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Conservation laws, propagation and control theory

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Preface

This thesis provides an overview of the papers [2, 36, 37, 38, 39, 40, 46, 64, 81, 113, 114, 115, 117, 118, 116].

Chapter 1 is the introduction of the thesis. It provides a quick introduction to the following topics.

1. Control theory, and more specifically the exact controllability problem, the asymptotic stabilization problem and the optimal control problem,
2. Hyperbolic conservation laws, or more specifically their physical origins, a description of their singular properties and finally the concept of entropy solutions,
3. The specific kind of problems considered in the Thesis which are at the interconnection between control theory and entropy solutions of hyperbolic conservation laws.

Chapter 2 focuses on the partial differential equations

$$\partial_t u + \partial_x \left(f(x, u) \right) = 0, \tag{1}$$

and

$$\partial_t U + f(x, \partial_x U) = 0. \tag{2}$$

It provides wellposedness results in large time with a more general hypothesis on f than are provided by the seminal results of Kruzkov in [95] and Crandall-Lions in [49]; though of course those papers are devoted to a much more general class of equations. In a second step, the formal correspondence between those equations — formally $u = \partial_x U$ — is provided, in all rigour, at the level of the semigroups.

All following chapters are devoted to control problems in the context of hyperbolic conservation laws. They are organized by the methodology used to tackle those problems. And, apart from the vanishing viscosity method, we expect that all the classical methods found in the literature are described.

In Chapter 3, we show how one can use the connection to Hamilton-Jacobi equation, and more importantly to the underlying optimal control problem for ordinary differential equations, to solve inverse design problems on one dimensional scalar hyperbolic conservation laws.

In Chapter 4, we provide asymptotic stabilization results using a boundary feedback control. We use Dafermos' theory of generalized characteristics from [51].

In Chapter 5, we start by describing the wave front tracking algorithm. It was initially introduced by Dafermos in [53] for scalar equations, then extended by Di Perna in [62] to 2×2 systems of conservation laws, and finally taken to the extreme by Bressan — see for instance [27, 31] — to show the uniqueness of the standard Riemann semigroup for $n \times n$ systems of conservation laws. In the scalar case, it was extensively used by Holden and its coauthors in particular for numerical purposes, starting with the article [85] from 1988. In the second part of the chapter, we use the front tracking algorithm to solve an exact controllability problem on scalar equations and an asymptotic stabilization problem for 2×2 systems of hyperbolic equations.

Finally, in Chapter 6, we explain how one can use Kato type inequalities to obtain both exact controllability results, stabilization results and robustness results in contexts that were — on the face of it — inaccessible to the previous techniques.

The thesis concludes by an expanded abstract of the Thesis in french for administrative purposes.

A secondary goal of the Thesis is to provide an overview of most methods used on entropy solutions of hyperbolic conservation laws for control purposes, with the notable exception of the method of vanishing viscosity. Chapter 2 does use viscous approximations to obtain existence results for equations (2) and (1) and then studies the associated singular problem using the compensated compactness method. Some of the open questions detailed in Chapter 6 are also connected to this approach. However, one should rather consult for instance [75, 91, 106] for control results in this context. An important point to make is that using the vanishing viscosity method for control purposes in the case of systems — following [25] — is a completely open question to which [123] might provide some interesting perspectives.

All figures in this Thesis were created thanks to Python [93], Numpy [83], Matplotlib [89] and Jupyter [94].

Chapitre 1

General Introduction

Most of the problems that will be described in this thesis are related to control theory in the framework of entropy solutions to conservation laws. Let us therefore begin by describing some general problems of control theory in a first section. We will then proceed by introducing hyperbolic conservation laws and their entropy solutions in a second section. Finally, we will consider the specifics of tackling control problems on such partial differential equations.

1.1 Control Theory

At the turn of the 17th century, Galileo Galilei described the necessity of introducing mathematics in the natural sciences with his famous quote “Philosophy [nature] is written in that great book which ever is before our eyes — I mean the universe — but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written”. One could roughly consider the efforts of Fermat and Descartes in “La géométrie” toward using mathematics to *describe* general curves through algebraic means as a first step in this program. The second step, then, would be the work of Newton when he used mathematical methods to *predict* physical phenomena using mathematics.

It is interesting to see that a third natural step could be considered to be the introduction of mathematics to understand the way one could *act* to influence nature in a particular way. The seminal work in this direction comes surprisingly later with Maxwell’s article “On Governors” dated 1868 [108]. There, he studied the stability of the Watt’s regulator for the steam engine, and toward that aim, he introduced the linearization criterion. Following this a huge amount of literature was created which we will collectively dub control theory.

1.1.1 Formal description of some problems

Let us introduce a few problems from this theory in a rather abstract setting. To this end, let us consider an abstract dynamical system :

$$\begin{cases} \dot{X}(t) = F(X(t), U(t)), \\ X(0) = X_0, \end{cases} \tag{1.1}$$

where $X \in \mathcal{X}$ is the state of the system while $U \in \mathcal{U}$ is the control. Note that a key aspect of the mathematical analysis consists in selecting the good spaces $\mathcal{X}$ and $\mathcal{U}$.

In concrete situations, the system (1.1) would be an ordinary differential equation or a partial differential equation. The idea is that we model a physical system — through X — on which we can only act by means of U. The question is of course to find an efficient way to select the control so that the system behaves in the desired manner.

To this end we will introduce three classical problems (there are many others).

1. The most straightforward one is that of *exact controllability*. It consists in driving the system, from an initial state, to a given target state, in a given time. More precisely : given two states $X_0 \in \mathcal{X}$, $X_1 \in \mathcal{X}$ and a time $T > 0$, is there a function $t \in [0, T] \mapsto U(t) \in \mathcal{U}$ such that the solution of (1.1) satisfies $X(T) = X_1$?
2. An alternative goal could be to drive the system while keeping a certain cost minimal, this is the *optimal control problem*. To provide a more precise description, let us introduce a time $T > 0$, a running cost function $R : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and a penalization function $P : \mathcal{X} \to \mathbb{R}$. To a particular control strategy $U : [0, T] \to \mathcal{U}$ one can associate its cost C through the formula

$$C(U) := \int_0^T R(X(t), U(t))dt + P(X(T)), \tag{1.2}$$

   where X is the solution associated to U through (1.1).

   The goal is then to determine a minimizer U_m of the functional C among all admissible control strategies.
3. Let us finally describe the problem of *asymptotic stabilization using a feedback control*. To that end, suppose that we have a couple $(X_e, U_e) \in \mathcal{X} \times \mathcal{U}$ such that $F(X_e, U_e) = 0$. What we mean is that we have an equilibrium point of the system. Those are natural solutions which are structurally important for (1.1). However, it should be clear that we can observe — when the evolution is free i.e. without active control — only those equilibrium state that are dynamically stable. For instance, when one think of the pendulum, it should be clear that the bottom position is stable while the top one is not, which is why we only observe the first one in the wild. To compensate, we could search for a control $t \in [0, T] \mapsto U(t) \in \mathcal{U}$ such that the solution to (1.1) satisfies $X(T) = X_e$ and $U(T) = U_e$. However, this strategy is actually horribly brittle. It is in fact very sensitive to perturbations and approximations : be it on the model, on the actuation by U or by our knowledge of the initial state X_0. We therefore prefer to use feedback loop controls. We need to provide a function $\mathbb{U} : \mathcal{X} \to \mathcal{U}$ satisfying $\mathbb{U}(X_e) = U_e$ and such that, for the so called "closed loop system" (whose evolution is now autonomous)

$$\begin{cases} \dot{X}(t) = F(X(t), \mathbb{U}(X(t))) \\ X(0) = X_0, \end{cases} \tag{1.3}$$

   the equilibrium point X_e is now asymptotically stable.

   In more precise term, this means that both following properties are satisfied.
   - For all $\epsilon > 0$, there exists $\nu > 0$ such that if X_0 is a state satisfying $\|X_0 - X_e\| \leq \nu$ and if X is a maximal solution of (1.3), then it is global in time and satisfies

$$\forall t \geq 0, \qquad \|X(t) - X_e\| \leq \epsilon.$$

   - For any initial state X_0, a maximal solution of (1.3) is global in time and satisfies

$$\|X(t) - X_e\| \underset{t\to+\infty}{\longrightarrow} 0.$$

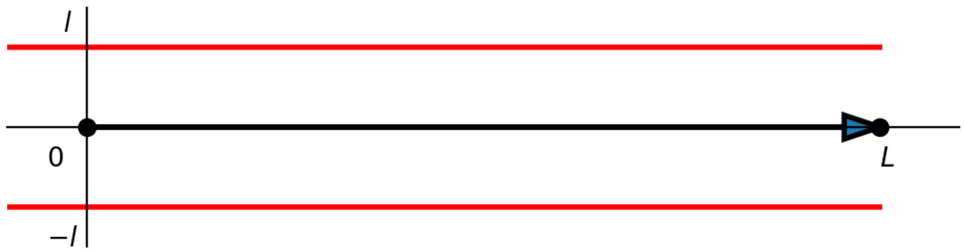

FIGURE 1.1 – Boundary of the road are in red

Note that those three problems admit numerous variants and combinations.

1.1.2 A concrete example

Let us now describe a very simple (and horribly idealized) example to illustrate those three notions. We consider the problem of driving on a road that is totally straightforward. We will reduce the vehicle to a point and it will be considered to advance at a fixed speed. The only control will be the direction of the vehicle. The state of the system will therefore be encoded by a point (x, y). Being on the road will mean satisfying $-l < y < l$, where $2l$ will be the width of the road. At this stage, we have $\mathcal{X} := \mathbb{R} \times [-l, l]$. For simplicity, we will consider the direction to be given by a function $t \mapsto u(t) \in \mathcal{U} := \mathbb{R}$. The dynamical equation of evolution is then

$$\begin{cases} \dot{x}(t) = v\cos(u(t)), \\ \dot{y}(t) = v\sin(u(t)), \end{cases} \tag{1.4}$$

where v will be the constant speed of the vehicle.

An instance of the exact controllability problem here would be to drive from the origin $(0, 0)$ to another point on the x-axis say $(L, 0)$ (with $L > 0$). Obviously, the control strategy $u : t \mapsto 0$ solves the problem for a time $T_m := \frac{L}{v}$. Note by the way that there is no solution for $T < T_m$, and that for a time $T > T_m$ there are multiple solutions. Look at Figure 1.1 for a geometric representation.

An instance of the optimal control problem — in fact related to the previous exact controllability problem — would be to keep the vehicle as close to the center of the road as possible and to end up as close as possible to $(L, 0)$. We could therefore look for a strategy $t \mapsto U(t)$ minimizing the quantity

$$C(U) := \int_0^T y(t)^2 dt + y(T)^2 + (x(T) - L)^2.$$

When $T = T_m$, it should be clear that the specific solution to the previous exact controllability problem is actually also a solution to this particular optimal control problem.

To understand now why the stabilization through a feedback control is interesting, one should focus on the meaning of the control strategy we have described so far. In common terms it basically translates to the following policy. Knowing the starting position and the final destination, one picks an angle for the wheel at the start and then keeps it locked. In theory, one could even shut his eyes and keep going for the correct duration.

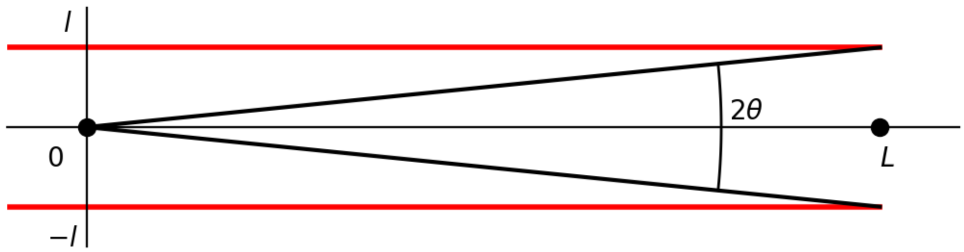

FIGURE 1.2 – Here θ is 3 minutes of arc

Obviously, no one in his right mind would dare proceed like this unless L is extremely small. To put this into perspective, for a road that is two meters wide — e.g. $l = 1$ — and one kilometer long — e.g. $L = 1000$ — the precision to which we have to pick the angle of the wheel is 3 minutes of arc (i.e. 0.0001 radians), see Figure 1.2 for a pictorial explanation. This is of course not reasonable for a human being. And even for an electro-mechanical system it could prove unnecessarily expensive to get something so precise.

An alternative strategy — that people actually use — is to move the wheel in accordance with the side of the road we occupy at each instant. Mathematically this would mean for instance taking $\mathbb{U}((x, y)) := -y/l$ which could mean that the closed loop evolution is now given by

$$\begin{cases} \dot{x}(t) = v \cos\left(-\frac{y(t)}{l}\right), \\ \dot{y}(t) = v \sin\left(-\frac{y(t)}{l}\right). \end{cases}$$

Now to understand the evolution of this system one just needs to realize that if $y(t) \in [-l, l]$ then we have

$$\frac{d\left(y^2\right)}{dt}(t) = 2vy(t)\sin(-\frac{y(t)}{l}) \leq 0,$$

therefore the quantity $y(t)^2$ is actually non-increasing. At the very least, if we start on the road, we stay on the road. It is relatively simple — using Gronwall's Lemma — to refine the argument to show that if $y(0) \in [-l, l]$, we have the bound

$$\forall t \geq 0, \quad y(t)^2 \leq y(0)^2 e^{-\frac{2\,v\,t\,\cos(1)}{l}},$$

which shows the exponential stability of $y = 0$. Note in passing that what we actually showed is that $\mathcal{L}(y) := y^2$ is a Lyapunov functional. See [45, Chapter 12, Section 1] for a general introduction to this tool.

To realize why the closed loop version is preferable to the open loop version, let us look at a perturbation on the control through a *parasite* function $t \mapsto p(t) \in [-\epsilon, \epsilon]$ with ϵ a relatively small positive number. The dynamics for the exact control strategy we proposed above — though generalized to a starting point (x_0, y_0) — is given by

$$\begin{cases} \dot{x}(t) = v \cos\left(p(t) + \arctan\left(\frac{-y_0}{L - x_0}\right)\right), \\ \dot{y}(t) = v \sin\left(p(t) + \arctan\left(\frac{-y_0}{L - x_0}\right)\right) \end{cases} \tag{OPEN LOOP}$$

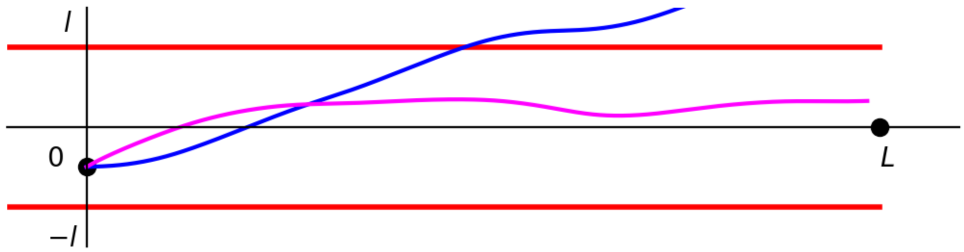

FIGURE 1.3 – dynamic of (CLOSED LOOP) in magenta and of (OPEN LOOP) in blue

while the one for the feedback control strategy is now

$$\begin{cases} \dot{x}(t) = \nu \cos\left(p(t) - \frac{y(t)}{l}\right), \\ \dot{y}(t) = \nu \sin\left(p(t) - \frac{y(t)}{l}\right). \end{cases} \tag{CLOSED LOOP}$$

Adding $\arctan\left(\frac{-y_0}{L-x_0}\right)$ in equation (OPEN LOOP) is exactly the open loop control strategy of picking the straight line connecting the initial point and the final point when the initial point is not necessarily $(0,0)$ but rather (x_0, y_0).

Rather than providing a full mathematical analysis, we will look at simulations for a perturbation limited to 5 degrees — $\epsilon = \pi/36$ above — to see why the closed loop strategy is more robust. We show one result of such a simulation in Figure 1.3. It was obtained using a simple explicit Euler scheme for equations (OPEN LOOP) and (CLOSED LOOP). As for the perturbation function p, we truncated the formula

$$\forall t \in [0,T], \quad p(t) := \sum_{k=0}^{\infty} \frac{\epsilon\, u_k}{2^{k+1}} \cos\left(\frac{k\,\nu\, t}{L}\right),$$

where $(u_k)_{k\geq 0}$ is a family of independent random variables uniformly distributed on $[-1,1]$. The actual script used for generating the formula can be found in Section A.1 of the Appendix. Note that to fit such a figure to the width of a page, we took a road that was much shorter $L = 10$, while keeping the same width $l = 1$, which makes the lack of the robustness of the open loop strategy even more terrifying.

The reason for the difference in robustness observed in Figure 1.3 is that, for the open loop the perturbations keep getting added to the state, while for the closed loop there is a self-correction mechanism.

The use of control mechanisms predates by far the mathematical analysis. One should consult [109] to see examples of feedback controls dating back to the antiquity, for instance in water clock mechanisms. For a more contemporary outlook, one can look at [23] which describes the feedback controls of mechanical nature used before 1930 in multiple domains : steam machines, textile industry, temperature regulation, etc. On the other hand, the second volume of Bennett's book [24] describes the emergence of electro-mechanical feedback controls starting around 1930 like the PID controller (Proportional, Integral, Derivative).

In the case where (1.1) is an ordinary differential equation, the theory is now very mature and many robust techniques exist to solve those problems. One could look for instance at [131] and [45] to learn more.

In this thesis, we will rather look at the case were (1.1) is a partial differential equation of evolution. And even more precisely to the specific case of hyperbolic conservation laws for which the generalization of finite dimensional techniques to partial differential equations usually fail. We will describe this type of equations in the next section before describing the specificities of control problems for them in the section closing the chapter.

1.2 Hyperbolic conservation laws and entropy solutions

Let us now try to describe both the importance and the specificity of hyperbolic conservation laws in the scope of partial differential equations theory.

1.2.1 Origins of conservation laws

For the sake of simplicity, we consider the time evolution of some matter distributed in one dimension of space. To that end, we need a density function $(t, x) \mapsto \rho(t, x)$ which is L^1_{loc}. Using this density, the quantity of the considered matter distributed in an interval $[x_0, x_1]$ at a time t will be given by $\int_{x_0}^{x_1} \rho(t, x)dx$. Additionally, we need a flux function $(t, x) \mapsto F(t, x)$, also L^1_{loc}.

The dynamics of the so called conservation law just can be summed up by saying that the quantity of matter in an interval (a, b) at time t_2 is equal to the quantity in the same interval at time t_1 modified by what went through the boundary as given by the flux function. In precise term, we request that for almost all $x_0 < x_1$ and almost all $t_1 < t_2$ we have

$$\int_{x_0}^{x_1} \rho(t_2, x)dx = \int_{x_1}^{x_2} \rho(t_1, x)dx + \int_{t_1}^{t_2} F(s, x_1)ds - \int_{t_1}^{t_2} F(s, x_2)ds \tag{1.5}$$

see Figure 1.4 for the visual interpretation.

If the density and the flux functions are more regular, say $\mathcal{C}^1$, it is clear — using the fundamental theorem of differential calculus — that equation (1.5) is actually equivalent to

$$\partial_t \rho(t, x) + \partial_x F(t, x) = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{1.6}$$

Now, to close the system and get a deterministic equation, one needs to provide a constitutive law connecting the flux to the density. Many possibilities exist and we may end up with very different partial differential equations.

