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Recurrence of Multidimensional Affine Recursions in the Critical Case

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Abstract
We prove, under various natural hypotheses, that the random multidimensional affine recursion $X_n = A_n X_{n-1} + B_n \in \mathbb{R}^d, n \geq 1$, is recurrent in the critical case. In particular we cover the cases where the matrices A_n are similarities, invertible, rank 1 or with non negative coefficients. These results are a consequence of a criterion of recurrence for a large class of affine recursions on $\mathbb{R}^d$, based on some moment assumptions of the so-called "reverse norm control random variable.

Keywords Affine recursion · Recurrence · Ladder epoch of a random walk · Markov walk

Mathematics Subject Classification (2020) 60J10 · 60J05 · 60B15

1 Introduction

We fix $d \geq 1$ and endow $\mathbb{R}^d$ with the euclidean norm $|\cdot|$ defined by $|x| := \sqrt{\sum_{i=1}^{d} x_i^2}$ for any column vector $x = (x_i)_{1\leq i\leq d}$.

We note $\mathcal{M}(d, \mathbb{R})$ the semigroup of $d \times d$-matrices with real-valued coefficients. For any $A \in \mathcal{M}(d, \mathbb{R})$ and $x \in \mathbb{R}^d$, we note Ax the image of x by the linear action of A and $\|A\|$ the norm of the matrix $A \in \mathcal{M}(d, \mathbb{R})$ defined by $\|A\| = \sup_{x\in\mathbb{R}^d, |x|=1} |Ax|$.

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Let $g_n = (A_n, B_n)$, $n = 1, 2, \ldots$ be independent and identically distributed random variables defined on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$ with distribution μ on $\mathcal{M}(d, \mathbb{R}) \times \mathbb{R}^d$. The distribution of the matrices A_n is denoted $\bar{\mu}$.

Let $(X_n)_{n\geq 0}$ be the "affine recursion" on $\mathbb{R}^d$ defined inductively by: for any $n \geq 0$

$$X_{n+1} = A_{n+1}X_n + B_{n+1}. \tag{1}$$

The process $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{R}^d$, and we are interested in its recurrence properties in the "critical case," that is, when the Lyapunov exponent $\gamma_{\bar{\mu}} := \lim_{n\to\infty} \dfrac{\mathbb{E}(\ln \|A_n \cdots A_1\|)}{n}$ of the product of the random matrices A_k equals 0.

When $X_0 = x$ for some fixed $x \in \mathbb{R}^d$, we set $X_n = X_n^x$. We say that the process $(X_n)_{n\geq 0}$ is *(topologically) recurrent* on $\mathbb{R}^d$ when there exits $K > 0$ such that, for any $x \in \mathbb{R}^d$,

$$\mathbb{P}(\liminf_{n\to+\infty} |X_n^x| \leq K) = 1.$$

In this article, we propose a general criterion that ensures that $(X_n)_{n\geq 0}$ is recurrent on $\mathbb{R}^d$. Then, we apply it to four interesting situations, depending on whether the matrices A_n are similarities, rank 1, invertible, or nonnegative matrices, with some restrictive hypotheses in each situation, which appear classically in the theory of product of random matrices and are labeled, respectively , **S**, **Rk1**, **I**, and **Nn** (see Sect. 3 for the precise lists). Our main statement is:

Theorem 1.1 *Assume that*

1. *Hypotheses* **S**, **Rk1**, **I**, *or* **Nn** *hold for the random matrices* A_n*;*
2. $\mathbb{E}\left((\ln^+ |B_1|)^\gamma\right) < +\infty$ *for some* $\gamma > 2$.

Then, for any $x \in \mathbb{R}^d$*, the Markov chain* $(X_n^x)_{n\geq 0}$ *is recurrent.*

In the case **I**, by using the action of the A_n on some suitable exterior product $\wedge^r \mathbb{R}^d$, $1 \leq r \leq d$ of $\mathbb{R}^d$, we may relax our assumptions and avoid the so-called hypothesis **I**-*proximality*. We refer to Theorem A.1 for details; the case of non-proximal invertible matrices requires some technical adaptations. We thought it would be interesting to deal first with the case where the proximality assumption is satisfied, proving first a criterium based on the properties of the action of products of the matrices A_n on the projective space $\mathbf{P}(\mathbb{R}^d)$ and using classical results on products of random matrices.

The stochastic recurrence equation $X_{n+1} = a_{n+1}X_n + b_{n+1}$ on $\mathbb{R}$, where the $(a_n, b_n)_{n\geq 1}$ are independent and identically distributed random variables with values in $\mathbb{R}_{*+} \times \mathbb{R}$, has been extensively studied, with special attention given to the existence of an invariant measure and its properties, especially its tails. This "random coefficients autoregressive process" occurs in different domains, in particular in economics, and has been studied intensively for several decades. We refer to the book by D. Buraczewski, E. Damek, and T. Mikosch [3] for a general survey of the topic and references therein. A rich literature covers the so-called contractive case $\mathbb{E}[\ln a_1] < 0$; this condition ensures that, when $\mathbb{E}[\ln^+ |b_1|] < +\infty$, there exists on $\mathbb{R}$ a unique invariant probability measure for the process $(X_n)_{n\geq 0}$. In the critical case $\mathbb{E}[\ln a_1] = 0$, the affine recursion is not positive recurrent [6]. Nevertheless, at the end of the 1990, P. Bougerol and L. Elie [4] (see also [2]) proved that in this case, there exists on $\mathbb{R}$ an invariant and unbounded Radon measure m for $(X_n)_{n\geq 0}$; furthermore, if $\mathbb{E}((\ln a_1)^{2+\epsilon}) < +\infty$, the process $(X_n)_n$ is topologically recurrent on $\mathbb{R}$. In a recent paper [1], G.Alsmeyer and A.Iksanov investigate the sharpness of these moment conditions.

The affine recursion $(X_n)_{n\geq 0}$ has also been considered in dimension $d \geq 2$. The random variables a_n and b_n above are replaced respectively by $d \times d$ random matrices A_n with real entries and random vectors B_n in $\mathbb{R}^d$. The equation (1) yields, for any $n \geq 0$,

$$X_n = A_n \cdots A_1 X_0 + \sum_{k=1}^{n} A_n \cdots A_{k+1} B_k.$$

The behavior of the chain $(X_n)_{n\geq 0}$ is thus related to the linear action of the left products $A_{n,k} := A_n \cdots A_k$, $1 \leq k \leq n$, of the matrices A_i, $i \geq 1$. A huge literature is devoted to these product of random matrices, mainly in the case where the A_k are either invertible or non negative; we refer here to [5, 13, 16, 17], and [18] with references therein. For $x \in \mathbb{R}^d$, $x \neq 0$, the study of the vector $A_{n,1}x$ is carried out via that of the behavior of the directions $A_{n,1}x/|A_{n,1}x|$, $n \geq 0$, when $A_{n,1}x \neq 0$ and that of the norm $|A_{n,1}x|$.

On the one hand, the control of the projective action of the matrices $A_{n,k}$ relies on some contraction properties of the closed semigroup $T_{\bar{\mu}}$ generated by the support of $\bar{\mu}$. This requires some restrictive assumptions: for instance, *proximality* and *strong irreducibility property*[1] in the case of products of invertible matrices [5], or existence of matrices with positive entries when the A_k is nonnegative [17]. On the other hand, the study of the norm $|A_{n,1}x|$ relies on the cocycle decomposition of its logarithm as

$$S_n(x) := \ln |A_{n,1}x| = \ln |A_1 x| + \ln \left| A_2 \left(\frac{A_1 x}{|A_1 x|} \right) \right| + \ldots + \ln \left| A_n \left(\frac{A_{n-1,1} x}{|A_{n-1,1} x|} \right) \right|,$$

which can be done as soon as $A_{k,1}x \neq 0$ for $k \geq 1$. The process $(\ln |A_{n,1}x|)_{n\geq 1}$ is a fundamental, even iconic, example of Markov walks on $\mathbb{R}$; the degree of dependence between the increments of this sum is controlled by the Markov chain $(A_{n,1}x/|A_{n,1}x|)_{n\geq 0}$, when it is well defined.

When $d \geq 2$, the contractive case for the affine recursion corresponds to the case where the Lyapunov exponent $\gamma_{\bar{\mu}}$ associated with the random matrices A_n is negative. The existence and uniqueness of an invariant probability measure are obtained following the same strategy as the one developed in dimension 1 [6, 9]. Various properties of this measure have been obtained in this case, based on results of product of random matrices (see [3], chap. 4, and references therein). As far as we know, the existence and uniqueness of an invariant Radon measure in the critical case $\gamma_{\bar{\mu}} = 0$ have been investigated only in the case of nonnegative matrices: the existence and uniqueness of an (infinite) invariant Radon measure are established in [7] when all the ratios $A_n(i, j)/A_n(k, \ell)$ are supposed to be bounded from above and below, uniformly in the alea ω and i, j, k, ℓ in $\{1, \ldots, d\}$. This ensures that the ratio $\|A_{n,1}\|/|A_{n,1}x|$ is

[1] More details on these notions are given in Section 3.

themselves $\mathbb{P}$-a.s. bounded from above, uniformly in n, in x (where x is an unit vectors with positive entries), and in the alea ω. This is a quite strong hypothesis, and one of the aims of the present paper is to relax it. The main new ingredient is the so-called *reverse norm control* random variable (see (3) for a precise definition), which allows to compare in a more flexible way the behavior of the two sequences $(\|A_{n,1}\|)_{n\geq 1}$ and $(|A_{n,1}x|)_{n\geq 0}$.

In Subsection 1.1, we fix the notations. Section 2 is devoted to a general criterion for recurrence of $(X_n)_{n\geq 0}$; this criterion is based on the properties of the action of products of the matrices A_n on the projective space $\mathbf{P}(\mathbb{R}^d)$. In Sect. 3, we analyze and establish these properties in the different models we consider in this paper: The matrices A_n are successively similarities, rank 1 matrices, invertible strongly irreducible, and proximal or nonnegative matrices. Section 4 is devoted to the study of the fluctuations of the process $(\ln|A_{n,1}v|)_{n\geq 1}$ in the rank 1 case. In Sect. A we explain how, in the invertible case, one can still obtain the recurrence of the process without the proximality assumption of the action of A_n on $\mathbf{P}(\mathbb{R}^d)$; this requires a slight extension of the main result of Sect. 2 (see Criterion A.2).

1.1 Tools and Notations

The recurrence of $(X_n)_{n\geq 0}$ is closely related to the contraction properties of elements of the semigroup generated by the support of $\bar{\mu}$. Assume $\mathbb{E}(\ln^+\|A_1\|) < +\infty$ and let $\gamma_{\bar{\mu}}$ be the Lyapunov exponent of the random walk $(A_{n,1})_{n\geq 1}$:

$$\gamma_{\bar{\mu}} := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}(\ln\|A_n\cdots A_1\|) \in [-\infty, +\infty).$$

Kingman's subadditive ergodic theorem ensures that this limit holds almost surely

$$\gamma_{\bar{\mu}} := \lim_{n\to+\infty} \frac{1}{n}\ln\|A_n\cdots A_1\| \quad \mathbb{P}-\text{a.s.} \tag{2}$$

From now on, we assume $\gamma_{\bar{\mu}} = 0$.

Let $\mathbf{P}(\mathbb{R}^d)$ denote the projective space of $\mathbb{R}^d$, that is, the quotient space $\mathbb{R}^d\setminus\{0\}/\sim$, where $x\sim y$ means that there exists $\lambda\neq 0$ such that $y=\lambda x$. We denote by $\bar{x}\in\mathbf{P}(\mathbb{R}^d)$ the class of the vector $x\in\mathbb{R}^d$.

For any non-null vectors x and y in $\mathbb{R}^d$, we note $x\wedge y$ their exterior product. The space $\mathbf{P}(\mathbb{R}^d)$ is endowed with the distance δ defined by: For any $\bar{x},\bar{y}\in\mathbf{P}(\mathbb{R}^d)$,

$$\delta(\bar{x},\bar{y}) := \frac{|x\wedge y|}{|x|\,|y|} = |\sin(\widehat{xy})|$$

where x and y are (any) representatives of $\bar{x}$ and $\bar{y}$, respectively, and $\widehat{xy}$ the angle between x and y.

The projective action of an invertible matrix A on $\mathbf{P}(\mathbb{R}^d)$ is defined by $A\cdot\bar{x} = \overline{Ax}$ for any $\bar{x}\in\mathbf{P}(\mathbb{R}^d)$ and any representative $x\neq 0$ of the element $\bar{x}$. Since the matrices we consider here are not necessarily invertible, it is convenient to introduce the space

$\mathcal{X} := \mathbf{P}(\mathbb{R}^d) \cup \{0\}$ and to define the "projective" action of elements of $\mathcal{M}(d, \mathbb{R})$ on $\mathcal{X}$ as follows: For any $A \in \mathcal{M}(d, \mathbb{R})$, $A \cdot 0 = 0$, and for any $\bar{x} \in \mathbf{P}(\mathbb{R}^d)$,

$$A \cdot \bar{x} = \begin{cases} \overline{Ax} & \text{when } Ax \neq 0, \\ 0 & \text{when } Ax = 0, \end{cases}$$

where $x \neq 0$ is an arbitrary representative of $\bar{x}$.

