In Section 2, we give background on relatively hyperbolic groups, transition kernels with their Green function and Martin boundary and we recall relative Ancona inequalities which roughly state that the random walk tracks relative geodesics with high probability in a relatively hyperbolic group.

In Section 3, we study the *first return kernel* $p_{\mathcal{H},r}(.,.)$ to a parabolic subgroup $\mathcal{H}$ of rank d. Assuming that the random walk is spectrally degenerate along $\mathcal{H}$ we give asymptotics for the jth derivative of the Green function associated to $p_{\mathcal{H},r}$ where $j = \lceil d/2 \rceil - 1$ (see Proposition 3.16).

In Section 4, we assume that parabolic subgroups are virtually abelian and show that the Martin boundary is stable in the sense of Definition 2.6 below (see Theorem 4.1). This had already been shown in [17] when the random walk is spectrally nondegenerate.

In Section 5, we assume that the Martin boundary of the full random walk is stable and the Martin boundary of the walk restricted to influential parabolic subgroups is reduced to a point at the spectral radius. Under these conditions, we prove that asymptotics for the jth derivative of the Green function of the full random walk are given by the analogous asymptotics for the transition kernels of the first return to the parabolic subgroups along which the walk is spectrally degenerate (see Theorem 5.4).

In Section 6, we gather the ingredients of the three previous section which give asymptotics for the jth derivative of the full Green function, where $j = \lceil d/2 \rceil - 1$ and d is the rank of spectral degeneracy of the walk. Theorem 1.3 follows, applying a Tauberian type theorem proved in [23].

Finally, in the two last sections, we present possible extensions of our main results. In Section 7, we show that whenever the parabolic subgroups are virtually abelian and the random walk is divergent and not spectrally degenerate along a parabolic subgroup of rank 5 or 6, the random walk is automatically spectrally positive recurrent. In Section 8, we explain exactly where we use that parabolic subgroups are virtually abelian and how our work might be generalized. We also show limitation in our overall strategy.

2. Random walks on relatively hyperbolic groups

2.1. **Relatively hyperbolic groups and relative automaticity.** Relatively hyperbolic groups have been studied by many authors with several equivalent viewpoints, see for instance [7], [13], [20], [30] just to name a few. We briefly recall here their definition, following the terminology of Bowditch.

2.1.1. *Limit set.* Consider a discrete group Γ acting by isometries on a Gromov-hyperbolic space X. Let $o \in X$ be fixed. Define the limit set $\Lambda\Gamma$ as the closure of Γo in the Gromov boundary ∂X of X. This set does not depend on o.

A point $\xi \in \Lambda\Gamma$ is called *conical* if there is a sequence $(\gamma_n)_n$ in Γ and distinct points ξ_1, ξ_2 in $\Lambda\Gamma$ such that, for all $\xi \neq \zeta$ in $\Lambda\Gamma$, the sequences $(\gamma_n\xi)_n$ and $(\gamma_n\zeta)_n$ converge to ξ_1 and ξ_2 respectively. A point $\xi \in \Lambda\Gamma$ is called *parabolic* if its stabilizer Γ_ξ in Γ is infinite and the elements of Γ_ξ fix only ξ in $\Lambda\Gamma$. A parabolic limit point ξ in $\Lambda\Gamma$ is said to be *bounded* if Γ_ξ acts cocompactly on $\Lambda\Gamma \setminus \{\xi\}$. The action of Γ on X is said to be *geometrically finite* if $\Lambda\Gamma$ only contains conical limit points and bounded parabolic limit points.

2.1.2. *Relatively hyperbolic groups.* Let Γ be a finitely generated groups and S be a fixed generating set. Let Ω_0 be a finite collection of subgroups, none of them being conjugate. Let Ω be the closure of Ω_0 under conjugacy.

The *relative graph* $\hat{\Gamma} = \hat{\Gamma}(S, \Omega_0)$ is the Cayley graph of Γ with respect to S and the union of all $\mathcal{H} \in \Omega_0$ [30]. It is quasi-isometric to the coned-off graph introduced by Farb in [20]. The distance $\hat{d}$ in $\hat{\Gamma}$ is called the *relative distance.* We also denote by $\hat{S}_n$ the sphere of radius n centered at e in $\hat{\Gamma}$. Also, we call *relative geodesic* a geodesic in $\hat{\Gamma}$.

Theorem 2.1 ([7])**.** *Using the previous notations, the following conditions are equivalent.*

(1) The group Γ has a geometrically finite action on a Gromov hyperbolic space X such that the parabolic limit points are exactly the fixed points of elements in Ω.
(2) The relative graph $\hat{\Gamma}(S, \Omega_0)$ is Gromov hyperbolic for the relative distance $\hat{d}$, and for all $L > 0$ and all $x \in \hat{\Gamma}$, there exist finitely many closed loops of length $L > 0$ which contains x.

When these conditions are satisfied, the group Γ is said to be relatively hyperbolic *with respect to Ω_0.*

Assume now that Γ is relatively hyperbolic with respect to Ω, and let X be a Gromov hyperbolic space on which Γ has a geometrically finite action whose parabolic subgroups are the element of Ω. The limit set $\Lambda\Gamma \subset \partial X$ is called the *Bowditch boundary* of Γ. It is unique up to equivariant homeomorphism.

Archetypal examples of relatively hyperbolic groups with respect to virtually abelian subgroups are given by finite co-volume Kleinian groups. In this case, the group acts via a geometrically finite action on the hyperbolic space $\mathbb{H}^n$ and the Bowditch boundary is the full sphere at infinity $\mathbb{S}^{n-1}$.

2.1.3. *Automatic structure.* The notion of relative automaticity was introduced by the first author in [15].

Definition 2.2. *A* relative automatic structure *- or shortly an* automaton *- for Γ with respect to the collection of subgroups Ω_0 and with respect to some finite generating set S is a directed graph $\mathcal{G} = (V, E_V, v_*)$ where the set of vertices V is finite, with a distinguished vertex v_* called the* starting vertex *and with a labeling map $\phi : E_V \to S \cup \bigcup_{\mathcal{H}\in\Omega_0} \mathcal{H}$ such that the following holds. If $\omega = (e_1, \dots, e_n)$ is a path of adjacent edges in $\mathcal{G}$, define $\phi(e_1, ..., e_n) = \phi(e_1) \dots \phi(e_n) \in \Gamma$. Then,*

- *no edge ends at v_*, except the trivial edge starting and ending at v_*,*
- *every vertex $v \in V$ can be reached from v_* in $\mathcal{G}$,*

- *for every path $\omega = (e_1, ..., e_n)$ of adjacent edges in $\mathcal{G}$, the path $e, \phi(e_1)$, $\phi(e_1, e_2)$, ..., $\phi(e_1, \dots, e_n)$ is a relative geodesic from e to $\phi(e_1, \dots, e_n)$, i.e. the image of $e, \phi(e_1), \phi(e_{1,2}), ..., \phi(e_1, \dots, e_n)$ in $\hat{\Gamma}$ is a geodesic for the metric $\hat{d}$,*
- *the extended map ϕ is a bijection between paths in $\mathcal{G}$ starting at v_* and elements of Γ.*

Theorem 2.3. [15, Theorem 4.2] *If Γ is relatively hyperbolic with respect to Ω, then for any finite generating set S and for any choice of a full family Ω_0 of representatives of conjugacy classes of elements of Ω, Γ is relatively automatic with respect to S and Ω_0.*

This statement is proved by adapting Cannon's standard method [9]. We first construct an automaton that encodes relative geodesics and then show that there exist finitely many relative cone-types, see [15, Definition 4.7, Proposition 4.9] for more details. To obtain a bijection between paths in the automaton and elements of Γ, one fixes an order on the union of S and all the $\mathcal{H} \in \Omega_0$, which allows to choose the smallest possible relative geodesics for the associated lexicographical order.

Relative automaticity is a key point in [16] to establish a local limit theorem in the spectrally nondegenerate case. We will use again this notion in Section 5.

2.2. **Transition kernels and Martin boundaries.** Let us give a general presentation of the notion of Martin boundaries. In what follows, E is a countable space endowed with the discrete topology and o is a fixed base point in E.

Definition 2.4. *A* transition kernel *p on E is a non-negative map $p : E\times E \to \mathbb{R}_{\geq 0}$ with* finite total mass*, i.e. such that*

$$\forall x \in E, \sum_{y\in E} p(x, y) < +\infty.$$

When for every x, the total mass is 1 (which we do not require), we call it a *probability transition kernel*. It then defines a *Markov chain on E*, i.e. a random process $(X_n)_{n\geq 0}$ such that $P(X_{n+1} = b|X_n = a) = p(a, b)$. In general, we say that p defines a *chain* on E.

If μ is a probability measure on a finitely generated group Γ, then the kernel $p_\mu(g, h) = \mu(g^{-1}h)$ is a probability transition kernel and the corresponding Markov chain is the μ-random walk.

Definition 2.5. *Let $p : E \times E \to \mathbb{R}_+$ be a transition kernel on E.*

- *The* Green function *associated to p is defined by*
$$G_p(x, y) = \sum_{n\geq 0} p^{(n)}(x, y) \in [0, +\infty],$$
where $p^{(n)}$ is the nth convolution power of p, i.e.
$$p^{(n)}(x, y) = \sum_{z_1,\dots,z_{n-1}\in E} p(x, z_1)p(z_1, z_2)\cdots p(z_{n-1}, y).$$
- *The chain defined by p is* finitely supported *if for every $x \in E$, the set of $y \in E$ such that $p(x, y) > 0$ is finite.*
- *The chain is* admissible *(or* irreducible*) if for every $x, y \in E$, there exists n such that $p^{(n)}(x, y) > 0$.*

- *The chain is* aperiodic *(or* strongly irreducible*) if for every $x, y \in E$, there exists n_0 such that $\forall n \geq n_0$, $p^{(n)}(x,y) > 0$.*
- *The chain is* transient *if the Green function is everywhere finite.*

Consider a transition kernel p defining an irreducible transient chain. For $y \in E$, define the Martin kernel based at y as

$$K_p(x,y) = \frac{G_p(x,y)}{G_p(o,y)}.$$

The *Martin compactification* of E with respect to p and o is a compact space containing E as an open and dense space, whose topology is described as follows. A sequence $(y_n)_n$ in E converges to a point ξ in the Martin compactification if and only if the sequence $(K(\cdot, y_n))_n$ converges pointwise to a function which we write $K(\cdot, \xi)$. Up to isomorphism, it does not depend on the base point o and we denote it by $\overline{E}_p$. We also define the *p-Martin boundary* (or *Martin boundary*, when there is no ambiguity) as $\partial_p E = \overline{E}_p \setminus E$. We refer for instance to [39] for a complete construction of the Martin compactification.

The Martin boundary contains a lot of information. It was first introduced to study non-negative harmonic functions. We use it here to prove our local limit theorem.

Let us now define the notion of stability for the Martin boundary, following Picardello and Woess [33]. Assuming that p is irreducible, the radius of convergence of the Green function $G_p(x,y)$ is independent of x and y. Denote it by R_p and for all $0 \leq r \leq R_\mu$ let us set $G_p(x,y|r) = G_{rp}(x,y)$, i.e.

$$G_p(x,y|r) = \sum_{n \geq 0} r^n p^{(n)}(x,y).$$

Also set $K(x,y|r) = K_{rp}(x,y)$. The Martin compactification, respectively the Martin boundary associated with $K(\cdot,\cdot|r)$, is called the r-Martin compactification, respectively the r-Martin boundary, and is denoted by $\overline{E}_{rp}$, respectively by $\partial_{rp} E$.

Definition 2.6. *The Martin boundary of E with respect to p is* stable *if the following conditions are satisfied.*

(1) For every $x, y \in E$, we have $G_p(x,y|R_p) < +\infty$ where R_p is the radius of convergence of the Green function.

(2) For every $0 < r_1, r_2 < R_p$, the sequence $(K(\cdot, y_n|r_1))_n$ converges pointwise if and only if $(K(\cdot, y_n|r_2))_n$ converges pointwise, i.e. the r_1 and r_2-Martin compactifications are homeomorphic. For simplicity we then write $\partial_p \Gamma$ for the r-Martin boundary whenever $0 < r < R_p$.

(3) The identity on Γ extends to a continuous and equivariant surjective map ϕ_p from $E \cup \partial_p E$ to $E \cup \partial_{R_p p} E$. We then write $K(x, \xi|R_p) = K(x, \phi_p(\xi)|R_p)$ for $\xi \in \partial_p E$.

(4) The map $(x, \xi, r) \in E \times \partial_p E \times (0, R_p] \mapsto K(x, \xi|r)$ is continuous with respect to (x, ξ, r).

We say that the Martin boundary is strongly stable *if it is stable and the second condition holds for every $0 < r_1, r_2 \leq R_p$; in this case, the map ϕ_p induces a homeomorphism from the r-Martin boundary to the R_p-Martin boundary.*

If p is the transition kernel of an admissible random walk on a finitely generated group which is *non-amenable*, it was shown by Guivarc'h in [26, p. 20, remark b] that the condition (1) is always satisfied. Note that non-elementary relatively hyperbolic groups are always non-amenable.

The Martin boundary of any finitely supported symmetric admissible random walk on a *hyperbolic group* is strongly stable. More generally, the Martin boundary of a finitely supported symmetric and admissible random walk on a *relatively hyperbolic group* is studied in [21], [18] and [17]. In particular, whenever the parabolic subgroups are virtually abelian, the homeomorphism type of the r-Martin boundary is described in [17]. It is proved there that the Martin boundary is strongly stable if and only if the random walk is spectrally non nondegenerate. We prove in Section 4 that when the parabolic subgroups are virtually abelian, the Martin boundary of a spectrally degenerate random walk is still stable (but not strongly stable).

Let us also mention a central computation which we use many times.

Lemma 2.7. *Let $p : E \times E \to \mathbb{R}_+$ be a transition kernel and for all $r \in [0, R_p]$, write again $G_p(x, y|r) = \sum_{n\geq 0} r^n p^{(n)}(x, y)$. Then*

$$\frac{d}{dr}\left(rG_p(x, y|r)\right) = \sum_{z\in E} G(x, z|r)G(z, y|r).$$

This result was first observed by Gouëzel and Lalley [23, Proposition 1.9], although it had been implicitly used before, see for instance [46, (27.6)]. The proof is a standard manipulation of power series. The generalization to higher derivatives is given by Lemma 5.1 below.

Standing assumptions. From now, and until the end of this paper, we fix a finitely generated group Γ relatively hyperbolic with respect to a finite collection of parabolic subgroups $\Omega_0 = \{\mathcal{H}_1, ..., \mathcal{H}_N\}$. We fix a finitely supported symmetric probability measure μ on Γ whose associated random walk is admissible and irreducible. In particular, we assume that the support S of μ is a finite generating set, which is fixed from now on; the distance on Γ is the word distance induced by S. This implies in particular that for all $x, y \in \Gamma$, if x and y are on the same geodesic in Γ, then there exists $n > 0$ such that $p^{(n)}(x, y) > 0$.

In the sequel, we denote by $\hat{\Gamma}$ the relative graph $\hat{\Gamma}(S, \Omega_0)$, by R_μ the inverse of the spectral radius of μ and by $G(x, y|r)$ the Green function, where $0 \leq r \leq R_\mu$ and $x, y \in \Gamma$. As already mentioned, since Γ is not amenable and μ is admissible, it follows from [26] that $G(x, y|R_\mu) < +\infty$ for all $x, y \in \Gamma$.

2.3. Relative Ancona inequalities.

For any set $A \subset \Gamma$, we set

$$G(x, y; A|r) = \sum_{n\geq 1} \sum_{g_1,...,g_{n-1}\in A} r^n \mu(x^{-1}g_1)\mu(g_1^{-1}g_2)...\mu(g_{n-2}^{-1}g_{n-1})\mu(g_{n-1}^{-1}y). \tag{2.1}$$

This quantity is called the *relative Green function* of paths staying in A except maybe at their beginning and end. Writing $A^c = \Gamma \setminus A$, the relative Green function $p_{A,r}(., .) := G(., .; A^c|r)$ is called the *first return kernel* to A.

For all $y \in \Gamma$ and $\eta > 0$, we denote by $B_\eta(y)$ the ball of center y and radius η in Γ, i.e. $B_\eta(y) = \{z \in \Gamma \mid d(y, z) \leq \eta\}$. We will use repeatedly the following results.

Proposition 2.8. [17, Corollary 3.7] *For every $\epsilon > 0$ and every $D \geq 0$, there exists η such that the following holds. For every x, y, z such that y is within D of a point* *on a relative geodesic from x to z and for every $r \leq R_\mu$,*

$$G(x, z; B_\eta(y)^c|r) \leq \epsilon G(x, z|r).$$

This proposition can be interpreted as follows : with high probability, a random path from x to z has to pass through a neighborhood of y, whenever y is on a relative geodesic from x to z. As a consequence, we have the following.

Proposition 2.9. *(Weak relative Ancona inequalities) For every $D \geq 0$, there exists C such that the following holds. For every x, y, z such that y is within D of a point on a relative geodesic from x to z and for every $r \leq R_\mu$,*

$$\frac{1}{C} G(x, y|r) G(y, z|r) \leq G(x, z|r) \leq C G(x, y|r) G(y, z|r).$$

This is proved by decomposing a trajectory from x to z according to its potential first visit to $B_\eta(y)$, where η is chosen such that $G(x, z; B_\eta(y)^c|r) \leq 1/2G(x, z|r)$ from Proposition 2.8, see precisely [21, Theorem 5.1, Theorem 5.2].