- If we take $F(t, x) := \frac{\rho(t,x)^2}{2}$, equation (1.6) is actually Burgers' equation, which is loosely inspired by the Euler equation from fluid dynamics, see (1.7) below.
- If we take $F(t, x) := -\kappa \partial_x \rho(t, x)$, equation (1.6) is now the classical heat equation in one dimension of space from Fourier [74].
- As a final example, if we take $F(t, x) := \nu\, \rho(t, x) \left(1 - \frac{\rho(t,x)}{\rho_m}\right)$, one gets the celebrated Lighthill-Whitham [104] and Richards [121] equation modelling traffic flow on a road.

In this thesis, we will focus on *hyperbolic* conservation laws, meaning that we will look for constitutive laws of the type $F(t, x) := f(x, \rho(t, x))$.

For the sake of simplicity, we focused on the evolution of a scalar quantity in one dimension of space, but we can consider a vectorial quantity in multiple space dimensions. For the most general setting one should look at [54, Chapter 1]. Finally, let us mention what is maybe

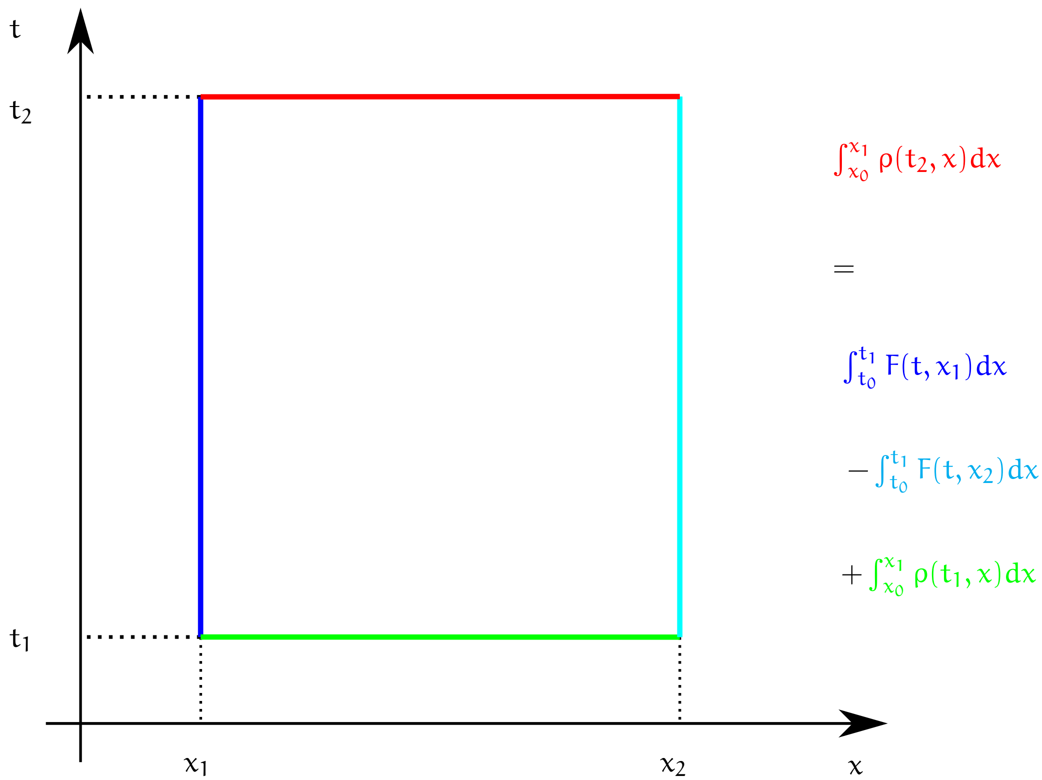

FIGURE 1.4 – Pictorial representation of equation (1.5)

the most fundamental equation of the domain, the Euler equations for gas dynamics in d-dimensions (compressible but isentropic) :

$$\begin{cases} \partial_t \rho(t, x_1, \dots, x_d) + \sum_{k=1}^{d} \partial_{x_k} m_k(t, x_1, \dots, x_d) = 0, \\ \partial_t m_1(t, x_1, \dots, x_d) + \partial_{x_1} P(\rho)(t, x_1, \dots, x_d) + \sum_{k=1}^{d} \partial_{x_k} \left(\frac{m_1 m_k}{\rho} \right) (t, x_1, \dots, x_d) = 0, \\ \dots, \\ \partial_t m_d(t, x_1, \dots, x_d) + \partial_{x_d} P(\rho)(t, x_1, \dots, x_d) + \sum_{k=1}^{d} \partial_{x_k} \left(\frac{m_d m_k}{\rho} \right) (t, x_1, \dots, x_d) = 0, \end{cases} \tag{1.7}$$

where P is the pressure law, ρ is the density of gas and m_k is the density of momentum in the x_k direction of space.

1.2.2 Paradise lost

In this section, we will focus on the Burgers' equation for the sake of simplicity. Given a function u_0 in $\mathcal{C}_c^\infty(\mathbb{R})$ we seek a regular function $(t, x) \mapsto u(t, x)$ such that :

$$\begin{cases} \partial_t u(t, x) + \partial_x \left(\frac{u^2}{2} \right) (t, x) = 0, \\ u(0, x) = u_0(x), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}. \tag{1.8}$$

We will proceed by analysis-synthesis.

Analysis. Let us suppose that we have a solution u of (1.8) that is in $\mathcal{C}^1([0,T]\times\mathbb{R})$, for some positive T. We can apply the chain rule and get

$$\forall t\in[0,T],\quad \forall x\in\mathbb{R},\qquad \partial_t u(t,x)+u(t,x)\,\partial_x u(t,x)=0,\ \text{ i.e. }\ (\partial_t+u(t,x)\partial_x)\,u(t,x)=0.$$

So we are saying that the directional derivative of u along the vector field $(t,x)\mapsto(1,u(t,x))$ is actually equal to zero. We can thus deduce that along the integral curves of said vector field, the function u is actually constant. Because of the first component of the vector field, it is clear that those integral curves are of the form $t\mapsto(t,p(t))$ where p is a solution of

$$\forall t\in[0,T],\quad \dot{p}(t)=u(t,p(t)).$$

However, the right-hand side of this differential equation is constant by construction, and thus the integral curves are straight lines.

Summing up this analysis we see that u satisfies :

$$\forall(t,x)\in[0,T]\times\mathbb{R},\quad u(t,x)=u_0(x-t\,u(t,x)). \tag{1.9}$$

Synthesis. Given a function u_0 in $\mathcal{C}_c^\infty(\mathbb{R})$, we consider a positive time T satisfying the condition $T\,\|u_0'\|_\infty<1$. We introduce the auxiliary function $\mathcal{A}$ defined by

$$\forall(t,x,v)\in[0,T]\times\mathbb{R}\times\mathbb{R},\quad \mathcal{A}(t,x,v):=v-u_0(x-t\,v). \tag{1.10}$$

It is clear that

$$\forall x\in\mathbb{R},\quad \mathcal{A}(0,x,u_0(x))=0. \tag{1.11}$$

At the same time,

$$\forall(t,x,v)\in[0,T]\times\mathbb{R}\times\mathbb{R},\quad \partial_v\mathcal{A}(t,x,v)=1-t\,u_0'(x-tv)\geq 1-T\|u_0'\|_\infty>0. \tag{1.12}$$

A connectedness argument and the Implicit Function theorem allow us to show that there exists a unique function u in $\mathcal{C}^\infty([0,T]\times\mathbb{R})$ satisfying

$$\forall(t,x,v)\in[0,T]\times\mathbb{R}\times\mathbb{R},\quad \mathcal{A}(t,x,u(t,x))=0. \tag{1.13}$$

Differentiating (1.13) we get

$$\begin{cases}\partial_t u(t,x)=-\frac{u_0'(x-t\,u(t,x))u(t,x)}{1+t\,u_0'(x-t\,u(t,x))},\\ \partial_x u(t,x)=\frac{u_0'(x-t\,u(t,x))}{1+t\,u_0'(x-t\,u(t,x))},\end{cases}$$

thus u clearly satisfies the first equation of (1.8). Taking $t=0$ in (1.13), we obtain the second equation of (1.8).

At this point we have proven the following Theorem.

Theorem 1

For any initial data u_0 in $\mathcal{C}_c^\infty(\mathbb{R})$, and for any time $T<1/\|u_0'\|_\infty$, there exists a unique solution to (1.8) defined on $[0,T]\times\mathbb{R}$.

Singularities Let us now exclude the uninteresting case $u_0 \equiv 0$ for which $u \equiv 0$. Thanks to the previous result, we can consider the maximal solution of (1.8), u, defined up to the time T^*. The reasoning done in the Analysis paragraph above can be rigorously applied and thus, adjusting (1.9), we get

$$\forall (t,x) \in \mathbb{R}^2, \quad 0 \leq t < T^* \implies u_0(x) = u(t, x + t\,u_0(x)). \tag{1.14}$$

The key fact is now that there exists (x_1, x_2) in $\mathbb{R}^2$ such that

$$x_1 < x_2 \text{ and } u_0(x_1) > u_0(x_2),$$

since the only non-decreasing function that is compactly supported is $x \mapsto 0$. But then for $t_p := -(x_2 - x_1)/(u_0(x_2) - u_0(x_1))$, we have

$$t_p > 0 \quad \text{and} \quad x_1 + t_p\,u_0(x_1) = x_2 + t_p\,u_0(x_2).$$

Calling x_p the common quantity on the right, we see that should $t_p < T^*$, we would have $u_0(x_1) = u(t_p, x_p) = u_0(x_2)$ which is contradictory, thus $T^* \leq t_p$. Consult Figure 1.5 for a graphical representation of this construction. Combined with equality (1.9), we have thus

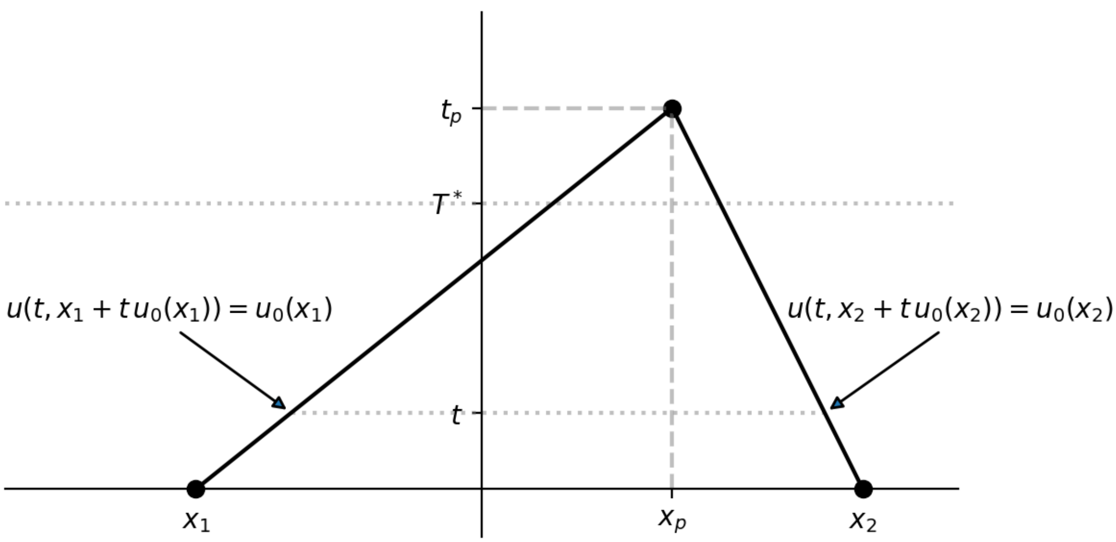

FIGURE 1.5 – Singularity for Burgers by crossing integral curves

proven the following complementary result.

Theorem 2

For any initial data u_0 in $\mathcal{C}_c^\infty(\mathbb{R})$, the maximal solution to (1.8) blows up in finite time T^*. At the same time we have

$$\forall (t,x) \in \mathbb{R}^2, \quad 0 \leq t < T^* \implies |u(t,x)| \leq \|u_0\|_\infty.$$

At this point we see that the formulation (1.6) seems to be the problem and not so much the original (1.5).

Geometric intuition Let us finish this subsection by providing some additional insight into what was described in the Analysis paragraph above. Introducing the integral curves of the vector field $(t, x) \mapsto (1, u(t, x))$ and looking at the evolution of u along them is a technique called the *method of characteristics*. The fact that the solution to (1.8) is constant on the integral curve, which turn out to be straight line can be visualized in the following way.

In formal term, if we introduce

$$\forall t \geq 0, \quad G_t := \{(x, y) \in \mathbb{R}^2 \,:\, y = u(t, x)\}, \tag{1.15}$$

and the transformations of $\mathbb{R}^2$

$$\forall t \geq 0, \quad T_t(x, y) := (x + t\,y, y), \tag{1.16}$$

then we have $G_t = T_t(G_0)$ as long as the solution is regular. Basically we move the points whose ordinate is y at speed y in direction x. One should look at figure 1.6 for the representation of the graph G_0 and of the transformation T_t. One should look at figure 1.7 for the evolution of the solution u using those transformations. Note, in particular, that the steepness increases in the part of the graph where the function is decreasing. Until, finally, we have a vertical slope, meaning a blow up of the derivative $x \mapsto \partial_x u(t, x)$ and also a discontinuity of $x \mapsto u(t, x)$. Note that if we apply T_t to G_0 for a time t greater than the blowup time, we end up with a set that is not the graph of a function. We refer to Section A.2 in the Appendix for the python code generating the figures.

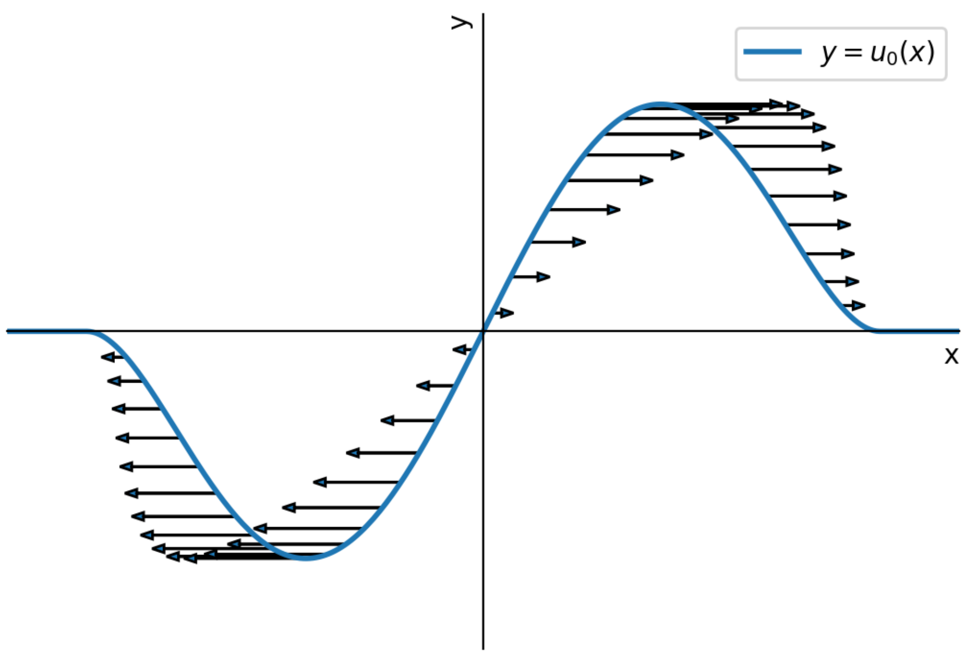

FIGURE 1.6 – The transform T_t as arrows and the graph G_0.

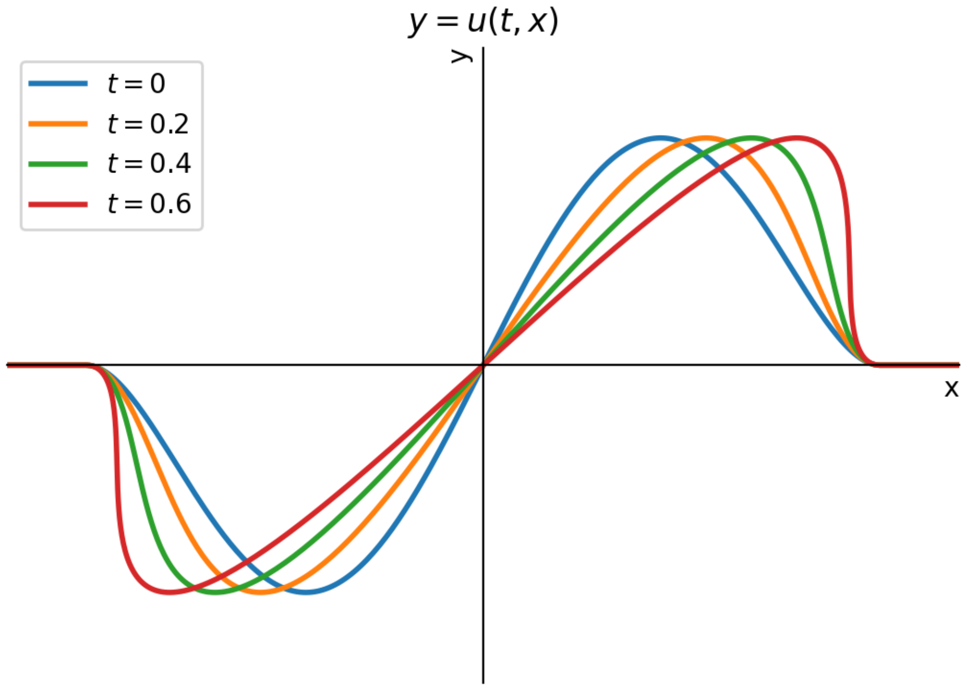

FIGURE 1.7 – The graphs G_t for multiple values of t.

1.2.3 Entropy — and non-entropic — solutions

In this section, we will describe how we are to get a global in time solution of

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u)(t,x) = 0, \\ u(0,x) = u_0(x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \tag{1.17}$$

where the function f will be regular enough (say $\mathcal{C}^1$).

Rankine-Hugoniot condition. As we saw in the previous section, if the goal is to get a solution of (1.17) for all time, one must look at less regular solution (possibly discontinuous). And to accommodate this, we need to use a weaker formulation of the equation ; for instance in the sense of (1.5).

It is straightforward to see that constant functions of t and x are solutions to (1.17). The next step in complexity is to consider Riemann initial data (following the paper [122] of Riemann himself). Precisely, given three real numbers x_c, u_l and u_r we consider the initial data

$$\forall x \in \mathbb{R}, \quad u_0(x) := \begin{cases} u_l & \text{if } x < x_c, \\ u_r & \text{if } x > x_c, \end{cases} \tag{1.18}$$

and look for a solution $(t,x) \mapsto u(t,x)$ satisfying (1.17) in the sense of (1.5). That is for all real numbers x_0, x_1, t_0 and t_1

$$\begin{cases} x_0 \le x_1 \\ 0 \le t_0 < t_1 \end{cases} \implies \int_{x_0}^{x_1} u(t_1, x)\, dx + \int_{t_0}^{t_1} f(u(t, x_1))\, dt - \int_{x_0}^{x_1} u(t_0, x)\, dx - \int_{t_0}^{t_1} f(u(t, x_0))\, dt = 0. \tag{1.19}$$

It is useful to realize that geometrically this means that the integral along the rectangle whose nodes are (x_0, t_0), (x_1, t_0), (x_1, t_1) and (x_0, t_1) of the differential form $u(t, x)\, dx - f(u(t, x))\, dt$ is equal to zero.