Observe that the action of $\mathcal{M}(d, \mathbb{R})$ on $\mathcal{X}$ is measurable but not continuous.

In order to analyze the linear action of $\mathcal{M}(d, \mathbb{R})$ on $\mathbb{R}^d$, we also consider the quantity $|A\bar{x}|$ defined by:

$$|A\bar{x}| = \begin{cases} \dfrac{|Ax|}{|x|} & \text{when } \bar{x} \in \mathbf{P}(\mathbb{R}^d) \quad (\text{where } x \neq 0 \text{ is an arbitrary representative of } \bar{x}), \\ 0 & \text{when } \bar{x} = 0. \end{cases}$$

Notice that this quantity is well defined since $\dfrac{|Ax|}{|x|}$ does not depend on the representative $x \neq 0$ of $\bar{x}$, when $\bar{x} \in \mathbf{P}(\mathbb{R}^d)$.

Let $g_n = (A_n, B_n)$, $n = 1, 2, \ldots$ be independent and identically distributed random variables defined on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$ with distribution μ on $\mathcal{M}(d, \mathbb{R}) \times \mathbb{R}^d$.

Let $(\overline{V}_n)_{n \geq 0}$ be the process on $\mathcal{X}$ defined by

$$\overline{V}_{n+1} = A_{n+1} \cdot \overline{V}_n$$

for any $n \geq 0$. This process is a Markov chain on $\mathcal{X}$ with transition probability kernel $\overline{P}$ defined by: For any bounded Borel function $\varphi : \mathcal{X} \to \mathbb{R}$ and any $\bar{v} \in \mathcal{X}$,

$$\overline{P}\varphi(\bar{v}) = \int_{\mathcal{M}(d,\mathbb{R})} \varphi(A \cdot \bar{v}) \bar{\mu}(\mathrm{d}A)$$

where $\bar{\mu}$ is the projection on $\mathcal{M}(d, \mathbb{R})$ of the probability measure μ on $\mathcal{M}(d, \mathbb{R}) \times \mathbb{R}^d$.

Under several type of conditions, it is possible to assume that the Markov chain $(\overline{V}_n)_{n \geq 0}$ admits at least one invariant measure ν on $\mathcal{X}$ whose support is contained in $\mathbf{P}(\mathbb{R}^d)$: It holds for instance when the random matrices A_n are invertible or when their entries are all positive. We explore here also other situations. Further criteria for the uniqueness of ν do exist, and we refer to [5] and reference therein.

Throughout the paper, we fix an invariant measure ν for the Markov chain $(\overline{V}_n)_{n \geq 0}$ with support included in $\mathbf{P}(\mathbb{R}^d)$ and assume that the distribution of $\overline{V}_0$ equals ν and that $\overline{V}_0$ is independent of the sequence $(A_n, B_n)_{n \geq 1}$.

For any $\bar{v} \in \mathbf{P}(\mathbb{R}^d)$ and $n \geq 1$, it holds $|A_{n,1}\bar{v}| \leq \|A_{n,1}\|$. In order to control more precisely the behavior of the sequences $(|A_n \cdots A_1 \bar{v}|)_{n \geq 1}$, we introduce the following random quantity, that we call the *reverse norm control* (RNC) random variable, defined by: For any $\bar{v} \in \mathbf{P}(\mathbb{R}^d)$ and $k \geq 1$,

$$C_k(\bar{v}) := \sup_{n \geq k} \frac{\|A_n \cdots A_k\|}{|A_n \cdots A_k \bar{v}|} \in [1, +\infty] \tag{3}$$

with the convention $\frac{0}{0} = 1$ and $\frac{c}{0} = +\infty$ for any $c > 0$. Notice that

$$|A_n \cdots A_k \bar{v}| \leq \|A_n \cdots A_k\| \leq C_k(\bar{v})\, |A_n \cdots A_k \bar{v}|.$$

In several classical situations, this RNC random variable is $\mathbb{P}$-a.s. finite. For instance, by [5, Proposition III.3.2] and under quite general assumptions, this is the case when the A_n are invertible; this also property holds for nonnegative matrices A_n, when v has positive entries.

We end this paragraph with the following definition, which plays an important role in the sequel of the present paper.

Definition 1.2 Let ν be an invariant probability measure for the Markov chain $(\overline{V}_n)_{n\geq 0}$ on $\mathbf{P}(\mathbb{R}^d)$ and assume that $\overline{V}_0$ has distribution ν.

One says that the sequence $(A_n)_{n\geq 1}$ satisfies the reverse norm control property of order $\beta > 0$ relatively to the measure ν if

$$\mathbb{E}\left((\ln^+ C_1(\overline{V}_0))^\beta\right) < +\infty.$$

2 A Criterion for Recurrence

The following classical result guarantees that the affine recursion $(X_n)_{n\geq 0}$ is recurrent in the contractive case.

Fact 2.1 *Assume $\gamma_{\bar{\mu}} < 0$ and $\mathbb{E}(\ln^+ |B_1|) < +\infty$. Then, the Markov chain $(X_n)_{n\geq 1}$ is recurrent on $\mathbb{R}^d$.*

For reader convenience, we present here a short proof.

Proof Observe that for any $m \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}\lim_{n\to+\infty} |g_n \circ \cdots \circ g_1(x) - g_n \circ \cdots \circ g_m(y)| &= \lim_{n\to+\infty} |A_n \cdots A_m(X^x_{m-1} - y)| \\ \leq \lim_{n\to+\infty} \|A_n \cdots A_m\|\, |X^x_{m-1} - y| &= 0 \quad \mathbb{P} - a.s.\end{aligned}$$

Hence, the random variable $L := \liminf_{n\to+\infty} |X^x_n| \in [0, +\infty]$ does not depend x; furthermore, for all $m \in \mathbb{N}$, it is independent of the σ-algebra $\mathcal{F}_m$ generated by the variables $g_1, \ldots g_m$. By Kolmogorov's $0 - 1$ law, this random variable L is constant $\mathbb{P}$-a.s. To prove the recurrence of $(X^x_n)_{n\geq 0}$, one just need to check that L is not infinite with positive probability.

By an easy induction using the definition of the affine recursion (1), we may write: For any $n \geq 1$ and $x \in \mathbb{R}^d$,

$$X^x_n = A_{n,1}x + B_{n,1}$$

where $A_{n,1} = A_n \cdots A_1$ and $B_{n,1} = \sum_{k=1}^{n} A_n \cdots A_{k+1} B_k$ (with the convention $A_n \cdots A_{n+1} = I$).

Since $\mathbb{E}(\ln^+ |B_1|) < +\infty$ and $\gamma_{\bar{\mu}} < 0$, by (2), it holds

$$\limsup_{k \to +\infty} |A_1 \cdots A_{k-1} B_k|^{1/k} = e^{\gamma_{\bar{\mu}}} < 1 \qquad \mathbb{P}\text{-a.s.} \tag{4}$$

Consequently, the series $(\overrightarrow{X}_n)_{n \geq 0}$, with $\overrightarrow{X}_n := \sum_{k=1}^{n} A_1 \cdots A_{k-1} B_k$ (corresponding to the action of the right products $g_1 \cdots g_n$ of the transformations g_k), is $\mathbb{P}$-a.s. absolutely convergent to some random variable $\overrightarrow{X}_\infty$ with values in $\mathbb{R}^d$. Since the random variables $\overrightarrow{X}_n$ and $X_n^0 = B_{n,1}$ have the same distribution, it yields, for $K > 0$,

$$\begin{aligned}
\mathbb{P}(\liminf_{n \to +\infty} |X_n^0| \leq K) &\geq \mathbb{P}\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} (|X_n^0| \leq K) \right) \\
&= \lim_{N \to +\infty} \mathbb{P}\left(\bigcup_{n \geq N} (|X_n^0| \leq K) \right) \\
&\geq \limsup_{N \to +\infty} \mathbb{P}(|X_N^0| \leq K) = \limsup_{N \to +\infty} \mathbb{P}(|\overrightarrow{X}_N| \leq K) = \mathbb{P}(|\overrightarrow{X}_\infty| \leq K)
\end{aligned}$$

as soon as K does not belong to the denumerable set of atoms of the distribution of $|\overrightarrow{X}_\infty|$. For such a K large enough, the last term is positive; hence, $\liminf_{n \to +\infty} |X_n^0| < +\infty$ $\mathbb{P}$-a.s.

□

Throughout this paper, we focus on the case when the Lyapunov exponent $\gamma_{\bar{\mu}}$ equals 0.

In the critical case, i.e., $\gamma_{\bar{\mu}} = 0$, this argument no longer works and should be modified accordingly.

From now on, we fix $\rho \in (0, 1)$ and, for any $\bar{v} \in \mathbf{P}(\mathbb{R}^d)$, we consider the random variable $\ell_1^{(\rho)}(\bar{v})$ defined by:

$$\ell_1^{(\rho)}(\bar{v}) := \inf\{n \geq 1 \mid |A_n \cdots A_1 \bar{v}| \leq \rho\} \in \mathbb{N} \cup \{+\infty\}.$$

Notice that the $\ell_1^{(\rho)}(\bar{v})$ are stopping times with respect to the natural filtration associated with the sequence $(A_n)_{n \geq 1}$.

Our main criterion is the following statement.

Criterion 2.2 *Let ν be a $\overline{\mu}$-invariant probability measure on $\mathbf{P}(\mathbb{R}^d)$ and $\overline{V}_0$ a random variable with distribution ν and independent of the sequence $(A_n, B_n)_{n \geq 1}$. Assume that there exist constants $\alpha > 0$, $\beta > 1 + \frac{1}{\alpha}$, and $\gamma \geq \max\{\frac{1}{\alpha}, 1\}$ such that the following hypotheses hold:*

A$_1(\alpha)$- $\mathbb{E}\left(\left(\ell_1^{(\rho)}(\overline{V}_0)\right)^{\alpha}\right) < +\infty$ *for any* $\rho \in (0, 1)$.
A$_2(\beta)$- $\mathbb{E}\left((\ln^+ C_1(\overline{V}_0))^{\beta}\right) < +\infty$. *(Reverse norm control property)*
B(γ)- $\mathbb{E}\left((\ln^+ |B_1|)^{\gamma}\right) < +\infty$.
Then, for any $x \in \mathbb{R}^d$, *the chain* $(X_n^x)_{n\geq 0}$ *is recurrent.*

We will see that typically $A_1(\alpha)$ hold for any $\alpha < \frac{1}{2}$. Thus, recurrence follows when $\beta > 3$ and $\gamma > 2$.

The random variables $C_1(\bar{v})$ are really relevant in dimension $d \geq 2$. Indeed, in dimension $d = 1$, the variable $C_1(\bar{v})$ equals 1.

Proof of criterion 2.2 We follow the strategy developed by L. Elie in [11]. Fix $\rho \in (0, 1)$ such that $\ln \rho < -\mathbb{E}\left(\ln C_1(\overline{V}_0)\right)$, and this is possible since $\beta > 1$ and hypothesis **A**$_2(\beta)$ yields $\mathbb{E}(\ln^+ C_1(\overline{V}_0)) < +\infty$.

Throughout the present proof, let $(\overline{V}^{(k)})_{k\geq 0}$ be a sequence of independent and identically distributed random variables with distribution ν and assume that this sequence is independent of $(A_n, B_n)_{n\geq 1}$. We set $\ell_0 = 0$ and, for any $k \geq 1$,

$$\ell_k = \inf\{n \geq \ell_{k-1} + 1 : |A_n \cdots A_{\ell_{k-1}+1}\overline{V}^{(k-1)}| \leq \rho\}.$$

Without loss of generality, we may assume $\overline{V}^{(0)} = \overline{V}_0$; let us emphasize that the random variables $\overline{V}_k = A_k \cdots A_1 \cdot \overline{V}_0, k \geq 1$, have also distribution ν but that the sequence $(\overline{V}_k)_{k\geq 0}$ is not i.i.d. The interest of the sequence $(\overline{V}^{(k)})_{k\geq 0}$ is to decompose the product $A_{\ell_n} \cdots A_1$ as a product of i.i.d. random variables as explained below.

Observe that the sub-process $(X_{\ell_n}^x)_{n\geq 0}$ is a stochastic dynamical system defined recursively by affine transformations. More precisely,

$$X_{\ell_n}^x = \widetilde{g}_n \circ \cdots \circ \widetilde{g}_1(x)$$

with

$$\widetilde{g}_k = (\widetilde{A_k}, \widetilde{B_k}) := g_{\ell_k} \circ \cdots \circ g_{\ell_{k-1}+1} = (A_{\ell_k} \ldots A_{\ell_{k-1}+1}, \sum_{i=\ell_{k-1}+1}^{\ell_k} A_{\ell_k} \cdots A_{i+1}B_i).$$

The random variables $\widetilde{g}_k, k \geq 1$, are independent and identically distributed, with distribution $\widetilde{\mu} = \mathrm{dist}(\widetilde{g}_1)$ on $\mathcal{M}(d, \mathbb{R}) \times \mathbb{R}^d$.