These inequalities were first proved by Ancona [2] in the context of hyperbolic groups for $r = 1$. The uniform inequalities up to the spectral radius were proved by Gouëzel and Lalley [23] for co-compact Fuchsian groups and then by Gouëzel [22] in general. For relatively hyperbolic groups, Gekhtman, Gerasimov, Potyagailo and Yang [21] proved them for $r = 1$. The uniform inequalities up to the spectral radius were then proved by the first author and Gekhtman [17]. They play a key role in the identification of the Martin boundary, see [21] for several other applications.

Let us mention that there exist *strong relative Ancona inequalities* (cf [16, Definition 2.14]), that are a key ingredients in [16] to prove a local limit theorem in the spectrally nondegenerate case. However, we do not need them in the present paper. We thus call *relative Ancona inequalities* the *weak relative Ancona inequalities.*

3. Asymptotics of the first return to parabolic Green functions

Throughout this section, we fix a parabolic subgroup $\mathcal{H} \in \Omega_0$. For $\eta \geq 0$, the η-neighborhood of $\mathcal{H}$ is denoted by $\mathcal{N}_\eta(\mathcal{H})$.

We introduce below the *first return transition kernel* to $\mathcal{N}_\eta(\mathcal{H})$. The main goal of this section is to establish asymptotics for the derivatives of the Green function associated to this first return kernel.

3.1. **First return transition kernel and spectral degeneracy.** For $r \leq R_\mu$, let $p_{\mathcal{H},\eta,r}(h, h') = G(h, h'; \mathcal{N}_\eta(\mathcal{H})^c|r)$ be the *first return kernel* to $\mathcal{N}_\eta(\mathcal{H})$, i.e.

$$p_{\mathcal{H},\eta,r}(h, h') = \sum_{n\geq 1} \sum_{\substack{g_1,\ldots,g_{n-1} \\ \notin \mathcal{N}_\eta(\mathcal{H})}} r^n \mu(h^{-1}g_1)\mu(g_1^{-1}g_2)...\mu(g_{n-2}^{-1}g_{n-1})\mu(g_{n-1}^{-1}h').$$

For simplicity, when $\eta = 0$, we write $p_{\mathcal{H},r} = p_{\mathcal{H},0,r}$.

The nth convolution power of $p_{\mathcal{H},\eta,r}$ is denoted by $p^{(n)}_{\mathcal{H},\eta,r}$ and the associated Green function, evaluated at t, is

$$G_{\mathcal{H},\eta,r}(h, h'|t) := \sum_{n\geq 1} p^{(n)}_{\mathcal{H},\eta,r}(h, h')t^n.$$

The radius of convergence $R_{\mathcal{H},\eta}(r)$ of this power series is the inverse of the spectral radius of the associated chain.

For simplicity, we write $R_{\mathcal{H},\eta} = R_{\mathcal{H},\eta}(R_\mu)$ and $R_{\mathcal{H}} = R_{\mathcal{H},0}(R_\mu)$. Recall the following definition from [17].

Definition 3.1. *The measure μ, or equivalently the random walk, is said to be* spectrally degenerate along $\mathcal{H}$ *if $R_{\mathcal{H}} = 1$.*

Remark 3.1. For suited random walks on a free product, there are several equivalent formulations of spectral degeneracy, some of them already pointed out in the work of Woess [46]: we refer to [19] for a proof of these equivalences. For general relatively hyperbolic groups, no simpler characterisation of spectral degeneracy is known.

Since $\mathcal{H}$ is fixed for the remainder of the section, we drop the index $\mathcal{H}$ in the notations. We now enumerate a list of properties satisfied by $p_{\eta,r}$ and $G_{\eta,r}$.

- Since the μ-random walk on Γ is invariant under the action of Γ, the transition kernel $p_{\eta,r}$ is $\mathcal{H}$-invariant, i.e. $p_{\eta,r}(hx, hy) = p_{\eta,r}(x, y)$ for every $h \in \mathcal{H}$ and every $x, y \in \mathcal{N}_\eta(\mathcal{H})$.
- By definition, $p_{\eta,r}(x, y) > 0$ if and only if there is a *first return path* in $\mathcal{N}_\eta(\mathcal{H})$ from x to y, i.e. if there exist $n \geq 0$ and a path $x, g_1, ..., g_n, y$ in Γ with positive probability such that $g_i \notin \mathcal{N}_\eta(\mathcal{H})$ for $i = 1, ..., n$.
- Therefore, $p_{\eta,r}$ is admissible, i.e. for every $x, y \in \Gamma$, there exists n such that $p^{(n)}_{\eta,r}(x, y) > 0$, see [18, Lemma 5.9] for a complete proof.

The following lemma shows that when x and y are in $\mathcal{N}_\eta(\mathcal{H})$, the Green function of $p_{\eta,r}$ at 1 equals the full Green function at r. The proof is straightforward, see [17, Lemma 4.4]. This property will be frequently used.

Lemma 3.2. *With the above notations, for every $\eta \geq 0$, every $x, y \in \mathcal{N}_\eta(\mathcal{H})$ and every $r \leq R_\mu$,*

$$G_{\eta,r}(x, y|1) = G(x, y|r),$$

For $x, y \in \Gamma$, we write $d_{\mathcal{H}}(x, y)$ the distance between the projections $\pi_{\mathcal{H}}(x)$ and $\pi_{\mathcal{H}}(y)$ of x and y respectively onto $\mathcal{H}$. Since projections on parabolic subgroups are well-defined up to a uniformly bounded error term, see [42, Lemma 1.15], $d_{\mathcal{H}}(x, y)$ is also defined up to a uniformly bounded error term. Letting $M \geq 0$, we say that $p_{\eta,r}$ has *exponential moments up to M* if for any $x \in \mathcal{N}_\eta(\mathcal{H})$, we have

$$\sum_{y \in \mathcal{N}_\eta(\mathcal{H})} p_{\eta,r}(x, y) \mathrm{e}^{M d_{\mathcal{H}}(x,y)} < +\infty.$$

The following lemma is the main reason for introducing $p_{\mathcal{H},\eta,r}$ with $\eta > 0$.

Lemma 3.3. [17, Lemma 4.6] *For every $M \geq 0$, then exists η_M such that for every $\eta \geq \eta_M$ and for every $r \leq R_\mu$, $p_{\eta,r}$ has exponential moments up to M.*

Note that η_M does not depend on r, hence, choosing the neighborhood of $\mathcal{H}$ large enough, all transition kernels $p_{\eta,r}$ have exponential moments up to M, uniformly in r.

3.2. **Spectrally degenerate and convergent random walks.** We give here more details on convergent random walks and the relation between this notion and spectral degeneracy. As aforementioned, the following result was proved in [15].

Proposition 3.4. [15, Proposition 5.8] *Let Γ be a relatively hyperbolic group and let μ be a finitely supported admissible and symmetric probability measure on Γ. Assume that μ is spectrally nondegenerate. Then, μ is divergent.*

This is in fact the non-trivial part of the proof that spectral nondegeneracy implies spectral positive recurrence, see also [15, Proposition 3.6, Proposition 3.7]. We deduce that if μ is convergent, then it is spectrally degenerate along some parabolic subgroup.

One of the main results of [15] states that

$$G^{(2)}(e,e|r) \asymp \left(G^{(1)}(e,e|r)\right)^3 \left(\sum_{\mathcal{H}\in\Omega_0} G^{(2)}_{\mathcal{H},r}(e,e|1)\right).$$

Assuming that μ is spectrally degenerate along some $\mathcal{H}$, we have $R_{\mathcal{H}}(R_\mu) = 1$, so it might happen that $G^{(2)}_{\mathcal{H},r}(e,e|1)$ is infinite at $r = R_\mu$. As a consequence, the factor $G^{(2)}_{\mathcal{H},r}(e,e|1)$ would influence the asymptotics of $G^{(2)}(e,e|r)$, as $r \to R_\mu$. Furthermore, if the random walk is convergent, then the factor $\left(G^{(1)}(e,e|r)\right)^3$ is uniformly bounded and so

$$G^{(2)}(e,e|r) \asymp \left(\sum_{\mathcal{H}\in\Omega_0} G^{(2)}_{\mathcal{H},r}(e,e|1)\right).$$

Thus, in this case, the asymptotics of $G^{(2)}(e,e|r)$ as $r \to R_\mu$ are roughly the same as the asymptotics of $\sum_{\mathcal{H}\in\Omega_0} G^{(2)}_{\mathcal{H},r}(e,e|1)$.

More generally, we prove below that in the convergent case, letting k be the smallest integer such that $G^{(k)}(e,e|R_\mu)$ is infinite, the asymptotics of $G^{(k)}(e,e|r)$ are governed by the asymptotics of $\sum_{\mathcal{H}\in\Omega_0} G^{(k)}_{\mathcal{H},r}(e,e|1)$, see precisely Proposition 5.3 and Theorem 5.4.

Next, we explain how convergent probability measures are constructed on free products, following the work of Candellero and Gilch [8]. Let Γ_0, Γ_1 be two finitely generated groups endowed respectively with finitely supported, admissible and symmetric probability measure μ_0 and μ_1. Assume that μ_0 is convergent on Γ_0. Consider the free product $\Gamma = \Gamma_0 * \Gamma_1$ and set

$$\mu_\alpha = \alpha\mu_1 + (1-\alpha)\mu_0.$$

Such a probability measure is called adapted to the free product structure and the random walk can only move towards one of the free factors at each step.

Proposition 3.5. *For α small enough, μ_α is convergent and spectrally degenerate along Γ_0, but not along Γ_1.*

Proof. We follow the strategy of Candellero and Gilch, which is based on the work of Woess [45], see also [46, Chapter 9]. We denote by $R(\alpha)$ the inverse of the spectral radius of μ_α and by R_i, $i = 0,1$, the inverse of the spectral radius of μ_i. We also set $\theta = R(\alpha)G(e,e|R(\alpha))$ and $\theta_i = R_iG_i(e,e|R_i)$. Finally, we define implicitly the functions Φ and Φ_i by the formulae

$$G(e,e|r) = \Phi(rG(e,e|r))$$

for every $r \leq R(\alpha)$, and

$$G_i(e,e|r) = \Phi_i(rG(e,e|r))$$

for every $r \leq R_i$. We then set $\Psi(t) = \Phi(t) - t\Phi'(t)$ and $\Psi_i(t) = \Phi_i(t) - t\Phi_i'(t)$. The functions Φ and Ψ are defined on an open neighborhood inside the complex plane of the interval $[0,\theta)$. Since $G(e,e|R(\alpha))$ is finite, these functions are also defined on $[0,\theta]$. We have by [46, Theorem 9.19]

$$\Phi(t) = \Phi_0((1-\alpha)t) + \Phi_1(\alpha t) - 1 \tag{3.1}$$

and

$$\Psi(t) = \Psi_0((1-\alpha)t) + \Psi_1(\alpha t) - 1. \tag{3.2}$$

As a consequence, Φ and Ψ are defined on $[0,\bar{\theta})$, where $\bar{\theta} = \min\{\theta_0/(1-\alpha), \theta_1/\alpha\}$. The key property of the functions Ψ_i is that assuming further that $G_i(e,e|R_i)$ is finite, then $\Psi_i(\theta_i) = 0$ if and only if $G_i'((e,e|R) = \infty$. Similarly, $\Psi(\theta) = 0$ if and only if $G'(e,e|R(\alpha)) = \infty$. This follows from the following expression given by [19, (2.6)], see also [46, (9.14)]. For $t = rG(e,e|r) < \theta$,

$$\Psi(t) = \frac{G(e,e|r)^2}{rG(e,e|r)' + G(e,e|r)}.$$

Similar expressions relate Ψ_i and G_i, G_i'. Finally, by [19, Corollary 2.5], $\Psi(\bar{\theta}) > 0$ if and only if the random walk is convergent, see also [46, Theorem 9.22].

Consequently, if μ_0 is convergent, then $G_0(e,e|R_0)$ and $G_0'(e,e|R_0)$ are finite, so that $\Psi_0(\theta_0) > 0$. Thus, θ_0 is finite and $\bar{\theta} = \theta_0/(1-\alpha) < \theta_1/\alpha$ for α small enough. In this case, the measure μ_α cannot be spectrally degenerate along Γ_1. Finally, $\Psi_1(\alpha\bar{\theta})$ converges to $\Psi_1(0) = 1$ and $\Psi_0((1-\alpha)\bar{\theta})$ to $\Psi_0(\theta_0) > 0$ as α tends to 0. Therefore, for small enough α, $\Psi(\bar{\theta})$ is positive and so μ_α is convergent. □

Remark 3.2. More generally, one may consider a relatively hyperbolic group Γ which has a parabolic subgroups $\mathcal{H}$ carrying a finitely supported, admissible and symmetric probability measure $\mu_{\mathcal{H}}$ which is convergent. Consider an auxiliary finitely supported, admissible and symmetric probability measure μ_* on Γ and set

$$\mu_\alpha = (1-\alpha)\mu_{\mathcal{H}} + \alpha\mu_*.$$

We conjecture that for small enough α, this measure μ_α is convergent and spectrally degenerate along $\mathcal{H}$.

This requires significant improvements of the proof of Proposition 3.5, which crucially uses (3.1) and (3.2) which both rely on the combinatorial structure of free products. Bypassing these two equations would require some kind of continuity of the Green function and its derivative in terms of the underlying probability measure, which in turn would require new material.

3.3. **Vertical displacement transition matrix.** Until the end of Section 3, we assume that $\mathcal{H}$ is virtually abelian of rank d and that μ is spectrally degenerate along $\mathcal{H}$. According to [17, Lemma 4.16], this implies that $R_\eta = 1$ for every $\eta \geq 0$, i.e. μ is spectrally degenerate along $\mathcal{N}_\eta(\mathcal{H})$. In the remainder of this section, we aim to obtain asymptotics as r tends to R_μ of the $\big(\lceil d/2 \rceil - 1\big)$th derivative of the Green function $G_{\eta,r}$ at 1, see Proposition 3.16 below.

We fix $\alpha \in (0,1)$ and consider the transition kernel $\tilde{p}_{\eta,r}$ defined by

$$\tilde{p}_{\eta,r}(x,y) = \alpha\delta_{x,y} + (1-\alpha)p_{\eta,r}(x,y).$$

Let $\tilde{G}_{\eta,r}$ be the corresponding Green function. Then, by [46, Lemma 9.2],

$$\tilde{G}_{\eta,r}(e,e|t) = \frac{1}{1-\alpha t} G_{\eta,r}\left(e,e\middle|\frac{(1-\alpha)t}{1-\alpha t}\right).$$

Thus, up to a constant that only depends on α and j, the jth derivative of $\tilde{G}_{\eta,r}$ and $G_{\eta,r}$ coincide at 1. Therefore, up to replacing $p_{\eta,r}$ by $\tilde{p}_{\eta,r}$ we can assume that, $p_{\eta,r}(x,x) > 0$ for every x so that the transition kernel $p_{\eta,r}$ is aperiodic. **We keep this assumption for all this section.**

By definition, there exists a subgroup of $\mathcal{H}$ of finite index which is isomorphic to $\mathbb{Z}^d$. Any section $\mathcal{H}/\mathbb{Z}^d \to \mathcal{H}$ allows us to identify $\mathcal{H}$ with $\mathbb{Z}^d \times F$ for some finite set F. As in [17] and [18], the group Γ can be $\mathcal{H}$-equivariantly identified with $\mathcal{H} \times \mathbb{N}$. Indeed, the parabolic subgroup $\mathcal{H}$ acts by left multiplication on Γ and the quotient is countable. We order elements in the quotient according to their distance to $\mathcal{H}$. It follows that

(1) $\mathcal{N}_\eta(\mathcal{H})$ can be $\mathbb{Z}^d$-equivariantly identified with $\mathbb{Z}^d \times \{1, ..., N_\eta\}$,
(2) if $\eta \leq \eta'$, then $N_\eta \leq N_{\eta'}$. In other words, the set $\mathbb{Z}^d \times \{1, ..., N_\eta\}$, identified with $\mathcal{N}_\eta(\mathcal{H})$, is a subset of $\mathbb{Z}^d \times \{1, ..., N_{\eta'}\}$, identified with $\mathcal{N}_{\eta'}(\mathcal{H})$.

Each element of $\mathcal{N}_\eta(\mathcal{H})$ can be written as (x, j), where $x \in \mathbb{Z}^d, j \in \{1, ..., N_\eta\}$. We also write $p_{j,j';r}(x, x') = p_{\eta,r}((x, j), (x', j'))$ for simplicity.