It is easy to see that if u satisfies (1.19) then for any fixed a the function $(t, x) \mapsto u(t, x-a)$ also satisfies (1.19), and of course the initial data is also translated. Thus we can suppose that in (1.18), $x_c = 0$.

In the same way, a simple calculation show that for a positive real number α, if u satisfies (1.19), then so does $(t, x) \mapsto u(\alpha t, \alpha x)$ (which is also defined on $\mathbb{R}^+ \times \mathbb{R}$). Of course the corresponding initial data is given correspondingly by $x \mapsto u_0(\alpha x)$.

This leads to searching u with the ansatz

$$\forall x \in \mathbb{R}, \quad u(t, x) := \begin{cases} u_l & \text{if } x < \gamma t \\ u_r & \text{if } x > \gamma t \end{cases} \tag{1.20}$$

where γ is to be determined.

At this point, picking $x_0 = \min(\gamma t_0, \gamma t_1)$ and $x_1 = \max(\gamma t_0, \gamma t_1)$ in (1.19) provides

$$\gamma (t_1 - t_0) (u_l - u_r) = (t_1 - t_0)(f(u_l) - f(u_r)),$$

(though one has to check for $\gamma > 0$ and $\gamma < 0$ separately). Look at Figure 1.8 for a graphical representation of this calculation.

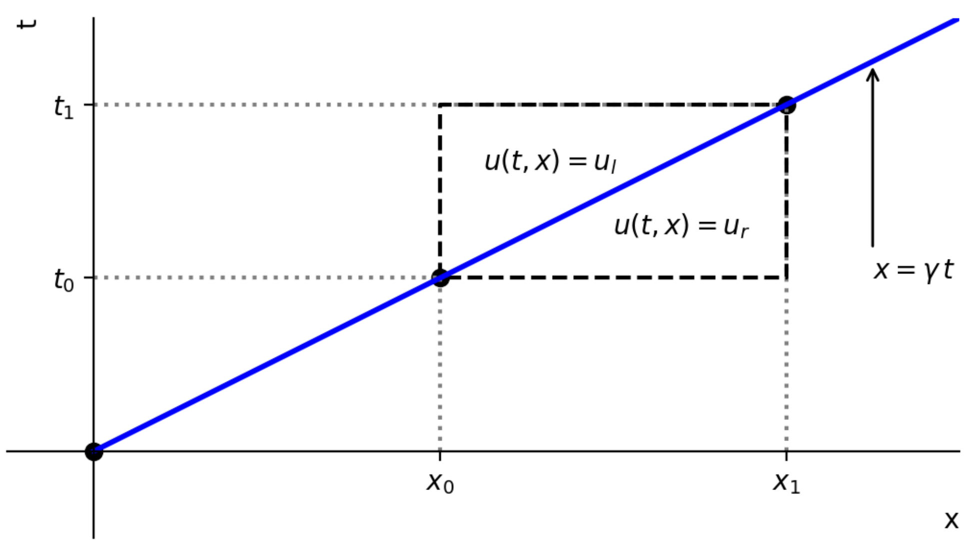

Figure 1.8 – Calculation for the Rankine-Hugoniot condition

At this point, we know that for a singularity to be admissible — i.e. for u defined by (1.20) to satisfy (1.19) — it is necessary for the Rankine-Hugoniot condition to hold :

$$\gamma = \frac{f(u_r) - f(u_l)}{u_r - u_l}. \tag{1.21}$$

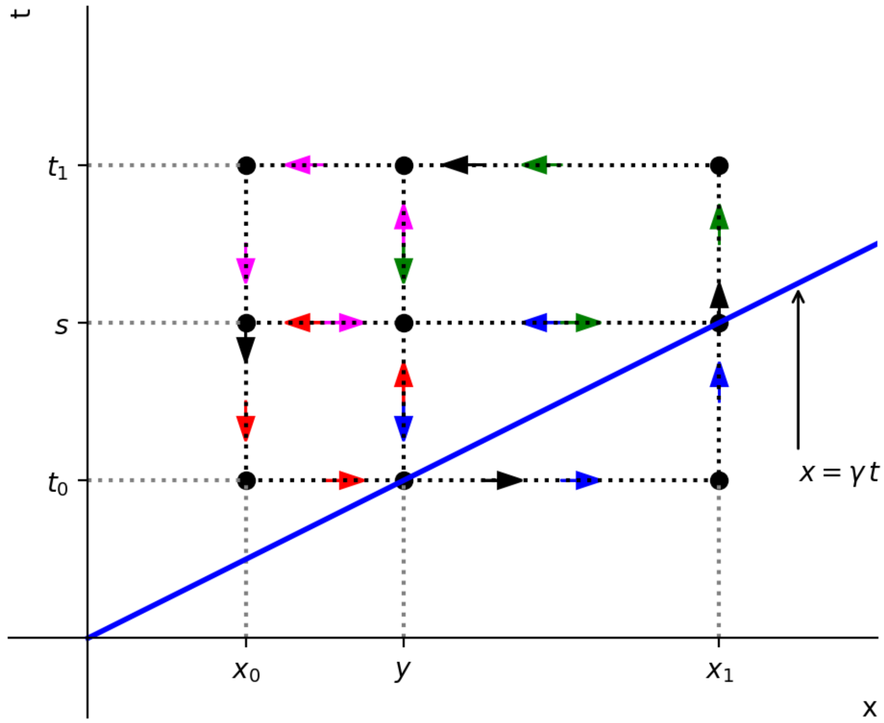

FIGURE 1.9 – Sufficiency of the Rankine-Hugoniot condition

It turns out that this condition is also sufficient. Indeed, following along Figure 1.9, we can decompose the quantities appearing in (1.19) using Chasles' decomposition and the fact that the internal edges in the picture are taken in both directions. More precisely, it should be clear that the integral of the form $\omega := u(t,x)\,dx - f(u(t,x))\,dt$ along the black rectangle is equal to the sum of the integrals along the blue, green, magenta and red rectangles. The previous analysis — that provided us with the Rankine-Hugoniot condition — shows that the integral along the blue rectangle is equal to zero. And using the fact that u is constant along the red, green and magenta rectangles, those integrals also vanish.

Entropy condition. At this point, we know how to propagate any discontinuity so as to get a solution to (1.17).

In fact, one can use those conditions to show the existence of a semigroup $(S_t^{pc})_{t\geq 0}$ which acts on the space of piecewise constant functions from $\mathbb{R}$ to $\mathbb{R}$ and solves equation (1.17) in the sense of equation (1.19). Once two discontinuities collide, we have a new Riemann data and we use Rankine-Hugoniot condition to know how this fused singularity propagate. This leads to pictures such as the one in Figure 1.10.

The next step would naturally be to extend this semigroup to more general functions. One is thus lead to consider the continuity property of said semigroup.

Let us consider three real numbers u_l, u_m and u_r satisfying

$$\gamma_l := \frac{f(u_l) - f(u_m)}{u_l - u_m} < \frac{f(u_r) - f(u_m)}{u_r - u_m} =: \gamma_r. \tag{1.22}$$

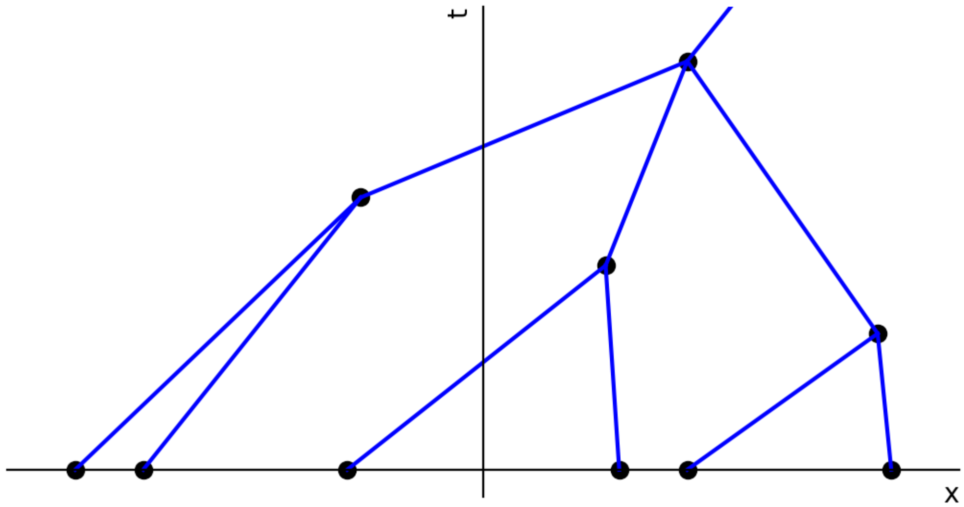

FIGURE 1.10 – Propagation of the discontinuities for piecewise constant data

For a positive real number ϵ one consider the initial data :

$$u_\epsilon(x) := \begin{cases} u_l & \text{if } x < 0 \\ u_m & \text{if } 0 < x < \epsilon \\ u_r & \text{if } \epsilon < x. \end{cases} \tag{1.23}$$

Using the previous results it should be clear that we have

$$(S_t^{pc} u_\epsilon)(x) := \begin{cases} u_l & \text{if } x < \gamma_l t \\ u_m & \text{if } \gamma_l t < x < \epsilon + \gamma_r t \\ u_r & \text{if } \epsilon + \gamma_r t < x. \end{cases} \tag{1.24}$$

(look at Figure 1.11 for the corresponding picture).

But letting ϵ go to 0^+, the limit satisfies :

$$\lim_{\epsilon \to 0^+} (S_t^{pc} u_\epsilon)(x) := \begin{cases} u_l & \text{if } x < \gamma_l t \\ u_m & \text{if } \gamma_l t < x < \gamma_r t \\ u_r & \text{if } \gamma_r t < x, \end{cases} \tag{1.25}$$

and this is clearly not

$$(S_t^{pc} \lim_{\epsilon \to 0^+} u_\epsilon)(x) := \begin{cases} u_l & \text{if } x < \gamma_m t \\ u_r & \text{if } \gamma_m t < x, \end{cases} \tag{1.26}$$

for $\gamma_m := (f(u_r) - f(u_l))/(u_r - u_l)$. Note however that both are solutions of (1.17) in the sense of (1.19) for the initial data given by (1.18). On one hand, the semigroup $(S_t^{pc})_{t \geq 0}$ is not continuous and therefore cannot be extended. On the other hand, there are multiple solutions to (1.17) in the sense of (1.19).

The way to solve this problem is to consider as admissible only those singularities satisfying the following so called entropy condition (technically Oleinik's version) :

$$\forall u \in \mathbb{R}, \quad \frac{f(u_l) - f(u)}{u_l - u} \geq \frac{f(u) - f(u_r)}{u - u_r}. \tag{1.27}$$

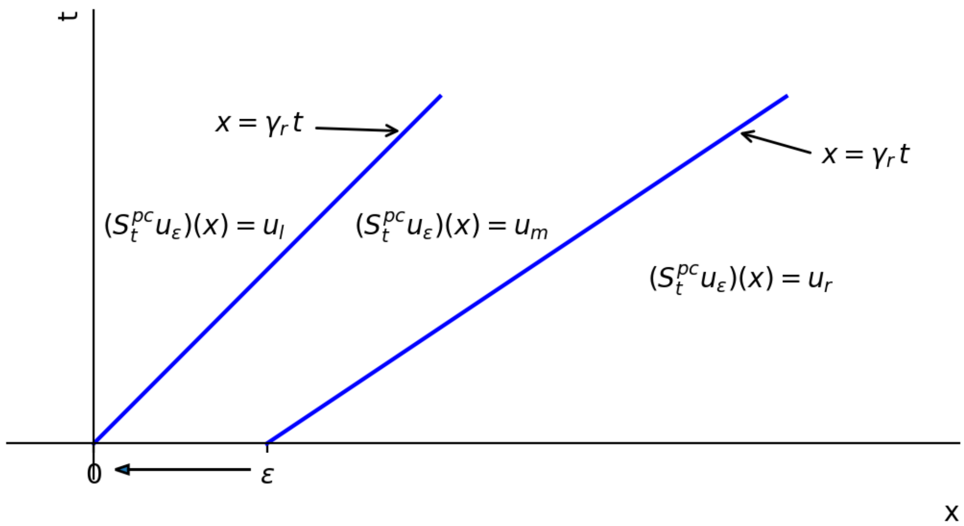

FIGURE 1.11 – Initial data showing the continuity problem of $(S_t^{pc})_{t\geq 0}$

For those singularities that do not satisfy said condition, there is actually a (somewhat) regular way to solve them using the characteristics method of Section 1.2.2.

Consider for instance the case of Burgers' equation 1.8 with an initial data given by the Riemann data from (1.18) with $u_l < u_r$ and $x_c = 0$. A simple calculation shows that the formula

$$\forall x \in \mathbb{R}, \quad \forall t > 0, \quad u(t,x) := \begin{cases} u_l & \text{if } x < t\,u_l \\ \frac{x}{t} & \text{if } t\,u_l \leq x \leq t\,u_r \\ u_r & \text{if } t\,u_r < x. \end{cases} \tag{1.28}$$

defines a regular solution, it is called a rarefaction wave. It can be found by plugging the ansatz $u(t,x) = w(x/t)$ in the equation following the scaling invariance we mentioned above. We refer to [29, Chapter 6] for the general situation. In Figure 1.12 and Figure 1.13 we represented the evolution of the rarefaction wave following the geometric technique we described in the last paragraph of the previous subsection. Look at Figure 1.14 to see the characteristic curve associated to the solution.

In principle, one could use admissible discontinuities and rarefaction waves to build a continuous semigroup of weak solutions. This actually leads to the wave front tracking method that is described in Chapter 5. We will now provide an equivalent way to pick solutions due to Kruzkov, see [95]. It is typical of the techniques used in Chapter 2 and Chapter 6.

Entropy solutions The starting point of this analysis is to realize that in many instances, the physical modeling leading to hyperbolic conservation laws actually neglected viscosity effects. Looking back at equation (1.5), this means that the flux $F(t,x)$ does not in fact depend only on $\rho(t,x)$ but also on $\partial_x \rho(t,x)$. However we can suppose that this additional term is actually small. This leads us to look at equation (1.17) as the limiting case of

$$\begin{cases} \partial_t u^\epsilon(t,x) + \partial_x \left(f(u^\epsilon(t,x))\right) = \epsilon\, \partial_{xx}^2 u^\epsilon(t,x), \\ u^\epsilon(0,x) = u_0(x), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.29}$$

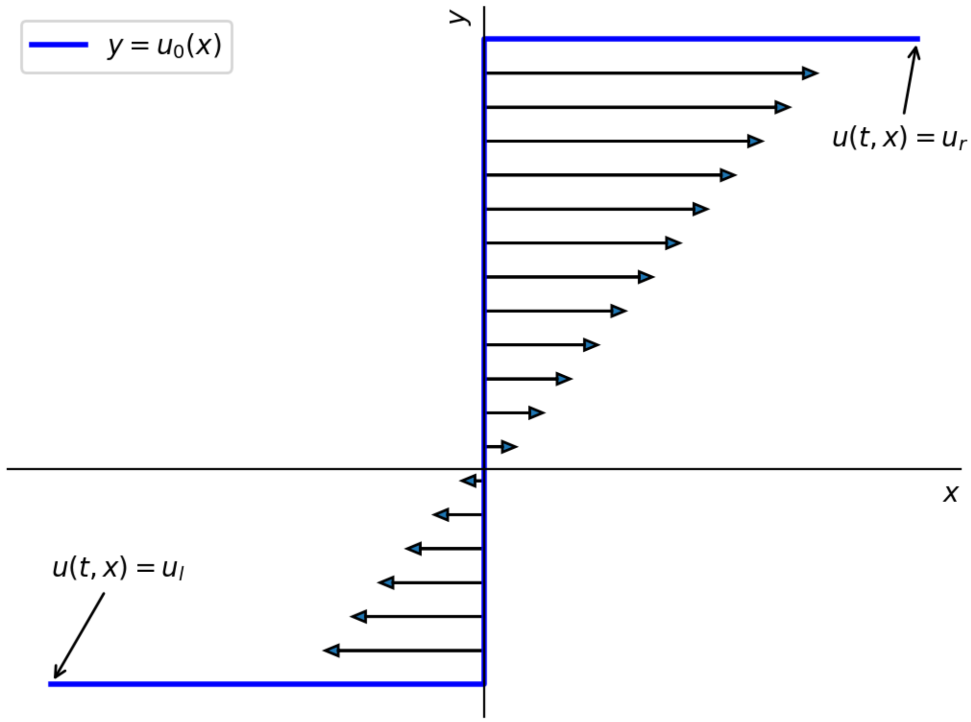

FIGURE 1.12 – Visualization of the rarefaction wave for Burgers' equation on the graph of u_0

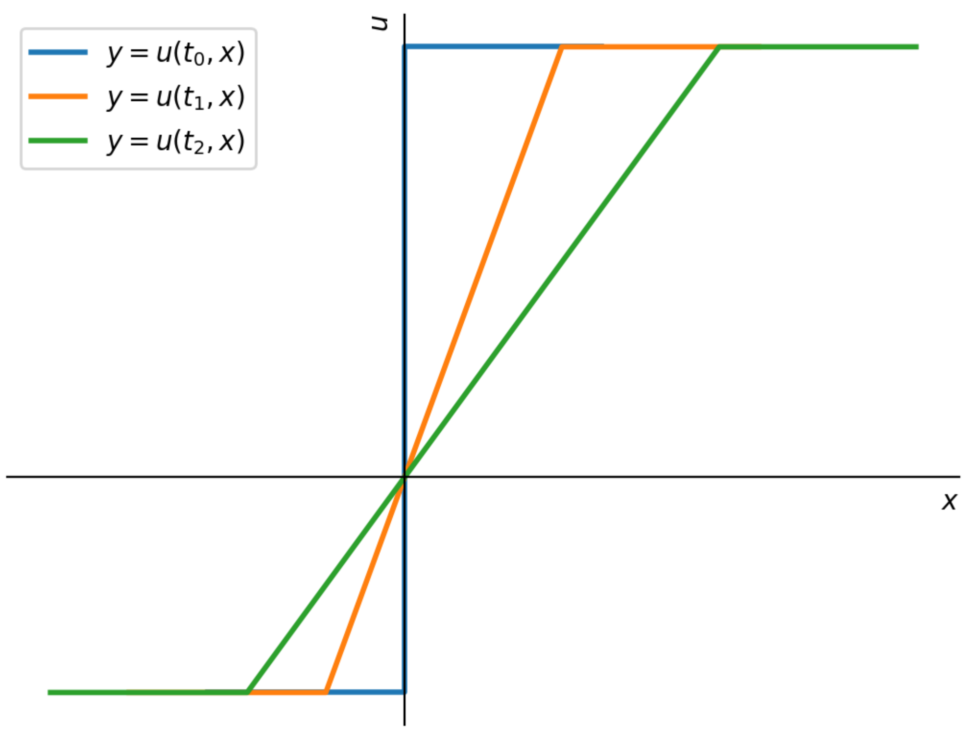

FIGURE 1.13 – Evolution of the rarefaction wave for Burgers' equation with $0 = t_0 < t_1 < t_2$

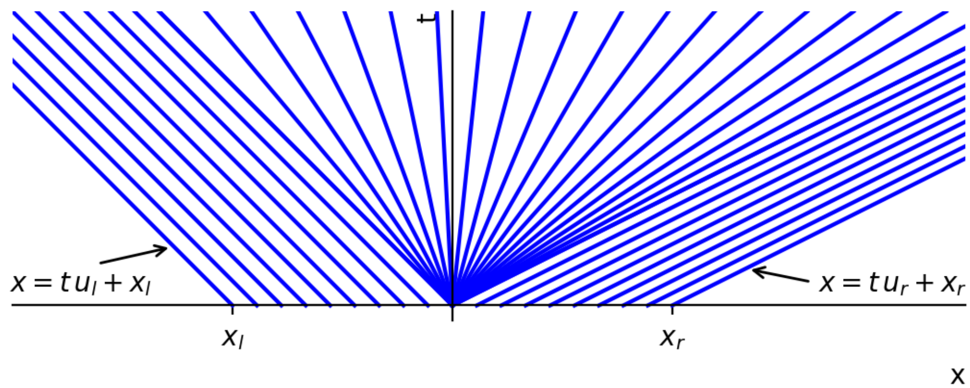

FIGURE 1.14 – Characteristic curves of the rarefaction wave for Burgers' equation

when ϵ goes to 0^+.