Let us now check that the sub-process $(X_{\ell_n})_{n\geq 0}$ satisfies the hypotheses of Fact 2.1. This will prove that $(X_{\ell_n})_{n\geq 0}$, hence $(X_n)_{n\geq 0}$, is recurrent.

We set $\alpha_k := \|A_{\ell_k} \cdots A_{\ell_{k-1}+1}\|$. By construction, the random variables $\alpha_k, k \geq 1$, are independent and identically distributed. We claim that they are log-integrable and have negative log-mean. In fact, by the definition of $C_1(\overline{V}_0)$, it holds

$$\|A_{\ell_1} \cdots A_1\| \leq C_1(\overline{V}_0)|A_{\ell_1} \cdots A_1\overline{V}_0| \leq \rho\, C_1(\overline{V}_0)$$

so that

$$\mathbb{E}(\ln \alpha_1) \leq \ln \rho + \mathbb{E}(\ln C_1(\overline{V}_0)) = \ln \rho + \mathbb{E}\left(\ln C_1(\overline{V}_0)\right) < 0.$$

Hence, the Lyapunov exponent of $\widetilde{\mu}$ satisfies

$$\gamma_{\widetilde{\mu}} = \lim_{n\to+\infty} \frac{1}{n} \ln \|A_{\ell_n,1}\| \leq \lim_{n\to+\infty} \frac{\ln \alpha_1 + \cdots + \ln \alpha_n}{n} \leq \ln \rho + \mathbb{E}\left(\ln C_1(\overline{V}_0)\right) < 0.$$

In order to check that $\widetilde{B_1} = B_{\ell_1,1}$ is log-integrable, we observe that

$$|B_{\ell_1,1}| \leq \sum_{i=1}^{\ell_1} \|A_{\ell_1} \cdots A_{i+1}\| |B_i| \leq \sum_{i=1}^{\ell_1} C_{i+1}(\overline{V}_i)|B_i| \quad \mathbb{P}\text{-a.s.}$$

Indeed, on the one hand, the inequalities $|A_{\ell_1} \cdots A_1 \overline{V}_0| \leq \rho$ and $|A_i \cdots A_1 \overline{V}_0| > \rho > 0$ for $1 \leq i < \ell_1$ yield

$$\begin{aligned}\|A_{\ell_1} \cdots A_{i+1}\| &\leq C_{i+1}(\overline{V}_i)|A_{\ell_1} \cdots A_{i+1} \overline{V}_i| \\ &= C_{i+1}(\overline{V}_i) \frac{|A_{\ell_1} \cdots A_1 \overline{V}_0|}{|A_i \cdots A_1 \overline{V}_0|} \leq C_{i+1}(\overline{V}_i) \frac{\rho}{\rho} = C_{i+1}(\overline{V}_i)\end{aligned}$$

for $1 \leq i < \ell_1$. On the other hand,

$$\|A_{\ell_1} \cdots A_{\ell_1+1}\| = |I| = 1 \leq C_{i+1}(\overline{V}_i).$$

Thus,

$$\ln^+ |B_{\ell_1,1}| \leq \ln^+ \ell_1 + \max_{1\leq i\leq \ell_1} \ln^+ C_{i+1}(\overline{V}_i) + \max_{1\leq i\leq \ell_1} \ln^+ |B_i| \quad \mathbb{P}\text{-a.s.} \tag{5}$$

Under hypothesis $A_1(\alpha)$, the random variable $\ln^+ \ell_1$ is integrable. The fact that

$$\mathbb{E}(\max_{1\leq i\leq \ell_1} \ln^+ |B_i|) < +\infty$$

whenever $\mathbf{B}(\gamma)$ is satisfied for some $\gamma \geq \max\{\frac{1}{\alpha}, 1\}$ is a rather classical result that has been used by several author (see, for instance, [10, Proposition 4]). However, this argument uses deeply the fact that the B_i are independent; it cannot be applied to the sequence $(C_{i+1}(\overline{V}_i))_{i\geq 0}$, which is stationary but not independent. To deal with the second term in the right-hand side of (5), we need to use Lemma 2.3, proved hereafter, which is more general but requires stronger moment. In particular, applying this Lemma with $Y_i = \ln^+ C_{i+1}(\overline{V}_i)$ and $\tau = \ell_1$, we can conclude that hypotheses $\mathbf{A}_1(\alpha)$ and $\mathbf{A}_2(\beta)$ with $\beta > 1+1/\alpha$ ensure that there exists a positive constant $C(\alpha, \beta)$ s.t.

$$\mathbb{E}\left(\max_{1\le i\le \ell_1} \ln^+ C_{i+1}(\overline{V}_i)\right) \le \ C(\alpha,\beta)\ \mathbb{E}(\ell_1^\alpha)^{\frac{\beta-1}{\beta}}\ \mathbb{E}\left(\ln^+ C_1(\overline{V}_0)^\beta\right)^{1/\beta} < +\infty.$$

□

Lemma 2.3 *Let $(Y_i)_{i\ge1}$ be a sequence of nonnegative random variables and τ a $\mathbb{N}$-valued random variable, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then, for $\alpha>0$ and $\beta>\frac{1+\alpha}{\alpha}$, there exists a constant $C=C(\alpha,\beta)$ s.t.*

$$\mathbb{E}\left(\max_{1\le i\le\tau} Y_i\right) \le C\ \mathbb{E}(\tau^\alpha)^{\frac{\beta-1}{\beta}}\ \sup_{i\ge1}\mathbb{E}\left(Y_i^\beta\right)^{1/\beta}. \tag{6}$$

Proof. For any $\delta>0$, $\left(\max_{1\le i\le\tau} Y_i\right) \le \max_{1\le i\le\tau}\left(\frac{Y_i}{i^\delta}\, i^\delta\right) \le \sup_{i\ge1}\left(\frac{Y_i}{i^\delta}\right)\tau^\delta.$

Hence, by Hölder inequality, $\mathbb{E}\left(\left(\max_{1\le i\le\tau} Y_i\right)\right) \le \mathbb{E}\left(\sup_{i\ge1}\left(\frac{Y_i^\beta}{i^{\delta\beta}}\right)\right)^{\frac{1}{\beta}} \mathbb{E}(\tau^{\delta q})^{\frac{1}{q}}$ where q satisfies $\frac{1}{\beta}+\frac{1}{q}=1$. Hence,

$$\mathbb{E}\left(\sup_{i\ge1}\left(\frac{Y_i^\beta}{i^{\delta\beta}}\right)\right) \le \mathbb{E}\left(\sum_{i\ge1}\frac{Y_i^\beta}{i^{\delta\beta}}\right) \le \sum_{i\ge1}\frac{\mathbb{E}(Y_i^\beta)}{i^{\delta\beta}} \quad \le \sup_{i\ge1}\mathbb{E}(Y_i^\beta)\times\left(\sum_{i\ge1}\frac{1}{i^{\delta\beta}}\right)$$

Take $\delta=\alpha/q=\alpha(\beta-1)/\beta$. Then, $\delta\beta=\alpha(\beta-1)>1$ if $\beta>(1+\alpha)/\alpha$. Inequality (6) follows with $C(\alpha,\beta)=\sum_{i\ge1}\frac{1}{i^{\alpha(\beta-1)}}<+\infty$.

□

3 Application to Significant Matrix Semigroups

We present here various situations where the matrices A_n belong to different and important classes of matrix semigroups: similarities, rank 1, invertible, and nonnegative matrices. In Subsect. 3.1, we present a list of hypotheses in each situations. In Subsections 3.2 and 3.3, we explain how these different conditions imply the recurrence of the process $(X_n)_{n\ge0}$.

In all these cases, there exists a $\bar{\mu}$-invariant probability ν on $\mathbf{P}(\mathbb{R}^d)$ and the affine recursion is controlled by the 1-dimensional random walk $(S_n)_{n\ge0}$, with Markovian increments, defined by:

$$S_0=0 \quad\text{and}\quad S_n=S_n(\overline{V}_0):=\ln|A_n\cdots A_1\overline{V}_0|=\sum_{k=1}^{n}\ln|A_k\overline{V}_{k-1}|. \tag{7}$$

The proof of our main theorem relies, in particular, on results concerning the fluctuations of this process $(S_n)_{n\ge0}$ and on the contraction properties on $\mathbf{P}(\mathbb{R}^d)$ of the random matrices $A_{n,1}, n \geq 1$. These results and properties require different moment and non-degeneracy assumptions according to the explored situation; the formulation of these hypotheses varies from case to case, but all of them ensure in particular that:

- The increments of $(S_n)_{n\geq 0}$ have at least *moment* of order 2, "from above" and "from below." The control "from above" can be seen in the fact that $\ln^+ |A_1|$ has at least moment of order 2. The one "from below" is reflected in the conditions called "reverse moments," which ensure that the random variable

$$\mathfrak{n}(A_1, \overline{V}_0) := \|A_1\|/|A_1\overline{V}_0| \ \in [1, +\infty]$$

  is sufficiently integrable; it implies in particular that the quantity $|A_1\overline{V}_0|$ is not too close to zero.
- The Markov walk $(S_n)_{n\geq 0}$ is *non-degenerate*, i.e., its variance $\sigma^2 := \lim_{n\to\infty} \mathrm{Var}(S_n)/n$ is nonzero. In particular, the process $(S_n)_{n\geq 0}$ is unbounded.
- The projective action of the matrices $A_{n,1}$ has nice "contraction" properties (except in case **S**, which in some sense a 1-dimensional situation); more precisely, there exists $\rho <\in [0, 1)$ and $c > 0$ such that

$$\forall \bar{u}, \bar{v} \in \mathbf{P}(\mathbb{R}^d), \forall n \geq 1 \qquad \mathbb{E}(\delta(A_{n,1} \cdot \bar{u}, A_{n,1} \cdot \bar{v})) \leq c\, \rho^n. \tag{8}$$

3.1 Hypotheses

3.1.1 Case S: The A_n are Similarities

We assume here that $A_n = a_n R_n$ with $a_n \in \mathbb{R}^{*+}$ and $R_n \in O(d, \mathbb{R})$.

In this case, the uniform distribution on $\mathbf{P}(\mathbb{R}^d)$ is $\bar{\mu}$-invariant and $S_n := \ln |A_n \cdots A_1\overline{V}_0| = \sum_{k=1}^{n} \ln a_k$, for any initial direction $\overline{V}_0$. This is the simplest of our example since it is controlled by the classical 1-dimensional random walk on $\mathbb{R}$ with i.i.d increments $\ln a_n$.

We consider the following hypotheses.

Hypotheses **S**

S-*Moments* : $\mathbb{E}((\ln a_n)^2) < +\infty$.

S-*Non-degeneracy* : $V(\ln a_1) > 0$.

S-*Centering*: $\mathbb{E}(\ln a_n) = 0$

3.1.2 Case Rk1: The A_n are Rank 1 Matrices

We assume here that there exists a sequence $(\tilde{w}_n, w_n, a_n)_{n\geq 1}$ of i.i.d. random variables with values in $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \times \mathbb{R}^{*+}$ such that $\mathrm{Im}(A_n) = \mathbb{R}w_n$, $\mathrm{Ker}(A_n) = \tilde{w}_n^{\perp}$ and $|A_n| = a_n$. In other words,

$$\forall v \in \mathbb{R}^d, \quad A_n v = a_n \langle \tilde{w}_n, v\rangle w_n$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$. In particular,

$$A_{n,1}v = a_n \cdots a_1 \langle \tilde{w}_n, w_{n-1}\rangle \cdots \langle \tilde{w}_1, v\rangle w_n. \tag{9}$$

Suppose that $\mathbb{P}(\langle \tilde{w}_{n+1}, w_n\rangle = 0) = 0$. Then, the distribution ν of the random variables $\overline{w_n}$ is a $\bar{\mu}$-invariant probability measure on $\mathbf{P}(\mathbb{R}^d)$. Indeed, let w_0 be a $\mathbb{S}^{d-1}$-valued random variable with distribution ν and independent of $(\tilde{w}_n, w_n, a_n)_{n\geq 1}$; then, $\overline{V}_0 = \overline{w_0}, \overline{V}_n = A_{n,1}\cdot\overline{w_0} = \overline{w_n}$ for $n \geq 1$ and $\overline{V}_n$ has also distribution ν. Since the former equality holds for any starting direction $\overline{w_0'}$ such that $\mathbb{P}(\langle \tilde{w}_1, w_0'\rangle = 0) = 0$, it also readily implies that ν is the unique $\bar{\mu}$-invariant probability measure on $\mathbf{P}(\mathbb{R}^d)$.