Definition 3.6. *Let $u \in \mathbb{R}^d$. The* vertical displacement transition matrix $F_r(u)$ *is defined as follows. for all $j, j' \in \{1, ..., N_\eta\}$, the (j, j') entry of $F_r(u)$ equals*

$$F_{j,j';r}(u) := \sum_{x \in \mathbb{Z}^d} p_{j,j';r}(0, x)\mathrm{e}^{u \cdot x}.$$

This transition matrix was introduced in [14] for $r = 1$. Many properties are derived there from the fact that it is strongly irreducible, i.e. there exists n such that every entry of $F_r(u)^n$ is positive. Since by [14, Lemma 3.2],

$$F_{j,j';r}(u)^n = \sum_{x \in \mathbb{Z}^d} p^{(n)}_{j,j';r}(0, x)\mathrm{e}^{u \cdot x},$$

where $p^{(n)}_{j,j';r}(0, x) = p^{(n)}_{\eta,r}((0, j), (x, j'))$, strong irreducibility is deduced from the fact that $p_{\eta,r}$ is aperiodic. Denote by $\mathcal{F}_r \subset \mathbb{R}^d$ the interior of the set of $u \in \mathbb{R}^d$ where $F_r(u)$ has finite entries. By the Perron-Frobenius Theorem [41, Theorem 1.1], the matrix $F_r(u)$ has a positive dominant eigenvalue $\lambda_r(u)$ on $\mathcal{F}_r$. Also, by [14, Proposition 3.5], the function λ_r is continuous and strictly convex on $\mathcal{F}_r$ and reaches its minimum at some value u_r.

By [46, Theorem 8.23], the spectral radius $\rho_\eta(r) = R_\eta(r)^{-1}$ satisfies

$$\rho_\eta(r) = \inf_u \lambda_r(u) = \lambda_r(u_r). \tag{3.3}$$

Actually, [46, Theorem 8.23] only deals with finitely supported transition kernels on $\mathbb{Z}^d \times \{1, ..., N\}$, but the statement remains valid for the transition kernel $p_{\eta,r}$ since this condition of finite support can be dropped, see [46, (8.24)].

Recall that our goal is to find asymptotics of the derivatives of the Green function $G_{\eta,r}$. These involve the quantity $\rho_\eta(r)$, so in view of (3.3), we need to study the function $\lambda_r(u_r)$.

First of all, we prove that u_r lies in some large ball, whose radius does not depend on r. Precisely, denote by $B(0, M)$ the closed ball of radius M and center 0 in $\mathbb{R}^d$. It follows from [18, (5), Proposition 4.6] that for large enough η, there exists a constant M such that, for every $u \in B(0, M)$, the matrix $F_r(u)$ has finite entries and the minimum of the function λ_r is reached at some $u_r \in B(0, M)$. In other words, $u_r \in B(0, M) \subset \mathcal{F}_r$ with M independent of r. This is a consequence of the fact that the transition kernel $p_{\eta,r}$ has arbitrary large exponential moments, up to taking η large enough. In what follows, we fix η large enough that satisfies this property.

Before studying differentiability of $r \mapsto \lambda_r(u_r)$, we recall how its continuity was established in [17]. Since η is fixed, the size of the matrices $F_r(u)$ is a fixed number, say K. We endow $M_K(\mathbb{R}^d)$ with a matrix norm. For fixed r, the function F_r is continuous in u. We then endow the space of continuous functions from $B(0,M)$ to $M_K(\mathbb{R}^d)$ with the norm $\|\cdot\|_\infty$. We also choose an arbitrary norm on $\mathbb{R}$ and endow the space of continuous functions from $B(0,M)$ to $\mathbb{R}$ with the norm $\|\cdot\|_\infty$. According to [17, Lemma 5.4, Lemma 5.5], the functions $r \mapsto F_r$ and $r \mapsto \lambda_r$ are continuous for these norms.

The function λ_r is strictly convex and reaches its minimum at u_r. We can deduce that $r \mapsto u_r$ is continuous and thus $r \mapsto \lambda_r(u_r)$ is also continuous.

3.4. **Differentiability of the parabolic spectral radius.** In order to establish the differentiability of the function $r \mapsto \rho_\eta(r)$, we first prove that the first return kernel is itself differentiable.

Lemma 3.7. *For every $x \in \mathcal{N}_\eta(\mathcal{H})$, the function $r \mapsto p_{\eta,r}(e,x)$ is continuously differentiable on $[0,R_\mu]$.*

Proof. Recall that $p_{\eta,r}(e,x) = G(e,x;\mathcal{N}_\eta(\mathcal{H})^c|r)$, so it can be expressed as a power series in r with positive coefficients a_n. These coefficients are at most equal to $\mu^{*n}(x)$. Since the random walk on Γ is convergent, it follows that

$$\sum_{n\geq 0} n a_n r^{n-1} \leq \sum_{n\geq 0} n\mu^{*n}(x) R_\mu^n < +\infty.$$

By monotonous convergence, the function $r \mapsto p_{\eta,r}(e,x)$ is thus continuously differentiable. □

For simplicity, we write $p_r = p_{\eta,r}$ and denote by p'_r the derivative of p_r at r; the kernels p_r and p'_r are both $\mathbb{Z}^d$-invariant transition kernels on $\mathbb{Z}^d \times \{1,...,N_\eta\}$. By Lemma 2.7, we have

$$\begin{aligned} \frac{d}{dr} r p_r(x,y) &= p_r(x,y) + r p'_r(x,y) \\ &= \sum_{z \notin \mathcal{N}_\eta(\mathcal{H})} G(x,z;\mathcal{N}_\eta(\mathcal{H})^c|r) G(z,y;\mathcal{N}_\eta(\mathcal{H})^c|r). \end{aligned} \tag{3.4}$$

The following statement can be thought of an enhanced version of relative Ancona inequalities for trajectories of the random walk avoiding horoballs.

Proposition 3.8. *Let $\eta \geq 0$ be fixed. There exists $C = C_\eta$ such that the following holds. Let $x \in \mathcal{N}_\eta(\mathcal{H})$ and let $y \notin \mathcal{N}_\eta(\mathcal{H})$. Consider a geodesic in Γ from y to $\mathcal{H}$ and denote by $\tilde{y}$ the point in Γ at distance η from $\mathcal{H}$ on this geodesic. Then,*

$$G(x,y;\mathcal{N}_\eta(\mathcal{H})^c|r) \leq C G(x,\tilde{y};\mathcal{N}_\eta(\mathcal{H})^c|r) G(\tilde{y},y;\mathcal{N}_\eta(\mathcal{H})^c|r).$$

Proof. Recall that for simplicity, we assume that the generating set S of Γ equals the support of μ. Let us fix x,y in Γ, a geodesic from y to $\mathcal{H}$ and $\tilde{y}$ satisfying the previous assumptions. If $G(x,y;\mathcal{N}_\eta(\mathcal{H})^c|r) = 0$, i.e. if there is no trajectory of the random walk from x to y staying outside $\mathcal{N}_\eta(\mathcal{H})$, then there is nothing to prove.

Otherwise there exists such a trajectory from x to y and we denote by $\tilde{x}$ the first point on this trajectory outside $\mathcal{N}_\eta(\mathcal{H})$. The following notion was introduced in [17, Definition 3.11].

Definition 3.9. *Let $k, c > 0$, $A \subset \Gamma$ and $y \in A$. The set A is (k,c)-*starlike around *y if for all $z \in A$, there exists a path of length at most $kd(y,z)+c$ staying in A.*

We now use the following version of Ancona inequalities for trajectories assigned to stay in starlike sets, which is given by [17, Proposition 3.12]. For every λ, c and for every $D \geq 0$ there exists $C > 0$ such that the following holds. For any $v \in \Gamma$ and any (λ, c)-starlike set A around v, if $u, w \in A$ are such that v is within D of a relative geodesic from u to w, then

$$C^{-1}G(u,w;A|r) \leq G(u,v;A|r)G(v,w;A|r) \leq CG(u,w;A|r)$$

for every $r \leq R_\mu$. To apply this to our setting, we need to prove that the connected component of $\tilde{x}$ in $\mathcal{N}_\eta(\mathcal{H})^c$ is starlike.

First of all, denote by x_0 the projection of x on $\mathcal{H}$. If γ is a trajectory of the random walk avoiding $\mathcal{N}_\eta(\mathcal{H})$ then $x_0^{-1}\gamma$ also avoids $\mathcal{N}_\eta(\mathcal{H})$. Consequently, up to translating x, y and $\tilde{y}$ by x_0^{-1}, we can assume that $x_0 = e$. In particular, $\tilde{x}$ lies in the ball $B(e, \eta+1)$.

Denote by $\mathcal{H}(\tilde{x})$ the set of $h \in \mathcal{H}$ such that there is a trajectory outside $\mathcal{N}_\eta(\mathcal{H})$ from $\tilde{x}$ to $h\tilde{x}$. Since the random walk is symmetric, $\mathcal{H}(\tilde{x})$ is a subgroup of $\mathcal{H}$.

Claim 3.10. *There exists $C \geq 0$ such that the following holds. Consider a trajectory of the random walk starting at $\tilde{x}$ and staying outside $\mathcal{N}_\eta(\mathcal{H})$. Let z be its endpoint and let z_0 be the projection of z on $\mathcal{H}$. Then there exists $h \in \mathcal{H}(\tilde{x})$ such that $d(h, z_0) \leq C$.*

Proof. Note that there exists finitely many connected components $\mathbf{c}_1, ..., \mathbf{c}_m$ in the complement of $\mathcal{N}_\eta(\mathcal{H})$ that project on $\mathcal{H}$ at e. Indeed, for every point z in $\mathcal{N}_\eta(\mathcal{H})^c$ projecting at e, we consider a geodesic from e to z. We denote by w the point on this geodesic at distance $\eta+1$ of e. Then, w and z lie in the same component of the complement of $\mathcal{N}_\eta(\mathcal{H})$. Since balls are finite and $w \in B(e, \eta+1)$, the number of such components is also finite.

We denote by $\mathcal{J}(\tilde{x})$ the subset of indices j such that there exists a trajectory outside $\mathcal{N}_\eta(\mathcal{H})$ from $\tilde{x}$ to some $h\mathbf{c}_j, h \in \mathcal{H}$. For every $j \in \mathcal{J}(\tilde{x})$, we fix such a particular trajectory γ_i and denote by w_j its endpoint and by h_j the corresponding point in $\mathcal{H}$. Therefore, $h_j^{-1}w_j \in \mathbf{c}_j$.

Consider now a trajectory γ starting at $\tilde{x}$ and staying outside $\mathcal{N}_\eta(\mathcal{H})$. Denote by z its endpoint and by z_0 the projection of z on $\mathcal{H}$. We construct a trajectory for the random walk from $\tilde{x}$ to $z_0h_j^{-1}\tilde{x}$. First, z necessarily lies in one of the components $z_0\mathbf{c}_j$, hence in particular, $j \in \mathcal{J}(\tilde{x})$. Consequently, $z_0^{-1}z$ and $h_j^{-1}w_j$ are both in $\mathbf{c}_j$ and so there exists a trajectory γ' joining these points staying outside $\mathcal{N}_\eta(\mathcal{H})$. The translated path $z_0\gamma'$ thus joins z and $z_0h_j^{-1}w_j$. Second, following backward $z_0h_j^{-1}\gamma_i$, we get a trajectory γ_i' from $z_0h_j^{-1}w_j$ to $z_0h_j^{-1}\tilde{x}$. All these trajectories stay outside $\mathcal{N}_\eta(\mathcal{H})$ and by concatenating γ, $z_0\gamma'$ and γ_i', we find the desired trajectory from $\tilde{x}$ to $z_0h_j^{-1}\tilde{x}$ staying outside $\mathcal{N}_\eta(\mathcal{H})$. We deduce that $z_0h_j^{-1} \in \mathcal{H}(\tilde{x})$. Finally, $d(z_0, z_0h_j^{-1}) = d(e, h_j^{-1})$ and h_j only depends on $\tilde{x}$, which in turn is contained in $B(e, \eta+1)$. Thus, this last quantity is uniformly bounded. □

Denote by $\tilde{y}'$ the point on the chosen geodesic from y to $\mathcal{H}$ which is at distance $\eta+1$ from $\mathcal{H}$. We now prove the following.

Claim 3.11. *There exist positive constants k,c only depending on η such that the connected component of $\tilde{x}$ in $\mathcal{N}_\eta(\mathcal{H})^c$ is (k,c)-starlike around $\tilde{y}'$.*

Proof. We have to prove that, for every $z \notin \mathcal{N}_\eta(\mathcal{H})$ that can be reached by a trajectory starting at $\tilde{x}$, there exists a path of length at most $kd(z,\tilde{y}')+c$ which stays outside $\mathcal{N}_\eta(\mathcal{H})$ and joins $\tilde{y}'$ from z.

Let z be such a point and $\tilde{z}$ be a point on a geodesic from z to $\mathcal{H}$ at distance $\eta+1$ from $\mathcal{H}$. Denote by z_0 and y_0, the respective projection of z and y on $\mathcal{H}$. By the above claim, we can find a trajectory of fixed length avoiding $\mathcal{N}_\eta(\mathcal{H})$, starting at some point $h\tilde{x}$ and ending at $\tilde{z}$. Moreover the distance between h and z_0 is uniformly bounded. Similarly, we can find a trajectory avoiding $\mathcal{N}_\eta(\mathcal{H})$ of fixed length, starting at $\tilde{y}'$ and ending at some $h'\tilde{x}$, where the distance between h' and y_0 is uniformly bounded.

Since $\mathcal{H}$ is virtually abelian, the subgroup $\mathcal{H}(\tilde{x})$ is quasi-isometrically embedded in Γ. Therefore, we can find a path $(h_j)_j$ from h to h' staying inside $\mathcal{H}(\tilde{x})$ with length at most $k_0 d(h,h')+c_0$ for some fixed constants k_0,c_0. By concatenating successive trajectories from $h_j\tilde{x}$ to $h_{j+1}\tilde{x}$, we can thus find a trajectory from $h\tilde{x}$ to $h'\tilde{x}$ of length at most $k_1 d(z_0,y_0)+c_1$ staying outside $\mathcal{N}_\eta(\mathcal{H})$, where $k_1,c_1>0$ only depend on η. Thus, there exist $c_2,k_2>0$ and a trajectory γ from z to $\tilde{y}'$ of length at most $k_2(d(z,\tilde{z})+d(z_0,y_0))+c_2$ and staying outside $\mathcal{N}_\eta(\mathcal{H})$. The distance formula [42, Theorem 3.1] shows that there exist positive constants c_3 and k_3 such that

$$d(z,\tilde{y}') \geq k_3^{-1}\left(d(z,z_0)+d(\tilde{y}',y_0)+d(z_0,y_0)\right)-c_3.$$

Thus, the length of γ is at most $kd(z,\tilde{y}')+c$, where k and c only depend on η. □

We now conclude the proof. Note that $\tilde{y}'$ is within a uniform bounded distance, depending only on η, of a point on a relative geodesic from $\tilde{x}$ to y. From relative Ancona inequalities for starlike sets [17, Proposition 3.12], it follows that

$$G(\tilde{x},y;\mathcal{N}_\eta(\mathcal{H})^c|r) \leq CG(\tilde{x},\tilde{y}';\mathcal{N}_\eta(\mathcal{H})^c|r)G(\tilde{y}',y;\mathcal{N}_\eta(\mathcal{H})^c|r). \tag{3.5}$$

Finally, the existence of one-step paths from $\tilde{y}'$ to $\tilde{y}$ and x to $\tilde{x}$ yields

$$G(x,y;\mathcal{N}_\eta(\mathcal{H})^c|r) \lesssim G(\tilde{x},y;\mathcal{N}_\eta(\mathcal{H})^c|r),$$
$$G(\tilde{x},\tilde{y}';\mathcal{N}_\eta(\mathcal{H})^c|r) \lesssim G(x,\tilde{y};\mathcal{N}_\eta(\mathcal{H})^c|r)$$

and

$$G(\tilde{y}',y;\mathcal{N}_\eta(\mathcal{H})^c|r) \lesssim G(\tilde{y},y;\mathcal{N}_\eta(\mathcal{H})^c|r).$$

Therefore, one can replace $\tilde{x}$ by x and $\tilde{y}'$ by $\tilde{y}$ in (3.5). □

Proposition 3.12. *Let $M\geq 0$. There exists η_M such that for every $\eta\geq\eta_M$ and for every $r\leq R_\mu$, the transition kernel p'_r has exponential moments up to M.*

Proof. In view of (3.4), it is enough to prove that for large enough η,

$$\sum_{y\notin\mathcal{N}_\eta(\mathcal{H})} G(e,y;\mathcal{N}_\eta(\mathcal{H})^c|r)G(y,x;\mathcal{N}_\eta(\mathcal{H})^c|r) \leq C\mathrm{e}^{-2M\|x_0\|},$$

where x_0 is the projection of x on $\mathbb{Z}^d$. For every $y\notin\mathcal{N}_\eta(\mathcal{H})$, denote by y_0 its projection on $\mathbb{Z}^d$ and let $\tilde{y}$ be the point at distance η from $\mathcal{H}$ on a geodesic from

y to $\mathcal{H}$. Also, let P_{y_0} be the set of points in Γ whose projection on $\mathbb{Z}^d$ is at y_0. Proposition 3.8 shows that

$$\sum_{y\notin\mathcal{N}_\eta(\mathcal{H})} G(e,y;\mathcal{N}_\eta(\mathcal{H})^c|r)G(y,x;\mathcal{N}_\eta(\mathcal{H})^c|r)$$
$$\lesssim \sum_{y_0\in\mathbb{Z}^d}\sum_{y\in P_{y_0}} G(e,\tilde{y};\mathcal{N}_\eta(\mathcal{H})^c|r)G(\tilde{y},x;\mathcal{N}_\eta(\mathcal{H})^c|r)$$
$$G(\tilde{y},y;\mathcal{N}_\eta(\mathcal{H})^c|r)G(y,\tilde{y};\mathcal{N}_\eta(\mathcal{H})^c|r).$$