We will look rigorously at such an approximation and at this — very singular — limit in Chapter 2. Let us just say for now that we expect to have almost everywhere pointwise convergence of the sequence and that for $\epsilon > 0$ the second order term prevents the formation of singularities in solutions of (1.29).

This mean that we can work with classical solutions in (1.29). The idea is then to consider a $\mathcal{C}^2(\mathbb{R}, \mathbb{R})$ function E — which we will call an entropy — and to multiply equation (1.29) by $E'(u^\epsilon(t, x))$. At the same time, we introduce Q — the so called entropy flux — the function associated to E by

$$\forall z \in \mathbb{R}, \quad Q(z) := \int_0^z f'(w)\, E'(w)\, dw. \tag{1.30}$$

We end up with u^ϵ satisfying

$$\partial_t E(u^\epsilon)(t, x) + \partial_x Q(u^\epsilon)(t, x) = \epsilon \left[\partial_{xx}^2 E(u^\epsilon)(t, x) - E''(u^\epsilon(t, x))(\partial_x u(t, x))^2\right]. \tag{1.31}$$

If E is a convex function, the last term if signed. If we multiply (1.31) by a non-negative function ϕ in $C_c^\infty(\mathbb{R}^2)$ and integrate over the upper half plane, after some integration by parts, we get :

$$\int_0^\infty \int_{-\infty}^\infty \partial_t \phi(t, x) E(u^\epsilon(t, x)) + \partial_x \phi(t, x) Q(u^\epsilon(t, x)) + \epsilon \partial_{xx}^2 \phi(t, x) E(u^\epsilon(t, x))\, dx dt$$
$$+ \int_{-\infty}^\infty E(u_0(x)) \phi(0, x)\, dx = \epsilon \int_0^\infty \int_{-\infty}^\infty E''(u^\epsilon(t, x))(\partial_x u^\epsilon(t, x))^2 \phi(t, x)\, dx\, dt \geq 0. \tag{1.32}$$

So if we have a almost everywhere convergence of $(u^\epsilon)_{\epsilon>0}$ to a function u — and a uniform L^∞ bound — we can apply the dominated convergence theorem on the left hand side and get

$$\int_0^\infty \int_{-\infty}^\infty \partial_t \phi(t, x) E(u(t, x)) + \partial_x \phi(t, x) Q(u(t, x))\, dx dt + \int_{-\infty}^\infty E(u_0(x)) \phi(0, x)\, dx \geq 0. \tag{1.33}$$

We will take this as our definition of an entropy solution of (1.17), to be satisfied for any $\mathcal{C}^2$ convex function E with the associated function Q and any test function ϕ non-negative in $\mathcal{C}_c^\infty(\mathbb{R}^2)$.

Kruzkov's paper [95] shows — in a much more general setting and along other results — that this notion of solution satisfies the entropy conditions previously obtained. Furthermore, he uses it to obtain a semigroup $(S_t)_{t\geq 0}$ which is actually contractive in $L^1(\mathbb{R})$.

A critical remark for much of the difficulties we will encounter later in this Thesis is that, contrary to classical solutions of (1.17), this notion of solution is not reversible in time. Indeed, the change of variable $t \mapsto T - t$ in the left hand side of (1.33) would lead to the opposite sign. We will discuss this phenomenon in more details in Chapter 3.

1.3 Propagation and control problems

Let us now try to describe how control theory and entropy solutions to hyperbolic conservation laws actually mix up and look at the kind of problems that will be the subject of the Thesis. We will first describe problems specific to partial differential equations before trying to adapt those that were described in Section 1.1.

1.3.1 Domain of spatial dependence

As could be intuited by the method of characteristics, a solution of (1.17) propagates the value taken by the initial data at a finite speed. Of course, that theory applies only to regular solutions. One can then wonder on which part of the initial data does the value $u(t,x)$ really depend. Let us introduce the notion of domain of dependence to formalize this question of propagation.

We consider the multidimensional scalar conservation law :

$$\begin{cases} \partial_t u + \sum_{k=1}^d \partial_{x_k} f_k(t,x,u) = 0, \\ u(0,x) = u_0(x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}^d. \tag{1.34}$$

Following Serre [127, Chapter 2, Section 3], we introduce the following.

Definition 1

Let u be an entropy solution to (1.34). The domain of dependence of u at a point (t,x) is the smallest compact set $\mathcal{D}_u(t,x)$ in $\mathbb{R}^d$ such that, for every bounded function w with compact support disjoint from $\mathcal{D}_u(t,x)$ and every positive but small enough ϵ, the solution v^ϵ of (1.34) with initial data $u_0 + \epsilon w$ actually coincides with u at (t,x) (in the sense of Lebesgue points).

In Kruzkov's paper [95] there is a rough estimation of such domain by pointing out that

$$\mathcal{D}_u(t,x) \subset B\left(x, t\sup\{\|(\partial_u f_1(t,x,v),\ldots,\partial_u f_d(s,x,v))\|_2 \,:\, s\in[0,t]\ x\in\mathbb{R}^d\ v\in\mathbb{R}\}\right). \tag{1.35}$$

A much sharper result is to be found in [120].

In Chapter 4, we will focus on a very precise technique due to Dafermos [51] that, in particular, perfectly answers this question in one dimension of space.

1.3.2 Inverse design

As we mentioned at the end of the previous section, the notion of entropy solution allow us to get an entropy semigroup $(S_t)_{t\geq 0}$ solving (1.17). We also explained that those solutions are not reversible in time. This leads to two very natural questions.

- The first is to determine the reachable states. Meaning that given a time positive time T we look for
$$R_T := \{v \in L^\infty(\mathbb{R}) \,:\, \exists u_0 \in L^\infty(\mathbb{R}) \quad S_T u_0 = v\}. \tag{1.36}$$
More concisely, we want to characterize the range of the semigroup.
- As a second step let us describe the inverse design problem. Given a reachable state, and since we don't have reversibility of the solutions, one could try to describe all the initial data leading to this particular state. More precisely, given a positive time T and a function v in $L^\infty(\mathbb{R})$ we want a description of
$$I_T(v) := \{u_0 \in L^\infty(\mathbb{R}) \,:\, S_T u_0 = v\}. \tag{1.37}$$

From a purely theoretical point of view, this is another way of trying to quantify the compactification properties of the semigroup, a topic discussed in [56, 8, 9].

From an applied viewpoint, the inverse design also occurs naturally. For instance, when a plane goes supersonic, it starts to produce sonic booms which, after propagating in the atmosphere, provoke a lot of perturbations at ground level. For this reason, the Concorde was mostly prohibited from flying at supersonic speed over land. A standard model to study the way the propagation occurs in the atmosphere is due to Whitham [136] and is related to a conservation law (in a relatively complicated way we will not try to describe here). The inverse design problem in this case leads to finding plane design such that the sonic boom wave is mostly attenuated when it arrives at ground floor. One should look at [80] for a much more precise description of the problem.

An alternative example is to be found in the context of traffic flow. When reversing time and space in the Lighthill-Whitham[104] and Richards [121] model, the inverse design problem can be translated in the following way. Given the measure of the traffic flow through a toll gate, is it possible to reconstruct the flow of vehicles backward from it and if so how ? For a related problem, one should consult [61].

1.3.3 Control problems

We will now describe how the exact controllability problem and the feedback stabilization problem look in the context of hyperbolic conservation laws. Note in relation to Section 1.1 that we will not investigate the optimal control problem in this context. The optimal control problem for ordinary differential equations will however occur as a tool in Chapter 3. The literature on optimal control problems in the context of entropy solutions to hyperbolic conservation laws is actually rather sizable. We refer therefore to [119] as an entry point to it and do not try to be exhaustive.

Exact controllability problem. The simplest and probably most natural way to state an exact controllability for hyperbolic conservation laws is to start with the initial boundary value problem for a scalar hyperbolic one dimensional conservation law :

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u)(t,x) = 0, \\ u(0,x) = u_0(x), \\ u(t,a) = u_l(t), \\ u(t,b) = u_r(t), \end{cases} \qquad t > 0, \quad a < x < b. \tag{1.38}$$

Then given a positive time T, an initial data u_0 and a target state u_1 one should look for boundary data u_l and u_r so that the solution u to (1.38) satisfies $u(T, \cdot) = u_1$.

We have left the spaces in which we search the initial state, target state, and the boundary controls blank on purpose. Indeed, in the context of control theory, one could try to bypass entropy solutions by additionally requesting that the controls prevent the blowup that we described in Subsection 1.2.2 from occurring. For a very successful instance of such an approach, we refer to [33, Theorem 1]. However one additional control — a source term depending only on t — was needed to obtain that result. In most situations, this approach seems to have a lot of defects.

- The additional requirement on the control might make the problem intractable.
- Only local results may be obtainable.
- Some natural targets are actually discontinuous.
- The proof usually proceed by solving the exact controllability problem for the viscous approximation to (1.38) in an uniform way. This is usually extremely tricky since the viscous term is usually instrumental to the controllability for the viscous approximation.

The authors of the pioneering paper [10] thus decided to work directly in the framework of entropy solutions we described in the previous section. The related literature has now grown, though it is not huge by any means [1, 7, 11, 12, 15, 30, 42, 60, 64, 75, 76, 77, 88, 91, 106, 99, 100, 101, 107, 114].

Russell's extension method. By design, we didn't discuss so far the initial boundary value problem (1.38). It turns out that, as it is stated, the problem is over-constrained. One should not expect the boundary conditions to hold for almost all time and still have wellposedness. Thinking at the characteristics method from Subsection 1.2.2, we would expect that the solution is equal to the boundary value at those points were the characteristics are pointing inward. It turns out that the situation is even more complicated and a great amount of literature was dedicated to this question, see for instance [97, 17, 65, 112, 135, 6, 41].

We will describe the meaning of those boundary conditions below when tackling the feedback control problem case. For now, we will explain how we can bypass this issue in the context of exact controllability. The idea, which is due to Russell [124], is the following. We fix the positive time T, the initial data u_0 in $L^\infty(a, b)$ and the target state u_1 in $L^\infty(a, b)$, but then we try to solve a different problem. Search for a function $v_0 \in L^\infty(\mathbb{R})$ extending u_0 i.e.

$$v_0(x) = u_0(x) \quad \text{for } x \text{ in } (a, b),$$

such that if v is the entropy solution of

$$\begin{cases} \partial_t v + \partial_x f(v)(t, x) = 0, \\ v(0, x) = v_0(x), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.39}$$

then we have

$$v(T, x) = u_1(x) \quad \text{for } x \text{ in } (a, b).$$

The hope is that if there is a good theory of existence and uniqueness for (1.38) and at the same time if we have a good notion of trace for solutions of (1.39) (using for instance the results of [134] or [110]), then we get back automatically a solution to the exact controllability problem by the taking the traces of v at $x = a$ and $x = b$ as control inputs u_l and u_r in (1.38).

In fact, the strategy is a bit more elaborate in that we apply this idea multiple times in order to proceed through different intermediary states.

Note that under this form, we have a strong connection to the inverse design problem : we are actually searching for an extension V_1 of u_1 outside of (a, b) such that the inverse design set $I_T(V_1)$ — associated to (1.39) through equation (1.37) — contains at least one extension V_0 of u_0 outside of (a, b).

A striking result. For classical PDEs, the way to solve the exact controllability problem usually involves a linearization argument. The linearized control problem itself is usually solved using the classical Hilbert Uniqueness Method of Jacques-Louis Lions, see [105, Chapter 1]. To understand the difficulty of providing a linearization framework for equation (1.38), one first need to realize that the formal linearized equation is

$$\partial_t \delta u + f'(u)\partial_x \delta u = 0.$$

The problem is that one expects u and its perturbation δu both to have discontinuities and almost surely at common point. This means that $f'(u)\partial_x u$ involves a Dirac delta applied to a discontinuous function at the point of discontinuity. There have been attempts to provide a rigorous answer to this problem — see for instance [55] — however, the following result casts a large shade over the prospects of this approach.

In [30], Bressan and Coclite showed that for the hyperbolic system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x (u\,\rho) = 0, \\ \partial_t u + \partial_x \left(\frac{u^2}{2} + \frac{K^2}{\gamma - 1} \rho^{\gamma - 1} \right) = 0, \end{cases} \tag{1.40}$$

for any interval (a, b) and for any positive number ϵ, there exists an initial data (ρ_0, u_0) whose total variation is less than ϵ such that any entropy solution (ρ, u) of (1.40) whose total variation remains small for all time has a dense set of discontinuities in x at any time t, meaning in particular that we have

$$\forall t \geq 0, \quad (x \mapsto (\rho(t, x), u(t, x))) \text{ is not constant on } (a, b).$$

However, the formal linearized system around a constant solution $(\bar{\rho}, \bar{u})$ of (1.40) can easily be shown to be exactly controllable.

This type of system seems to fail the linearization paradigm or at least the classical application in Banach space. Indeed for Frechet space topology, the Nash-Moser Theorem asks for a property to hold locally and not just at the point of linearization for it to transfer to the non-linear system. One should consult the survey of Hamilton [82] to see why this is necessary in this functional setting.

In any case, this means that we are condemned for the foreseeable future to find ad hoc methods to solve the exact controllability problem in the framework of entropy solutions.

Feedback stabilization. As we saw in Section 1.1, exact controllability results might be appealing from a mathematical point of view. However from an application point of view, we often prefer to consider the problem of feedback control and asymptotic stabilization.

In the context of equation (1.38), the simplest goal is the following. Given a constant state $\bar{u}$ in $\mathbb{R}$, we are looking for $\mathbb{U}_l$ and $\mathbb{U}_r$ two functionals defined on $L^\infty(a, b)$ and taking values in $\mathbb{R}$ such that the new system

$$\begin{cases} \partial_t u + \partial_x f(u)(t,x) = 0, \\ u(t,a) = \mathbb{U}_l(u(t,\cdot)), \\ u(t,b) = \mathbb{U}_r(u(t,\cdot)), \\ u(0,x) = u_0(x), \end{cases} \quad t > 0, \quad x \in (a,b), \tag{1.41}$$

satisfies the following properties.

- The constant solution $x \mapsto \bar{u}$ is stable :

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall u_0 \in L^\infty(a,b), \quad \|u_0 - \bar{u}\|_\infty \leq \delta \implies \forall t > 0, \ \|u(t,\cdot) - \bar{u}\|_\infty \leq \epsilon. \tag{1.42}$$

- The constant solution $x \mapsto \bar{u}$ is attractive :

$$\forall u_0 \in L^\infty(a,b), \quad \|u(t,\cdot) - \bar{u}\|_\infty \underset{t \to +\infty}{\to} 0. \tag{1.43}$$

Note that we worked with the L^∞ norm but this can be adapted. Of course we should have an existence result for the closed loop system and any maximal solution should be global and satisfy (1.42) and (1.43).

In the framework of regular solutions, the literature on this problem is gigantic. We will only point to the book [20] as an entry point to the literature.

Once again avoiding the framework of entropy solutions is rather unsatisfying.

- It might be self limiting since it imposes additional constraints on the controls.
- Only local results should be accessible because of the form of the controls.
- We are interested in feedback controls because they are supposed to be more robust with respect to perturbations. Clearly the emergence of singularities is such a perturbation so our feedback controls should be ready to tackle them.
- Some interesting targets are the stationary shock waves which are themselves discontinuous. There are hybrid methods (see for instance [21, 84]) designed to work with the regular solutions framework and such targets. However they still suffer from the other three limitations.

Sadly, in this setting it should be clear that one cannot use Russell's extension method to short circuit the boundary conditions. Indeed, the controls computed with that method depend directly on the initial data.

The simplest way to consider the meaning of $u(t,a) = u_l(t)$ in (1.38) is the following. We say that the value $u(t,a)$ is admissible with respect to the boundary condition $u_l(t)$ if when we solve

$$\begin{cases} \partial_s v(s,x) + \partial_x f(v)(s,x) = 0 \\ v(0,x) = u(t,a) & \text{if } x > 0 \\ v(0,x) = u_l(t) & \text{if } x < 0, \end{cases} \quad s > 0, \quad x \in \mathbb{R}, \tag{1.44}$$

then we have

$$\forall s > 0, \quad \forall x > 0, \qquad v(s,x) = u(t,a).$$

The rough idea is that the waves generated by solving the Riemann problem splitting the boundary data and the actual trace of the solution have to leave the domain. Note that as was the case in Section 1.2 we can actually formulate those boundary conditions in term of integral terms extending the Kruzkov formulation (1.33).

Let us describe formally the argument. We start, once again, with a viscous approximation to (1.38)

$$\begin{cases} \partial_t u^\epsilon(t,x) + \partial_x f(u^\epsilon)(t,x) = \epsilon \partial_{xx}^2 u^\epsilon(t,x), \\ u^\epsilon(0,x) = u_0(x), \\ u^\epsilon(t,a) = u_l(t), \\ u^\epsilon(t,b) = u_r(t), \end{cases} \qquad t > 0, \quad a < x < b. \tag{1.45}$$

This system satisfies the boundary condition for all time t thanks to the second order term. Once again we introduce a regular convex entropy E and its associated entropy flux Q connected by formula (1.30). We multiply the first equation in (1.45) by $E(u^\epsilon(t,x))$ to get

$$\partial_t E(u^\epsilon)(t,x) + \partial_x Q(u^\epsilon)(t,x) = \epsilon \partial_{xx}^2 E(u^\epsilon)(t,x) - \epsilon E'(u^\epsilon(t,x))(\partial_x u^\epsilon(t,x))^2. \tag{1.46}$$

We now consider a non-negative test function ϕ in $\mathcal{C}^\infty(\mathbb{R}^2)$, multiply the previous equation by ϕ, integrate on $\mathbb{R} \times (a,b)$ and do some integration by parts to get

$$\begin{aligned} &\int_0^\infty \int_a^b E(u^\epsilon(t,x))\, \partial_t \phi(t,x) + Q(u^\epsilon(t,x))\partial_x \phi(t,x) + \epsilon E(u^\epsilon(t,x))\partial_{xx}^2 \phi(t,x)\, dxdt \\ &\quad + \int_0^\infty \phi(t,a)\left(Q(u_l(t)) - \epsilon \partial_x E(u^\epsilon)(t,a)\right) - \phi(t,b)\left(Q(u_r(t)) - \epsilon \partial_x E(u^\epsilon)(t,b)\right)\, dt \\ &\qquad + \int_0^\infty \epsilon E(u_l(t))\partial_x \phi(t,a) - \epsilon E(u_r(t))\partial_x \phi(t,b)\, dt \\ &\qquad\qquad + \int_a^b E(u_0(x))\phi(0,x)\, dx \geq 0. \end{aligned} \tag{1.47}$$

At this point let us recall that we expect to only have $u^\epsilon \to u$ almost everywhere. Therefore in the expression above the terms in ϵ in the first line and in the third line vanish as soon as we have some L^∞ bound uniform in ϵ. The new difficulty comes from the term in ϵ from the second line. They correspond to the appearance of boundary layers in the vanishing viscosity limit. We refer to Otto [112] for a discussion of the new ideas involved in resolving this difficulty.