The 1-dimensional random walk $(S_n)_{n\geq 0}$, which controls the behavior of the affine recursion (9) can be written as:

$$S_n(\overline{V}_0) = \ln |A_{n,1}\overline{V}_0| = \sum_{k=1}^{n} \ln a_k |\langle \tilde{w}_k, w_{k-1}\rangle|. \tag{10}$$

This is a Markov random walk with stationary increments $Y_k := \ln a_k |\langle \tilde{w}_k, w_{k-1}\rangle|, k \geq 1$. Observe that Y_k and Y_{k+1} are not independent, but Y_k and Y_ℓ are independent as soon as $|k-\ell| \geq 2$. The ergodic theorem ensures that $(S_n(\overline{V}_0)/n)_n$ converges $\mathbb{P}$-a.s. to $\mathbb{E}(Y_1)$, and this yields

$$\gamma_{\bar{\mu}} = \mathbb{E}(Y_1) = \mathbb{E}(\ln |a_1\langle \tilde{w}_1, w_0\rangle|).$$

Notice atlast that $\mathfrak{n}(A_1, \overline{V}_0) := \|A_1\|/|A_1\overline{V}_0| = 1/|\langle \tilde{w}_1, w_0\rangle|$.
We introduce the following assumptions:

Hypotheses **Rk1**

Rk1-*Moments*: *There exists* $\varepsilon > 0$ *such that* $\mathbb{E}\left(|\ln a_1|^{2+\varepsilon}\right) < +\infty$.

Rk1-*Reverse moments*: *There exists* $\varepsilon > 0$ *such that* $\mathbb{E}\left(|\ln|\langle \tilde{w}_1, w_0\rangle||^{3+\varepsilon}\right) < +\infty$.

Rk1-*Non-degeneracy*: $\sigma^2 = V(Y_2) + 2\mathrm{cov}(Y_1, Y_2) > 0$, *with* $Y_2 := \ln a_2 + \ln|\langle \tilde{w}_2, w_1\rangle|$ *and* $Y_1 := \ln a_1 + \ln|\langle \tilde{w}_1, w_0\rangle|$.

Rk1-*Centering*: $\mathbb{E}(\ln a_1) + \mathbb{E}(\ln|\langle \tilde{w}_1, w_0\rangle|) = 0$.

At the first glance, this rank 1 situation may seem a little artificial, but it is not the case. Indeed, when the A_n are either invertible or nonnegative (see the cases **I** and **Nn** below), the closure in $\mathcal{M}(d,\mathbb{R})$ of the semigroup generated by the support of $\bar{\mu}$ contains a rank 1 matrix. More precisely, if the Lyapunov exponent $\gamma_{\bar{\mu}}$ equals 0, the sequence $(A_{n,1})_{n\geq 0}$ is recurrent in $\mathcal{M}(d,\mathbb{R})$ and, under standard hypothesis, its cluster points either equal 0 or have rank 1. Furthermore, the rank 1 situation explored here is a handy toy model to understand how to extend results obtained for products of invertible matrices to other semigroups of matrices.

3.1.3 Case I: the A_n are Invertible

We assume here that the random variables $A_n, n \geq 1$, are $\mathrm{Gl}(d,\mathbb{R})$-valued, where $\mathrm{Gl}(d,\mathbb{R})$ equals the set of $d \times d$ invertible matrices with real coefficients. We assume

$d \geq 2$, the case $d = 1$ corresponds to a scalar affine recursion (which can be viewed as a particular case of **S**) and is treated under the additional non-degeneracy condition.

Product of random matrices in $\mathrm{Gl}(d, \mathbb{R})$ have been widely studied. We refer in particular to the works of Guivarch [16] and Le Page [18] and to the books of Bougerol and Lacroix [5] or Benoist and Quint [8].

Following [5], we now introduce the following hypotheses:

Hypotheses **I**

I-*(Exponential) Moments*: *There exists* $\varepsilon > 0$ *such that* $\mathbb{E}(\|A_1\|^\varepsilon) < +\infty$.

I-*(Exponential) Reverse moments*: *There exists* $\varepsilon > 0$ *such that* $\mathbb{E}\left(\|A_1^{-1}\|^\varepsilon\right) < +\infty$.

I-*Proximality*: *The semigroup* $T_{\overline{\mu}}$ *generated by the support of* $\overline{\mu}$ *contains a proximal element, that is a matrix* A*, which admits an eigenvalue* $\lambda \in \mathbb{R}$*, which has multiplicity one and whose all other eigenvalues have modulus* $< |\lambda|$.

If $A \in T_{\overline{\mu}}$ is proximal, then the sequence $(A^{2n}/\|A^{2n}\|)_{n\geq 1}$ converges to a rank 1 matrix.

I-*Strong irreducibility*: *There exists no proper finite union of subspaces of* $\mathbb{R}^d$*, which is invariant with respect to all elements of* $T_{\overline{\mu}}$.

I-*Centering*: $\gamma_{\bar{\mu}} = 0$.

Before presenting the last example considered in this article, let us make a few comments on the assumptions **I**.

First, as announced in the introduction, notice that by [8, Lemma 4.13] assumption, **I**-*proximality* is always satisfied when considering the action of the A_n on some suitable exterior product $\wedge^r\mathbb{R}^d$, $1 \leq r \leq d$. Therefore, this hypothesis with $r = 1$ is not required to get the recurrence of $(X_n)_{n\geq 0}$. As a first step, we add it here to explain how case **I** can be treated as the other cases by using criterion 2.2. We refer the reader to sect. A for an extension of this criterion.

Secondly, hypotheses **I**-*(exponential) moments* and **I**-*(exponential) reverse moments* imply that the random variable $\mathfrak{n}(A_1, \overline{V}_0) := \|A_1\|/|A_1\overline{V}_0| \leq \|A_1\|\|A_1^{-1}\|$ has an exponential moment of order $\varepsilon/2$.

Forty years ago, E. le Page [18] proved that, under hypotheses **I**-*(exponential) moments and reverse moments*, **I**-*proximality* and **I**-*strong irreducibility*, for any $\bar{v} \in \mathbf{P}(\mathbb{R}^d)$, the process $(S_n(v) = \ln|A_{n,1}v|)_n$ satisfies a central limit theorem (CLT). He established in particular that the "asymptotic" variance

$$\sigma^2 := \lim_{n\to+\infty} \frac{\mathbb{E}(\ln|A_{n,1}x| - n\gamma_{\bar{\mu}})^2}{n}$$

is nonzero (see also [5, Chapter V-8, Theorem 5.1]). This corresponds to the *non-degeneracy* condition in cases **S** and **Rk1**. His proof relies on the following "mean contraction property": For $\epsilon \in (0, 1)$ small enough, there exist $\rho_\epsilon \in (0, 1)$ and $c_\epsilon > 0$ such that, for any $\bar{u}, \bar{v} \in \mathbb{P}(\mathbb{R}^d)$,

$$\mathbb{E}(\delta^\epsilon(A_{n,1} \cdot \bar{u}, A_{n,1} \cdot \bar{v})) \leq c_\epsilon\, \rho_\epsilon^n\, \delta^\epsilon(\bar{u}, \bar{v}). \tag{11}$$

(see [5, Chapter V-2, Proposition 2.3]). In particular, this implies (8).

More recent works have significantly relaxed moment hypotheses to prove similar results. In particular, Y.Benoist and J-F. Quint [8] proved the CLT under the optimal second moment condition $\mathbb{E}((\ln \|A_1\|)^2+(\ln(\|A_1^{-1}\|)^2) < +\infty$ and a work in progress of A. Péneau [19] should guarantee the projective contraction property (8) without any moment assumption.

In the present paper, we still need exponential moments since we rely on results of [15] for fluctuation properties of S_n. However, it is reasonable to think that future developments of this theory will ensure that the second-order moments are sufficient to obtain recurrence of affine recursion.

3.1.4 Case Nn: The A_n are Nonnegative

Another interesting case that has been widely studied in the last year is the product of matrices with nonnegative coefficient. In this case, the measure μ is no longer supported by a group but lives in the semigroup $\mathcal{S}$ of matrices with nonnegative entries such that each column contains at least one positive entry. Notice that the action of T_μ on the full projective space is no longer continuous. However, the semigroup $\mathcal{S}$ acts continuously on the simplex $\mathbb{X}$ of nonnegative vectors of norm 1; this action possesses nice contraction properties, which yield results similar to the one obtains in the case of invertible matrices. From a historical point of view, this case was studied first; we refer to the works by Furstenberg and Kesten [13] and Hennion [17].

The set $\mathbb{X} = \mathbb{S}^{d-1} \cap (\mathbb{R}^+)^d$ can be identified with the subset of the projective space $\mathbf{P}(\mathbb{R}^d)$ whose elements correspond to half lines in the cone $(\mathbb{R}^+)^d$. Observe for any $A \in \mathcal{S}$ and $v \in \mathbb{X}$, the element $A \cdot v := Av/|Av|$ belongs to $\mathbb{X}$.

Recent papers on random product of nonnegative matrices have proved that is useful in this setting to endow $\mathbb{R}^d$ with the L^1-norm $|\cdot|_1$ and to provide $\mathbb{X}$ with a special distance $\mathfrak{d}$ that is strongly contracted by the action of the elements of $\mathcal{S}$ (the reader can find in [17, Section 10] and [7, Section 2.2] a precise description of the properties of the distance $\mathfrak{d}$). In the present paper, in order to remain consistent with the other cases, we prefer to use the euclidean norm $|\cdot|$ on $(\mathbb{R}^+)^d$ and the distance δ on $\mathbb{X}$. We can translate known results in the present setting, using the facts that

- $|x| \leq |x|_1 \leq \sqrt{d}|x|$ for any $x \in \mathbb{R}^d$;
- The metrics δ and $\mathfrak{d}$ satisfies $\delta(\bar{u}, \bar{v}) \leq |u - v| \leq 2\mathfrak{d}(u, v)$ for any $u, v \in \mathbb{X}$ and induce the same topologies on $\mathbb{X}$. Nevertheless, $\mathfrak{d}$ is not equivalent with neither other metrics $|\cdot|$ and δ on $\mathbb{X}$.

For any $A = (A(i, j))_{1\leq i,j\leq d} \in \mathcal{S}$, set $v(A) := \min_{1\leq j\leq d}\Big(\sum_{i=1}^{d} A(i, j)\Big)$. This quantity is of interest since $\frac{v(A)}{\sqrt{d}}\, |x| \leq |Ax| \leq \|A\|\, |x|$ for any $x \in \mathbb{R}^d_+$, with $v(A) > 0$; in particular, $\mathfrak{n}(A, v) \leq \frac{\sqrt{d}\|A\|}{v(A)}$.

Following [20], we now introduce the following hypotheses:

Hypotheses Nn

Nn-*Exponential moments*: *There exists* $\varepsilon > 0$ *such that* $\mathbb{E}\,(\|A_1\|^{\varepsilon}) < +\infty$.

Nn-*Reverse moments*: *There exists* ε *such that* $\mathbb{E}\left(\left(\ln \frac{\|A_1\|}{v(A_1)}\right)^{6+\varepsilon}\right) < +\infty$.

Nn-*Contraction property* : *There exists* $n_0 \geq 1$ *such that* $\mu^{\star n_0}(\mathcal{S}^+) > 0$.
Nn-*Irreducibility*: *There exists no affine subspaces* $\mathcal{A}$ *of* $\mathbb{R}^d$ *such that* $\mathcal{A} \cap \mathcal{C}$ *is non-empty and bounded and invariant under the action of all elements of the support of* μ.
Nn-*Centering*: $\gamma_{\bar{\mu}} = 0$.

The affine recursion generated by nonnegative matrices has been investigated in [7] under the restrictive condition that the A_n are $\mathcal{S}_\delta$-valued for some $\delta > 0$, where $\mathcal{S}_\delta$ denotes the set of matrices A in $\mathcal{S}$ such that $A(i, j) \geq \delta A(k, l)$ for any $1 \leq i, j, k, l \leq d$. Here, we break free from this strong assumption.

Let us now give a few comments on the above hypotheses.

By [17], under hypotheses **Nn**-*exponential moments*, *reverse moments*, *contraction property*, and *irreducibility*, the "asymptotic" variance

$$\sigma^2 := \lim_{n\to+\infty} \frac{\mathbb{E}(\ln |A_{n,1}x| - n\gamma_{\bar{\mu}})^2}{n}$$

is nonzero (which corresponds to condition *non-degeneracy* in the cases **S** and **Rk1**) and the sequence $\left(\frac{1}{\sqrt{n}}(\ln |A_{n,1}x| - n\gamma_{\bar{\mu}})\right)_{n\geq 1}$, $x \neq 0$, converges in distribution to the normal distribution $\mathcal{N}(0, \sigma^2)$.

At last, under hypothesis **Nn**-*contraction property*, there exist $\rho \in (0, 1)$ and $c > 0$ such that, for any $u, v \in \mathbb{X}$,

$$\mathbb{E}(\mathfrak{d}(A_{n,1} \cdot u, A_{n,1} \cdot v)) \leq c\, \rho^n\, \mathfrak{d}(u, v). \tag{12}$$

This property is of crucial interest below (see Lemma 3.3).

3.2 On the Moments of the Ladder Time $\ell_1^{(\rho)}(\bar{v})$

We will now demonstrate that in the above four situations, there exist positive reals α and β, with $\beta > 1 + \frac{1}{\alpha}$ such that conditions $\mathbf{A}_1(\alpha)$ and $\mathbf{A}_2(\beta)$ of criterion 2.2 are satisfied.

We prove here the following statement concerning fluctuations of random walks.

Proposition 3.1 *Under hypotheses* **S**, **Rk1**, **I**, *or* **Nn**, *the condition* $\mathbf{A}_1(\alpha)$ *holds for any* $\alpha \in (0, 1/2)$.