Since the random walk is convergent, Lemma 2.7 yields

$$\sum_{y\in P_{y_0}} G(\tilde{y},y;\mathcal{N}_\eta(\mathcal{H})^c|r)G(y,\tilde{y};\mathcal{N}_\eta(\mathcal{H})^c|r) \lesssim G'(e,e|R_\mu) < +\infty.$$

Consequently,

$$\sum_{y\notin\mathcal{N}_\eta(\mathcal{H})} G(e,y;\mathcal{N}_\eta(\mathcal{H})^c|r)G(y,x;\mathcal{N}_\eta(\mathcal{H})^c|r)$$
$$\lesssim \sum_{y_0\in\mathbb{Z}^d} G(e,\tilde{y};\mathcal{N}_\eta(\mathcal{H})^c|r)G(\tilde{y},x;\mathcal{N}_\eta(\mathcal{H})^c|r).$$

By Lemma 3.3, for η large enough, it holds that $G(e,\tilde{y};\mathcal{N}_\eta(\mathcal{H})^c|r) \lesssim \mathrm{e}^{-4M\|y_0\|}$ and $G(\tilde{y},x;\mathcal{N}_\eta(\mathcal{H})^c|r) \lesssim \mathrm{e}^{-2M\|y_0-x_0\|} \lesssim \mathrm{e}^{2M\|y_0\|-2M\|x_0\|}$. As a consequence,

$$\sum_{y\notin\mathcal{N}_\eta(\mathcal{H})} G(e,y;\mathcal{N}_\eta(\mathcal{H})^c|r)G(y,x;\mathcal{N}_\eta(\mathcal{H})^c|r) \lesssim \left(\sum_{y_0\in\mathbb{Z}^d} \mathrm{e}^{-2M\|y_0\|}\right)\mathrm{e}^{-2M\|x_0\|}$$

This concludes the proof, since $\sum_{y_0\in\mathbb{Z}^d}\mathrm{e}^{-2M\|y_0\|} < +\infty$. □

Proposition 3.12 allows us to describe the regularity of the map $(r,u)\mapsto F_r(u)$ on $(0,R_\mu)\times\mathring{B}(0,M)$ where M is the constant which appears at the end of the previous subsection and $\mathring{B}(0,M)$ is the open ball with center 0 and radius M. We fix η such that both p_r and p'_r have exponential moments up to M.

Lemma 3.13. *The function $(r,u)\mapsto F_r(u)$ is continuously differentiable on the open set $(0,R_\mu)\times\mathring{B}(0,M)$. Its derivative extends continuously to $(0,R_\mu]\times\mathring{B}(0,M)$.*

Proof. It suffices to prove that for every j,j', $F_{j,j';r}(u)$ is continuously differentiable. By definition,

$$F_{j,j';r}(u) = \sum_{x\in\mathbb{Z}^d} p_{j,j';r}(0,x)\mathrm{e}^{u\cdot x}.$$

For all $x\in\mathbb{Z}^d$, the function $f:(r,u)\mapsto p_{j,j';r}(0,x)\mathrm{e}^{u\cdot x}$ is continuously differentiable and its derivative is given by

$$\nabla_{r,u}f(r,u) = p'_{j,j';r}(0,x)\mathrm{e}^{u\cdot x}v_r + p_{j,j';r}(0,x)\mathrm{e}^{u\cdot x}v_u(x),$$

where $v_r = (1,(0,...,0))$ and $v_u(x) = (0,x)$. Then,

$$\|\nabla_{r,u}f\|_\infty \leq \sup_r p'_{j,j';r}(0,x)\mathrm{e}^{M\|x\|} + \|x\|\sup_r p'_{j,j';r}(0,x)\mathrm{e}^{M\|x\|}.$$

Lemma 3.3 and Proposition 3.12 show that this quantity is summable. By dominated convergence, we deduce that the function $(r,u)\mapsto F_{j,j';r}(u)$ is continuously differentiable and its derivative equals

$$\nabla_{r,u} F_{j,j';r}(u) = \sum_{x \in \mathbb{Z}^d} p'_{j,j';r}(0,x) \mathrm{e}^{u \cdot x} v_r + p_{j,j';r}(0,x) \mathrm{e}^{u \cdot x} v_u(x).$$

This expression extends continuously to $(0, R_\mu] \times \mathring{B}(0, M)$. □

We can now prove that the function $r \mapsto \rho_\eta(r)$ is differentiable on $(0, R_\mu]$ and compute the value of its derivative.

Proposition 3.14. *The function $r \mapsto \rho_\eta(r)$ is continuously differentiable on $(0, R_\mu]$ and its derivative is given by $\rho'_\eta(r) = \lambda'_r(u_r)$.*

Proof. The function $F \mapsto \lambda$ is analytic on the set where F has a unique dominant eigenvalue. Thus, $(r, u) \mapsto \lambda_r(u)$ is continuously differentiable on $(0, R_\mu) \times \mathring{B}(0, M)$ and its derivative extends continuously to $(0, R_\mu] \times \mathring{B}(0, M)$. Moreover, it follows from [14, Proposition 3.5] that for all r, the Hessian of the map $u \mapsto \lambda_r(u)$ is positive definite. Therefore, the implicit function theorem shows that the function $r \mapsto u_r$ is continuously differentiable on $(0, R_\mu]$, and so is the function $r \mapsto \lambda_r(u_r)$. Moreover,

$$\rho'_\eta(r) = \nabla_u \lambda_r(u_r) \cdot u'_r + \lambda'_r(u_r).$$

Since λ_r is strictly convex and reaches its minimum at u_r, we have $\nabla_u \lambda_r(u_r) = 0$, hence the desired formula. □

Lemma 3.15. *For any $r \leq R_\mu$, we have $\rho'_\eta(r) \neq 0$.*

Proof. We just need to show that $\lambda'_r(u) \neq 0$ for any $u \in \mathring{B}(0, M)$. For a strongly irreducible matrix F, denote by C and by ν right and left eigenvectors associated to the dominant eigenvalue λ. By the Perron-Frobenius Theorem [41, Theorem 1.1], they both have positive coefficients. Moreover, one can normalize them such that we have $\nu \cdot C = 1$ and such that $F \mapsto C$ and $F \mapsto \nu$ are analytic functions, see [14, Lemma 3.3]. In particular, denoting by $C_r(u)$ and $\nu_r(u)$ right and left eigenvectors of $F_r(u)$ associated with the eigenvalue $\lambda_r(u)$, we get that the maps $(r, u) \mapsto C_r(u)$ and $(r, u) \mapsto \nu_r(u)$ are continuously differentiable and satisfy $\nu_r(u) \cdot C_r(u) = 1$. Therefore,

$$\lambda_r(u) = \nu_r(u) \cdot F_r(u) \cdot C_r(u)$$

and so

$$\lambda'_r(u) = \lambda_r(u) \Big(\nu'_r(u) \cdot C_r(u) + \nu_r(u) \cdot C'_r(u) \Big) + \nu_r(u) \cdot F'_r(u) \cdot C_r(u).$$

Differentiating in r the expression $\nu_r(u) \cdot C_r(u) = 1$, we get

$$\nu'_r(u) \cdot C_r(u) + \nu_r(u) \cdot C'_r(u) = 0$$

and so

$$\lambda'_r(u) = \nu_r(u) \cdot F'_r(u) \cdot C_r(u).$$

Since $p'_r(e, x)$ is non-negative for every $x \in \Gamma$, the matrix F'_r has non-negative entries. Also, it cannot be equal to the null matrix since $p'_r(e, x)$ is positive for at least some x. Moreover, $C_r(u)$ and $\nu_r(u)$ both have positive entries, hence $\lambda'_r(u)$ is positive. □

3.5. **Asymptotics of $G_r^{(j)}$.** We set $j = \lceil d/2 \rceil - 1$. Applying the previous results, we prove the following statement; this is a crucial step in the proof of our main theorem.

Proposition 3.16. *Assume that μ is spectrally degenerate along $\mathcal{H}$. If η is large enough, then the following holds. As $r \to R_\mu$,*

- *if d is even then*
$$G_{\eta,r}^{(j)}(e,e|1) \sim C \ \log\left(\frac{1}{R_\mu - r}\right),$$
- *if d is odd then*
$$G_{\eta,r}^{(j)}(e,e|1) \sim \frac{C}{\sqrt{R_\mu - r}}.$$

Proof. We recall the following local limit theorem on $\mathbb{Z}^d$. For every r, we have that $p_{\eta,r}^{(n)}(e,e) \sim C_{\eta,r} R_\eta(r)^{-n} n^{d/2}$ as n tends to infinity. Its proof for fixed r is standard, but we can be more precise. By [14, Proposition 3.14] applied to the kernel $R_\eta(r)p_{\eta,r}$, there exists C_r such that $C_r^{-1}R_\eta(r)^n p_{\eta,r}^{(n)} n^{d/2} - 1$ converges to 0 as $n \to +\infty$. Moreover, the convergence is uniform on r and the function $r \mapsto C_r$ is continuous. Consequently, the quantity C_r remains bounded away from 0 and infinity. Fix $\epsilon > 0$. Assume first that d is even, so that $j = d/2 - 1$. Then, for large enough n, say $n \geq n_0$, independently of r, we have
$$\left| C_r^{-1} n^j p_{\eta,r}^{(n)} - n^{-1}\rho_\eta(r)^n \right| \leq \epsilon n^{-1} \rho_\eta(r)^n.$$
Consequently,
$$\left| \sum_{n \geq n_0} \left(n^j p_{\eta,r}^{(n)} - C_r n^{-1} \rho_\eta(r)^n \right) \right| \leq C_r \epsilon \sum_{n\geq 0} n^{-1} \rho_\eta(r)^n.$$
Thus,
$$\left| \sum_{n\geq 0} n^j p_{\eta,r}^{(n)} - C_r \sum_{n \geq 0} n^{-1}\rho_\eta(r)^n \right| \leq \sum_{n \leq n_0 - 1} n^j p_{\eta,r}^{(n)} + C_r \sum_{n \leq n_0 - 1} n^{-1} \rho_\eta(r)^n$$
$$+ \epsilon C_r \sum_{n\geq 0} n^{-1}\rho_\eta(r)^n.$$
Note that
$$C_r \sum_{n\geq 0} n^{-1} \rho_\eta(r)^n = C_r \ \log\left(\frac{1}{1-\rho_\eta(r)}\right).$$
Since $\rho_\eta(r)$ converges to 1 as $r \to R_\mu$, this last quantity tends to infinity as r converges to R_μ. In particular, this proves that
$$\sum_{n \geq 0} n^j p_{\eta,r}^{(n)} \underset{r \to R_\mu}{\sim} C_r \ \log\left(\frac{1}{1-\rho_\eta(r)}\right),$$
and so
$$G_{\eta,r}^{(j)}(e,e|1) \underset{r \to R_\mu}{\sim} C'_r \ \log\left(\frac{1}{1-\rho_\eta(r)}\right). \tag{3.6}$$

By Proposition 3.14, there exists $\alpha \in \mathbb{R}$ such that $\rho_\eta(r) = 1+\alpha(r-R_\mu)+o(r-R_\mu)$. Lemma 3.15 yields $\alpha \neq 0$, hence $1-\rho_\eta(r) \sim \alpha(R_\mu - r)$. Combined with (3.6), this concludes the proof. The case where d is odd is treated in the same way. □

4. Stability of the Martin boundary

This section is dedicated to the proof of the *stability* of the Martin boundary (see Definition 2.6). In the case where the random walk is spectrally degenerate, this had not been dealt with before. We adapt here the arguments of [14], [17] and [18] in our context. The central result of this section is the following one.

Theorem 4.1. *Let Γ be a relatively hyperbolic group with respect to virtually abelian subgroups. Let μ be an admissible, symmetric and finitely supported probability measure on Γ. Then, the Martin boundary of (Γ,μ) is stable.*

Proof. Recall that the Martin boundary is stable if it satisfies the four conditions given by Definition 2.6. Let us first explain why the three first conditions are already known to be satisfied.

- As already mentioned, for all $x, y \in \Gamma$, since μ is admissible and Γ is non-amenable it follows from Guivarc'h [26] that $G(x,y|R_\mu) < +\infty$, i.e. condition (1) of stability is satisfied.
- From [17, Theorem 1.2], conditions (2) and (3) of stability are satisfied for any admissible random walk on a relatively hyperbolic group with virtually abelian parabolic subgroups : the homeomorphism type of $\Gamma \cup \partial_{r\mu}\Gamma$ does not depend on $r \in (0, R_\mu)$ and we denote $\partial_{r\mu}\Gamma$ by $\partial_\mu\Gamma$. Moreover, there exists an equivariant surjective and continuous map $\phi_\mu : \Gamma \cup \partial_\mu\Gamma \to \Gamma \cup \partial_{R_\mu\mu}\Gamma$ that extends the identity of Γ.
- It also follows from [17, Theorem 1.2] the r-Martin boundary of the random walk is identified with the r-*geometric boundary* of Γ, defined as follows. When $r < R_\mu$, the r-geometric boundary is constructed from the Bowditch boundary of Γ where each parabolic limit point ξ is replaced with the visual boundary of the corresponding parabolic subgroup. Equivalently, it is the Gromov boundary $\partial\hat{\Gamma}$ to which has been attached at each parabolic limit point $\xi_{\mathcal{H}}$ fixed by $\mathcal{H}$ the visual boundary of $\mathcal{H}$. At $r = R_\mu$, the r-geometric boundary is given by the same construction, with the following modification. The parabolic limit points are replaced with the visual boundary of the corresponding parabolic subgroup only when the random walk is spectrally nondegenerate along the underlying parabolic subgroup.

We are hence left with showing that the map

$$(x, y, r) \in \Gamma \times \Gamma \cup \partial_\mu\Gamma \times (0, R_\mu] \mapsto K(x, y|r)$$

is continuous, where for $\xi \in \partial_\mu(\Gamma)$, we write $K(x,\xi|R_\mu) = K(x,\phi_\mu(\xi)|R_\mu)$. Notice that this property is proved in [17, Theorem 1.3] in the case where the random walk is spectrally nondegenerate. Thus, what is left to prove is continuity at (x,ξ,R_μ), where ξ is in the preimage of a parabolic limit point such that μ is spectrally degenerate along the corresponding parabolic subgroup.

By using the geometric interpretation of the r-Martin boundaries mentioned before, we need to check that, for any $x \in \Gamma$, any sequence $(y_n)_n$ in $\Gamma \cup \partial_\mu\Gamma$ which converges to a point ξ in the geometric boundary of a parabolic subgroup $\mathcal{H}$ along which the random walk is spectrally degenerate and any $(r_n)_n$ which converges to R_μ, the sequence $(K(x, y_n|r_n))_n$ converges to $K(x, \xi|R_\mu)$.

Let us denote by $\pi_{\mathcal{H}}(y_n)$ the projection of y_n on $\mathcal{H}$. Since (y_n) converges to a point in $\partial\mathcal{H}$ the sequence $(\pi(y_n))$ goes to infinity. As a consequence, for large enough n, the point $\pi_{\mathcal{H}}(y_n)$ is within a bounded distance of a relative geodesic from x to y_n for every fixed x. Fix $\epsilon > 0$. By Proposition 2.8, there exists $\eta \geq 0$ such that $G(x, y_n; B_\eta(\pi_{\mathcal{H}}(y_n))^c|r_n) \leq \epsilon G(x, y_n|r_n)$. Hence, up to an error of order ϵ, we can reduce to trajectories that do enter in $B_\eta(\pi_{\mathcal{H}}(y_n))$. This allows us to replace $K(x, y_n|r)$ by $K(x, u_n|r)$ where $u_n \in B_\eta(\pi_{\mathcal{H}}(y_n))$ and so to assume that y_n stays in a fixed neighborhood $\mathcal{N}_\eta(\mathcal{H})$. We refer to [18, Proposition 5.5] or [17, Proposition 4.3] for more details. Now, this fixed neighborhood can be identified with $\mathbb{Z}^d \times \{1, ..., N\}$ as in Section 3.3. Theorem 4.1 is thus a direct consequence of Proposition 4.2 below which gives the convergence of $(K(x, y_n|r_n))_n$. $\square$

Proposition 4.2. *Let p be a $\mathbb{Z}^d$-invariant transient kernel on $E = \mathbb{Z}^d \times \{1, ..., N\}$. Assume that p is irreducible and aperiodic and has exponential moments. Then, the Martin boundary is stable and the function*

$$(x, y, r) \in E \times E \cup \partial_p E \times (0, R_p] \mapsto K(x, y|r)$$

is continuous.

The proof of this theorem relies on two lemmas. For any $v \in \mathbb{R}^d$, we write $\langle v \rangle \in \mathbb{Z}^d$ the vector with integer entries which is closest (for the Euclidean distance) to v, choosing the first in lexicographical order in case of ambiguity.