Due to those technical difficulties, the literature on feedback controls in the framework of entropy solutions is still relatively tiny [26, 46, 64, 66, 67, 113, 116]. Chapters 4, 5 and 6 provide such results using a variety of different approaches.

Chapitre 2

The Cauchy Problem

This chapter follows closely [37] and to a lesser extent [39].

2.1 The Homogeneous Case

When trying to build a wellposedness theory for scalar conservation laws like

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i} f_i(u)(t,x) = 0, \tag{2.1}$$

one has multiple methods at its disposal as a quick look at [54, Chapter 6] can attest :

- method of vanishing viscosity,
- semigroup theory,
- layering method,
- relaxation method,
- kinetic theory.

However, whatever the method, one cannot help but see the fundamental role played by the family of stationary solutions provided by the constant functions from $\mathbb{R}$ to $\mathbb{R}$. They are of a fundamental nature in particular to get a priori bound for any contractive semigroup. Indeed, following Kruzkov [95, Theorem 3] or [50], it can be shown that any contractive semigroup in $L^1(\mathbb{R})$ has to be order preserving. More precisely if $(S_t)_{t\geq 0}$ is a semigroup on L^1 for any u_0 and v_0 in L^1 we have

$$\forall t \geq 0, \quad \|S_t u_0 - S_t v_0\|_{L^1} \leq \|u_0 - v_0\|_{L^1}, \tag{2.2}$$

then we can show that for any u_0 and v_0 in L^1 we have

$$u_0 \leq v_0 \text{ almost everywhere} \implies \forall t \geq 0, \quad S_t u_0 \leq S_t v_0 \text{ almost everywhere.} \tag{2.3}$$

The idea being that a family of stationary solutions will provide a priori bound using (2.3). To be complete, the argument leading from (2.2) to (2.3) can actually be localized. This allows us to use this idea for the family of constant solutions which are of course not integrable. We will provide the exact argument in the following section since it will be instrumental there, more precisely look at Proposition 3.

The fact that constants are actually solutions is directly related to the invariance by translation of the semigroup of solution. This property was taken for granted in Subsection 1.2.3

when discussing the solution to Riemann data. It turns out that the situation is more complicated. Following the study of discontinuous fluxes (for traffic flow purposes), meaning

$$\begin{cases} \partial_t u + \partial_x f_l(u)(t,x) = 0, & t > 0, \quad x < 0 \\ \partial_t u + \partial_x f_r(u)(t,x), & t > 0, \quad x > 0, \end{cases} \tag{2.4}$$

it was actually realized that there are multiple contractive semigroups of weak solutions of such equations, see for instances [13, 14]. The very rough idea is that when solving (2.4), we use the entropy solution when the singularity is at point that is not 0, and at 0 we use any Riemann solver that is order preserving. When $f_l \neq f_r$ in (2.4) this actually breaks the invariance by translation and prevent the problem which lead to the notion of entropy condition in Subsection 1.2.3. At this point, we have multiple semigroups of weak solutions for a given conservation law. Of course, when the flux does not depend on the spatial variable explicitly, it feels reasonable to suppose that the entropy semigroup can be characterized as the only one that is invariant by all translations. However when the flux depends explicitly on x, this doesn't seem to apply.

2.2 The Non-homogeneous Case

2.2.1 Setup and first result

Compared to the previous section, we will now consider the situation where the flux depends explicitly on the space variable :

$$\begin{cases} \partial_t u(t,x) + \partial_x f(x, u(t,x)) = 0, \\ u(0,x) = u_0(x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}. \tag{2.5}$$

On the face of it, the results of Kruzkov [95] provide a perfectly satisfactory theory to deal with this equation. However, the paper deals with multiple dimensions of space and with source terms. The generality of the framework necessitates hypothesis that are rather too strong when translated to the case of (2.5) :

$$f \in \mathcal{C}^3(\mathbb{R}^2), \tag{2.6}$$

$$\forall K \in \mathbb{R}, \qquad \sup_{(x,u)\in\mathbb{R}\times[-K,K]} |\partial_u f(x,u)| < +\infty, \tag{2.7}$$

$$\sup_{x\in\mathbb{R}} |\partial_x f(x,0)| < +\infty, \qquad \sup_{(x,u)\in\mathbb{R}^2} (-\partial^2_{xu} f(x,u)) < +\infty. \tag{2.8}$$

To see why, let us adapt the Lighthill-Whitham [104] and Richards [121] model to the situation where at some point on the road, we have a transition of one lane to two, and at the same time the speed limit changes. This corresponds to picking in (2.5)

$$f(x,u) := V(x)u(t,x)\left(1 - \frac{u(t,x)}{U(x)}\right), \tag{2.9}$$

where $V(x)$ represents the maximal speed of vehicles at the position x and $U(x)$ the maximum density of vehicles at the same point. We therefore expect V and U to be relatively regular, bounded and to have derivatives in x with compact support, but this clearly does not lead to the last requirement of (2.8).

We proposed therefore in [37] to use the following hypothesis on f :

$$f \in \mathcal{C}^3(\mathbb{R}), \tag{2.10}$$

$$\exists X > 0, \quad \forall (x,u) \in \mathbb{R}^2, \quad |x| > X \implies \partial_x f(x,u) = 0, \tag{2.11}$$

$$\forall h \in \mathbb{R}, \quad \exists \mathcal{U}_h \in \mathbb{R}, \quad \forall (x,u) \in \mathbb{R}^2, \quad |f(x,u)| \leq h \implies |u| \leq \mathcal{U}_h, \tag{2.12}$$

$$\text{for a.e. } x \in \mathbb{R}, \quad \{u \in \mathbb{R} : \partial^2_{uu} f(x,u) = 0\} \text{ has empty interior.} \tag{2.13}$$

At the same time, we defined an entropy solution along the lines of Kruzkov (with a very light modification for the initial data).

Definition 2

Given an initial data u_0 in $L^\infty(\mathbb{R})$, a function $u \in L^\infty((0,T)\times\mathbb{R})$ is an entropy solution to (2.5) when for any real number k and any non-negative test function ϕ in $\mathcal{C}^1_c(\mathbb{R}^2)$ we have

$$\int_0^\infty \int_{-\infty}^\infty \left(|u(t,x) - k|\partial_t \phi(t,x) + \Phi(x,u(t,x),k)\partial_x \phi(t,x)\right)\, dxdt$$
$$- \int_0^{+\infty} \int_{-\infty}^\infty \operatorname{sign}(u(t,x) - k)\partial_x f(x,k)\phi(t,x)\, dxdt$$
$$+ \int_{-\infty}^\infty |u_0(x) - k|\phi(0,x)\, dx \geq 0, \tag{2.14}$$

where the entropy flux Φ is defined by

$$\forall (x,u,k) \in \mathbb{R}^3, \quad \Phi(x,u,k) := \operatorname{sign}(u-k)(f(x,u) - f(x,k)). \tag{2.15}$$

We managed to show that we still got a good wellposedness theory.

Theorem 3: Colombo-Perrollaz-Sylla [37]

There exists a semigroup $(S_t)_{t\geq 0}$ acting on $L^\infty(\mathbb{R})$ such that the following are satisfied.

- The map $(t,x) \mapsto (S_t u_0)(x)$ is the unique maximal entropy solution to (2.5) according to Definition 2.
- The map $t \mapsto S_t u_0$ is continuous with respect to the $L^1_{loc}(\mathbb{R})$ topology,
- The semigroup $(S_t)_{t\geq 0}$ is contractive in L^1_{loc} in the sense that for two initial data u_0 and v_0 we have

$$\forall t \geq 0, \quad \forall R \geq 0, \quad \int_{-R}^{R} |(S_t u_0)(x) - (S_t v_0)(x)| dx \leq \int_{-R-Lt}^{R+Lt} |u_0(x) - v_0(x)| dx. \tag{2.16}$$

  with L being a bound on the speed of propagation given by

$$L := \sup\{|\partial_u f(x,w)| : x \in \mathbb{R}, \quad |w| \leq C\}, \tag{2.17}$$

$$C := \max(\|(t,x)\mapsto (S_t u_0)(x)\|_{L^\infty((0,+\infty)\times\mathbb{R})}, \|(t,x)\mapsto (S_t v_0)(x)\|_{L^\infty((0,+\infty)\times\mathbb{R})}) < +\infty. \tag{2.18}$$

2.2.2 Sketch of the proof

The uniqueness and the properties of the semigroup can be obtained a priori, by marginally tweaking the ideas of Kruzkov [95]. For the existence part however the situation is different. We also start with the viscous approximation :

$$\begin{cases} \partial_t u^\epsilon(t,x) + \partial_x f(x, u^\epsilon(t,x)) = \epsilon \partial_{xx}^2 u^\epsilon(t,x), \\ u^\epsilon(0,x) = u_0(x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \tag{2.19}$$

for ϵ positive but possibly small. The existence of regular solutions to this equation is obtained with rather classical techniques taking inspiration from [86, Appendix B] and relying heavily on the second order term. The difficult part is to get a limit for $\epsilon \to 0$.

L^∞ bound. The first step is to get the following uniform bound in ϵ.

Proposition 1: Colombo-Perrollaz-Sylla [37]

For a Lipschitz initial data u_0, there exists a constant C such that any solution u^ϵ of (2.19) satisfies

$$\forall \epsilon > 0, \quad \|u^\epsilon\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \leq C. \tag{2.20}$$

This is accomplished by slightly tweaking the classical Bernstein method, see [129] for an excellent overview of the general method. Note that Hypothesis (2.12) is instrumental here.

Compensated compactness. The second step is to show that, with the uniform bound from Proposition 1, we do have convergence.

Proposition 2: Colombo-Perrollaz-Sylla [37]

Consider $u_0 : \mathbb{R} \to \mathbb{R}$ a Lipschitz function, there exists a function $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that

$$u^\epsilon(t,x) \underset{\epsilon \to 0}{\to} u(t,x) \quad \text{for a.e. } (t,x). \tag{2.21}$$

This part is accomplished using the compensated compactness method of Tartar [132]. We mostly adapted the proof strategy from [90, Chapter 5] to take into account the spatial dependency of the flux. Hypothesis (2.13) is instrumental here.

Semigroup extension. At this point, we have defined the family $(S_t)_{t\geq 0} u_0$ **only** for Lipschitz functions u_0. We can show that $(t,x) \mapsto (S_t u_0)(x)$ is indeed the unique entropy solution, but we do not have a semigroup yet. The next step is therefore to extend the functions $(S_t)_{t\geq 0}$ to $L^\infty(\mathbb{R})$. We use the classical result on extending uniformly continuous functions and the contractive properties of the $(S_t)_{t\geq 0}$ in L^1_{loc} from the next proposition.

Proposition 3: Colombo-Perrollaz-Sylla [37]

Consider two entropy solutions u and v of (2.5) in $L^\infty((0,+\infty)\times\mathbb{R})$. Then for any positive numbers T and R we have :

$$\int_{-R}^{R} |u(T,x)-v(T,x)|dx \le \int_{-R-LT}^{R+LT} |u(0,x)-v(0,x)|dx, \tag{2.22}$$

$$\int_{-R}^{R} (u(T,x)-v(T,x))^+dx \le \int_{-R-LT}^{R+LT} (u(0,x)-v(0,x))^+dx, \tag{2.23}$$

where L is the maximum speed of propagation, meaning :

$$L := \sup\{|\partial_u f(x,w)| \,:\, x\in\mathbb{R}, \quad |w|\le C\},$$
$$C := \max(\|u\|_{L^\infty((0,T)\times\mathbb{R})}, \|v\|_{L^\infty((0,T)\times\mathbb{R})}).$$

Those a priori estimates respectively follow from the following Kato type inequalities :

$$\partial_t|u-v| + \partial_x\left(\operatorname{sign}(u-v)(f(x,u)-f(x,v))\right)\le 0,$$
$$\partial_t(u-v)^+ + \partial_x\left(\operatorname{sign}((u-v)^+)(f(x,u)-f(x,v))\right)\le 0.$$

The fundamental ideas for the derivation are due to Kruzkov [95] and we will go into the details in Chapter 6.

Conclusion. The last difficulty is to show that the extension still provides entropy solutions. This is done by passing to the limit in Definition 2 thanks to the Dominated Convergence theorem. But of course the tricky part is not the pointwise convergence but the domination part. To that end we proved the following a priori estimate.

Proposition 4: Colombo-Perrollaz-Sylla [37]

For any constant C there exists a constant M such that for any Lipschitz initial data $u_0 : \mathbb{R}\to\mathbb{R}$ we have

$$\|u_0\|_{L^\infty(\mathbb{R})}\le C \implies \forall t\ge 0, \quad \|S_t u_0\|_{L^\infty(\mathbb{R})}\le M. \tag{2.24}$$

Let us first provide a heuristic explanation on why we can reasonably expect this bound to hold. The characteristics equation of (2.5) is the system :

$$\begin{cases}\dot q = \partial_2 f(q,p),\\ \dot p = -\partial_1 f(q,p).\end{cases}$$

It is a Hamiltonian system and a simple calculation shows that f is preserved along solutions : $f(q(t),p(t)) = f(q(0),p(0))$. But since $p(t)=u(t,q(t))$ and $p(0)=u(0,q(0))$, we can reasonably expect a solution u of (2.5) to satisfy

$$\forall (t,x)\in\mathbb{R}^+\times\mathbb{R}, \quad -\|f(\cdot,u_0(\cdot))\|_\infty \le f(x,u(t,x)) \le \|f(\cdot,u_0(\cdot))\|_\infty,$$

which would provide a good a priori bound on the equation when combined with the coercivity hypothesis (2.12). Had we the convexity in u of the flux f we could use the theory of generalized characteristics — anticipating on Chapter 4 — to adapt the previous remark as a proof.

In [37], we adopted a different approach allowing us to bypass this hypothesis. First let us remark that thanks to (2.23) we can see that for any entropy solutions u and v of (2.5) we have

$$u(0,\cdot) \leq v(0,\cdot) \implies \forall t > 0, \quad u(t,\cdot) \leq v(t,\cdot).$$

To prove Proposition 4, it is therefore sufficient to build sufficiently many stationary solutions in $L^\infty(\mathbb{R})$ that will serve as *fences*. In the spatially homogeneous case, the family of constant solutions is obviously up to the task. Here it is more complicated, one should consult [37, Section 3.2] for the precise construction.

The idea is to rectify the level sets $\{(x,u) \in \mathbb{R}^2 \,:\, f(x,u) = c\}$ which are not necessarily functions graph by allowing for jumps at appropriate points. To use this approach one need some tools from differential topology. In particular, the program can be completed only for fluxes f with appropriate geometric properties. Those specific fluxes turn out to be dense — thanks to some arguments coming from transversality theory — for the topology of uniform convergence. One may thus use compensated compactness arguments again to pass to the limit with on the flux in (2.5). Note that in all this, the coercivity hypothesis (2.12) is fundamental while hypothesis (2.13) seems to be a technical requirements from the compensated compactness method.

2.2.3 Complementary results

It turns out that there is another type of equations, intimately connected with (2.5), that is the following Hamilton-Jacobi equation :

$$\begin{cases} \partial_t U(t,x) + f(x, \partial_x U(t,x)) = 0, \\ U(0,x) = U_0(x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \tag{2.25}$$

whose origin we will actually discuss in the next Chapter.

At a formal level, it should be clear by taking the derivative in x of (2.25) that if U is a regular solution of (2.25) then $\partial_x U(t,x)$ is expected to be a regular solution to (2.5). Of course, this formal argument does not apply since we have shown in Subsection 1.2.2 that singularities are expected to be generic for conservation laws. It is therefore also the case for this equation, otherwise the previous argument would be rigorous.

It turns out that there is also a theory of weak solutions for (2.25). They are called viscosity solutions and were introduced by Crandall and Lions in [49]. Let us provide their definition which, in appearance, seem totally unrelated to the notion of entropy solutions.

Definition 3

Consider U a continuous function defined on $\mathbb{R}^+ \times \mathbb{R}$ and satisfying the following properties.

- U is a subsolution of (2.25) meaning that for any test function ϕ in $\mathcal{C}^1(\mathbb{R}^2)$ and for any point (t_0, x_0) in $\mathbb{R}^+_* \times \mathbb{R}$, should $U - \phi$ have a local maximum at (t_0, x_0) then
$$\partial_t \phi(t_0, x_0) + f(x_0, \partial_x \phi(t_0, x_0)) \leq 0.$$
- U is a supersolution of (2.25) meaning that for any test function ϕ in $\mathcal{C}^1(\mathbb{R}^2)$ and for any point (t_0, x_0) in $\mathbb{R}^+_* \times \mathbb{R}$, should $U - \phi$ have a local minimum at (t_0, x_0) then
$$\partial_t \phi(t_0, x_0) + f(x_0, \partial_x \phi(t_0, x_0)) \geq 0.$$
- For any point x in $\mathbb{R}$ we have $U(0, x) = U_0(x)$.

Surprisingly, it turns out that this definition can also be motivated by a vanishing viscosity argument.

Consider U^ϵ a solution to

$$\partial_t U^\epsilon(t, x) + f(x, \partial_x U^\epsilon(t, x)) - \epsilon \partial^2_{xx} U^\epsilon(t, x) = 0. \tag{2.26}$$

If we have a regular test function ϕ and a point (t_0, x_0) in $\mathbb{R}^+_* \times \mathbb{R}$ then

- if $U^\epsilon - \phi$ has a maximum at (t_0, x_0) then of course we know :
$$\partial_t U^\epsilon(t_0, x_0) = \partial_t \phi(t_0, x_0),\ \ \partial_x U^\epsilon(t_0, x_0) = \partial_x \phi(t_0, x_0),\ \ \partial^2_{xx} U^\epsilon(t_0, x_0) \leq \partial^2_{xx} \phi(t_0, x_0),$$
and combining with (2.26) we get
$$\partial_t \phi(t_0, x_0) + f(x_0, \partial_x \phi(t_0, x_0)) - \epsilon \partial^2_{xx} \phi(t_0, x_0) \leq 0,$$
- if $U^\epsilon - \phi$ has a minimum at (t_0, x_0) then of course we know :
$$\partial_t U^\epsilon(t_0, x_0) = \partial_t \phi(t_0, x_0),\ \ \partial_x U^\epsilon(t_0, x_0) = \partial_x \phi(t_0, x_0),\ \ \partial^2_{xx} U^\epsilon(t_0, x_0) \geq \partial^2_{xx} \phi(t_0, x_0),$$
and combining with (2.26) we get
$$\partial_t \phi(t_0, x_0) + f(x_0, \partial_x \phi(t_0, x_0)) - \epsilon \partial^2_{xx} \phi(t_0, x_0) \geq 0.$$

Passing to the limit when $\epsilon \to 0$, we arrive at the condition we requested on the test functions.