Proof In the case **S**, the above statement is a consequence of classical results on fluctuations of random walks on $\mathbb{R}$ for which it is well known that, for any $\rho > 0$ and $\bar{v} \in \mathbf{P}(\mathbb{R}^d)$, there exists a constant $c_\rho > 0$ s.t. $\mathbb{P}(\ell_1^{(\rho)}(\bar{v}) > n) \sim c_\rho/\sqrt{n}$ as $n \to +\infty$ [12].

Similar results do exist in the case **I** and **Nn** by [15] and [20], respectively (with $v \in \mathbb{X}$ in the case **Nn**).

The case **Rk1** is a consequence of a general result of fluctuations of Markov walks on $\mathbb{R}$ proved in [14]; we explain in Sect. 4 how this general result can be applied to the **Rk1** situation.

□

3.3 On the Reverse Norm Control Property

By the above subsection, the hypothesis $\mathbf{A}_1(\alpha)$ holds for any $\alpha \in (0, 1/2)$. We now check that, for some $\beta > 1 + \frac{1}{\alpha} > 3$, the reverse norm control (RNC) property $\mathbf{A}_2(\beta)$ holds under hypotheses **S**, **Rk1**, **I**, or **Nn**.

Let us fix a basis $\mathbf{e} = (e_i)_{i=1,..d}$ of $\mathbb{R}^d$. There exits a constant $c_{\mathbf{e}}$ such that $\|A\| \leq c_{\mathbf{e}} \sum_{i=1}^d |Ae_i|$ for any $A \in \mathcal{M}(d, \mathbb{R})$; hence, for any $\bar{v} \in \mathbf{P}(\mathbb{R}^d)$,

$$C_1(\bar{v}) := \sup_{n\geq 1} \frac{\|A_{n,1}\|}{|A_{n,1}\bar{v}|} \leq c_{\mathbf{e}} \sum_{i=1}^d \sup_{n\geq 1} \frac{|A_{n,1}\bar{e}_i|}{|A_{n,1}\bar{v}|} = c_{\mathbf{e}} \sum_{i=1}^d C_1(\bar{e}_i, \bar{v})$$

where, for any $\bar{u}, \bar{v} \in \mathbf{P}(\mathbb{R}^d)$, we set $C_1(\bar{u}, \bar{v}) := \sup_{n\geq 1} \frac{|A_{n,1}\bar{u}|}{|A_{n,1}\bar{v}|}$. Then, since $\ln^+$ is non-decreasing and $\ln^+(ab) \leq \ln^+(a) + \ln^+(b)$,

$$\ln^+ C_1(\bar{v}) \leq \ln^+ c_{\mathbf{e}} + \sum_{i=1}^d \ln^+ C_1(\bar{e}_i, \bar{v}).$$

Hence, the RNC property $A_2(\beta)$ holds as soon as $\mathbb{E}((\ln^+ C_1(\bar{u}, \overline{V}_0))^\beta) < +\infty$ for any u in a set of generators of $\mathbf{e}$ of $\mathbb{R}^d$.

Now, the quantity $\frac{|A_{n,1}\bar{u}|}{|A_{n,1}\bar{v}|}$ may be decomposed as: $\prod_{k=1}^n \frac{|A_k\bar{u}_{k-1}|}{|A_k\bar{v}_{k-1}|}$, with $\bar{u}_0 = \bar{u}, \bar{v}_0 = \bar{v}$ and $\bar{u}_k = A_k \cdot \bar{u}_{k-1}, \bar{v}_k = A_k \cdot \bar{v}_{k-1}$ for any $k \geq 1$. Hence,

$$\ln^+ C_1(\bar{u}, \bar{v}) \leq \sum_{k=1}^{+\infty} \ln^+ \left(\frac{|A_k\bar{u}_{k-1}|}{|A_k\bar{v}_{k-1}|} \right). \tag{13}$$

We will see in the following proposition that this sum converges in the appropriate L^p spaces in the different situations presented above.

Let us point out why the reasons of such convergence are different in each of the situations explored here. In the case of similarities **S**, it holds $\frac{|A_n\bar{u}|}{|A_n\bar{v}|} = 1$ for all $\bar{u}, \bar{v} \in \mathbf{P}(\mathbb{R}^d)$; thus, all the terms of the series in the right-hand side of (13) vanish. In the cases **Rk1**, **I**, and **Nn**, the convergence is due to the fact that the distance between $\bar{u}_k$ and $\overline{V}_k$ goes to zero at exponential speed.

Proposition 3.2 *Let ν be a μ invariant measure supported by $\mathbb{S}^{d-1}$ and $\overline{V}_0 \sim \nu$ then, under hypotheses* **S**, **Rk1**, **I**, *or* **Nn**, *the RNC property $\boldsymbol{A}_2(\beta)$ holds (i.e., $\mathbb{E}((\ln^+ C_1(\overline{V}_0))^\beta) < +\infty$) for $\beta > 3$.*

Proof • Case **S**. The case **S** is straightforward since $C_1(v) = 1$ for any $v \in \mathbf{P}(\mathbb{R}^d)$ in this case. The other cases rely on the contraction property satisfied by the process $(\overline{V}_k)_{k\geq 0}$. Let us first propose a general criterion and then apply it in these 3 remaining situations.

• Case **Rk1**. In this case, $\bar{u}_k = \overline{V}_k = \overline{w}_k$ for any $k \geq 1$ so that

$$\ln^+ C_1(\bar{u}, \bar{v}) \leq \ln^+ \left(\frac{|A_1 \bar{u}|}{|A_1 \bar{v}|}\right) \leq \ln^+ \left(\frac{|\langle \tilde{w}_1, u\rangle|}{|\langle \tilde{w}_1, v\rangle|}\right)) \leq \ln^+ \left(\frac{1}{|\langle \tilde{w}_1, v\rangle|}\right).$$

Thus, $\mathbf{A}_2(\beta)$ is a direct consequence of hypothesis **Rk1**-*Reverse moments* with $3 < \beta < 3 + \varepsilon$.

To deal with cases **I** and **Nn**, we use the following lemma.

Lemma 3.3 *We assume that there exists an invariant mesure ν for the chain $(\overline{V}_k)_{k\geq 0}$ with support included in $\mathbf{P}(\mathbb{R}^d)$, constants $\beta > 2$, $C > 0$, $\rho \in (0, 1)$*

(i) $\mathbb{E}\left(\left(\ln^+ \dfrac{\|A_1\|}{|A_1 \overline{V}_0|}\right)^b\right)$ for some $b > 2\beta$ where $\overline{V}_0$ has distribution ν;

(ii) $\mathbb{E}\left(\delta(A_{k,1} \cdot \bar{u}, A_{k,1} \cdot \overline{V}_0)\right) \leq C\rho^k$ for some fixed $\bar{u} \in \mathbf{P}(\mathbb{R}^d)$.

Then, $\mathbb{E}\left((\ln^+ C_1(\bar{u}, \overline{V}_0))^\beta\right) < +\infty$.

Proof. First, for any $\beta \geq 1$, inequality (13) yields

$$\mathbb{E}\left((\ln^+ C_1(\bar{u}, \overline{V}_0))^\beta\right)^{1/\beta} \leq \sum_{k\geq 1} \mathbb{E}\left(\left(\ln^+ \frac{|A_k \bar{u}_{k-1}|}{|A_k \overline{V}_{k-1}|}\right)^\beta\right)^{1/\beta}$$

Note that, for any $A \in M(d, \mathbb{R})$ and any two directions $\bar{u}, \bar{v} \in \mathbf{P}(\mathbb{R}^d)$ with representative vectors $u, v \in \mathbb{S}^{d-1}$,

$$\ln^+ \frac{|A\bar{u}|}{|A\bar{v}|} \leq \ln^+ \frac{\|A\|}{|A\bar{v}|} = \ln^+ \mathfrak{n}(A, \bar{v})$$

with $\mathfrak{n}(A, \bar{v}) := \|A\|/|A\bar{v}| \in [1, +\infty]$. Furthermore,

$$\ln^+ \frac{|A\bar{u}|}{|A\bar{v}|} \leq \ln^+ \frac{|Av| + |A(u - v)|}{|Av|} \leq \ln^+(1 + \mathfrak{n}(A, v)|u - v|)$$

and

$$\ln^+ \frac{|A\bar{u}|}{|A\bar{v}|} \leq \ln^+ \frac{|A(-v)| + |A(u + v)|}{|A\bar{v}|} \leq \ln^+(1 + \mathfrak{n}(A, \bar{v})|u + v|)$$

so that

$$\ln^+ \frac{|A\bar{u}|}{|A\bar{v}|} \leq \ln^+(1 + \mathfrak{n}(A, \bar{v}) \min(|u - v|, |u + v|)) \leq \sqrt{2}\, \mathfrak{n}(A, \bar{v})\, \delta(\bar{u}, \bar{v}). \quad (14)$$

In particular, for any $k \geq 1$, setting $a_k = \rho^{-k/2\beta}$,

$$\begin{aligned}
\left(\ln^+ \frac{|A_k \bar{u}_{k-1}|}{|A_k \overline{V}_{k-1}|}\right)^\beta &= \left(\ln^+ \frac{|A_k \bar{u}_{k-1}|}{|A_k \overline{V}_{k-1}|}\right)^\beta \mathbf{1}_{[\mathfrak{n}(A_k, \overline{V}_{k-1}) > a_k]} \\
&\quad + \left(\ln^+ \frac{|A_k \bar{u}_{k-1}|}{|A_k \overline{V}_{k-1}|}\right)^\beta \mathbf{1}_{[\mathfrak{n}(A_k, \overline{V}_{k-1}) \leq a_k]} \\
&\leq \left(\ln^+ \mathfrak{n}(A_k, \overline{V}_{k-1})\right)^\beta \mathbf{1}_{[\mathfrak{n}(A_k, \overline{V}_{k-1}) > a_k]} \\
&\quad + \left(\sqrt{2}\, \mathfrak{n}(A_k, \overline{V}_{k-1})\, \delta(\bar{u}_{k-1}, \overline{V}_{k-1})\right)^\beta \mathbf{1}_{[\mathfrak{n}(A_k, \overline{V}_{k-1}) \leq a_k]} \\
&\leq \left(\ln^+ \mathfrak{n}(A_k, \overline{V}_{k-1})\right)^\beta \frac{\left(\ln^+ \mathfrak{n}(A_k, \overline{V}_{k-1})\right)^{\beta'}}{(\ln^+ a_k)^{\beta'}} \\
&\quad + 2^{\beta/2}\, a_k^\beta\, \delta(\bar{u}_{k-1}, \overline{V}_{k-1})^\beta \\
&\leq \left(\frac{2\beta}{-(\ln \rho)}\right)^{\beta'} \frac{\left(\ln^+ \mathfrak{n}(A_k, \overline{V}_{k-1})\right)^{\beta+\beta'}}{k^{\beta'}} \\
&\quad + 2^{\beta/2} \rho^{-k/2} \delta(\bar{u}_{k-1}, \overline{V}_{k-1})
\end{aligned}$$

since $\delta(\bar{u}_{k-1}, \overline{V}_{k-1}) \leq 1$ and $\beta > 1$. Consequently, there exist positive constants $C(\beta, \beta', \rho)$ and C' such that

$$\begin{aligned}
&\mathbb{E}\left(\left(\ln^+ \frac{|A_k \bar{u}_{k-1}|}{|A_k \overline{V}_{k-1}|}\right)^\beta\right)^{1/\beta} \\
&\leq C(\beta, \beta', \rho) \left(\frac{\mathbb{E}\left(\ln^+ \mathfrak{n}(A_1, \overline{V}_0)^{\beta+\beta'}\right)}{k^{\beta'}} + \rho^{-k/2}\mathbb{E}(\delta(\bar{u}_{k-1}, \overline{V}_{k-1}))\right)^{1/\beta} \\
&\leq C(\beta, \beta', \rho) \left(\frac{\mathbb{E}\left(\ln^+ \mathfrak{n}(A_1, \overline{V}_0)^{\beta+\beta'}\right)}{k^{\beta'}} + C'\, \rho^{k/2}\right)^{1/\beta}.
\end{aligned}$$

Since $\beta' = b - \beta > \beta$, the sequence $\left(\mathbb{E}\left(\left(\ln^+ \frac{|A_k \bar{u}_{k-1}|}{|A_k \overline{V}_{k-1}|}\right)^\beta\right)^{1/\beta}\right)_{k\geq 1}$ is summable; hence, $\ln^+ C_1(\overline{V}_0)$ has a moment of order β.

□

Let us now apply this lemma.

• Case **I**. Under hypotheses **I**, the assumptions of Lemma 3.3 are satisfied for any $\bar{u} \in \mathbf{P}(\mathbb{R}^d)$. Indeed, by [5], proximality, strong irreductibility, and exponential moments hypotheses imply condition *(ii)* of Lemma 3.3 (i.e., uniform exponential contraction in mean) for every $\bar{u}$ and $\bar{v} \in \mathbf{P}(\mathbb{R}^d)$.