The next lemma is technical and is inspired by [14, Lemma 3.28]. It is used in a particular case. Namely, p_n is the convolution power of the transition kernel of first return to $\mathcal{N}_\eta(\mathcal{H})$, β_v is the derivative of the eigenvalue λ_v, α_v is some suited factor and Σ_v is the Hessian matrix associated with p_η. The fact that the quantity a_n defined in the lemma uniformly converges to 0 is a consequence of [14, Proposition 3.16].

Lemma 4.3. *Let $p_n(x)$ be a sequence of real numbers, depending on $x \in \mathbb{Z}^d$. Let $K \subset \mathbb{R}^N$ be a compact set and $v \mapsto \alpha_v$ and $v \mapsto \beta_v$ two continuous functions on K, with $\alpha_v \in \mathbb{R}$ and $\beta_v \in \mathbb{R}^d$. Let Σ_v be a positive definite quadratic form on $\mathbb{R}^d$, that depends continuously on $v \in K$. Define*

$$a_n(x, v, \gamma) = \left(\frac{\|x - n\beta_v\|}{\sqrt{n}}\right)^\gamma \left((2\pi n)^{\frac{d}{2}} p_n(x) \mathrm{e}^{v \cdot x} - \alpha_v \mathrm{e}^{-\frac{1}{2n}\Sigma_v(x - n\beta_v)}\right).$$

Denote by $g(x)$ the sum over n of the $p_n(x)$. If $(a_n)_n$ converges to 0 uniformly in $x \in \mathbb{Z}^d$, $v \in K$ and $\gamma \in [0, 2d]$, then, for $x \in \mathbb{Z}^d$ and for $v \in K$ such that $\beta_v \neq 0$, it holds as t tends to infinity,

$$(2\pi t)^{\frac{d-1}{2}} \ \Sigma_v(\beta_v) \ g(\langle t\beta_v \rangle - x) \mathrm{e}^{v \cdot (\langle t\beta(v) \rangle - x)} = \alpha_v + o(1)$$

where the term $o(1)$ is bounded by an asymptotically vanishing quantity which does not depend on v.

Proof. It is proved in [14, Lemma 3.28] that with the same assumptions, *assuming moreover that* $\beta_v \neq 0$ for every $v \in K$,

$$(2\pi t)^{\frac{d-1}{2}} g(\langle t\beta_v \rangle - x) \mathrm{e}^{v \cdot (\langle t\beta(v) \rangle - x)} \underset{t \to \infty}{\longrightarrow} \frac{\alpha_v}{\Sigma_v(\beta_v)},$$

the convergence being uniform in v. The proof is the same as in Ney and Spitzer [29, Theorem 2.2]. This is basically what we need to prove here, except that we want to move the factor $\Sigma_v(\beta_v)$ to the left hand-side and keep uniform convergence over v. By technical manipulations, when summing p_n over n, Ney and Spitzer reduce the proof to bounding $\mathrm{e}^{-y^2\Sigma_v(\beta_v)}$ by an integrable function. The key ingredient is that $\Sigma_v(\beta_v)$ is uniformly bounded from below by some constant β, so that

$$\mathrm{e}^{-y^2\Sigma_v(\beta_v)} \leq \mathrm{e}^{-y^2\beta}. \tag{4.1}$$

The function on the right-hand side is indeed integrable on $[0, +\infty)$ and the bound is uniform in v, which allow Ney and Spitzer to prove uniform convergence in v.

In our context, we do not necessarily have $\beta_v > 0$ for all $v \in K$ and so we cannot use compactness of K to ensure that $\Sigma(\beta_v)$ is uniformly bounded from below. However, what we need to prove is that $\Sigma_v(\beta_v)(2\pi t)^{\frac{d-1}{2}}g(\langle t\beta_v\rangle - x)\mathrm{e}^{\langle t\beta(v)\rangle - x}$ converges uniformly in v and we can replace (4.1) by

$$\Sigma_v(\beta_v)\mathrm{e}^{-y^2\Sigma_v(\beta_v)} \leq C\frac{1}{1+y^2}.$$

The right-hand side is again an integrable function and so we can conclude exactly like in the proof of Ney and Spitzer [29, Theorem 2.2]. □

As announced, β_v plays the role of $\nabla\lambda_r(u)$, hence this result yields the necessary asymptotics of the Green function for every r, u such that $\nabla\lambda_r(u) \neq 0$. On the other hand, when $\nabla\lambda_r(u) = 0$, we need the following lemma.

Lemma 4.4. *With the same notations as in Proposition 4.3, assume that for every $\gamma \in [0, d-1]$, a_n converges to 0, uniformly in $x \in \mathbb{Z}^d$ and $v \in K$. Then, for $x \in \mathbb{Z}^d$ and for $v \in K$ such that $\beta_v = 0$, as y tends to infinity, we have*

$$g(y-x)\mathrm{e}^{v\cdot(y-x)} \sim \alpha_v C_d \frac{1}{\|\Sigma(y)\|^{\frac{d-2}{2}}},$$

where C_d only depends on the rank d.

Proof. Define

$$\tilde{g}(y) = \sum_{n\geq 1} \frac{1}{(2\pi n)^{d/2}}\alpha_v \mathrm{e}^{-\frac{1}{2n}\Sigma_v(y)}.$$

Setting $t_n = \frac{n}{\Sigma_v(y-x)}$, we have $\Delta_n := t_n - t_{n-1} = \frac{1}{\Sigma_v(y-x)}$ which tends to 0 as y tends to infinity. Thus,

$$\frac{1}{\alpha_v}(2\pi)^{d/2}\Sigma_v(y-x)^{\frac{d-2}{2}}\tilde{g}(y-x) = \sum_{n\geq 1} t_n^{-d/2}\mathrm{e}^{-\frac{1}{2t_n}}\Delta_n$$

is a Riemannian sum of $\int_0^{+\infty} t^{-d/2}\mathrm{e}^{-\frac{1}{2t}}dt = C'_d$. Consequently, we just need to show that

$$\frac{g(y-x)e^{v\cdot(y-x)}}{\tilde{g}(y-x)} \underset{y\to\infty}{\longrightarrow} 1.$$

Equivalently, we prove that

$$g(y-x)e^{v\cdot(y-x)} - \tilde{g}(y-x) = o\left(\|y\|^{-(d-2)}\right).$$

We set

$$\alpha_n = \sup_{y\in\mathbb{Z}^d} \sup_{\gamma\in[0,d-1]} \left(\frac{\|y-x\|}{\sqrt{n}}\right)^{\gamma} \left|(2n\pi)^{d/2} p_n(y-x)\mathrm{e}^{v\cdot(y-x)} - \alpha_v \mathrm{e}^{-\frac{1}{2n}\Sigma_v(y-x)}\right|.$$

By assumption, $(\alpha_n)_n$ converges to 0 as n tends to infinity. Let $\epsilon > 0$. Then for $n \geq n_0$, $\alpha_n \leq \epsilon$. We have

$$\|y-x\|^{d-2} \left|g(y-x)e^{v\cdot(y-x)} - \tilde{g}(y-x)\right| \leq \frac{1}{\|y-x\|(2\pi)^{d/2}} \sum_{n=1}^{n_0-1} \frac{\alpha_n}{n^{1/2}}$$
$$+ \frac{1}{\|y-x\|(2\pi)^{d/2}} \sum_{n=n_0}^{\|y-x\|^2} \frac{\alpha_n}{n^{1/2}} + \frac{\|y-x\|^{d-2}}{(2\pi)^{d/2}} \sum_{n>\|y-x\|^2} \frac{\alpha_n}{n^{d/2}}.$$

The first term in the right-hand side converges to 0 as $\|y\|$ tends to infinity. The second term is bounded by

$$\frac{\epsilon}{\|y-x\|(2\pi)^{d/2}} \int_0^{\|y-x\|^2} t^{-1/2} dt \lesssim \epsilon.$$

The last term is bounded by

$$\frac{\epsilon\|y-x\|^{d-2}}{(2\pi)^{d/2}} \int_{\|y-x\|^2}^{+\infty} t^{-d/2} dt \lesssim \epsilon.$$

This concludes the proof. □

All the ingredients are gathered to end the proof of Proposition 4.2. Roughly speaking, Lemma 4.3 yields the convergence of the Martin kernels, as r tends to R_μ, $r < R_\mu$, and y tends to a point ξ in the Martin boundary. Lemma 4.4 gives in turn the convergence of the Martin kernels for fixed $r = R_\mu$ and y converging to the same ξ. Since these two limits coincide, this proves continuity of the map $(x, y, r) \mapsto K(x, y|r)$.

Proof of Proposition 4.2. For all $u \in \mathbb{R}^d$ and $r \in [0, R_\mu]$, let $F_r(u)$ be the vertical displacement transition matrix defined in Section 3.3 and let $\lambda(r, u)$ be its dominant eigenvalue. Let K be the set of pairs (r, u) such that $r \in [0, R_\mu]$ and u satisfies that $\lambda(u) = 1$. Since $r \mapsto \lambda_r$ is a continuous function from $[0, R_\mu]$ to the set of continuous functions of u, the set K is compact.

Recall that for fix $k, j \in \{1, ..., N\}$, we write $p_{k,j}(x, y) = p((x, k), (y, j))$, for every $x, y \in \mathbb{Z}^d$. According to [14, Proposition 3.14, Proposition 3.16], for every k, j in $\{1, ..., N\}$, the kernel $p_{k,j}$ satisfies the assumptions of Lemma 4.3 and Lemma 4.4 if we define

$$\alpha_{(r,u)} := \frac{1}{\det(\Sigma_u)} C_r(u)_k \nu_r(u)_j,$$

$\beta_{u,r} = \nabla\lambda_r(u)$ and $\Sigma_{(r,u)}$ to be the inverse of the quadratic form associated to the Hessian of the eigenvalue $\lambda(u)$ and where $C_r(u)$ and $\nu_r(u)$ are right and left eigenvectors associated to $\lambda_r(u)$.

Consider ξ in the boundary $\partial\mathcal{H}$ of $\mathcal{H}$ and let $((y_n, j_n))_n$ be a sequence in E which converges to ξ, i.e. $(y_n)_n$ tends to infinity and $(y_n/\|y_n\|)_n$ converges to ξ. Also write $e = (0, k_0)$ for the basepoint of E.

Assuming that $r < R_\mu$, [14, Lemma 3.24] shows that the map

$$u \in \{u, \lambda_r(u) = 1\} \mapsto \frac{\nabla \lambda_r(u)}{\|\nabla \lambda_r(u)\|}$$

is a homeomorphism between $\{u, \lambda_r(u) = 1\}$ and $\mathbb{S}^{d-1}$. Thus, there exists $u_{r,n}$ such that

$$\frac{y_n}{\|y_n\|} = \frac{\nabla \lambda_r(u_{r,n})}{\|\nabla \lambda_r(u_{r,n})\|}.$$

As explained in the previous section, [18, (5), Proposition 4.6] shows that the set $\{u, \lambda_r(u) = 1\}$ is contained in a fixed ball $B(0, M)$. Therefore, $u_{r,n}$ is bounded and so is $\|\nabla \lambda_r(u_{r,n})\|$. Setting $t_n = \|y_n\|/\|\nabla \lambda_r(u_{r,n})\|$, we see that $(t_n)_n$ tends to infinity and that

$$y_n = t_n \nabla \lambda_r(u_{r,n}).$$

Consider now a sequence $(r_n)_n$ converging to R_μ, $r_n < R_\mu$. By Lemma 4.3,

$$\begin{aligned}(2\pi t_n)^{\frac{d-1}{2}} & G((x,k),(y_n,j_n)|r_n)\mathrm{e}^{u_{r_n,n}\cdot(y-x)} \\ &= \frac{1}{\Sigma_{r_n,u_{r,n}}(\nabla\lambda_r(u_{r,n}))}\left(\frac{C_{r_n}(u_{r_n,n})_k \nu_{r_n}(u_{r_n,n})_{j_n}}{\det(\Sigma_{r_n,u_{r_n,n}})} + o(1)\right)\end{aligned}$$

and

$$\begin{aligned}(2\pi t_n)^{\frac{d-1}{2}} & G(e,(y_n,j_n)|r_n)\mathrm{e}^{u_{r_n,n}\cdot y} \\ &= \frac{1}{\Sigma_{r_n,u_{r,n}}(\nabla\lambda_r(u_{r,n}))}\left(\frac{C_{r_n}(u_{r_n,n})_{k_0} \nu_{r_n}(u_{r_n,n})_{j_n}}{\det(\Sigma_{r_n,u_{r_n,n}})} + o(1)\right)\end{aligned}$$

Consequently,

$$K((x,k),(y_n,j_n)|r_n) = \mathrm{e}^{u_{r,n}\cdot x}\frac{\frac{C_{r_n}(u_{r_n,n})_k \nu_{r_n}(u_{r_n,n})_{j_n}}{\det(\Sigma_{r_n,u_{r_n,n}})} + o(1)}{\frac{C_{r_n}(u_{r_n,n})_{k_0} \nu_{r_n}(u_{r_n,n})_{j_n}}{\det(\Sigma_{r_n,u_{r_n,n}})} + o(1)}.$$

By [17, Lemma 5.6], the sequence $(u_{r_n,n})_n$ converges to u_{R_μ} as $(y_n/\|y_n\|)_n$ tends to ξ and $(r_n)_n$ tends to R_μ. Note that the limit does not depend on ξ. Indeed, u_{R_μ} is a point such that $\lambda_{R_\mu}(u_{R_\mu}) = 1$ and by (3.3), the minimum of λ_{R_μ} is 1. Since λ_{R_μ} is strictly convex, the point u_{R_μ} is unique. The $o(1)$ term above is uniform in r, hence $(K((x,k),(y_n,j_n)|r_n))_n$ converges to $\mathrm{e}^{u_{R_\mu,\xi}\cdot x}\frac{C_{R_\mu}(u_{R_\mu,\xi})_k}{C_{R_\mu}(u_{R_\mu,\xi})_{k_0}}$.

Assume now that $r = R_\mu$ is fixed and $(y_n)_n$ converges to ξ. We apply Lemma 4.4 to the same parameters α_v, β_v, Σ_v to deduce that

$$K((x,k),(y_n,j_n)|R_\mu) \sim \mathrm{e}^{u_{R_\mu,\xi}\cdot x}\frac{C_{R_\mu}(u_{R_\mu,\xi})_k}{C_{R_\mu}(u_{R_\mu,\xi})_{k_0}}.$$

Thus, as $(y_n, j_n) \to \xi$ and $r_n \to R_\mu$ with $r_n < R_\mu$, the limit of the two sequences $(K((x,k),(y_n,j_n)|r_n))_n$ and $(K((x,k),(y_n,j_n)|R_\mu))_n$ coincide. □

5. Asymptotics of the full Green function

The purpose in this section is to show that, for a convergent random walk on a relatively hyperbolic group whose Martin boundary is stable, the asymptotics of the derivatives of the full Green function are given by the asymptotics of the derivatives of the Green functions associated to the first return kernels to dominant parabolic subgroups. The precise statement is given in Theorem 5.4 below. This is the last crucial step before the proof of the local limit theorem. Note that throughout this section, we do not need to assume that parabolic subgroups are virtually abelian.

If $x \in \Gamma$, we define $I_x^{(k)}(r)$ by
$$I_x^{(k)}(r) = \sum_{x_1,...,x_k \in \Gamma} G(e,x_1|r)G(x_1,x_2|r)...G(x_{k-1},x_k|r)G(x_k,x|r).$$

For $x = e$, we write $I^{(k)}(r) = I_e^{(k)}(r)$. These quantities are related to the derivatives of the Green function by the following result. We inductively define
$$F_{1,x}(r) = \frac{d}{dr}(rG_r(e,x))$$
and
$$F_{k,x}(r) = \frac{d}{dr}(r^2 F_{k-1,x}(r)), k \geq 2.$$

The following lemma generalizes Lemma 2.7 and is valid for any transition kernel.

Lemma 5.1. [15, Lemma 3.2] *For every $x \in \Gamma$ and $r \in [0, R_\mu]$,*
$$F_{k,x}(r) = k! r^{k-1} I_x^{(k)}(r).$$

The following result is a direct consequence of this lemma.

Proposition 5.2. *For every $k \geq 1$, $x \in \Gamma$ and $r \leq R_\mu$,*
$$I_x^{(k)}(r) \asymp G(e,x|r) + G'(e,x|r) + ... + G^{(k)}(e,x|r).$$
Moreover, if k is the smallest integer such that $I^{(k)}(R_\mu) = +\infty$ — or equivalently such that $G^{(k)}(e,e|R_\mu) = +\infty$ — then, as $r \to R_\mu$,
$$I_x^{(k)}(r) \sim C G^{(k)}(e,x|r).$$

For any parabolic subgroup $\mathcal{H}$ of Γ and any $\eta \geq 0$, we also set for every $x \in \mathcal{N}_\eta(\mathcal{H})$ and $r \in [0, R_\mu]$
$$I_{\mathcal{H},\eta,x}^{(k)}(r) = \sum_{x_1,...,x_k \in \mathcal{N}_\eta(\mathcal{H})} G(e,x_1|r)G(x_1,x_2|r)...G(x_{k-1},x_k|r)G(x_k,x|r).$$

Again, if $x = e$, we write $I_{\mathcal{H},\eta}^{(k)}(r) = I_{\mathcal{H},\eta,e}^{(k)}(r)$. Since $G(x,x'|r) = G_{\mathcal{H},\eta,r}(x,x'|1)$ for any $x, x' \in \mathcal{N}_\eta(\mathcal{H})$, the quantities $I_{\mathcal{H},\eta,x}^{(k)}(r)$ are related to the derivatives of $G_{\mathcal{H},\eta,r}$ at 1 by the same formulae as in Lemma 5.1.