Now since (2.26) and (2.19) both have regular solutions, we actually have rigorously

$$\partial_x U_0 = u_0 \implies \partial_x U^\epsilon = u^\epsilon, \tag{2.27}$$

and then taking the vanishing viscosity limit as described in the previous subsection we have that the derivative of viscosity solutions are entropy solutions.

This program was actually completed in [37] were we proved the following results.

Theorem 4: Colombo-Perrollaz-Sylla [37]

There exists a semigroup $(S_t^{HJ})_{t\geq 0}$ defined and taking values in Lip($\mathbb{R}$) — the space of globally Lipschitz functions on $\mathbb{R}$ — and such that the following are satisfied.

- The map $(t,x) \mapsto (S_t^{HJ}U_0)(x)$ is the unique maximal viscosity solution to (2.25) according to Definition 3.
- The map $t \mapsto S_t^{HJ}U_0$ is continuous with respect to the $L^\infty(\mathbb{R})$ topology,
- The semigroup $(S_t^{HJ})_{t\geq 0}$ is contractive in L^∞_{loc} in the sense that for two initial data u_0 and v_0 we have

$$\forall t \geq 0, \quad \forall R \geq 0, \quad \max_{|x|\leq R}\Big((S_t^{HJ}U_0)(x) - (S_t^{HJ}V_0)(x)\Big) \leq \max_{|x|\leq R}(U_0(x) - V_0(x)). \tag{2.28}$$

with L given by

$$L := \sup\{|\partial_u f(x,w)| \,:\, x \in \mathbb{R}, \quad |w| \leq C\}, \tag{2.29}$$

$$C := \max\Big(\|(t,x) \mapsto \partial_x(S_tU_0)(x)\|_{L^\infty((0,+\infty)\times\mathbb{R})}, \\ \|(t,x) \mapsto \partial_x(S_tV_0)(x)\|_{L^\infty((0,+\infty)\times\mathbb{R})}\Big) < +\infty. \tag{2.30}$$

And the semigroups are connected through the following statement.

Theorem 5: Colombo-Perrollaz-Sylla [37]

Given u_0 in $L^\infty(\mathbb{R})$ and U_0 in Lip($\mathbb{R}$) then we have

$$\partial_x U_0 = u_0 \implies \forall t \geq 0, \quad S_t^{HJ}u_0 = \partial_x(S_tU_0), \tag{2.31}$$

where the equalities are taken to mean at almost every point of $\mathbb{R}$.

We can rephrase the above relations with the following commutative diagram.

$$\begin{array}{ccc} U_o & \longrightarrow & S_t^{HJ}U_o \\ \partial_x \downarrow & & \downarrow \partial_x \\ u_o & \longrightarrow & S_t u_o \end{array}$$

2.3 Discussion

The multidimensional case. As we saw, the result of Kruzkov — by its generality itself — can turn out to be a little too restrictive in specific situations. We explained how we could find some more reasonable assumptions in the one dimensional case when the flux depends on x. A natural question is thus to look into the situation for the multidimensional case where there is no source term and the flux depends on the space variable. It is rather natural to have this situation when dealing with crowd dynamics for instance. See [35] for more on the subject.

Interpretation of the Hamilton-Jacobi connection. It is of interest to mention that the proof of existence of the vanishing viscosity limit for (2.25) is actually much simpler than for (2.19). The compactness is a simple application of the Arzela-Ascoli theorem, using some equi-Lipschitz estimate. And passing to the limit in the notion of viscosity solution is actually directly possible with the uniform convergence provided. A related remark is that in the correspondence of Theorem 5, we send semigroup on semigroup as functions, but the continuity properties and the stability properties do not correspond. On one hand, this shows that there are hidden regularization effects for Hamilton-Jacobi equations that are not usually studied. On the other hand, it looks as if there is an underlying concept of solution on scalar conservation laws, which lead to the same solution as the one described by Kruzkov but which would be easier to manipulate. The question would therefore be to access this notion of solution without using the Hamilton-Jacobi connection.

Classification of the semigroups. As we mentioned in the first part of this chapter, it feels reasonable to believe that among the infinite family of contractive semigroups of weak solutions of a scalar conservation law whose flux is independent of x, the entropy semigroup is the only one that is invariant by the group of spatial translation. This would lead to an intrinsic characterization of the entropy semigroup inside this family. The next step would be to find an alternative characterization when the flux *does* depend on x.

Invariance in time. We described in the first part of the Chapter how we could find multiple coherent solution of the wellposedness problem as soon as we forsake the invariance by translation. It might be of interest to see what we could get when removing the semigroup aspect — i.e. removing the invariance by time translation — while still keeping a solution in the sense of Hadamard. This would mean having a family of functions $(S_t)_{t\geq 0}$ which are continuous and such $(t,x) \mapsto (S_t u_0)(x)$ is a weak solution, but we would not require $S_t \circ S_s = S_{t+s}$ anymore.

Furthermore, using the wave front tracking method and non-standard Riemann solvers at precise points in spacetime, it is possible to construct weak solutions that are compactly supported in spacetime in the spirit of [126, 130, 57, 58]. Would it be possible to get them as particular trajectories of those previous families of coherent solutions in the sense of Hadamard or are they purely sporadic solutions ?

Chapitre 3

Exploiting the Hamilton-Jacobi connection

This chapter is concerned with [36, 38, 40].

3.1 Connection between the Hamilton-Jacobi equation and an optimization problem

We saw in the previous section that the conservation conservation law (2.5) and the Hamilton-Jacobi equation (2.25) are connected through Theorem 5. One could thus try to solve the problems described in Section 1.3 by working at the level of the Hamilton-Jacobi equation.

The true advantage of working with the Hamilton-Jacobi equation is actually when the flux $(x,u) \mapsto f(x,u)$ is strongly convex in u (i.e. f' is an increasing diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$). Which, by the way, also implies that f satisfies Hypothesis (2.12) and Hypothesis (2.13).

To see those advantages, let us start by introducing the function $L : \mathbb{R}^2 \to \mathbb{R}$ as the Legendre transform of f :

$$\forall (x,q) \in \mathbb{R}^2, \quad L(x,q) := \sup_{p \in \mathbb{R}} \left(p\, q - f(x,p) \right). \tag{3.1}$$

We will then look at the following optimization problem :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \quad V(t,x) := \inf_{\substack{\gamma(t)=x \\ \gamma \in \mathrm{Lip}(0,t)}} \left(\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + U_0(\gamma(0)) \right). \tag{3.2}$$

The surprising result is the following.

Theorem 6: Crandall-Lions [48]

The value function V of the previous optimization problem is the viscosity solution of (2.25).

Let us describe heuristically the proof of this classical result (and refer to [18, 16] for a rigorous presentation).

The first step is to obtain the following result called the Dynamic Programming Principle by Richard Bellman (see [22]). It is based on the heuristic that for a minimizer to be optimal from $t = t_0$ to $t = t_1$, it has to be optimal from $t = t_0$ to $t = t_i$ and optimal from from $t = t_i$ to $t = t_1$.

Proposition 5

For any positive times t_0, t_1 and for any point x in $\mathbb{R}$, if $t_0 < t_1$ the function V satisfies :

$$V(t_1, x) = \inf_{\substack{\gamma(t_1)=x \\ \gamma \in \mathrm{Lip}(0,t_1)}} \left(\int_{t_0}^{t_1} L(\gamma(s), \dot{\gamma}(s))\, ds + V(t_0, \gamma(t_0)) \right). \tag{3.3}$$

From this point, the *very informal* argument is as follows :

$$\begin{aligned} V(t+\epsilon, x) &= \inf_{\gamma} \left(\int_{t}^{t+\epsilon} L(\gamma(s), \dot{\gamma}(s))\, ds + V(t, \gamma(t)) \right) \\ &= \inf_{\gamma} \left(\epsilon L(\gamma(t+\epsilon), \dot{\gamma}(t+\epsilon)) + V(t, \gamma(t+\epsilon) - \epsilon\dot{\gamma}(t+\epsilon)) + o(\epsilon) \right) \\ &= \inf_{p \in \mathbb{R}} \left(\epsilon L(x, p) + V(t, x) - \epsilon \partial_x V(t, x) p + o(\epsilon) \right), \end{aligned}$$

introducing $p = \dot{\gamma}(t+\epsilon)$. We can reorganize the last equality to get :

$$\begin{aligned} \frac{V(t+\epsilon, x) - V(t, x)}{\epsilon} &= \inf_{p \in \mathbb{R}} \left(L(x, p) - p\partial_x V(t, x) \right) && (3.4) \\ &= -\sup_{p \in \mathbb{R}} \left(p\partial_x V(t, x) - L(x, p) \right) && (3.5) \\ &= -f(x, \partial_x V(t, x)), && (3.6) \end{aligned}$$

since when f is strongly convex, the Legendre transform (3.1) is an involution. Passing to the limit $\epsilon \to 0^+$ we get the Hamilton-Jacobi equation (2.25) as announced.

We will use this connection between conservation laws and optimization problems to solve the inverse design problem — described in Subsection 1.3.2 — in the next two sections. Note however that this connection was used to attack the exact controllability problem by some authors, for instance in [10].

3.2 The Homogeneous Case

In this section, the flux f does not depend on x and is supposed to be strongly convex. This leads to some big simplifications in the optimization problem from the previous section.

The Lax-Hopf formula. First this actually implies that L — defined in (3.1) — is also independent of x and also strongly convex. But then using Jensen inequality, one can see that for any Lipschitz function γ defined on $[0,t]$ we have

$$\begin{aligned}\int_0^t L(\dot\gamma(s))\,ds &= t\int_0^t L(\dot\gamma(s))\,\frac{ds}{t}\\ &\geq t\,L\left(\int_0^t \dot\gamma(s)\frac{ds}{t}\right)\\ &\geq tL\left(\frac{\gamma(t)-\gamma(0)}{t}\right).\end{aligned}$$

Therefore it should be clear — considering the possibility that γ might be linear — that the viscosity solution of

$$\begin{cases}\partial_t U(t,x) + f(\partial_x U(t,x)) = 0, \\ U(0,x) = U_0(x),\end{cases} \quad t>0, \quad x\in\mathbb{R}, \tag{3.7}$$

actually satisfies

$$\forall (t,x)\in\mathbb{R}^+\times\mathbb{R}, \quad U(t,x) = \inf_{y\in\mathbb{R}}\left(t\,L\left(\frac{x-y}{t}\right) + U_0(y)\right). \tag{3.8}$$

And therefore one can deduce that the entropy solution u of

$$\begin{cases}\partial_t u(t,x) + \partial_x f(u)(t,x) = 0, \\ u(0,x) = u_0(x),\end{cases} \quad t>0, \quad x\in\mathbb{R}, \tag{3.9}$$

satisfies

$$\forall (t,x)\in\mathbb{R}^+\times\mathbb{R}, \quad u(t,x) = (f')^{-1}\left(\frac{x-y(t,x)}{t}\right)$$
$$\text{where } y(t,x)\in\underset{z\in\mathbb{R}}{\operatorname{argmin}}\left(t\,L\left(\frac{x-z}{t}\right) - \int_0^z u_0(\xi)d\xi\right). \tag{3.10}$$

This is actually called the Lax-Hopf formula, it was derived by Hopf in [87] for the case $f(u) = u^2/2$ and by Lax in [96] for the general case. Remarkably long before the theory of viscosity solutions and the connection to optimal control problem was put forward.

Reachability condition. Now it was shown in [96] that the minimizer function $y(t,x)$ is actually non-decreasing in x, which actually shows that an entropy solution to (3.9) satisfies the so called Oleinik condition (from the paper [111]) :

$$\forall (t,x_1,x_2)\in\mathbb{R}_*^+\times\mathbb{R}\times\mathbb{R}, \quad x_1<x_2 \implies f'(u(t,x_2)) - f'(u(t,x_1)) \leq \frac{x_2-x_1}{t}. \tag{3.11}$$

This provides a necessary condition for a state to be reachable through the entropy semigroup in time t. In fact it is the key to solving the full inverse design problem.

Theorem 7: Colombo-Perrollaz [36]

Let us fix the precise representative w of an element of $L^\infty(\mathbb{R})$ (see [71, Chapter 1, Section 7]) and a positive time T. We define the auxiliary function $\pi : \mathbb{R} \to \mathbb{R}$ through

$$\forall x \in \mathbb{R}, \quad \pi(x) := x - T f'(w(x)). \tag{3.12}$$

Then calling $(S_t)_{t\geq 0}$ the semigroup generated by (3.7), the inverse design set

$$I_T(w) := \{u_0 \in L^\infty(\mathbb{R}) \,:\, S_T u_0 = w\} \tag{3.13}$$

is non empty if and only if π is non-decreasing.

The proof of the previous statement is actually related to the more explicit construction that follows.

Proposition 6: Colombo-Perrollaz [36]

If π — defined in (3.12) — is non-decreasing then the entropy solution χ of

$$\begin{cases} \partial_s \chi + \partial_y f(\chi) = 0, \\ \chi(0, y) = w(-y), \end{cases}$$

is isentropic : for any entropy entropy flux couple (E, Q) we have

$$\partial_s E(\chi) + \partial_y Q(\chi) = 0.$$

As a consequence $\rho : (t, x) \mapsto \chi(T - t, -x)$ is an entropy solution of

$$\partial_t \rho + \partial_x f(\rho) = 0.$$

And finally $u_0^* := \rho(0, \cdot) \in I_T(w)$.

Note that we refer to [36] for the previous results because it is difficult to find a proper attribution in the literature. But it is clearly older.

Inverse design problem. This particular solution turns out to be the key to solve the inverse design problem. In [36], we provided a complete characterisation of the inverse design set. We will not provide it here since it is actually much more readable to work directly with the Hamilton-Jacobi equation. And we will do precisely so in the next section when dealing with the non-homogeneous problem. We will, however, provide a result describing the geometrical/topological structure of the set of inverse designs, which is in fact a consequence of the characterization.

Theorem 8: Colombo-Perrollaz [36]

Let us once again fix the precise representative w of an element of $L^\infty(\mathbb{R})$ and a positive time T. Then $I_T(w) \neq \emptyset$ implies

1. $I_T(w)$ convex,
2. $I_T(w)$ is a cone of vertex u_0^*,
3. $I_T(w)$ is a F_σ set for the L^1_{loc} topology.

Furthermore

1. $I_T(w)$ singleton if and only if additionally $w \in \mathcal{C}^0$,
2. otherwise it is unbounded in L^∞, but once localized in L^∞ it is closed for the L^1_{loc} topology,
3. there is no extremal facet of finite dimension besides $\{u_0^*\}$.

The last remark means that given any initial data u_0 in $I_T(w)$ different from u_0^* and any positive natural number d, we can write u_0 as a barycenter of a d-dimensional simplex whose vertices are in $I_T(w)$. In particular, we see that given a point of $I_T(w)$ different than u_0^* there is an infinite number of dimensions in which we can move freely while staying in $I_T(w)$. Alternatively, there are also an infinite number of dimensions in which we cannot move if we want to stay in $I_T(w)$.

3.3 The Non-homogeneous Case

Let us now go back to the case where f does depend on x; though not outside of an interval $[-X, X]$ since we rely on the results of the previous chapter. We will also require that $u \mapsto f(x, u)$ is strongly convex for any u.

We can solve the inverse design problem in a manner that is at first reminiscent of the previous section.

Setup and first results. Let us start by describing the replacement to the function π introduced in Theorem 7. The method of characteristics for the equation

$$\partial_t u + \partial_x \left(f(x, u)\right) = 0, \tag{3.14}$$

involves the following system of ordinary differential equations

$$\begin{cases} \dot{q}(t) = \partial_u f(q(t), p(t)) \\ \dot{p}(t) = -\partial_x f(q(t), p(t)). \end{cases} \tag{3.15}$$

A straightforward computation shows that for a regular solution u of (3.14), and a solution (q, p) of (3.15) if for some t_0 we have $p(t_0) = u(t_0, q(t_0))$ then

$$\forall t \quad p(t) = u(t, q(t)).$$

Let us fix w in $L^\infty(\mathbb{R})$ and a positive time T. Given a real x, consider (q, p) the maximal solution of (3.14) satisfying $q(T) = x$ and $p(T) = w(x)$. With the hypothesis we made on f at the beginning of the section we can show that the solution is global in time. We then define $\pi(x) := q(0)$.

We can now provide the analogous result to Theorem 7.

Theorem 9: Colombo-Perrollaz-Sylla [38]

Let us fix the precise representative w of an element of $L^\infty(\mathbb{R})$ and a positive time T. Then calling $(S_t)_{t\geq 0}$ the semigroup generated by (3.14) — following the results of Chapter 2 — the inverse design set

$$I_T(w) := \{u_0 \in L^\infty(\mathbb{R}) \,:\, S_T u_0 = w\} \tag{3.16}$$

is non empty if and only if π is non-decreasing.

When $I_T(w)$ is not empty, it turns out that there exists again a particular initial data u_0^* — that we will describe later — that provides an analogous result to Theorem 8.

Theorem 10: Colombo-Perrollaz-Sylla [38]

Let us once again fix the precise representative w of an element of $L^\infty(\mathbb{R})$ and a positive time T. Then $I_T(w) \neq \emptyset$ implies

1. $I_T(w)$ convex,
2. $I_T(w)$ is a cone of vertex u_0^*,
3. $I_T(w)$ is a F_σ set for the L^1_{loc} topology.
4. there is no extremal facet of finite dimension besides $\{u_0^*\}$.

At this point, it would seem that the non-homogeneous case is just an extension of the homogeneous case with more complicated objects and proofs. It turns out that the situation is more complicated. In particular, the initial data u_0^* cannot be computed by reversing time like it was in Proposition 6. More precisely we have the following result.

Consider f defined by

$$\forall (x,u) \in \mathbb{R}^2, \quad f(x,u) := \begin{cases} \frac{u^2}{2} + 1 - (1-x^2)^4 & \text{if } |x| \leq 1, \\ \frac{u^2}{2} + 1 & \text{otherwise.} \end{cases} \tag{3.17}$$

Theorem 11: Colombo-Perrollaz-Sylla [38]

There exists a function w in $L^\infty(\mathbb{R})$ and a time T such that $I_T(w) \neq \emptyset$ but for any initial data $u_0 \in I_T(w)$ we have an entropy-entropy flux couple (E, Q) such that

$$\partial_t E(S_t u_0) + \partial_x Q(x, S_t u_0) \neq 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}). \tag{3.18}$$

Roughly speaking there is a state that is reachable but never in a reversible — i.e. isentropic — way. In fact, we can even show that there is a minimum entropy production for reaching the state. Looking at the Hamilton-Jacobi connection, this is due to the possibility of encountering focal points in the underlying minimization problem.

Alternative construction. It turns out that it is actually much more convenient to work with the Hamilton-Jacobi equation

$$\partial_t U(t,x) + f(x, \partial_x U(t,x)) = 0, \tag{3.19}$$

its generated semigroup from Theorem 4, $(S_t^{HJ})_{t\geq 0}$ and the inverse designs set :

$$\forall T > 0, \quad \forall W \in \mathrm{Lip}(\mathbb{R}), \quad I_T^{HJ}(W) := \{U_0 \in \mathrm{Lip}(\mathbb{R}) \, : \, S_T^{HJ} U_0 = W\}. \tag{3.20}$$

It is then just a matter of applying Theorem 5 to transfer all results to the conservation law (3.14). Of course, Theorem 9 and Theorem 10 also have an equivalent for equation (3.19). Note however that π stays the same. The key point is that the vertex of the cone of inverse designs — say U_0^* — can be accessed directly through the optimization problem.