As a matter of fact, the existence of exponential moments allows a more direct proof, since it yields to the following inequality: There exist constants $C > 0$ and

$\rho \in [0, 1)$ s.t.

$$\mathbb{E}\left(\left(\ln^+ C_1(\bar{u}, \overline{V}_0)\right)^p\right) \le C\, \mathbb{E}(\mathfrak{n}(A_1, \overline{V}_0)^\epsilon) \sum_{n=1}^{+\infty} \rho^n < +\infty.$$

As mentioned in Sect. 3.1.3, some work in progress [19] seems able to relax this strong moment assumption, ensuring that condition *(ii)* of Lemma 3.3 holds without moment condition. If this is confirmed, Lemma 3.3 can be applied only under polynomial moments.

• Case **Nn** In this case, under hypotheses **Nn**, the support of ν is included in $\mathbb{X}$. By (12), we have that for all $u, v \in \mathbb{X}$,

$$\mathbb{E}\left(\delta(A_{k,1}\cdot \bar{u}, A_{k,1}\cdot \bar{v})\right) \le 2\, \mathbb{E}\left(\mathfrak{d}(A_{k,1}\cdot u, A_{k,1}\cdot v)\right) \le 2\, c\, \rho^k\, \mathfrak{d}(u, v) \le 2\, c\, \rho^k$$

Hence, the assumptions of Lemma 3.3 are satisfied for any $u \in \mathbb{X}$, taking $b > 6$ in order to get $\beta > 3$.

□

Proof of Theorem 1.1 Theorem 1.1 is a direct consequence of criterion 2.2. Indeed, in the four cases explored here:

• Assumption $\mathbf{A}_1(\alpha)$ is satisfied for $\alpha \in (0, 1/2)$, by Proposition 3.1;

• In the case **S**, assumption $\mathbf{B}(\beta)$ is satisfied for any $\beta > 0$ since the RNC random variable vanishes in this case.

In the case **Rk1**, this assumption holds for some $\beta > 0$ by hypothesis **Rk1**-*reverse moments*.

In the two remaining cases, this is a consequence of Proposition 3.2.

• The last assumption $\mathbf{B}(\gamma)$ does not depend on the nature of the matrices A_k. □

We end this section by discussing the irreducibility assumptions of the linear part.

Remark 3.4 : **optimality of the irreducibility assumptions**

Here, we give two examples of transient affine recursions on $\mathbb{R}^2$ in the critical case where the irreducibility assumptions on the linear part $\overline{\mu}$ fail.

Example 1(reducible linear part)

It is clear that Theorem 1.1 fails if we only assume that supp$(\overline{\mu})$ is reducible as one can see by taking $A_1 = I_2$ almost surely. Here is another less trivial example. Let μ be a probability measure on Aff$(\mathbb{R}^2)$ such that $\overline{\mu}$ is supported on the set of 2×2-diagonal matrices $A = \begin{bmatrix} \alpha(A) & 0 \\ 0 & \alpha^{-1}(A) \end{bmatrix}$ of determinant one. This is a reducible model as each of the coordinate axis is invariant under the support of $\overline{\mu}$. Assume that $\int \log \alpha(A) d\overline{\mu}(A) = 0$, $\overline{\mu}(\{A : \alpha(A) = 1\}) < 1$ and $\mu(\{g : g\mathcal{A} = \mathcal{A}\}) < 1$ for every proper non-empty affine subspace $\mathcal{A}$ of $\mathbb{R}^2$. Assume also that μ has a moment of order $2 + \delta$ for some $\delta > 0$. Then, the process $(X_n)_{n\in\mathbb{N}}$ is transient. Indeed, setting $X_n^0 := (x_n, y_n)$, we notice that the diagonal structure of $A_n \cdots A_1$ implies that both x_n and y_n are critical one-dimensional affine recursions with respective linear part $\alpha(A_n \cdots A_1)$ and $\alpha^{-1}(A_n \cdots A_1)$. Each of these affine recursions satisfies assumptions (H) of Babillot–Bougerol–Elie's local contraction's theorem [2, Theorem 3.1]. The aforementioned theorem yields that for any $K > 0$, almost surely, $\alpha(A_n \cdots A_1)\mathbf{1}_{|x_n|\leq K} \underset{n\to+\infty}{\to} 0$ and $\alpha^{-1}(A_n \cdots A_1)\mathbf{1}_{|y_n|\leq K} \underset{n\to+\infty}{\to} 0$. It follows that, almost surely, $|X_n^0| \underset{n\to+\infty}{\to} +\infty$; hence, the process $(X_n)_{n\in\mathbb{N}}$ is topologically transient.

Example 2 (Irreducible but not strongly irreducible linear part)

Consider a probability measure μ on $\mathrm{Aff}(\mathbb{R}^2)$ whose projection $\overline{\mu}$ on $\mathrm{GL}_2(\mathbb{R})$ satisfies $\mathrm{Supp}(\overline{\mu}) = \left\{ \begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ with $\lambda > 1$. It is immediate that the support of $\overline{\mu}$ is irreducible but not strongly (as it permutes the two coordinate axis). Assume also that there is no proper affine subspace of $\mathbb{R}^2$ fixed by all elements of the support of μ. The process $(X_{2n})_{n\in\mathbb{N}}$ is the affine recursion of $\mathbb{R}^2$ induced by the probability measure $\mu^{\star 2}$ on $\mathrm{Aff}(\mathbb{R}^2)$. Observe that $\overline{\mu}^{*2}$ is supported on $\{I_2, \mathrm{diag}(\lambda, \lambda^{-1}), \mathrm{diag}(\lambda^{-1}, \lambda)\}$ and gives equal mass to the two diagonal non identity matrices. It follows that for every $n \in \mathbb{N}$, $A_{2n} \cdots A_1 = \begin{bmatrix} \lambda^{S_n} & 0 \\ 0 & \lambda^{-S_n} \end{bmatrix}$, where $(S_n)_n$ is a centered random walk on $\mathbb{Z}$. Hence, from the strong law of large numbers, $\gamma_{\overline{\mu}} = 0$. It is easy to check that there is no proper affine subspace of $\mathbb{R}^2$ fixed by all elements of μ^{*2}.

By Example 1. above, we deduce that the process $(X_{2n})_{n\in\mathbb{N}}$ is transient. Since $X_{2n+1} = A_{2n+1}X_{2n} + B_{2n+1}$ for every $n \in \mathbb{N}$, and since the support of μ is compact, we deduce that $|X_n| \underset{n\to+\infty}{\to} +\infty$ $\mathbb{P}$-almost surely. Hence, the process $(X_n)_{n\in\mathbb{N}}$ is topologically transient.

4 On the Fluctuations of $(\ln |A_{n,1}v|)_{n\geq 1}$ in the rank 1 case

The process $(\ln |A_{n,1}v|)_{n\geq 1}$ fits into the general context of Markov walks, that is random walks whose increments are governed by an underlying Markov chain $(Z_n)_{n\geq 0}$. The fluctuations of Markov walks have been studied in [14], Theorem 2.2, under some general hypotheses on the chain $(Z_n)_{n\geq 0}$. There are several ways to apply this result; we present here one suitably adapted to the situation.

We denote here by $\mathbb{R}^d_\pm$ the quotient space $\mathbb{R}^d/\sim$ where $z \sim z'$ means $z = z'$ or $z = -z'$, for any z, z' in $\mathbb{R}^d$; the class of z equals $\pm z := \{z, -z\}$. The one-to-one correspondence

$$\begin{array}{ccc} \mathbb{R} \times \mathbf{P}(\mathbb{R}^d) & \to & \mathbb{R}^d_\pm \setminus \{0\} \\ (\lambda, \bar{v}) & \mapsto & \pm e^\lambda v \end{array}$$

(with $v \in \mathbb{S}^{d-1}$ a representative of $\bar{v}$) allows to identify $\mathbb{R} \times \mathbf{P}(\mathbb{R}^d)$ with $\mathbb{R}^d_\pm \setminus \{0\}$. In order to simplify the notation, the element $\pm z \in \mathbb{R}^d_\pm$ is also denoted z and is identified with $(\ln |z|, \bar{z})$ when $z \neq 0$.

Let $(Z_n)_{n\geq 0}$ be the Markov chain on $\mathbb{R}^d_\pm$ defined recursively by

$$Z_n := \pm A_n Z_{n-1}/|Z_{n-1}| \text{ if } Z_{n-1} \neq 0 \text{ and } Z_n = 0 \text{ if } Z_{n-1} = 0$$

and whose transition kernel P is defined by: For any Borel function $\varphi : \mathbb{R}^d_\pm \to \mathbb{R}^+$ and any $z \in \mathbb{R}^d_\pm$,

$$P\varphi(z) := \begin{cases} \int_{\mathcal{M}(d,\mathbb{R})} \varphi\left(\pm A\frac{z}{|z|}\right) \bar{\mu}(\mathrm{d}A) & \text{if } z \neq 0, \\ \varphi(0) & \text{if } z = 0. \end{cases}$$

Lets us emphasize that $\overline{Z_n} = \overline{V_n}$ for any $n \geq 0$, where $(\overline{V}_n)_{n\geq 0}$ is the Markov chain on the projective space defined in the previous sections. When $Z_n \neq 0$,

$$Z_n \cong (\ln|Z_n|, \overline{Z_n}) = (\ln|A_n\overline{Z}_{n-1}|, \overline{Z_n}).$$

For any probability measure ν on $\mathbf{P}(\mathbb{R}^d)$, let us denote $\bar{\mu} * \nu$ the probability measure on $\mathbb{R}^d_\pm$ defined by: For any Borel function $\varphi : \mathbb{R}^d_\pm \to \mathbb{R}^+$,

$$\bar{\mu} * \nu(\varphi) := \int \varphi(\pm Av)\bar{\mu}(dA)\nu(d\bar{v}), .$$

When ν is $\bar{\mu}$-invariant, the measure $\bar{\mu} * \nu$ is a stationary probability measure for $(Z_n)_{n\geq 0}$ with support in $\mathbb{R}^d_\pm \setminus 0$.

For $z \in \mathbb{R}^d_\pm \setminus \{0\}$, set $f(z) := \ln|z|$. When $Z_0 \neq 0$, the quantity $\ln|A_{n,1}Z_0|$, $n \geq 0$, may be decomposed as:

$$S_n(\overline{Z}_0) = \ln|A_{n,1}\overline{Z}_0| = \sum_{k=1}^{n} \ln\left|A_k\overline{Z}_{k-1}\right| = \sum_{k=1}^{n} f(Z_k).$$

The process $S_n(\overline{Z}_0)$ is a Markov walk on $\mathbb{R}$ driven by the Markov chain $(Z_n)_n$ on $\mathbb{R}^d_\pm$.

Under hypotheses **Rk1**, the distribution of $\overline{w}_0$ is the unique invariant measure ν for the process $(\overline{Z}_n)_{n\geq 0}$ on $\mathbf{P}(\mathbb{R}^d)$. We assume that Z_0 has distribution $\bar{\mu} * \nu$.

Proposition 4.1 *Under hypotheses* **Rk1**, *we have* $\mathbb{E}(\ell_\rho(\overline{Z}_0)^\alpha) < +\infty$ *for* $\alpha < 1/2$.

Proof Let $\varepsilon > 0$ given by hypothesis **Rk1**-*Moments* and fix $\delta > 1$ and $p > 2$ such that $\delta p < 2 + \varepsilon$. For all $z \in \mathbb{R}^d_\pm \setminus \{0\}$, set

$$N(z) := \sup_{n\geq 0} \left(P^n|f|^{\delta p}(z)\right)^{1/p} = \sup_{n\geq 0} \mathbb{E}_z\left(|\ln|Z_n||^{\delta p}\right)^{1/p} \in [0, \infty].$$

Observe that, under hypothesis **Rk1**, for any nonnegative Borel function $\varphi : \mathbb{R}^d_\pm \to \mathbb{R}_+$, the sequence $(P^n\varphi(z))_{n\geq 0}$ converges "very quickly" toward $\overline{\mu} * \nu(\varphi)$. Indeed, $\overline{Z}_n = \overline{w}_n$ for any $n \geq 0$; thus, $\overline{Z}_n$ is independent from the starting point in $\mathcal{X}_0 := \left\{z \in \mathbb{R}^d_\pm \mid \mathbb{P}(|A_1 z| = 0) = 0\right\}$. This yields, for any $z \in \mathcal{X}_0$ and $n \geq 2$,

$$P^n\varphi(z) = \mathbb{E}_z(\varphi(\pm A_n Z_{n-1}/|Z_{n-1}|)) = \mathbb{E}(\varphi(\pm A_1 Z_0/|Z_0|)) = \overline{\mu} * \nu(\varphi). \qquad (15)$$

We claim that this ensures that for all $n \in \mathbb{N}$ and $z \in \mathbb{R}^d_\pm$,

$$P^n N(z) \leq 3N(z) \text{ and } (P^n N^p(z))^{1/p} \leq 3N(z). \tag{16}$$

Indeed, $N(z) = +\infty$ when $z \notin \mathcal{X}_0$ and the inequalities are trivially satisfied; when $z \in \mathcal{X}_0$, the identity (15) implies that $N(z) \leq |f|^\delta(z) + \left(P|f|^{\delta p}(z)\right)^{1/p} + (\overline{\mu} * \nu(|f|^{\delta p}))^{1/p}$. Then, by Jensen inequality, since $1/p < 1$,

$$\begin{aligned} P^n N(z) &\leq P^n|f|^\delta(z) + P^n\left(P|f|^{\delta p}\right)^{1/p}(z) + (\overline{\mu} * \nu(|f|^{\delta p}))^{1/p} \\ &\leq \left(P^n|f|^{\delta p}(z)\right)^{1/p} + \left(P^{n+1}|f|^{\delta p}(z)\right)^{1/p} + (\overline{\mu} * \nu((|f|^{\delta p}))^{1/p} \leq 3N(z). \end{aligned}$$