We now fix a finite set $\{\mathcal{H}_1, ..., \mathcal{H}_N\}$ of representatives of conjugacy classes of the parabolic subgroups. For $\eta \geq 0$, we define
$$J_\eta^{(k)}(r) = \sum_{p=1}^{N} I_{\mathcal{H}_p,\eta}^{(k)}(r) \tag{5.1}$$
and $J^{(k)}(r) = J_0^{(k)}(r), k \geq 1$.

Proposition 5.3. *Consider a finitely generated relatively hyperbolic group Γ and a finitely supported symmetric and admissible probability measure μ on Γ. Assume that the random walk is convergent, i.e. $I^{(1)}(R_\mu)$ is finite. Let k be the smallest integer such that $J^{(k)}(R_\mu)$ is infinite. Then, the quantity $I^{(j)}(R_\mu)$ is finite for every $j < k$ and for every $\eta \geq 0$,*
$$I^{(k)}(r) \asymp J_\eta^{(k)}(r)$$
where the implicit constant only depends on η.

Proof. Clearly, we have $J^{(j)}(r) \lesssim I^{(j)}(r)$ for every j. Also, by [15, Lemma 5.7], the sum $I^{(j)}$ is bounded by some quantity that only depends on all the $I^{(l)}(r)$, $l < j$ and on all the $J^{(l)}(r)$, $l \leq j$. Thus, by induction, $I^{(j)}(R_\mu)$ is finite for every $j < k$ and $I^{(k)}(r) \lesssim J^{(k)}(r)$. Finally, $J^{(j)}(r) \asymp J^{(j)}_\eta(r)$, where the implicit constant only depends on η. □

The purpose of this section is to prove the following theorem that refines Propostion 5.3. Its assumptions are satisfied as soon as the parabolic subgroups are virtually abelian, according to Theorem 4.1 and [17, Proposition 4.3].

Theorem 5.4. *Consider a finitely generated relatively hyperbolic group Γ and a finitely supported symmetric and admissible probability measure μ on Γ. Assume that the random walk is convergent, i.e. $I^{(1)}(R_\mu)$ is finite. For any parabolic subgroup $\mathcal{H}$ of Γ and any $r \leq R_\mu$, let $p_{\mathcal{H},r}$ be the first return kernel to $\mathcal{H}$ associated with $r\mu$.*

Assume that the following holds.

- *The Martin boundary is stable and the function*
$$(x, y, r) \in \Gamma \times \Gamma \cup \partial_\mu \Gamma \times (0, R_\mu] \mapsto K(x, y|r)$$
is continuous.
- *The 1-Martin boundary of $(\mathcal{H}, p_{\mathcal{H},R_\mu})$ is reduced to a point for all $\mathcal{H}$ such that the random walk is spectrally degenerate along $\mathcal{H}$.*

Let k be the smallest integer such that $J^{(k)}(R_\mu)$ is infinite. Then, for every $\eta \geq 0$, there exists a constant C_η such that as $r \to R_\mu$,
$$I^{(k)}(r) \sim C_\eta J^{(k)}_\eta(r).$$

The next two subsections are dedicated to the proof of this theorem.

5.1. **Asymptotics of the second derivative.** We start with proving Theorem 5.4 when $k = 2$, i.e. $J^{(1)}(R_\mu)$ is finite and $J^{(2)}(R_\mu)$ is infinite. We first consider the case $\eta = 0$.

Claim 5.5. *Under the assumptions of Theorem 5.4, if $k = 2$, then there exists a positive constant C such that*
$$I^{(2)}(r) \sim CJ^{(2)}(r).$$

5.1.1. *Step 1. $I^{(2)}(r)$ via Birkhoff sums.* We write $H(x, y|r) = G(x, y|r)G(y, x|r)$. The purpose of this paragraph is to express $I^{(2)}(r)$ up to a bounded error as
$$\sum_{n\geq 0} \sum_{k=0}^{n-1} \sum_{x\in \hat{S}_n} H(e, x|r)\Psi_r\big(T^k[e, x]\big),$$
where the function Ψ_r is introduced below. See precisely Proposition 5.8.

Many computations are analogous to [16, Section 4, Section 5] and we first make some general comments about our strategy. In [16], the first author introduced the transfer operator
$$\mathcal{L}_r(f)(\tilde{x}) = \sum_{T\tilde{y}=\tilde{x}} \frac{H(e, y|r)}{H(e, x|r)} f(\tilde{y}).$$
Here, $\tilde{x}$ and $\tilde{y}$ are words accepted by the automaton $\mathcal{G}$ encoding relative geodesics given by Theorem 2.3, x and y are the corresponding elements of Γ and T is the

left shift on this automaton. Roughly speaking, we choose a fixed relative geodesic $[e,x]$ from e to x for every $x\in\Gamma$. Then, $T\tilde{y}=\tilde{x}$ means that $[e,x]$ is obtained from $[e,y]$ by deleting the first increment. Denoting by $\emptyset$ the empty word, we see that

$$H(e,e|r)\mathcal{L}_r^n(f)(\emptyset)=\sum_{x\in\hat{S}_n}H(e,x|r)f(x).$$

With this terminology, we thus want to express $I^{(2)}(r)$ as the sum of the iterates of $\mathcal{L}_r$ of the Birkhoff sum of Ψ_r.

However, spectral nondegeneracy is used in crucial ways in [16]. First, to prove that the spectral data of $\mathcal{L}_r$ vary continuously in r, second to check that the different functions involved in the proofs have sufficient regularity to control the convergence of the iterates of $\mathcal{L}_r$. As a consequence, we cannot use thermodynamic formalism as in [16] in the present paper. On the other hand, assuming convergence of the random walk simplify many computations. For instance, a formula analogous to the one we want to prove in this paragraph is given by [16, Proposition 5.3], but the error term, rather than being bounded, involves $I^{(1)}(r)$ which explodes at R_μ. To overcome this difficulty, the analogous function Ψ_r introduced in [16] is more sophisticated than the one that we use below.

Thus, if we skip the spectral degeneracy assumption, we cannot use the operator $\mathcal{L}_r$ and this is the main difference with [16]. Nevertheless, for our purpose, rough asymptotic estimates of $I^{(1)}(r)$ are sufficient.

We now start the proof of Claim 5.5. By definition, $I^{(1)}(r)=\sum_{x\in\Gamma}H(e,x|r)$. We introduce the function Φ_r defined by

$$\Phi_r(x)=\sum_{y\in\Gamma}\frac{G(e,y|r)G(y,x|r)}{G(e,x|r)}.$$

We then have

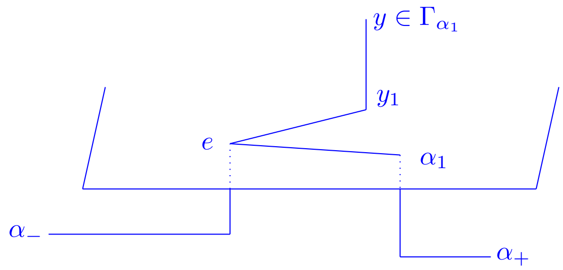
$$I^{(2)}(r)=\sum_{x\in\Gamma}H(e,x|r)\Phi_r(x).$$

We fix a finite generating set S. Using the automaton $\mathcal{G}$ encoding relative geodesics given by Theorem 2.3, for any $x\in\Gamma$, we choose a relative geodesic $[e,x]$ from e to x. Also, we write $\Omega_0=\{\mathcal{H}_1,...,\mathcal{H}_N\}$ and $\mathcal{H}_0=S$ so that each increment of a relative geodesic is in one of the $\mathcal{H}_j$. Distinct parabolic subgroups have finite intersections, see for instance [13, Lemma 4.7], so up to enlarging $\mathcal{H}_0$ to a bigger finite set, we can assume that all $\mathcal{H}_j$ are disjoint. This is not needed, but this might help readability.

Consider a finite relative geodesic

$$\alpha=(\alpha_{-k_1}\alpha_{-k_1+1}...,\alpha_0,\alpha_1,...,\alpha_{k_2})$$

such that $\alpha_0=e$. We denote by Γ_{α_1} the set of elements $y\in\Gamma$ such that y_1 and α_1 lie in the same $\mathcal{H}_j$; that is the increments of $[e,y]$ and α with index 1 belong to the same parabolic subgroup, as suggested by the following picture.



We then define $\Psi_r(\alpha)$ by

$$\Psi_r(\alpha) = \sum_{y \in \Gamma_{\alpha_1}} \frac{G(\alpha_-, y|r)G(y, \alpha_+|r)}{G(\alpha_-, \alpha_+|r)},$$

where $\alpha_- = \alpha_{-k_1}$ and $\alpha_+ = \alpha_{k_2}$ are the left and right extremities of α. Let T be the left shift on relative geodesics, so that $T^k\alpha$ is the relative geodesic $\alpha(k)^{-1}\alpha$. Our first goal is to prove the following.

Proposition 5.6. *Let $x \in \hat{S}_n$. Then*

$$\Phi_r(x) = \sum_{k=0}^{n-1} \Psi_r\big(T^k[e,x]\big) + O(n).$$

Proof. Set $[e,x] = (e, x_1, x_2, ..., x_n)$. Fix $k \leq n-1$. Let Γ_k be the set of elements y such that the projection of y on $[e,x]$ in the relative graph $\hat{\Gamma}$ equals x_k. If there are several projections, we choose the one which is the closest to e. Let j_k be such that $x_k^{-1}x_{k+1} \in \mathcal{H}_{j_k}$.

We set, for $k \leq n-1$,

$$\Sigma_k = \sum_{y \in \Gamma_k} \frac{G(e,y|r)G(y,x|r)}{G(e,x|r)} - \Psi_r(T^k[e,x]).$$

Consider some $y \in \Gamma_k$, $k \leq n-1$, so that the projection of the relative geodesic $x_k^{-1}y$ on $T^k[e,x] = x_k^{-1}[e,x]$ equals e. Consider also the relative geodesic $[e, x_k^{-1}y]$ and write $[e, x_k^{-1}y] = (e, z_1, ..., z_m)$. Note that the condition $z_1 \in \mathcal{H}_{j_k}$ exactly means that $x_k^{-1}y \in \Gamma_{(T^k[e,x])_1}$. Consequently, in Σ_k, we are left with the elements $y \in \Gamma_k$ such that $z_1 \notin \mathcal{H}_{j_k}$. We prove that the contribution of such y is bounded. Roughly speaking, those y project on $\mathcal{H}_{j_k}$ near e. Precisely, we prove the following.

Claim 5.7. *If $z_1 \notin \mathcal{H}_{j_k}$, then any relative geodesic from $z_m = x_k^{-1}y$ to $x_k^{-1}x$ passes within a bounded distance of e.*

Proof. By [42, Lemma 1.15], there exists $L \geq 0$ such that if the projection in the Cayley graph of Γ of z_m on $\mathcal{H}_{j_k}$ is at distance at least L from e, then the relative geodesic $[e, x_k^{-1}y]$ contains an edge in $\mathcal{H}_{j_k}$. Since $[e, x_k^{-1}y]$ is a relative geodesic, This edge cannot be z_1. Thus, the projection of z_m on $\mathcal{H}_{j_k}$ is within L of e. By [15, Lemma 4.16], we also know that any relative geodesic $[z_m, x_k^{-1}x]$ from z_m to $x_k^{-1}x$ passes within a bounded distance of $x_k^{-1}x_{k+1}$. If $d(e, x_k^{-1}x_{k+1}) \leq L$, this proves that $[z_m, x_k^{-1}x]$ passes within a bounded distance of e. On the contrary, if $d(e, x_k^{-1}x_{k+1}) \geq L$, then using [42, Lemma 1.15] once again, such a geodesic has an edge in $\mathcal{H}_{j_k}$. By [42, Lemma 1.13], the entrance point in $\mathcal{H}_{j_k}$ is within a bounded distance of the projection of z_m on $\mathcal{H}_{j_k}$, which is itself at a distance at most L of e.

Thus, in any case, any relative geodesic from z_m to $x_k^{-1}x$ passes within a bounded distance of e. $\square$

By relative Ancona inequalities, we deduce that

$$G(y,x|r) = G(x_k^{-1}y, x_k^{-1}x|r) \asymp G(x_k^{-1}y, e|r)G(e, x_k^{-1}x|r) = G(y, x_k|r)G(x_k, x|r).$$

Now, using again [15, Lemma 4.16], since $y \in \Gamma_k$, any relative geodesic from x_k^{-1} to $z_m = x_k^{-1}y$ passes within a bounded distance of e. Thus, similarly, relative Ancona inequalities yield

$$G(e,y|r) \asymp G(e,x_k|r)G(x_k,y|r).$$

Finally, by definition, x_k is on a relative geodesic from e to x, hence

$$G(e,x|r) \asymp G(e,x_k|r)G(x_k,x|r).$$

We deduce that whenever $z_1 \notin \mathcal{H}_{j_k}$, it holds

$$\frac{G(e,y|r)G(y,x|r)}{G(e,x|r)} \lesssim G(x_k,y|r)G(y,x_k|r).$$

Therefore, Σ_k is bounded by

$$\sum_{\substack{y\in\Gamma_k \\ z_1\notin\mathcal{H}_{j_k}}} G(x_k,y|r)G(y,x_k|r) \lesssim \sum_{y\in\Gamma} G(x_k,y|r)G(y,x_k|r) = I^{(1)}(r).$$

Now, we treat the case $k = n$. Any $y \in \Gamma_n$ projects on $[e,x]$ at x. By [15, Lemma 4.16], any relative geodesic from e to y passes within a bounded distance of x. Thus, weak relative Ancona inequalities yield this time

$$\frac{G(e,y|r)G(y,x|r)}{G(e,x|r)} \lesssim G(x,y|r)G(y,x|r).$$

Thus, we get

$$\sum_{y\in\Gamma_n} \frac{G(e,y|r)G(y,x|r)}{G(e,x|r)} \lesssim I^{(1)}(r).$$

Since the random walk is convergent, $I^{(1)}(r)$ is uniformly bounded. Summing over k from 0 to n, we thus have

$$\left|\Phi_r(x) - \sum_{k=0}^{n-1}\Psi_r\big(T^k[e,v]\big)\right| \lesssim n,$$

which is the expected bound. $\square$

We deduce the following.

Proposition 5.8. *We have*

$$I^{(2)}(r) = \sum_{n\geq 0}\sum_{k=0}^{n-1}\sum_{x\in\hat{S}_n} H(e,x|r)\Psi_r\big(T^k[e,x]\big) + O(1).$$

Proof. By Proposition 5.6,

$$\left|I^{(2)}(r) - \sum_{n\geq 0}\sum_{k=0}^{n-1}\sum_{x\in\hat{S}_n} H(e,x|r)\Psi_r\big(T^k[e,x]\big)\right| \lesssim \sum_{n\geq 0} n \sum_{x\in\hat{S}_n} H(e,x|r).$$

We now prove that $\sum_{x\in\hat{S}_n} H(e,x|R_\mu)$ decays exponentially fast in n, using that the random walk is convergent. In order to simplify the proof, we use thermodynamic formalism.

As explained above, following [16], the sum $\sum_{x\in\hat{S}_n} H(e,x|r)$ can be written as the value at the empty sequence of the nth iterate of the transfer operator $\mathcal{L}_r$ applied to the constant function equal to 1, see [16, Section 6.1] for more details. Moreover, by [16, Lemma 4.3], the Markov shift associated with the automaton $\mathcal{G}$ encoding relative geodesics has finitely many images and by [16, Lemma 4.5, Lemma 4.7], the transfer operator $\mathcal{L}_r$ has finite pressure and is semisimple. Thus, by [16, Theorem 3.5], we have $\sum_{x\in\hat{S}_n} H(e,x|r)\sim C\mathrm{e}^{P(r)}$, where $P(r)$ is the maximal pressure of $\mathcal{L}_r$. Since the random walk is convergent, we must have $P(R_\mu)<0$. Therefore, the sum $\sum_{n\geq 0} n\sum_{x\in\hat{S}_n} H(e,x|R_\mu)$ is finite. This concludes the proof, since $H(e,x|r)\leq H(e,x|R_\mu)$. □

5.1.2. *Step 2. $J^{(2)}(r)$ via Birkhoff sums.* By Proposition 5.8, our Claim 5.5 is a direct consequence of the following statement.