Working toward that end, let us introduce

$$\mathcal{R}_T := \{q \in \mathcal{C}^1([0,T]) \, : \, \exists p \in \mathcal{C}^1([0,T]), \quad (q,p) \text{ solves } (3.15)\}. \tag{3.21}$$

Given a Lipschitz function W and a positive time T, we almost have (see [38] for the precise details) that π is non-decreasing if and only if the function U_0^* defined by

$$\forall x \in \mathbb{R}, \quad U_0^*(x) := \sup_{\substack{q \in \mathcal{R}_T \\ q(0)=x}} W(q(T)) - \int_0^T L(q(s), \dot{q}(s)) ds, \tag{3.22}$$

is in $I_T^{HJ}(W)$.

Furthermore, once we have π and U_0^* we have a perfect characterization of the inverse design set.

Theorem 12: Colombo-Perrollaz-Sylla [38]

Given a Lipschitz function W and a positive time T such that π — defined using $w = \partial_x W$ — is non-decreasing. For any Lipschitz function U_0, $U_0 \in I^{HJ}(W)$ if and only if

$$\begin{cases} \forall x \in \mathbb{R}, & U_0(x) \geq U_0^*(x), \\ \forall y \in \mathbb{R}, & U_0(\pi(y)) = U_0^*(\pi(y)). \end{cases} \tag{3.23}$$

Let us now describe the ideas involved in proving the previous results.

Reduction of the trajectories. First starting from the definition of the value function in (3.2) and Theorem 6 we know that

$$\forall t \geq 0, \ \forall x \in \mathbb{R}, \quad (S_t^{HJ} U_0)(x) = \inf_{\substack{\gamma(t)=x \\ \gamma \in \mathrm{Lip}(0,t)}} \left(\int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds + U_0(\gamma(0)) \right).$$

We saw in the previous section that when there is no x dependency, it is enough to consider straight lines in the previous infimum, thanks to the convexity and to Jensen inequality. When there is a dependency in x the situation is more tricky but we still have

$$\forall x \in \mathbb{R}, \quad (S_T U_0)(x) := \min_{\substack{q \in \mathcal{R}_T \\ q(T)=x}} \int_0^T L(q(s), \dot{q}(s)) ds + U_0(q(0)), \tag{3.24}$$

where $\mathcal{R}_T$ was defined in (3.21). This reduction is based on proving that the infimum is actually a minimum, and then applying the Pontryagin maximum principle. We refer to [103] for a delightful introduction to this tool and to [98] for reference purposes. Note that $\{q \in \mathcal{R}_T : q(T) = x\}$ is indeed a one parameter set due to (3.15).

Motivation for U_0^*. At this point we see that $S_T U_0$ is characterized by

$$\forall q \in \mathcal{R}_T, \quad (S_T U_0)(q(T)) \leq U_0(q(0)) + \int_0^T L(q(s), \dot{q}(s)) ds, \tag{3.25}$$

and

$$\forall x \in \mathbb{R}, \quad \exists y \in \mathbb{R}, \quad \exists q \in \mathcal{R}_T, \quad \begin{cases} q(T) = x, \\ q(0) = y, \\ (S_T U_0)(x) = U_0(y) + \int_0^T L(q(s), \dot{q}(s)) ds. \end{cases} \tag{3.26}$$

The key point here is that (3.25) is symmetric in 0 and T. Moving the integral term to the left hand side, we arrive at the Definition (3.22) of U_0^*. It guarantees that (3.25) is satisfied with W instead of $S_T U_0^*$. The difficulty is of course to show that (3.26) is also satisfied. Because what follows from the definition of U_0^* is actually :

$$\forall y \in \mathbb{R}, \quad \exists x \in \mathbb{R}, \quad \exists q \in \mathcal{R}_T, \quad \begin{cases} q(T) = x, \\ q(0) = y, \\ W(x) = U_0(y) + \int_0^T L(q(s), \dot{q}(s)) ds, \end{cases} \tag{3.27}$$

so we have to reverse the quantifiers. This is where the hypothesis on π — and thus on W — is critical.

We will not go into more details here but will just mention a remarkable fact that is key. In the formula (3.24) there exists a minimizer depending only on $S_T U_0$. It is the solution q of

$$\begin{cases} \dot{q}(t) = \partial_2 f(q(t), p(t)), \\ \dot{p}(t) = -\partial_1 f(q(t), p(t)), \\ q(T) = x, \\ p(T) = \partial_x W(x^-). \end{cases}$$

3.4 Discussion

Numerics. As we explained in Subsection 1.3.2, the inverse design problem is also motivated by concrete applications. An important goal is thus to develop robust numerical methods to approximate the set of inverse designs so as to solve optimization problems on it. The difficulty is quite apparent given the geometry/topology of the inverse design set as described in Theorem 8. We refer to [80] for a general discussion on the subject, and to [102] for some more recent attempts.

Using vanishing viscosity. In a sequence of papers [68, 69, 70], Esteve and Zuzua provided an alternative approach to the problem. They focused on the viscous forward in time and backward in time approximations of the Hamilton-Jacobi equation :

$$\begin{cases} \partial_t U^\epsilon + H(\partial_x U^\epsilon) = \epsilon \partial^2_{xx} U^\epsilon, \\ U^\epsilon(0,x) = U_0(x), \end{cases} \tag{FORWARD}$$

$$\begin{cases} \partial_t V^\epsilon + H(\partial_x V^\epsilon) = -\epsilon \partial^2_{xx} V^\epsilon, \\ V^\epsilon(T,x) = V_0(x). \end{cases} \tag{BACKWARD}$$

Passing to the limit in the operators $U_0 \mapsto U(T,\cdot)$ and $V_0 \mapsto V(0,\cdot)$ provided them with two operators, say F_T and B_T which allowed them to characterize the inverse designs set. This method also works to the multi dimensional setting for the Hamilton-Jacobi equation. In fact, our definition of U_0^* in (3.22), show that we could have introduced it by solving a Hamilton-Jacobi equation backward in time. Furthermore they managed to show that $F_T \circ B_T$ provides the closest reachable state in the L^2 topology. The non-homogeneous case is however more tricky as we saw in the last section and as can be glimpsed reading the paper by Barron-Cannarsa-Jensen-Sinestrari [19]. Another dangerous point is that the role of the convexity — and thus the connection to the optimal control problem — seems a bit more hidden with this approach.

More general settings. Once the flux is not convex anymore, the situation is more tricky. First of all when using some toy fluxes which are polygonal but not convex, it is relatively easy to find inverse designs sets that are not convex anymore. A structure theorem equivalent to Theorem 8 is thus not easy to conjecture.

From a technical point of view, we still have the connection to the Hamilton-Jacobi equations thanks to the result of Chapter 2. But there is no underlying optimal control problem anymore. In some cases, it is possible to get a connection to a differential game problem, see [92]. For a running cost and a dynamic evolution function satisfying the Isaacs condition, it so happens that it is always possible to go from the differential game to some Hamilton-Jacobi equations, and thus to conservation laws. However, it is not clear at all if the inverse is possible, since we don't have an equivalent to the involution of the Legendre transform (for the optimal control problem case). To be clearer, it always possible to go back from the Hamilton-Jacobi equation to a differential game problem. However it is not clear if it possible to do so while making sure that the running cost satisfies the Isaacs condition.

Let us provide an horribly flawed calculation that does make the point. Roughly speaking a differential game will look like the following. Let us fix three regular functions $F : \mathbb{R}^3 \to \mathbb{R}$, $L : \mathbb{R}^3 \to \mathbb{R}$ and $K : \mathbb{R} \to \mathbb{R}$. We consider for a fixed (t_0, x_0) in $(0, +\infty) \times \mathbb{R}$ we define two value functions through

$$\begin{cases} U^-(t_0,x_0) := \sup\limits_{j_1} \inf\limits_{j_2} \int_0^{t_0} L(x(t), j_1(t), j_2(t))dt + K(x(0)), \\ U^+(t_0,x_0) := \inf\limits_{j_2} \sup\limits_{j_1} \int_0^{t_0} L(x(t), j_1(t), j_2(t))dt + K(x(0)), \end{cases}$$

where the function j_1 is the strategy of the first player and the function j_2 that of the second player, and x is given by

$$\begin{cases} \dot{x}(t) = F(x(t), j_1(t), j_2(t)), \\ x(t_0) = x_0, \end{cases}$$

We always have $U^- \leq U^+$, and using the same ideas than in Section 3.1 we can show that U^- is supersolution of

$$\begin{cases} \partial_t U + H^-(x, \partial_x U) = 0, \\ U(0, \cdot) = K, \end{cases}$$

while U^+ is a subsolution of

$$\begin{cases} \partial_t U + H^+(x, \partial_x U) = 0, \\ U(0, \cdot) = K, \end{cases}$$

where H^- and H^+ are defined by

$$\begin{cases} H^-(x, p) := \inf_a \sup_b pF(x, a, b) - L(x, a, b), \\ H^+(x, p) := \sup_b \inf_a pF(x, a, b) - L(x, a, b). \end{cases}$$

The Isaacs condition is the hypothesis that $H^- = H^+$. The comparison result [37, Theorem 2.8] between viscosity subsolution and supersolution and the Isaacs condition then implies $U^- = U^+$ allowing to solve the original differential game problem.

So the precise questions we were asking before are the following. What are the flux functions $(x, u) \mapsto f(x, u)$ for which we can we find F and L such that in the previous construction we end up with $H^+ = f = H^-$? And if this is not all fluxes, does this class have better properties ?

Chapitre 4

Generalized Characteristics

This chapter is mostly concerned with [115, 116]. Note however that the classical theory of characteristics was also used in [113, 117, 118, 2] for control purposes.

4.1 Boundary conditions

The goal of this section is to describe — more precisely than we did in 1.3.3 — how we can take boundary conditions into account.

The following notation will be frequently used.

$$\forall (a,b) \in \mathbb{R}^2, \quad I(a,b) := [\min(a,b), \max(a,b)]\,. \tag{4.1}$$

Let us explain how we relax the Dirichlet boundary conditions for the equation $\partial_t u + \partial_x f(u) = 0$. To this end, we need to introduce

$$\forall a \in \mathbb{R}, \quad \begin{cases} \mathcal{A}_l(a) := \{b \in \mathbb{R} \,:\, \forall k \in I(a,b) \quad \operatorname{sign}(b-a)(f(b)-f(k)) \leq 0\}, \\ \mathcal{A}_r(a) := \{b \in \mathbb{R} \,:\, \forall k \in I(a,b) \quad \operatorname{sign}(b-a)(f(b)-f(k)) \geq 0\}. \end{cases} \tag{4.2}$$

The boundary data u_l and u_r will be two BV_{loc} functions of time defined on $\mathbb{R}^+$. The initial data u_0 is in $BV(0,L)$. Rather than asking for $u(t,0^*) = u_l(t)$ and $u(t,L^-) = u_r(t)$ to hold for almost all time, we require instead

$$\begin{cases} \partial_t u(t,x) + \partial_x(f(u))(t,x) = 0, \\ u(t,0^+) \in \mathcal{A}_l(u_l(t)), \\ u(t,L^-) \in \mathcal{A}_r(u_r(t)), \\ u(0,x) = u_0(x). \end{cases} \qquad t > 0, \quad x \in (0,L). \tag{4.3}$$

Thanks to [97] we will still get the existence and uniqueness of an entropy solution. Furthermore, the unique entropy solution u will be such that the mappings $x \mapsto u(t,x)$ will be in $BV(0,L)$ for almost all time. This in particular implies that we will a have pointwise boundary trace of u for almost all time, so the boundary conditions in (4.3) *do* make sense.

Remark 4.1.1. Let us analyze the meaning of the boundary conditions. Given two real numbers a and b, having $b \in \mathcal{A}_l(a)$ is equivalent to requiring

- either $b = a$,

- or if $b \neq a$ we clearly have

$$\forall k \in I(a,b), \quad \text{sign}(b-a) = \text{sign}(b-k).$$

and thus $b \in \mathcal{A}_l(a)$ becomes equivalent to requiring

$$\forall k \in I(a,b), \quad |b-k|\frac{f(b)-f(k)}{b-k} \leq 0. \tag{4.4}$$

Which looking back at Oleinik's entropy condition (1.27) means that the solution to the Riemann problem with a on the left and b on the right generates waves only on the left part of the upper plane.

So in the end, when $u(t,0^+) \in \mathcal{A}_l(u_l(t))$, either the trace is equal to the boundary condition or the solution to the Riemann problem between the two has all its waves leaving $(0,L)$.

The same kind of interpretation holds for the boundary condition at $x = L$. One should consult [65] for a follow up on this kind of description.

In the following, we will always consider a flux f which will be $\mathcal{C}^2$ and strongly convex. This will provide a regularization effect such that if the initial data is just L^∞ then at any positive time the entropy solution satisfies $x \mapsto u(t,x)$ is in BV, see [54, Theorem 11.2.2] for a proof. The sets $\mathcal{A}_l$ and $\mathcal{A}_r$ are also simpler under this condition on f.

4.2 Definition and properties of generalized characteristics

In this section, we will explain how to extend the method of characteristics introduced in Subsection 1.2.2 to deal with entropy solutions of a scalar conservation law whose flux is convex. This is the theory of generalized characteristics due to Dafermos [51]. We start with the classical definition, avoiding the question of the boundary.

Definition 4: Dafermos [51]

- If γ is an absolutely continuous function defined on an interval $(a,b) \subset \mathbb{R}^+$ and with values in $(0,L)$, we say that γ is a generalized characteristic of (4.3) when

$$\dot{\gamma}(t) \in I(f'(u(t,\gamma(t)^-)), f'(u(t,\gamma(t)^+))) \qquad dt \text{ a.e.}.$$

  This is the classical characteristic ODE taken in the weak sense of Filippov [72].
- A generalized characteristic γ is said to be genuine on (a,b) if :

$$u(t,\gamma(t)^+) = u(t,\gamma(t)^-) \quad dt \text{ a.e.}.$$

Let us recall a first result holding for all generalized characteristics.

Theorem 13: Dafermos [51]

- For any (t, x) in $(0, +\infty) \times (0, L)$ there exists at least one generalized characteristic γ defined on an interval (a, b) such that $a < t < b$ and $\gamma(t) = x$.
- If γ is a generalized characteristics defined on (a, b) then for almost all t in (a, b) :

$$\dot{\gamma}(t) = \begin{cases} f'(u(t, \gamma(t))) & \text{if } u(t, \gamma(t)^+) = u(t, \gamma(t)^-), \\ \frac{f(u(t,\gamma(t)^+)) - f(u(t,\gamma(t)^-))}{u(t,\gamma(t)^+) - u(t,\gamma(t)^-)} & \text{if } u(t, \gamma(t)^+) \neq u(t, \gamma(t)^-). \end{cases}$$

And now more specifically we recall a result on genuine characteristics.

Theorem 14: Dafermos [51]

- If γ is a genuine generalized characteristics on (a, b) — with $\gamma(a), \gamma(b) \in (0, L)$ — then there exists a $\mathcal{C}^1$ function v defined on (a, b) such that :

$$\begin{gathered} u(b, \gamma(b)^+) \leq v(b) \leq u(b, \gamma(b)^-), \\ u(t, \gamma(t)^+) = v(t) = u(t, \gamma(t)^-) \quad \forall t \in (a, b), \\ u(a, \gamma(a)^-) \leq v(a) \leq u(a, \gamma(a)^+). \end{gathered} \tag{4.5}$$

  Furthermore (γ, v) satisfy the classical ODE equation :

$$\begin{cases} \dot{\gamma}(t) = f'(v(t)), \\ \dot{v}(t) = 0, \end{cases} \qquad \forall t \in (a, b). \tag{4.6}$$

- Two genuine characteristics may intersect only at their endpoints.
- If γ_1 and γ_2 are two generalized characteristics defined on (a, b), then we have :

$$\forall t \in (a, b), \quad (\gamma_1(t) = \gamma_2(t) \Rightarrow \forall s \geq t,\ \gamma_1(s) = \gamma_2(s)).$$

- For any (t, x) in $\mathbb{R}^+ \times (0, L)$ there exist two generalized characteristics χ^+ and χ^- called maximal and minimal and associated to v^+ and v^- by (4.6), such that if γ is a generalized characteristic going through (t, x) then

$$\begin{gathered} \forall s \leq t, \qquad \chi^-(s) \leq \gamma(s) \leq \chi^+(s), \\ \chi^+ \text{ and } \chi^- \text{ are genuine on } \{s < t\}, \\ v^+(t) = u(t, x^+) \qquad \text{and} \qquad v^-(t) = u(t, x^-). \end{gathered}$$

Note that, so far, every property dealt only with the interior of $\mathbb{R}^+ \times [0, L]$. The following result describe the influence of the boundary conditions on the generalized characteristics.

Theorem 15: Perrollaz [116]

Let u be the unique entropy solution of (4.3) and consider χ a genuine characteristic on an interval $[a, b]$ such that

$$\forall t \in (a, b), \qquad \chi(t) \in (0, L),$$

then we know from Theorem 14 above that there is a constant $\nu \in \mathbb{R}$ such that

$$\forall t \in [a, b], \qquad \dot{\chi}(t) = f'(\nu)$$

and

$$\forall t \in (a, b), \qquad u(t, \gamma(t)) = \nu.$$

But then we have

$$\chi(a) = 0 \Rightarrow u_l(a^+) \leq \nu \leq u_l(a^-), \tag{4.7}$$

$$\chi(a) = L \Rightarrow u_r(a^-) \leq \nu \leq u_r(a^+). \tag{4.8}$$

The existence of the limits is a consequence of u_l and u_r being regulated.

The main difficulty of this result is that the boundary conditions from (4.3) hold only for almost all time. Therefore we can't directly use them at the exact time where a genuine characteristic touches the boundary. Furthermore, if one think of a rarefaction wave spreading from the boundary, it should be clear that the result cannot be improved.

4.3 Stabilization of constant solutions

For the sake of simplicity, let us consider in this section — and the next — the Burgers' equation :

$$\partial_t u + \partial_x \left(u^2/2\right) = 0. \tag{4.9}$$

Note however that the results of those sections apply to fluxes f which are strongly convex as shown in [115, 116].

Clearly we have a simple family of stationary solutions $(\bar{u}_m)_{m \in \mathbb{R}}$ defined by

$$\forall x \in \mathbb{R}, \quad \bar{u}_m(x) = m. \tag{4.10}$$

Thanks to the comparison principle, an entropy solution u of (4.9) associated to an initial data u_0 satisfies :

$$\forall t \geq 0, \quad \operatorname{essinf} u_0 \leq u(t, x) \leq \operatorname{essup} u_0, \text{ for almost every } x. \tag{4.11}$$

Subtracting the constant solution $\bar{u}_m$ to the previous inequality, we see that any of those stationary solutions is stable in L^∞ norm.

Looking at the results of [54, Chapter 11 Section 5], we actually have a stronger result. If $u_0 - \bar{u}_m$ is integrable we actually have asymptotic stability at polynomial speed.

In the presence of a boundary, we actually have the following improved result.

Theorem 16: Perrollaz [115]

Fix a positive L and a stationary solution $\bar{u}_m$ with m positive. For any initial data u_0 in $L^\infty(0, L)$ the entropy solution u of

$$\begin{cases} \partial_t u(t,x) + \partial_x(u^2/2)(t,x) = 0, \\ u(0,x) = u_0(x), \\ u(t, L^-) \in [0, +\infty[, \\ u(t, 0^+) \in]-\infty, -m] \cup \{m\}, \end{cases} \quad t > 0, \quad x \in (0, L), \tag{4.12}$$

satisfies :

$$\forall t \geq \left(-W_{-1}(-e^{-3})\right) \frac{L}{m}, \quad u(t,x) = m, \text{ for almost every } x. \tag{4.13}$$

Here W_{-1} is the second real branch of the Lambert function and $-W_{-1}(-e^{-3}) \approx 4.505241495792883$, see [43] for more information.