Similarly, by the L_p-triangle inequality,

$$\begin{aligned} \left(P^n N^p(z)\right)^{1/p} &\leq \left(P^n|f|^{\delta p}\right)^{1/p}(z) + \left(P^n\left(P|f|^{\delta p}\right)^{\frac{1}{p}p}(z)\right)^{1/p} + (\overline{\mu} * \nu(|f|^{\delta p}))^{1/p} \\ &\leq \left(P^n|f|^{\delta p}(z)\right)^{1/p} + \left(P^{n+1}|f|^{\delta p}(z)\right)^{1/p} + (\overline{\mu} * \nu(|f|^{\delta p}))^{1/p} \\ &\leq 3N(z). \end{aligned}$$

Consequently, it is of interest to consider the subset $\mathcal{X}_N$ of $\mathbb{R}^d_\pm$ defined by

$$\mathcal{X}_N := \left\{z \in \mathbb{R}^d_\pm \mid N(z) < +\infty\right\}.$$

Under hypothesis **Rk1**, the set $\mathcal{X}_N$ has full $\bar{\mu} * \nu$ measure since

$$\mathbb{E}_{\bar{\mu}*\nu}(N(Z_0)) \leq 3\mathbb{E}(|\log|A_1\overline{V}_0||^{\delta p})^{1/p} < +\infty$$

in this case. Thus, we can restrict the Markov operator P to spaces of functions restricted to $\mathcal{X}_N$. Let us consider the space $\mathcal{B}$ of Borel functions $\mathbb{C}$ defined by:

$$\mathcal{B} = \left\{\varphi : \mathcal{X}_N \to \mathbb{C} \mid \underbrace{\sup_{z \in \mathcal{X}_N} \frac{|\varphi(z)|}{1 + N(z)}}_{=:|\varphi|_\mathcal{B}} < +\infty\right\} \tag{17}$$

The space $(\mathcal{B}, |\cdot|_\mathcal{B})$ is a Banach space over $\mathbb{C}$.

We now restate hypotheses M1-M5 introduced in [14] and check that under hypotheses **Rk1**, they are satisfied on the Banach space $\mathcal{B}$ by the kernel P and the function $f : \mathbb{R}^d_\pm \setminus \{0\} \to \mathbb{R}, z \mapsto f(z) := \ln|z|$. This allows us to apply Theorem 2.2 in [14] and conclude the proof of Proposition 3.1.

Hypothesis M1

M1.1. The constant function 1 *belongs to* $\mathcal{B}$.

This is a trivial consequence of the fact that $1 \le 1 + N$.

M1.2. For any $z \in \mathcal{X}_N$, the Dirac mass δ_z at z belongs to the topological dual space $\mathcal{B}'$ of $\mathcal{B}$, endowed with the norm $|\Phi|_{\mathcal{B}'} = \sup_{\substack{\varphi\in\mathcal{B}\\ \varphi\neq 0}} \frac{|\Phi(\varphi)|}{|\varphi|_{\mathcal{B}}}$.

Indeed, by definition of the norm on $\mathcal{B}$, for any $z \in \mathcal{X}_N$,

$$|\delta_z|_{\mathcal{B}'} = \sup_{\substack{\varphi\in\mathcal{B}\\ \varphi\neq 0}} \frac{|\varphi(z)|}{|\varphi|_{\mathcal{B}}} \le 1 + N(z) < +\infty.$$

M1.3. The Banach space $\mathcal{B}$ is included in $\mathbb{L}^1(P(z,\cdot))$, for any $z \in \mathcal{X}_N$.

For any $\varphi \in \mathcal{B}$ and $z \in \mathcal{X}_N$,

$$\begin{aligned}\int |\varphi(z_1)|P((z, dz_1) &\le |\varphi|_{\mathcal{B}} \int (1 + N(A_1 z/|z|))\bar{\mu}(dA_1)\\ &= |\varphi|_{\mathcal{B}}(1 + PN(z)) \le |\varphi|_{\mathcal{B}}(1 + 3N(z)) < +\infty.\end{aligned}$$

M1.4. There exists $\epsilon_0 > 0$ such that, for any $\varphi \in \mathcal{B}$, the function $e^{itf}\varphi$ belongs to $\mathcal{B}$ for every $t \in]-\epsilon_0, \epsilon_0[$.

This is a direct consequence of the equality $|e^{itf}\varphi| = |\varphi|$ valid for any $t \in \mathbb{R}$. Property *M1.4.* thus holds for any $\epsilon_0 > 0$.

Hypothesis M2

M2.1. The map $\varphi \mapsto P\varphi$ is a bounded operator on $\mathcal{B}$.

Indeed, for any $\varphi \in \mathcal{B}$ and $z \in \mathcal{X}_N$,

$$|P\varphi(z)| \le \mathbb{E}_z(|\varphi(Z_1)|) \le |\phi|_{\mathcal{B}}\mathbb{E}_z(1 + N(Z_1)) \le 3|\varphi|_{\mathcal{B}}(1 + N(z))$$

, which implies $P\varphi \in \mathcal{B}$ and $|P|_{\mathcal{B}} \le 3$.

M2.2. There exist constants $C_P > 0$ and $\kappa \in [0, 1)$ such that $P = \Pi + Q$, where Π is a one-dimensional projector and Q a bounded operator on $\mathcal{B}$ with spectral radius < 1 and satisfying $\Pi Q = Q\Pi = 0$.

For any $\varphi \in \mathcal{B}$, set $\Pi(\varphi) := \overline{\mu} * \nu(\varphi)$. The equality $\Pi P = P\Pi = \Pi$ holds since $\overline{\mu} * \nu$ is P-invariant.

Thus, with $Q := P - \Pi$, it holds $\Pi Q = Q\Pi = 0$ and $P^n = \Pi + Q^n$. Furthermore, equality (15) may be rewritten as $P^n = \Pi$; thus, $Q^n = 0$, for $n \ge 2$.

Hypothesis M3

For any $t \in]-\epsilon_0, \epsilon_0[$, any $\varphi \in \mathcal{B}$ and $z \in \mathcal{X}_N$, let $P_t\varphi(z) := \mathbb{E}_z(e^{itf(Z_1)}\varphi(Z_1))$. Then, the map $\varphi \mapsto P_t\varphi$ is a bounded operator on $\mathcal{B}$ such that

$$\sup_{\substack{n\ge 1\\ |t|\le\epsilon_0}} \|P_t^n\|_{\mathcal{B}\to\mathcal{B}} < +\infty.$$

This hypothesis is satisfied in a obvious way here since $|e^{itf}\varphi|_{\mathcal{B}} = |\varphi|_{\mathcal{B}}$. Hence, $\|P_t^n\|_{\mathcal{B}\to\mathcal{B}} = \|P^n\|_{\mathcal{B}\to\mathcal{B}} \le 3$ for any $n \ge 0$ and $t \in \mathbb{R}$.

Hypothesis M4 (Local integrability). *There exist bounded nonnegative functions $N_0, N_1, N_2, \ldots$ in $\mathcal{B}$ and constants $c, \beta, \gamma > 0$ and $p_0 > 2$ such that, for any $z \in \mathcal{X}_N$ and $k \geq 1$,*

1. $\max \left\{ |f(z)|^{1+\gamma}, \|\delta_z\|_{\mathcal{B}'}, \sup_{n\geq 1} \left(\mathbb{E}_z^{1/p_0}(N_0(Z_n)^{p_0}) \right) \right\} \leq c(1 + N_0(z))$;

2. $N_0(z)\mathbf{1}_{\{N_0(z)\geq k\}} \leq N_k(z)$;

3. $|\Pi(N_k)| \leq \dfrac{c}{k^{1+\beta}}$.

Recall that $\delta > 1$ and $p > 2$ are fixed and satisfy $\delta p < 2 + \varepsilon$ where ε is given by hypothesis **Rk1**-*moments*. We set $N_0 := N$, $N_k := N(z)\mathbf{1}_{\{N(z)\geq k\}}, k \geq 1$, and $p_0 = p, c = 3, \beta = p - 2, \gamma = \delta - 1$. With these functions and constants, hypothesis *M4* is satisfied since:

M4.1) $|f(z)|^{1+\gamma} = (P^0|f|^{\delta p})^{1/p} \leq N(z) \leq c(1 + N(z))$, $|\delta_z|_{\mathcal{B}'} \leq 1 + N(z)$ by *M1.2* and $\mathbb{E}_z^{1/p}(N(Z_n)^p) = (P^n N^p(z))^{1/p} \leq 3N(z)$ for any $n \geq 1$, by (16);

M4.2) This condition is satisfied since $N_k := N(z)\mathbf{1}_{\{N(z)\geq k\}}, k \geq 1$;

M4.3) For any $k \geq 1$,

$$
\begin{aligned}
\Pi(N_k) &= \mathbb{E}_{\mu*\nu}(N(Z_0)\mathbf{1}_{\{N(Z_0)\geq k\}}) \\
&\leq \mathbb{E}_{\mu*\nu}\left(N(Z_0)\frac{N(Z_0)^{p-1}}{k^{p-1}} \right) \\
&\leq \frac{\mathbb{E}_{\mu*\nu}(N(Z_0)^p)}{k^{p-1}} \leq 3\frac{\mathbb{E}_{\mu*\nu}(|\ln|Z_0|^{\delta p})}{k^{1+(p-2)}} \quad \text{with} \quad \mathbb{E}_{\mu*\nu}(|\ln|Z_0|^{\delta p}) < +\infty.
\end{aligned}
$$

Hypothesis M5. *We suppose that* $\gamma_{\bar{\mu}} = \lim_{n\to+\infty} \dfrac{1}{n}\mathbb{E}_z(f(Z_n)) = 0$ *and*

$$
\sigma^2 := \lim_{n\to+\infty} \frac{1}{n} V_z\left(\sum_{k=1}^n f(Z_k) \right) > 0.
$$

Equality (10) and hypothesis **Rk1**-*centering* yield $\gamma_{\bar{\mu}} = \mathbb{E}(\ln a_1) + \mathbb{E}(\ln|\langle \tilde{w}_2, w_1\rangle|) = 0$.

Let us now compute the asymptotic variance of S_n. Since $f(Z_k)$ and $f(Z_\ell)$ are independent when $\ell \geq k + 2$, it holds

$$
\begin{aligned}
\frac{1}{n}V_z\left(\sum_{k=1}^n f(Z_k) \right) &= \frac{1}{n}\sum_{k=1}^n V_z(f(Z_k)) + \frac{2}{n}\sum_{k=1}^{n-1} \mathrm{cov}_z(f(Z_k), f(Z_{k+1})) \\
&\overset{n\to+\infty}{\longrightarrow} \quad V(f(Z_2)) + 2\,\mathrm{cov}(f(Z_2), f(Z_3)).
\end{aligned}
$$

Notice that the asymptotic variance does not depend on z since the random variables $(f(Z_k), f(Z_{k+1}))$ are independent of $Z_0 = z$ and have the same distribution as $(f(Z_2), f(Z_3))$ when $k \geq 2$; furthermore, it is nonzero by **Rk1**-*non-degeneracy* assumption.

Finally, under hypotheses **Rk1**, the chain $(Z_n)_{n\geq 0}$ and the function f satisfy conditions M1-M5 of [14]; hence, Theorem 2.4.2 in [14] is valid in our context, which readily implies that there exists a constant $c_\rho > 0$ such that, for any $z \in \mathcal{X}_N$ with modulus 1,

$$\mathbb{P}(\ell_1^{(\rho)}(\bar{z}) > n) \leq \frac{c_\rho}{\sqrt{n}}(1 + N(z)).$$

In particular, if Z_0 has distribution $\bar{\mu} * \nu$, modulus 1 and $\overline{Z}_0 = \overline{V}_0$ has distribution ν, we obtain

$$\mathbb{P}\left(\ell_1^{(\rho)}(\overline{V}_0) > n\right) \leq \frac{c_\rho}{\sqrt{n}}\bar{\mu} * \nu(1 + N).$$

This imply that condition $\mathbf{A}_1(\alpha)$ holds for any $\alpha < \frac{1}{2}$.

□

A Appendix: Generalization to Non-Proximal Invertible Matrices

When the matrices A_n are invertible, it is possible to generalize the previous results in the case where hypothesis **I**-*proximality* fails. We only exclude the degenerate case where $T_{\bar{\mu}}$ is a sub-semigroup of similarities, corresponding to a scalar cocycle already covered by case **S**. By [8, Theorem 4.11], the remaining cases satisfy a non-degenerate central limit theorem.