Proposition 5.9. *With the same notations as before,*

$$\sum_{n\geq 0}\sum_{k=0}^{n-1}\sum_{x\in\hat{S}_n} H(e,x|r)\Psi_r\big(T^k[e,x]\big)\sim CJ^{(2)}(r).$$

Proof. Using the automaton $\mathcal{G}$ encoding relative geodesics, we uniquely decompose $x\in\hat{S}_n$ as $x=x_2hx_1$, where $x_1\in\hat{S}_{n-k-1}$, $h\in\mathcal{H}_j$ for some j and $x_2\in\hat{S}_k$. If y is fixed, we write X_y^j (resp. X_j^y) for the set of elements z of relative length j that can precede (resp. follow) y in the automaton $\mathcal{G}$. We also write X_y, respectively X^y for the set of all elements z that precede, respectively follow y, without specifying the length of z. Writing $y=h'x'$ and setting $\alpha=T^k[e,x]$, the fact that $y\in\Gamma_\alpha$ can be reformulated as $h'\in\Gamma_h$, where Γ_h is the set of elements h' such that h and h' lie in the same $\mathcal{H}_j$. We thus need to study

$$\widetilde{\sum}:=\sum_{k\geq 0}\sum_{n\geq k+1}\sum_{x_1\in\hat{S}_{n-k-1}}\sum_{h\in X_{x_1}^1}\sum_{x_2\in X_h^k} H(e,x_1|r)\frac{H(e,x_2hx_1|r)}{H(e,x_1|r)}$$
$$\sum_{h'\in\Gamma_h}\sum_{x'\in X^{h'}}\frac{G(x_2^{-1},h'x'|r)G(h'x',hx_1|r)}{G(x_2^{-1},hx_1|r)}.$$

We reorganize the sub-sum over h,x_2,h',x' as

$$\sum_{h\in X_{x_1}^1}\sum_{h'\in\Gamma_h} G(e,h'|r)G(h',h|r)G(h,e|r)\sum_{x_2\in X_h^k}\sum_{x'\in X^{h'}} H(e,x_2|r)H(e,x'|r)$$
$$\chi_r(h,h',x_1,x_2,x'),$$

where the function χ_r is defined by

$$\chi_r(h,h',x_1,x_2,x')=\frac{G(hx_1,x_2^{-1}|r)}{G(e,x_2^{-1}|r)G(h,e|r)G(x_1,e|r)}$$
$$\frac{G(x_2^{-1},h'x'|r)}{G(x_2^{-1},e|r)G(e,h'|r)G(e,x'|r)}\frac{G(h'x',hx_1|r)}{G(x',e|r)G(h',h|r)G(e,x_1|r)} \tag{5.2}$$

so that

$$\widetilde{\sum} = \sum_{k\geq 0}\sum_{n\geq 0}\sum_{x_1\in \hat{S}_n} H(e,x_1|r) \sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h} G(e,h'|r)G(h',h|r)G(h,e|r)$$
$$\sum_{x_2\in X^k_h}\sum_{x'\in X^{h'}} H(e,x_2|r)H(e,x'|r)\chi_r(h,h',x_1,x_2,x')$$

The quantity $J^{(2)}$ appears in the sub-sum $\sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h} G(e,h'|r)G(h',h|r)G(h,e|r)$.
It remains to control the sub-sum

$$\sum_{x_2\in X^k_h}\sum_{x'\in X^{h'}} H(e,x_2|r)H(e,x'|r)\chi_r(h,h',x_1,x_2,x').$$

Proposition 5.10. *The functions χ_r are bounded uniformly in $r\in[0,R_\mu]$. Moreover, as r tends to R_μ, χ_r uniformly converges to χ_{R_μ}.*

Proof. The proof relies on the stability of the Martin boundary. By the relative Ancona inequalities, every quotient in the definition of χ_r is uniformly bounded. Thus, we just need to prove that each of them uniformly converges to some limit as $r\to R_\mu$.

We start with the first term and we fix $\epsilon>0$. By Proposition 2.8, there exists η, independent of r, such that

$$G(hx_1,x_2^{-1};B_\eta(h)^c|r)\leq \epsilon G(hx_1,x_2^{-1}|r).$$

Decomposing a trajectory that does enter $B_\eta(h)$ according to its first passage hu in this ball, we deduce that for every $r\leq R_\mu$,

$$\left|\frac{G(hx_1,x_2^{-1}|r)}{G(e,x_2^{-1}|r)G(h,e|r)G(x_1,e|r)} - \sum_{u\in B_\eta(e)}\frac{G(x_1,u;B_\eta(e)^c|r)G(hu,x_2^{-1}|r)}{G(e,x_2^{-1}|r)G(h,e|r)G(x_1,e|r)}\right|\lesssim \epsilon$$

and so we just need to prove that $\sum_{u\in B_\eta(e)}\frac{G(x_1,u;B_\eta(e)^c|r)}{G(e,x_2^{-1}|r)}\frac{G(hu,x_2^{-1}|r)}{G(h,e|r)G(x_1,e|r)}$ converges as r tends to R_μ, uniformly in x_1,h,x_2. We study separately the two ratios which appear in this sum.

Lemma 5.11. *For fixed η, the ratio $\frac{G(x_1,u;B_\eta(e)^c|r)}{G(x_1,e|r)}$ converges to $\frac{G(x_1,u;B_\eta(e)^c|R_\mu)}{G(x_1,e|R_\mu)}$, uniformly in x_1 and $u\in B_\eta(e)$.*

Proof. By finiteness of $B_\eta(e)$, it is sufficient to prove that the convergence is uniform in x_1. Since the function $(x,y,r)\mapsto K(x,y|r)$ is continuous, the ratio $\frac{G(x_1,u|r)}{G(x_1,e|r)}$ uniformly converges to $\frac{G(x_1,u|R_\mu)}{G(x_1,e|R_\mu)}$. Conditioning on the last passage through $B_\eta(e)$ before u, we have

$$G(x_1,u|r)=G(x_1,u;B_\eta(e)^c|r)+\sum_{v\in B_\eta(e)} G(x_1,v|r)G(v,u;B_\eta(e)^c|r),$$

where, as $r\to R_\mu$,

(i) $\frac{G(x_1,v|r)}{G(x_1,e|r)}$ uniformly converges to $\frac{G(x_1,v|R_\mu)}{G(x_1,e|R_\mu)}$;
(ii) $G(v,u;B_\eta(e)^c|r)$ uniformly converges to $G(v,u;B_\eta(e)^c|R_\mu)$. □

Similarly, to study the behavior of the ratio $\frac{G(hu,x_2^{-1}|r)}{G(e,x_2^{-1}|r)G(h,e|r)}$ as $r \to R_\mu$, we use Proposition 2.8 and replace $G(hu, x_2^{-1}|r)$ by

$$\sum_{v \in B_{\eta'}(e)} G(hu, v; B_{\eta'}(e)^c|r)G(v, x_2^{-1}|r),$$

where η' only depends on ϵ and η. As above, we check that both $\frac{G(hu,v;B_{\eta'}(e)^c|r)}{G(h,e|r)}$ and $\frac{G(v,x_2^{-1}|r)}{G(e,x_2^{-1}|r)}$ uniformly converge, as $r \to R_\mu$.

This shows uniform convergence of the first quotient in the definition of χ_r. We deal similarly with the two other ones to conclude. □

Let $\epsilon > 0$. Since $\sum_x H(e, x|R_\mu)$ is finite, if $|R_\mu - r|$ is small enough, we have

$$\begin{aligned}&\sum_{k\geq 0}\sum_{n\geq 0}\sum_{x_1\in \hat{S}_n} H(e,x_1|r)\sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h} G(e,h')G(h',h)G(h,e)\\ &\quad\sum_{x_2\in X^k_h} H(e,x_2|r)\sum_{x'\in X^{h'}} H(e,x'|r)\big|\chi_r(h,h',x_1,x_2,x') - \chi_{R_\mu}(h,h',x_1,x_2,x')\big|\\ &\quad\leq \epsilon \sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h} G(e,h')G(h',h)G(h,e) \leq \epsilon J^{(2)}(r).\end{aligned}$$

We can thus replace χ_r by χ_{R_μ} in the expression of $\widetilde{\sum}$.

Proposition 5.12. *Consider a parabolic subgroup $\mathcal{H}$ along which the random walk is spectrally degenerate. As $h, h' \in \mathcal{H}$ tend to infinity and $d(h, h')$ tends to infinity, the function*

$$\sum_{k\geq 0}\sum_{x_2\in X^k_h}\sum_{x'\in X^{h'}} H(e,x_2|r)H(e,x'|r)\chi_{R_\mu}(h,h',x_1,x_2,x')$$

converges to a function $\tilde{\chi}_{\mathcal{H}}(x_1)$. Moreover, the convergence is uniform in x_1 and $r \in [0, R_\mu]$ and the function $\tilde{\chi}_{\mathcal{H}}$ is bounded.

We first prove the following lemma.

Lemma 5.13. *If D is large enough and if $d(e, h) > D$, the set of x_2 that can precede h lying in a fixed parabolic subgroup $\mathcal{H}$ is independent of h.*

Proof. Let h_1 and h_2 be in the same parabolic subgroup $\mathcal{H}$, such that both have length bigger than D. Assume that x can precede h_1. By [15, Lemma 4.11], if D is large enough, the concatenation of x and h_2 is a relative geodesic. Now, consider elements $\tilde{x}$ and $\tilde{h}$ in $\mathcal{H}$ such that $\tilde{x}\tilde{h} = xh_2$, hence $\tilde{x}\tilde{h}h_2^{-1}h_1 = xh_1$ so that $\tilde{x} \geq x$ in the lexicographical order. If $\tilde{x} > x$, then the concatenation of $\tilde{x}$ and $\tilde{h}$ is bigger than the concatenation of x and h_2. Otherwise, $\tilde{x} = x$ and so $\tilde{h} = h_2$. Therefore x can precede h_2. □

The same proof does not apply to elements x' that can follow h'. However, decomposing elements of Γ as $h'x'$ and choosing the inverse lexicographical order rather than the original lexicographical order, we get similarly the following result.

Lemma 5.14. *If D is large enough and if $d(e, h') > D$, the set of x' that can follow h' lying in a fixed parabolic subgroup $\mathcal{H}$ is independent of h'.*

We can now prove Proposition 5.12. We use here that the Martin boundary of the first return kernel $p_{\mathcal{H},R_\mu}$ is reduced to a point.

Proof. By Lemmas 5.13 and 5.14, it is enough to prove that $\chi_{R_\mu}(h,h',x_1,x_2,x')$ converges to a function, as h,h' tend to infinity and $d(h,h')$ tends to infinity, uniformly in x_1,x_2,x'. Uniformity is proved using Proposition 2.8, as in the proof of Proposition 5.10. Thus, we just need to prove that for fixed u and v, the ratio $\frac{G(hu,v|R_\mu)}{G(h,e|R_\mu)}$ converges as h tends to infinity. We write

$$\frac{G(hu,v|R_\mu)}{G(h,e|R_\mu)}=\frac{G(hu,v|R_\mu)}{G(hu,e|R_\mu)}\frac{G(hu,e|R_\mu)}{G(h,e|R_\mu)}=\frac{G(hu,v|R_\mu)}{G(hu,e|R_\mu)}\frac{G(u,h^{-1}|R_\mu)}{G(e,h^{-1}|R_\mu)}.$$

Both hu and h^{-1} tend to infinity. Since we assume that the Martin boundary of the first return kernel to $\mathcal{H}$ is reduced to a point, both ratios $\frac{G(hu,v|R_\mu)}{G(hu,e|R_\mu)}$ and $\frac{G(u,h^{-1}|R_\mu)}{G(e,h^{-1}|R_\mu)}$ converge, as h tends to infinity. □

To simplify notations, we set

$$\tag{5.3}\begin{aligned}\tilde{I}^{(2)}(r)=\sum_{n\geq 0}\sum_{x_1\in\hat{S}_n}H(e,x_1|r)\sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h}G(e,h'|r)G(h',h|r)G(h,e|r)\\ \sum_{k\geq 0}\sum_{x_2\in X^k_h}\sum_{x'\in X^{h'}}H(e,x_2|r)H(e,x'|r)\chi_{R_\mu}(h,h',x_1,x_2,x').\end{aligned}$$

We also write $\mathcal{H}_h$ for the parabolic subgroup containing h. When h lies in several parabolic subgroups, we arbitrarily choose one of them and, if possible, we choose one along which the random walk is spectrally degenerate. Recall that the intersection of two parabolic subgroups is finite, hence $\mathcal{H}_h$ is uniquely defined if $d(e,h)$ is large enough. Finally, if the random walk is spectrally nondegenerate along $\mathcal{H}$, we set $\tilde{\chi}_{\mathcal{H}}(x_1)=1$.

Proposition 5.15. *Let $\epsilon>0$. If $|r-R_\mu|$ is small enough, then*

$$\left|\tilde{I}^{(2)}(r)-\sum_{n\geq 0}\sum_{x_1\in\hat{S}_n}H(e,x_1|r)\sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h}G(e,h'|r)G(h',h|r)G(h,e|r)\tilde{\chi}_{\mathcal{H}_h}(x_1)\right|$$
$$\lesssim \epsilon J^{(2)}(r).$$

Proof. There exists D_ϵ such that if the random walk is spectrally degenerate along $\mathcal{H}_h$ and if $d(e,h),d(e,h'),d(h,h')\geq D_\epsilon$.

$$\left|\sum_{k\geq 0}\sum_{x_2\in X^k_h}\sum_{x'\in X^{h'}}H(e,x_2|r)H(e,x'|r)\chi_{R_\mu}(h,h',x_1,x_2,x')-\tilde{\chi}_{\mathcal{H}_h}(x_1)\right|\leq\epsilon.$$

Now, the remaining sum

$$\sum_{n\geq 0}\sum_{x_1\in\hat{S}_n}H(e,x_1|r)\sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h}G(e,h'|r)G(h',h|r)G(h,e|r)$$

is bounded by $J^{(2)}(r)$. Therefore, we can assume that either the random walk is spectrally nondegenerate along $\mathcal{H}$, or that one of $d(e,h),d(e,h'),d(h,h')$ is at most D_ϵ. On the one hand, the sub-sum in (5.3) over the h and h' such that, either $d(e,h)<D_\epsilon$ or $d(e,h')<D_\epsilon$ or $d(h,h')<D_\epsilon$, is uniformly bounded. Thus, it can be bounded by $\epsilon J^{(2)}(r)$ if r is close enough to R_μ. On the other hand, the

sub-sum over the h and h' such that $d(e,h), d(e,h'), d(h,h') \geq D_\epsilon$ but the random walk is spectrally nondegenerate along $\mathcal{H}_h$ is also uniformly bounded and can thus be bounded as well by $\epsilon J^{(2)}(r)$ if r is close enough to R_μ. $\square$

To conclude the proof, we only need to prove that

$$\sum_{n\geq 0}\sum_{x_1\in \hat{S}_n} H(e,x_1|r)\sum_{h\in X^1_{x_1}}\sum_{h'\in\Gamma_h} G(e,h'|r)G(h',h|r)G(h,e|r)\tilde{\chi}_{\mathcal{H}_h}(x_1) \sim CJ^{(2)}(r).$$

The double sum over x_1 and h that can precede x_1 is exactly the sum over every element of relative length $1+\hat{d}(e,x_1)$, so we can replace this double sum by a double sum over h and x_1 that can follow h. We thus need to prove that

$$\sum_{h}\sum_{h'\in\Gamma_h} G(e,h'|r)G(h',h|r)G(h,e|r)\sum_{n\geq 0}\sum_{x_1\in X^h_n} H(e,x_1|r)\tilde{\chi}_{\mathcal{H}_h}(x_1) \sim CJ^{(2)}(r).$$

By Lemma 5.14, if $d(e,h)$ is large enough, then the set of x_1 that can follow h is independent of h. This concludes the proof, since for fixed D, the sub-sum over the h such that $d(e,h)\leq D$ is uniformly bounded. $\square$

This concludes the proof of Claim 5.5. To end the proof of Theorem 5.4 in the case $k=2$, we only need to show the following result.

Proposition 5.16. *Under the assumptions of Theorem 5.4, if $k=2$, then for every $\eta\geq 0$, there exists a positive constant C_η such that*

$$J^{(2)}_\eta(r) \sim C_\eta J^{(2)}(r).$$

Denote by E_η the set of $x\in\Gamma$ such that x is in $\mathcal{N}_\eta(\mathcal{H})$ for some $\mathcal{H}$. Recall that for a relative geodesic α such that $\alpha(0)=e$, the set Γ_{α_1} contains all the elements $x\in\Gamma$ such that x_1 and α_1 lie in the same $\mathcal{H}_j$. Setting

$$\Psi^\eta_r(\alpha) = \sum_{y\in\Gamma_{\alpha_1}} \frac{G(\alpha_-,y|r)G(y,\alpha_+|r)}{G(\alpha_-,\alpha_+|r)} 1_{\alpha_-^{-1}\alpha_+\in E_\eta}1_{y\in E_\eta}$$

the proof of Proposition 5.16 is exactly the same as the one of Claim 5.5, replacing Ψ_r by Ψ^η_r.

5.2. **Higher derivatives.** We now consider the general case, i.e. $J^{(j)}(R_\mu)$ is finite for every $j<k$ and $J^{(k)}$ is infinite. We introduce the function $\Phi^{(k)}_r$ and $\Psi^{(k)}_r$ defined for any $x\in\Gamma$ by

$$\Phi^{(k)}_r(x) = \sum_{y_1,\ldots,y_{k-1}} \frac{G(e,y_1|r)G(y_1,y_2|r)...G(y_{k-1},v|r)}{G(e,v|r)}.$$

and, for any relative geodesic α such that $\alpha(0)=e$,

$$\Psi^{(k)}_r(\alpha) = \sum_{y_1,\ldots,y_{k-1}\in\Gamma_{\alpha_1}} \frac{G(\alpha_-,y_1|r)G(y_1,y_2|r)...G(y_{k-1},\alpha_+|r)}{G(\alpha_-,\alpha_+|r)}.$$

As above, it holds

$$I^{(k)}(r) = \sum_{x\in\Gamma} H(e,v|r)\Phi^{(k)}_r(x).$$

We have the following.

Proposition 5.17. *There exists D_k such that, for any $x \in \hat{S}_n$,*

$$\Phi_r^{(k)}(x) = \sum_{j=0}^{n-1} \Psi_r^{(k)}\big(T^j[e,v]\big) + O(n^{D_k}).$$

Proof. Like in the proof of Proposition 5.6, we consider the set Γ_l of elements y such that the projection of y on $[e,x]$ is at x_l. If all the y_j do not lie in the same Γ_l, then

$$\sum_{y_1,...,y_{k-1}} \frac{G(e,y_1|r)G(y_1,y_2|r)...G(y_{k-1},x|r)}{G(e,x|r)}$$

is bounded by a quantity only involving the $J^{(j)}(r)$, $j < k$, which are uniformly bounded. Consequently, we can restrict the sum in the definition of $\Phi_r^{(k)}$ to the y_j lying in the same Γ_l and then, the remainder of the proof of Proposition 5.6 can be reproduced. □

As in Proposition 5.8, we get

$$I^{(k)}(r) = \sum_{n\geq 0}\sum_{l=0}^{n-1}\sum_{x\in\hat{S}_n} H(e,x|r)\Psi_r^{(k)}\big(T^l[e,x]\big) + O(1)$$

and we conclude like in the case $k = 2$. This proves Theorem 5.4. □

6. Proof of the local limit theorem

In this section, we prove Theorem 1.3. Recall that for fixed x in Γ,

$$I_x^{(k)}(r) = \sum_{y_1,...,y_k} G(e,y_1|r)...G(y_k,x|r).$$

Proposition 6.1. *Under the assumptions of Theorem 5.4, for every $\eta \geq 0$ and every $x \in \Gamma$, there exists $C_{\eta,x}$ such that*

$$I_x^{(k)}(r) \sim C_{\eta,x} J_\eta^{(k)}(r).$$

Sketch of proof. We use the same arguments as in Theorem 5.4. Let us briefly outline the case $k = 2$.

We have

$$I_x^{(2)}(r) = \sum_{y,z\in\Gamma} G(e,y|r)G(y,z|r)G(z,e|r)\frac{G(z,x|r)}{G(z,e|r)}.$$

We introduce the function $\Phi_{r,x}$ defined by

$$\Phi_{r,x}(z) = \sum_{y\in\Gamma} \frac{G(e,y|r)G(y,z|r)}{G(e,z|r)}\frac{G(z,x|r)}{G(z,e|r)},$$

i.e. $\Phi_{r,x}(z) = \Phi_r(z)\frac{G(z,x|r)}{G(z,e|r)}$. We then have

$$I_x^{(2)}(r) = \sum_{z\in\Gamma} H(e,z|r)\Phi_{r,x}(z).$$

We can then reproduce the same proof, replacing the function χ_r defined in (5.2) by the function $\chi_{r,x}$ defined by

$$\chi_{r,v}(h,h',z_1,z_2,z') = \frac{G(hz_1,z_2^{-1}|r)}{G(e,z_2^{-1}|r)G(h,e|r)G(z_1,e|r)}$$
$$\frac{G(z_2^{-1},h'z'|r)}{G(z_2^{-1},e|r)G(e,h'|r)G(e,z'|r)}\frac{G(h'z',hz_1|r)}{G(z',e|r)G(h',h|r)G(e,z_1|r)}\frac{G(z_2hz_1,x|r)}{G(z_2hz_1,e|r)}.$$

Thus, $\chi_{r,x} = \chi_r \frac{G(z_2hz_1,x|r)}{G(z_2hz_1,e|r)}$ and it is bounded by a constant that only depends on x. Since we assume that the Martin boundary is stable and the function

$$(x,y,r) \in \Gamma\times\Gamma\cup\partial_\mu\Gamma\times(0,R_\mu] \mapsto K(x,y|r)$$

is continuous, the family $(\chi_{r,x})_r$ uniformly converges to $\chi_{R_\mu,x}$, as r tends to R_μ. This is enough to reproduce the proof of Theorem 5.4. □

We now prove Theorem 1.3 and so we assume in particular that parabolic subgroups are virtually abelian. By Theorem 4.1 and [17, Proposition 4.3], the assumptions of Theorem 5.4 are satisfied. Combining Proposition 3.16 and Proposition 6.1, we get the following result.

Corollary 6.2. *For every x, there exists $C_x > 0$ such that, as $r \to R_\mu$*

$$G^{(j)}(e,x|r) \sim \frac{C_x}{\sqrt{R_\mu - r}} \quad \text{if } d \text{ is odd,}$$
$$\text{and} \quad G^{(j)}(e,x|r) \sim C_x \log\left(\frac{1}{R_\mu - r}\right) \quad \text{if } d \text{ is even.}$$

In both cases, we deduce that $p^{(n)}(e,x) \sim C_x R_\mu^{-n} n^{-d/2}$; let us explain this last step. The odd case is proved exactly like [23, Theorem 9.1]. The method is based on a *Tauberian theorem* of Karamata and also applies to the even case.

Let us give a complete proof of the even case for sake of completeness. A function f is called *slowly varying* if for every $\lambda > 0$, the ratio $f(\lambda x)/f(x)$ converges to 1 as x tends to infinity. Combining Corollary 6.2 and [4, Corollary 1.7.3] (applied with the slowly varying function log) one gets

$$\sum_{k=0}^{n} k^j R_\mu^k \mu^{*k}(x) \sim C'_x \log(n).$$

Moreover, by [23, Corollary 9.4],

$$n^j R_\mu^n(\mu^{*n}(e) + \mu^{*n}(x)) = q_n(x) + O\big(\mathrm{e}^{-cn}\big), \tag{6.1}$$

where $c > 0$ and the sequence $(q_n(x))_n$ is non-increasing. We first deduce that

$$\sum_{k=0}^{n} k^j R_\mu^k q_k(e) \sim C_0 \log(n).$$

The same proof as in [23, Lemma 9.5] yields

$$n^j R_\mu^n q_n(e) \sim C_1 n^{-1}$$

and so, using again (6.1),

$$n^j R_\mu^n \mu^{*n}(e) \sim C_1 n^{-1}.$$

Recall that $j = d/2 - 1$, so that

$$\mu^{*n}(e) \sim C_1 R_\mu^{-n} n^{-d/2}.$$

Since $G^{(j)}(e,e|r) + G^{(j)}(e,x|r) \sim (C_e + C_x)\log 1/(R_\mu - r)$ as $r \to R_\mu$, we can again apply [4, Corollary 1.7.3] to deduce that

$$\sum_{k=0}^{n} k^j R_\mu^k q_k(x) \sim C_x^{(0)} \log(n).$$

The fact that $(q_n(x))_n$ is non-increasing readily implies

$$n^j R_\mu^n q_n(x) \sim C_x^{(1)} n^{-1}$$

as above. Finally, using (6.1) once again,

$$\mu^{*n}(x) \sim C_x^{(2)} R_\mu^{-n} n^{-d/2}.$$

Since C_1 is positive and the random walk is admissible, the quantity $C_x^{(2)}$ is also positive. This concludes the proof of Theorem 1.3 when the random walk is aperiodic. If not, then note that it is symmetric so its period must be 1 or 2, hence the desired estimates for p_{2n} and p_{2n+1} follow. Finally, Corollary 1.4 follows from Theorem 1.3 and [22, Proposition 4.1]. □

7. Divergence and spectral positive recurrence

In this section, we describe the relationship between divergence and spectral positive recurrence of random walks on relatively hyperbolic groups. As before, assume that Γ is a finitely generated relatively hyperbolic group with respect to virtually abelian subgroups and fix a finite set Ω_0 of representatives of conjugacy classes of parabolic subgroups.

Recall that a random walk on Γ is called spectrally positive recurrent if it is divergent and has finite Green moments, i.e. using the notations defined in (5.1), if $J^{(2)}(R_\mu)$ is finite. By definition, this occurs when $I^{(2)}_{\mathcal{H}}(R_\mu) < +\infty$ for all parabolic subgroup $\mathcal{H} \in \Omega_0$. We prove here that under some assumptions, divergence automatically implies spectral positive recurrence, that is, assuming that $I^{(1)}(R_\mu)$ is infinite, we have for all $\mathcal{H}$ that $I^{(2)}_{\mathcal{H}}(R_\mu)$ is finite.

If the random walk driven by μ is spectrally nondegenerate along $\mathcal{H}$, then we automatically have $I^{(2)}_{\mathcal{H}}(R_\mu) < +\infty$. This is because $I^{(2)}_{\mathcal{H}}$ is finite if and only if the second derivative of $t \mapsto G_{\mathcal{H},R_\mu}(e,e|t)$ is finite at 1, which obviously holds if $R_{\mathcal{H}}(R_\mu) > 1$. Furthermore, by [17, Proposition 6.1], if $\mathcal{H}$ has rank $d \leq 4$, then μ cannot be spectrally degenerate along $\mathcal{H}$. Thus, $I^{(2)}_{\mathcal{H}}(R_\mu) = +\infty$ can only happen if $d \geq 5$.

Next, assume that μ is spectrally degenerate along $\mathcal{H}$. Up to taking η large enough, the first return kernel $p_{\mathcal{H},\eta,R_\mu}$ has exponential moments. Therefore, the local limit theorem (1.1) implies that whenever the rank of $\mathcal{H}$ is at least 7, then $G_{\mathcal{H},\eta,R_\mu}(e,e|1)$, $G'_{\mathcal{H},\eta,R_\mu}(e,e|1)$ and $G''_{\mathcal{H},\eta,R_\mu}(e,e|1)$ are all finite. We deduce that $I^{(2)}_{\mathcal{H},\eta}(R_\mu)$ is finite, hence $I^{(2)}_{\mathcal{H}}(R_\mu)$ is also finite. To conclude, $I^{(2)}_{\mathcal{H}}(R_\mu) = +\infty$ can only happen if $d = 5$ or 6 and the random walk is spectrally degenerate along $\mathcal{H}$. We gather these comments in the following proposition.

Proposition 7.1. *Let Γ be a a finitely generated relatively hyperbolic group Γ with respect to virtually abelian subgroups. Let μ be a finitely supported, admissible, symmetric probability measure on Γ. Assume that for every parabolic subgroup $\mathcal{H}$ of rank 5 or 6, the random walk is spectrally nondegenerate along $\mathcal{H}$. Then the random walk is spectrally positive recurrent* if and only if *it is divergent.*

In a separate paper [19], we show that there exist examples of spectrally degenerate random walks on relatively hyperbolic groups with parabolic subgroups being virtually abelian of rank $d = 5$ or 6, which are *divergent but not spectrally positive recurrent*. They show exotic local limit theorems which are neither of the form (1.1) nor (1.2). We construct such examples on free products of abelian groups.

Remark 7.1. Note that these exotic examples contradict [8, Lemma 4.5]. Unfortunately the proof of this lemma shows a subtle gap. A previous version of our article contained a similar gap, which leaded to the (wrong) conclusion that any divergent random walk is spectrally positive recurrent as soon as parabolic subgroups are virtually abelian.

8. Beyond virtually abelian parabolic subgroups

In this final section, we explain exactly where we use the assumption that parabolic subgroups are virtually abelian and how our results could be generalized.

Our overall strategy can be decomposed into **two main steps**. We assume that μ is convergent and we let k be the first integer such that $I^{(k)}(R_\mu)$ is infinite, which is also the first integer such that $J^{(k)}(R_\mu)$ is infinite. **First**, by Proposition 3.16, for large enough η, $J_\eta^{(k)}(r)$ is asymptotic either to $-C\ \log(R_\mu - r)$ or to $C(R_\mu - r)^{-1/2}$. **Second**, by Proposition 6.1, for every η and every $x \in \Gamma$, there exists $C_{\eta,x}$ such that $I_x^{(k)}(r) \sim C_{\eta,x} J_\eta^{(k)}(r)$ as $r \to R_\mu$.

For the first step, we need an accurate control of the derivatives of $G_{\mathcal{H},\eta,r}$ as r tends to R_μ. Our argument relies on enhanced Ancona inequalities (Proposition 3.8) whose proof uses virtually abelian parabolic subgroups, to ensure that any subgroup of $\mathcal{H}$ quasi-isometrically embeds into Γ. Moreover, a large part of our arguments in Section 3.4 uses the expansion of the Green function on virtually abelian groups.

For the second step, the assumptions of Proposition 6.1 are that

- the 1-Martin boundary of $(\mathcal{H}, p_{\mathcal{H},R_\mu})$ is reduced to a point for every influential parabolic subgroup $\mathcal{H}$;
- the Martin boundary is stable.

By [17, Proposition 4.10], if $\mathcal{H}$ is any virtually nilpotent parabolic subgroup and μ is spectrally degenerate along $\mathcal{H}$, then the Martin boundary of $(\mathcal{H}, p_{\mathcal{H},R_\mu})$ is reduced to a point. However, we can only prove that the Martin boundary of (Γ, μ) is stable whenever parabolic subgroups are virtually abelian. Indeed, if $\mathcal{H}$ is a virtually nilpotent parabolic subgroup, the homeomorphism type of the r-Martin boundary of $\mathcal{H}$ for r below the inverse of the spectral radius is not known. The stability of this Martin boundary seems hard to show without first studying its homeomorphism type.

Nevertheless, we expect that our results hold for virtually nilpotent parabolic subgroups and we now briefly explain why.

We first point out that finitely supported random walks on virtually nilpotent groups and virtually abelian groups present the same type of expansions for the

Green function. Indeed, let $\mathcal{H}_0$ be a finitely generated virtually nilpotent group and let μ_0 be a finitely supported, admissible, aperiodic probability measure on $\mathcal{H}_0$. Denote by G_0 the Green function associated with μ_0 and by R_0 its radius of convergence, i.e. the inverse of the spectral radius of μ_0. Assuming further that μ_0 is symmetric, we have that $R_0 = 1$ by amenability.

Virtually nilpotent finitely generated groups have polynomial growth and the exact degree of the growth d_0 was identified independently by Bass [3] and Guivarc'h [25] as the homogeneous dimension of the group. By a celebrated result of Gromov [24], the converse is true and groups of polynomial growth are exactly virtually nilpotent groups.

By the landmark results of Alexopoulos, we have that for every fixed $x \in \mathcal{H}_0$,

$$\mu_0^{*n}(x) \sim C_x R^{-n_0} n^{-d_0/2},$$

where d_0 is the homogeneous dimension of $\mathcal{H}_0$, see precisely [1, Corollary 1.17, Corollary 1.18]. We deduce that $j_0 = \lceil d_0/2 \rceil - 1$ is the first integer such that $G_0^{(j_0)}(e, e|R_0)$ is infinite. We also deduce from Karamata's Tauberian theorem [4, Corollary 1.7.3] that as r tends to R_0, we have $G_0^{(j_0)}(e, x|r) \sim -C'_x \log(1 - r)$ if d_0 is even and $G_0^{(j_0)} \sim C'_x(1 - r)^{-1/2}$ if d_0 is odd. We can thus expect that our first step above still holds if parabolic subgroups are virtually nilpotent. However, when considering the first return kernel to a parabolic subgroup $\mathcal{H}$, we only get a transition kernel which has exponential moments. We thus need an extension of Alexopoulos' results which are only proved under a finite support assumption.

Also, in the special case of adapted random walks on free products, our results extend to virtually nilpotent parabolic subgroups as we now show.

Consider a free product $\Gamma = \Gamma_0 * \Gamma_1$ and assume that Γ_i are virtually nilpotent. Let μ_i be a finitely supported, admissible and symmetric probability measure on Γ_i and define the adapted probability measure $\mu_\alpha = (1 - \alpha)\mu_0 + \alpha\mu_1$ on Γ. Let d_i be the homogeneous dimension of Γ_i. We can use the arguments of Candellero and Gilch [8] in the convergent case to deduce that

$$\mu_\alpha^{*n}(x) \sim C_x R^{-n} n^{-d/2},$$

where d is the minimum of d_i such that μ_α is spectrally degenerate along Γ_i. Indeed, the assumptions of [8, Theorem 3.1] only involve the expansions of the Green functions of μ_0 and μ_1 and we explained above that virtually nilpotent groups and virtually abelian groups present the same type of expansions for the Green function.

To conclude, let us mention that our approach is unlikely to extend to any class of parabolic subgroups. Indeed, in order to compare the singularities of the Green function with those of the induced Green functions on parabolic subgroups, we crucially use that there exists an integer k such that $J^{(k)}(R_\mu)$ is infinite, or equivalently such that $I^{(k)}(R_\mu)$ is infinite. However, this property may fail, as proved in the recent work [12, Example 5.3].

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