Theorem A.1 *Assume that the matrices $A_n, n \geq 1$, are invertible and satisfy hypotheses*

I-*(exponential) moments and reverse moments,* **I**-*strong irreducibility , and* **I**-*centering.*

Assume moreover that $T_{\overline{\mu}}$ is not conjugate to a sub-semigroup of $\mathbb{R}^* O_d(\mathbb{R})$. Then, if there exists $\gamma > 2$ such that $\mathbb{E}\left((\ln^+ |B_1|)^\gamma\right) < +\infty$, the Markov chain $(X_n^x)_{n\geq 0}$ is recurrent.

Proof Let T_μ be the sub-semigroup of $Gl(d, \mathbb{R})$ generated by the support of μ. Let $r = r(T_\mu) \geq 1$ be the *proximal dimension* of T_μ in $\mathbb{R}^d$, that is the least rank of a nonzero element M of the closure

$$\overline{\mathbb{R}T_\mu} := \{M \in \mathcal{M}(d, \mathbb{R}) \mid M = \lim_{n\to+\infty} \lambda_n M_n \ with \ \lambda_n \in \mathbb{R}, M_n \in T_\mu\}.$$

Let us now consider the action of $\mathrm{Gl}(d, \mathbb{R})$ on the r-exterior product $\wedge^r \mathbb{R}^d$ and denote by $\| \wedge^r A\|$ the related operator norm of $A \in \mathrm{Gl}(d, \mathbb{R})$. By [8, Lemma 4.13], there exists a T_μ-invariant subspace $\mathcal{W}_r$ of $\wedge^r \mathbb{R}^d$ such that the action of T_μ on $\mathcal{W}_r$ is proximal (i.e., T_μ contains a proximal element) and strongly irreducible. Moreover, there exists $C \geq 1$ such that, for any $A \in T_\mu$, one has

$$C^{-1}\|A\|^r \leq \| \wedge^r A\|_{\mathcal{W}_r} \leq \|A\|^r \tag{18}$$

(where $\| \wedge^r A\|_{\mathcal{W}_r}$ denotes the norm of the restriction of $\wedge^r A$ to $\mathcal{W}_r$).

We claim that the action of the random matrices A_n on $\mathcal{W}_r$ by the standard representation $\mathrm{Gl}(d, \mathbb{R}) \to \mathrm{Gl}(d, \wedge^r \mathbb{R})$ satisfies hypotheses **I**. Indeed, by (18),

(i) Assumptions **I**-*(exponential) moments and reverse moments* on $\mathbb{R}^d$, with exponent $\varepsilon > 0$, imply:

$$\mathbb{E}\left(\| \wedge^r A_1\|_{\mathcal{W}_r}^{\varepsilon/r}\right) \leq \mathbb{E}\left(\|A_1\|^{r\varepsilon/r}\right) < +\infty \text{ and } \mathbb{E}\left(\| \wedge^r A_1^{-1}\|_{\mathcal{W}_r}^{\varepsilon/r}\right) \leq \mathbb{E}\left(\|A_1^{-1}\|^{r\varepsilon/r}\right) < +\infty.$$

In other words, the same hypotheses hold for the standard representation on $\mathcal{W}_r$, with exponent ε/r.

(ii) The Lyapunov exponent $\gamma_{\bar{\mu}}^{\mathcal{W}_r}$ of $\wedge^r A_n$ on $\mathcal{W}_r$ equals $r\gamma_{\bar{\mu}}$, since by (18)

$$r\frac{\mathbb{E}(\ln \|A_{n,1}\|)}{n} - \frac{\ln C}{n} \leq \frac{\mathbb{E}(\ln \| \wedge^r A_{n,1}\|_{\mathcal{W}_r})}{n} \leq r\frac{\mathbb{E}(\ln \|A_{n,1}\|)}{n}.$$

In particular, the centering hypothesis holds for the action of the A_n on $\mathbb{R}^d$ if and only if it holds for the action of $\wedge^r A_n$ on $\mathcal{W}_r$.

Consequently, assumptions **I** hold on $\mathcal{W}_r$ and there exits a unique stationary probability measure $\nu_{\mathcal{W}_r}$ on $\mathbf{P}(\mathcal{W}_r)$. Let $\overline{W}_0$ be a random variable with distribution $\nu_{\mathcal{W}_r}$.

As in sect. 2, we consider the ladder times $\ell_1^{(\rho),\mathcal{W}_r}$, $0 < \rho < 1$ and the reverse norm coefficients $C_k^{\mathcal{W}_r}$ associated with the standard representation on $\mathcal{W}_r$ as follows: For any $\overline{w} \in \mathbb{P}_1(\mathcal{W}_r)$,

$$\ell_1^{(\rho),\mathcal{W}_r}(\overline{w}) := \inf\{n \geq 1 \mid |A_n \cdots A_1\overline{w}|_{\mathcal{W}_r} \leq \rho\} \text{ and } C_k^{\mathcal{W}_r}(\overline{w}) := \sup_{n\geq k} \frac{\| \wedge^r A_n \cdots A_k\|_{\mathcal{W}_r}}{|A_n \cdots A_k\overline{w}|_{\mathcal{W}_r}}.$$

These ladder times are almost surely finite since, by Theorem 4.11 of [8] together with (18), the log-norm cocycle on $\mathcal{W}_r$ is centered with positive variance. In order to control the norms $\|A_n \dots A_k\|$ from the random variables $C_k^{\mathcal{W}_r}(\overline{w})$, we introduce a "generalized reverse norm control coefficient" defined as follows:

$$C_k^{\mathbb{R}^d,\mathcal{W}_r}(\overline{w}) := \sup_{n\geq k} \frac{\|A_n \cdots A_k\|}{|A_n \cdots A_k\overline{w}|_{\mathcal{W}_r}^{1/r}}.$$

Observe that, by (18),

$$C_k^{\mathbb{R}^d,\mathcal{W}_r}(\overline{w}) = \sup_{n\geq k} \frac{\|A_{n,k}\|}{|A_{n,k}\overline{w}|_{\mathcal{W}_r}^{1/r}} \leq C^{1/r} \sup_{n\geq k} \frac{\| \wedge^r A_{n,k}\|^{1/r}}{|A_{n,k}\overline{w}|_{\mathcal{W}_r}^{1/r}} = C^{1/r}\left(C_k^{\mathcal{W}_r}(\overline{w})\right)^{1/r}.$$

Proposition 3.1 and 3.2 enable to control the moments of the random variables $\ell_1^{(\rho),\mathcal{W}_r}(\overline{W}_0)$ and $C_k^{\mathcal{W}_r}(\overline{W}_0)$ and ensure that, for $\alpha < 1/2$ and $\beta > 3$,

$$\mathbb{E}\left(\left(\ell_1^{(\rho),\mathcal{W}_r}(\overline{W}_0)\right)^{\alpha}\right) < +\infty\, \mathbb{E}\left((\ln^+ C_1^{\mathcal{W}_r}(\overline{W}_0))^{\beta}\right) < +\infty \text{ and } \mathbb{E}\left(\left(\ln^+ C_1^{\mathbb{R}^d,\mathcal{W}_r}(\overline{W}_0)\right)^{\beta}\right) < +\infty.$$

We conclude the proof using the following criterion, which slightly generalizes criterion 2.2. □

Criterion A.2 *Let $(\mathcal{W}, |\cdot|_{\mathcal{W}})$ be a $\mathbb{R}$-vector space and consider a representation π of $\mathrm{Gl}(d,\mathbb{R})$ on $\mathrm{Gl}(\mathcal{W})$. For simplicity, we write $Aw = \pi(A)w$ for any $w \in \mathcal{W}$.*

Let $\nu_{\mathcal{W}}$ be a $\overline{\mu}$-invariant probability measure on the protective space $\mathbf{P}(\mathcal{W})$, of $\mathcal{W}$ and let $\overline{W}_0$ be a random variable with distribution $\nu_{\mathcal{W}}$.

For $\overline{w} \in \mathbb{P}(\mathcal{W})$, $0 < \rho < 1$ and $r > 0$, let

$$\ell_1^{(\rho),\mathcal{W}}(\overline{w}) := \inf\{n \geq 1 \mid |A_n \cdots A_1 \overline{w}|_{\mathcal{W}} \leq \rho\} \text{ and } C_k^{\mathbb{R}^d,\mathcal{W}}(\overline{w}) := \sup_{n \geq k} \frac{\|A_n \cdots A_k\|}{|A_n \cdots A_k \overline{w}|_{\mathcal{W}}^{1/r}} \tag{19}$$

Assume that there exist constants $\alpha > 0$, $\beta > 1 + \frac{1}{\alpha}$, and $\gamma \geq \max\{\frac{1}{\alpha}, 1\}$ such that the following hypotheses hold:

$\mathbf{A}_1^{\mathcal{W}}(\alpha)$- $\mathbb{E}\left(\left(\ell_1^{(\rho),\mathcal{W}}(\overline{W}_0)\right)^\alpha\right) < +\infty$ *for any $\rho \in (0,1)$.*

$\mathbf{A}_2^{\mathcal{W}}(\beta)$- $\mathbb{E}\left((\ln^+ C_1^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_0))^\beta\right) < +\infty$ *for some $r > 0$.*

$\mathbf{B}(\gamma)$- $\mathbb{E}\left((\ln^+ |B_1|)^\gamma\right) < +\infty$.

Then, for any $x \in \mathbb{R}^d$, the chain $(X_n^x)_{n\geq 0}$ is recurrent.

Proof of criterion A.2 The proof follows the same strategy as the one of criterion 2.2. We present the main steps.

Let $(\overline{W}^{(k)})_{k\geq 0}$ be a sequence of independent and identically distributed random variables with values in $\mathbf{P}(\mathcal{W})$ and distribution $\nu_{\mathcal{W}}$. Assume that this sequence is independent of $(A_n, B_n)_{n\geq 1}$. As in the proof of criterion 2.2, we can suppose $\overline{W}^{(0)} = \overline{W}_0$ without loss of generality. Fix $0 < \rho < 1$ such that $\ln \rho < -r\mathbb{E}\left(\ln C_1^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_0)\right)$. We set $\ell_0 = 0$ and, for any $k \geq 1$,

$$\ell_k := \inf\{n \geq \ell_{k-1} + 1 : |A_n \cdots A_{\ell_{k-1}+1} \overline{W}^{(k-1)}|_{\mathcal{W}} \leq \rho\}.$$

Then, the sub-process $(X_{\ell_n}^x)_{n\geq 0}$ is a multidimensional affine recursion corresponding to the i.i.d. affine random transformations $\widetilde{g}_k$ with the same law as $\widetilde{g}_1$ given by:

$$\widetilde{g}_1 = (\widetilde{A}_1, \widetilde{B}_1) := (A_{\ell_1} \ldots A_1, \sum_{i=1}^{\ell_1} A_{\ell_1} \cdots A_{i+1} B_i).$$

Let us now check that the sub-process $(X_{\ell_n})_{n\geq 0}$ satisfies the hypotheses of Fact 2.1; this will prove that $(X_{\ell_n})_{n\geq 0}$, hence $(X_n)_{n\geq 0}$, is recurrent.

On the one hand, the Lyapunov exponent of the distribution $\tilde{\mu}$ of $\widetilde{g}_1$ is negative:

$$\begin{aligned}
\gamma_{\tilde{\mu}} &\leq \mathbb{E}(\log \|A_{\ell_1} \cdots A_1\|) \leq \mathbb{E}\left(\log(C_1^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_0) |A_{\ell_1} \cdots A_1 \overline{W}_0|_{\mathcal{W}}^{1/r})\right) \\
&\leq \frac{\ln \rho}{r} + \mathbb{E}\left(\ln C_1^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_0)\right) < 0.
\end{aligned}$$

On the other hand, in order to check that $\widetilde{B_1} = B_{\ell_1,1}$ is log-integrable, we observe that

$$|B_{\ell_1,1}| \leq \sum_{i=1}^{\ell_1} \|A_{\ell_1} \cdots A_{i+1}\| |B_i| \leq \sum_{i=1}^{\ell_1} C_{i+1}^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_i)|B_i| \quad \mathbb{P}\text{-a.s.}$$

where $\overline{W}_i := A_{i,1} \cdot \overline{W}_0$. Indeed,

$$\begin{aligned}\|A_{\ell_1} \cdots A_{i+1}\| &\leq C_{i+1}^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_i)|A_{\ell_1} \cdots A_{i+1}\overline{W}_i|_{\mathcal{W}}^{1/r} \\ &= C_{i+1}^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_i)\frac{|A_{\ell_1} \cdots A_1\overline{W}_0|_{\mathcal{W}}^{1/r}}{|A_i \cdots A_1\overline{W}_0|_{\mathcal{W}}^{1/r}} \leq C_{i+1}^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_i)\frac{\rho^{1/r}}{\rho^{1/r}} = C_{i+1}^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_i).\end{aligned}$$

Consequently,

$$\ln^+ |B_{\ell_1,1}| \leq \ln^+ \ell_1 + \max_{1\leq i\leq \ell_1} \ln^+ C_{i+1}^{\mathbb{R}^d,\mathcal{W}}(\overline{W}_i) + \max_{1\leq i\leq \ell_1} \ln^+ |B_i| \quad \mathbb{P}\text{-a.s.}$$

and we can conclude as in criterion 2.2 by using Lemma 2.3. $\square$

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Declarations

Conflict of interest The authors declare no conflict of interest.

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