Note that the boundary conditions in (4.12) are $u(t, 0^+) \in \mathcal{A}_l(m)$ and $u(t, L^-) \in \mathcal{A}_r(m)$ from Section 4.1 specialized to the case of Burgers' equation. Let us provide the key steps of the proof.

Dispersion. For any initial data u_0 in $L^\infty(0, L)$, consider u the entropy solution of (4.12). Let us fix a positive time t_0 and a point x_0 in $(0, L)$. If we consider γ one of the extremal backward characteristic originating from (t_0, x_0) then using Theorem 14 and Theorem 15 we have the alternative :

- the starting point is on one of the boundary, thus $u(t_0, x_0) = m$,
- the starting point is at $t = 0$, but then geometrically we see that $0 \leq x_0 - t_0 u(t_0, x_0) \leq L$, and therefore we have

$$-\frac{L}{t_0} \leq u(t_0, x_0) \leq \frac{L}{t_0}. \tag{4.14}$$

  Roughly speaking : the biggest parts — in absolute value — of the initial data are the quickest to leave the domain.

Invasion. From the previous analysis, we see that once $m > L/t_0$, the trace $u(t_0, +)$ cannot be lower than $-m$ thus looking at the boundary condition at $x = 0$ in (4.12) we have

$$\forall t > \frac{L}{m}, \quad u(t, 0^+) = m. \tag{4.15}$$

At this point, let us define

$$\forall t \geq 0, \quad p(t) := \sup\{x \in (0, L) : \forall y \in (0, x) \quad u(t, y) = m\}. \tag{4.16}$$

Considering equation (4.14), it is possible — using once again Theorem 13, Theorem 14 and Theorem 15 — to show that :

$$\forall t \geq \frac{L}{m}, \quad 0 < p(t) < L \implies \dot{p}(t) \geq \frac{m - L/t}{2}.$$

Finally, we see that for t_c solution of

$$\int_{L/m}^{t_c} \frac{m - L/s}{2} ds = L,$$

we have indeed

$$\forall t \geq t_c, \quad u(t,x) = m, \text{ for a.e. } x.$$

And we can obtain the formula for t_c from a straightforward calculation.

4.4 Stabilization of stationary shocks

As previously said, the results of this section apply to strongly convex flux, but for the sake of simplicity we provide them for Burgers' equation only.

Besides the previous family of constant solutions there is another family of stationary solutions $(\bar{u}_{\alpha,m})_{\substack{\alpha\in(0,L)\\ m\in(0,+\infty)}}$.

Definition 5

Let us consider $m > 0$ and $\alpha \in (0, L)$, we define

$$\forall (t,x) \in \mathbb{R} \times (0,L), \qquad \bar{u}_{\alpha,m}(t,x) := \begin{cases} m & \text{if } x < \alpha, \\ -m & \text{if } x \geq \alpha. \end{cases} \tag{4.17}$$

It should be clear that applying $u_l(t) = m$ and $u_r(t) = -m$ at the boundary will not allow for asymptotic stabilization because of the different possible values of α. More precisely with the techniques of the previous section it is possible to show that for any initial data u_0 in $L^\infty(0,L)$ there exists $\alpha \in [0,L]$ and a positive time T such that the entropy solution u of

$$\begin{cases} \partial_t u + \partial_x (u^2/2) = 0, \\ u(t,0^+) \in \mathcal{A}_l(m), \\ u(t,L^-) \in \mathcal{A}_r(-m), \\ u(0,x) = u_0(x), \end{cases}$$

satisfies

$$\forall t \geq T, \quad \forall x \in (0,L), \quad u(t,x) = \bar{u}_{\alpha,m}(x).$$

The problem is of course that α depends on u_0. We will propose in the following an active feedback control to solve this issue.

We suppose that we have now fixed an interval $[0,L]$, a position α in $(0,L)$ and a positive real number m.

Definition 6

Let us consider three positive numbers ϵ, δ, ν (those will be parameters to be tuned later on). We will suppose that $[\alpha-\delta,\alpha+\delta]\subset(0,L)$ and define the functions.

$$\forall z\in\mathbb{R},\qquad \mathcal{H}_{\epsilon,\nu}(z):=\begin{cases}-\epsilon & \text{if } z\le-\nu,\\ \epsilon\frac{z}{\nu} & \text{if } -\nu\le z\le\nu\ .\\ \epsilon & \text{if } \nu\le z\end{cases}\tag{4.18}$$

$$\forall v\in L^1(0,L),\qquad \mathcal{M}_{\alpha,\delta}(v):=\frac{1}{2\delta}\int_{\alpha-\delta}^{\alpha+\delta}(v(x)-\bar{v}_{\alpha,m})dx.\tag{4.19}$$

The operator $\mathcal{M}$ represent a measure on the system that we have access to. It should be considered as some kind of mean value of the solution near the position α. Which is of course, where we want to place the singularity to get asymptotic stabilization of $\bar{u}_{\alpha,m}$. By analogy when dealing with water canals and the Saint-Venant equation — more on this in Chapter 6 — we have access to the height of buoys at certain positions of the canal.

We will now be interested in the solutions of the following closed loop system

$$\begin{cases}\partial_t u+\partial_x(u^2/2)=0,\\ u(t,0)\in\mathcal{A}_l(m-\mathcal{H}_{\epsilon,\nu}(\mathcal{M}_{\alpha,\delta}(u(t,\cdot)))),\\ u(t,L)\in\mathcal{A}_r(-m),\\ u(0,x)=u_0(x)\end{cases}\tag{4.20}$$

We actually have wellposedness of the dynamical system and asymptotic stabilization of the stationary solution $\bar{u}_{\alpha,m}$.

Theorem 17: Perrollaz [116]

Given L, α, m and δ we can find ϵ and ν small enough and positive constants C_1 and C_2 such that

- for any $u_0\in BV(0,L)$ the system (4.20) has a unique maximal entropy solution u,
- this entropy solution is global in time,
- we have asymptotic stabilization in the sense that

$$\forall t\ge 0,\qquad \|u(t,.)-\bar{u}_{\alpha,m}\|_{L^1(0,L)}\le C_1e^{-C_2t}\|u_0-\bar{u}_{\alpha,m}\|_{L^1(0,L)}.\tag{4.21}$$

The existence and uniqueness part come from a fixed point argument on the value of $t\mapsto\mathcal{M}_{\alpha,\nu}(u(t,\cdot))$ and previous results on the open loop initial boundary value problem.

Let us give a very high level overview of the proof of asymptotic stabilization.

1. The first two steps of the proof are actually the same as in the previous section : dispersion of the large waves then invasion from the right and left boundary. The difference is that from the left boundary we actually slightly modulate the invasive value above and below the value of the target m. After the first two steps we end up with $x\mapsto u(t,x)$ taking value close to m at the left of a position $p(t)$ and taking exactly the value $-m$ when $x>p(t)$.

2. The next step is to show that this function p is absolutely integrable in time and satisfy a delayed differential equation with uncertain delay. The delay is basically the time it takes for information to go from the left boundary $x = 0$ to the position of the singularity, i.e. $p(t)$. This step once again relies entirely on Theorem 14 and Theorem 15.
3. It is then possible to provide a sort of Lyapunov functional on p and — using the delayed differential equation — we can show that p converges exponentially fast to α.
4. Finally, plugging this in the system (4.20), we can transform this exponential convergence of p, to the one of the entropy solution.

4.5 Discussion

Space homogeneity. A first question would be to generalize Theorem 17 to the case where f depends on the space variable x but is still convex in u. Dafermos original paper [51] does actually cover this case, thus only Theorem 15 would have to be extended. Of course one would have to replace the families of stationary solutions considered here. The results of [39] should be instrumental in this.

Non-convex flux. The use of the generalized characteristics is mostly restricted to the case where the flux function is convex. Indeed Theorem 14 is very much a consequence of the fact that when the flux is convex no rarefaction wave is generated at positive time. There is some results dealing with the case where f has one inflexion point, see [52] for instance, but the situation doesn't look very hopeful. An alternative outlook would be to use the method introduced in [73], the key insight is that it is more promising to compare conservation laws with the continuum equation

$$\partial_t u + \partial_x (a(t, x) u) = 0,$$

than with the transport equation

$$\partial_t u + a(t, x) \partial_x u = 0.$$

More precisely they show that for a weak solution u of $\partial_t u + \partial_x f(u) = 0$ and for any real number k, if we define

$$\forall (a, b) \in \mathbb{R}^2, \quad \Phi(a, b) := \begin{cases} \frac{f(a) - f(b)}{a - b} & \text{if } a \neq b, \\ f'(a) & \text{otherwise,} \end{cases}$$

the differential equation

$$\begin{cases} \dot{x}(t) = \Phi(u(t, x(t)), k), \\ x(t_0) = x_0 \end{cases}$$

taken in the sense of Filippov [72] has a unique forward solution for any (t_0, x_0) if and only if we have

$$\partial_t |u - k| + \partial_x \text{sign}(u - k) \Phi(u, k) \leq 0,$$

in the distributional sense.

And of course, Kruzkov's definition of an entropy solution is to have the distributional inequality for all number k.

Multidimensional case. For the multidimensional setting, the question is also rather complicated, though the techniques of Chapter 6 do provide interesting perspectives. However, from the propagation point of view, the most hopeful prospects seem to come from the methodology of Pogodaev in [120]. He manages to use the differential inclusion generalizing the one from the generalized characteristics to build sophisticated test functions. He then proceed to plug them in the Kruzkov estimate comparing two entropy solutions u_1 and u_2 of $\partial_t u + \mathrm{div}(f(u)) = 0$

$$\partial_t |u_1 - u_2| + \mathrm{div}\left(\mathrm{sign}(u_1 - u_2)(f(u_1) - f(u_2))\right) \leq 0.$$

This allows him to obtain much more precise estimates about the domain of propagation. In a sense, this is just a variant of Kruzkov original estimates, but with more sophisticated test functions. This could prove useful as an a priori analysis tool to study asymptotic stabilization.

Systems. In the case of systems, the generalized characteristics are also very difficult to deploy. First of all, a result with the same scope as Theorem 14 is clearly out of question. If only because even with genuinely non-linear families, rarefaction waves can be created at positive times. The difficulty of analyzing the propagation domain for systems can be seen by looking at the paper of Glimm and Lax [78]. Still the situation might not be completely hopeless. Indeed, there has been interesting work accomplished by Trivisa in [133] which could provide a useful guide for the applications of generalized characteristics to control problems on systems.

Chapitre 5

Wave Front Tracking

This chapter is concerned with [114, 46]

5.1 The gist of the Algorithm

As we saw in Subsection 1.2.3, using just the Rankine-Hugoniot condition, we can build a semigroup $(S_t^{pc})_{t\geq 0}$ on the space of the piecewise constant functions such that $(t,x) \mapsto (S_t^{pc}u_0)(x)$ is a weak solution of

$$\partial_t u(t,x) + \partial_x f(u)(t,x) = 0. \tag{5.1}$$

However as we explained, this semigroup does not have good continuity properties and thus cannot be extended on a larger class of functions.

However Dafermos introduced a modification in [53] allowing us to use piecewise constant functions and still pass to the limit at the end. His work focused on the case where $f : \mathbb{R} \to \mathbb{R}$ but was then extended by Di Perna in [62] to the case of $f : \mathbb{R}^2 \to \mathbb{R}^2$. Finally Bressan managed to adapt the construction to deal with the case where $f : \mathbb{R}^n \to \mathbb{R}^n$ in [27]. It allowed him not only to provide an alternative existence result than the one obtained by Glimm in [79] but also to show the existence of a contractive semigroup of solutions and the uniqueness of said semigroup in a series of papers [28, 32, 31]. In the scalar case, Holden and its co-authors have long used the methods extensively, in particular for numerical purposes starting from the paper [85]. The standard references for the wave front tracking algorithm are Bressan's book [29], and Holden and Risebro's book [86]. Let us explain the idea on the case of equation (5.1), when the flux f is uniformly convex.

But first let us recall the definition of the total variation functional and of the space BV($\mathbb{R}$) since those will prove useful in the following. For a positive integer n let us call $\mathcal{I}_n$ the set of increasing functions from $\{0,\dots,n\}$ to $\mathbb{R}$. Given a function $w : \mathbb{R} \to \mathbb{R}$ we define the total variation of w as

$$\mathrm{TV}(w) := \sup_{\substack{n\geq 1 \\ x\in\mathcal{I}_n}} \sum_{k=1}^{n} |w(x(k)) - w(x(k-1))|, \tag{5.2}$$

with the possibility of taking the value $+\infty$.

The function space BV($\mathbb{R}$) can now be defined as

$$\mathrm{BV}(\mathbb{R}) := \{w : \mathbb{R} \to \mathbb{R} \,:\, \mathrm{TV}(w) < +\infty\}. \tag{5.3}$$

We refer the reader to [29, Chapter 2 Section 4] for a description of the good properties of BV($\mathbb{R}$). In particular, let us mention that keeping the total variation under control for family is the key to get compactness thanks to Helly's theorem.

We will proceed in the following way.

1. We will build a particular family of semigroups $(S_t^\epsilon)_{t\geq 0}$ — for ϵ positive but small — acting on the space of piecewise constant functions.
2. Those semigroups will have the same continuity problem described before. However we will manage to pass to the limit $\epsilon \to 0$ and get a family of functions $(S_t)_{t\geq 0}$. And then $(t,x) \mapsto (S_t u_0)(x)$ will satisfy the entropy inequalities of Kruzkov.
3. This family will not be yet a semigroup. While the S_t will still be defined on the space of piecewise constant functions, they will take value in the space BV($\mathbb{R}$).
4. However the family will have good continuity properties — in L^1_{loc} — allowing us to extend them to the space BV($\mathbb{R}$).
5. At this point we will get a semigroup that is contractive for the L^1 norm.

Given u_0 a piecewise constant function and ϵ a small positive number, the function $(t,x) \mapsto (S_t^\epsilon u_0)(x)$ will still be obtained by patching up solutions of Riemann data, using the fact that we expect invariance of the semigroup by translation in space and in time.

We just need to detail the solution for the following initial data

$$\forall x \in \mathbb{R}, \quad u_0(x) := \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0. \end{cases} \tag{5.4}$$

Now using the fact that f is convex there are only two situations.

- If $u_l > u_r$ we can just propagate the singularity using the Rankine-Hugoniot condition :

$$\forall (t,x) \in \mathbb{R}_*^+ \times \mathbb{R}, \quad (S_t^\epsilon u_0)(x) = \begin{cases} u_l & \text{if } \frac{x}{t} < \frac{f(u_r)-f(u_l)}{u_r-u_l} \\ u_r & \text{if } \frac{x}{t} > \frac{f(u_r)-f(u_l)}{u_r-u_l}. \end{cases} \tag{5.5}$$

- If however $u_l < u_r$, we saw in Subsection 1.2.3 that the *good* solution was a rarefaction wave that was not piecewise constant. We will therefore use the small parameter ϵ to discretize this solution. To this end, we introduce

$$n := \left\lfloor \frac{u_r - u_l}{\epsilon} \right\rfloor + 1, \tag{5.6}$$

$$\forall i \in \{0, \dots, n\}, \quad v_i := \left(1 - \frac{i}{n}\right) u_l + \frac{i}{n} u_r, \tag{5.7}$$

$$\forall j \in \{1, \dots, n\}, \quad \gamma_j := \frac{f(v_j) - f(v_{j-1})}{v_j - v_{j-1}}. \tag{5.8}$$

  We can now define

$$\forall (t,x) \in \mathbb{R}_*^+ \times \mathbb{R}, \quad (S_t^\epsilon u_0)(x) = \begin{cases} v_0 & \text{if } \frac{x}{t} < \gamma_1, \\ v_i & \text{if } \gamma_i < \frac{x}{t} < \gamma_{i+1} \text{ and } 1 \leq i \leq n-1, \\ v_n & \text{if } \gamma_n < \frac{x}{t}. \end{cases} \tag{5.9}$$

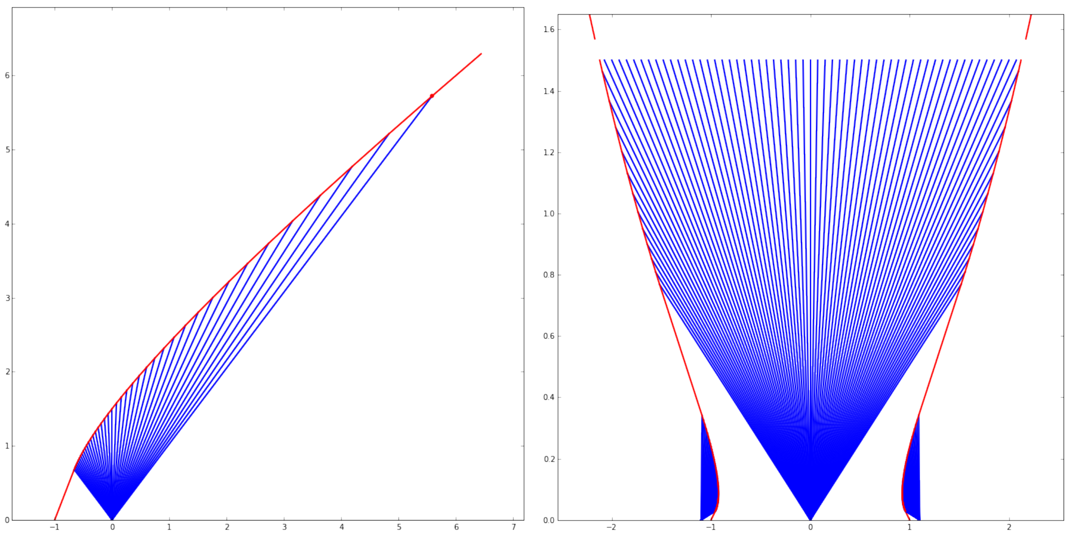

FIGURE 5.1 – Examples of front tracking approximations in the x-t plane for Burgers' equation with rarefactions in blue and shocks in red.

To see how the fronts of such a solution might look let us refer to Figure 5.1.

It should be clear that the Rankine-Hugoniot condition is verified at each singular point, therefore we have built a weak solution to (5.1).

Looking at the construction it should be clear that the discretization of the rarefaction wave does not increase the total variation. In the same way, collisions of two waves only lead to the total variation decreasing. It is thus possible to show that

$$\forall t > 0, \quad \mathrm{TV}(S_t^\epsilon u_0) \leq \mathrm{TV}(u_0), \tag{5.10}$$

$$\forall t > 0, \quad \|S_t^\epsilon u_0\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}. \tag{5.11}$$

This is enough to show that for a fixed u_0 that is piecewise constant — and therefore in $\mathrm{BV}(\mathbb{R})$ — and for any positive time t, the family $(S_t^\epsilon u_0)_{\epsilon>0}$ is compact in $L^1(\mathbb{R})$ thanks to Helly's theorem. At the same time, the propagation speed is bounded by

$$\max(|f'(-\|u_0\|_{L^\infty(\mathbb{R})})|, |f'(\|u_0\|_{L^\infty(\mathbb{R})})|).$$

Thus we can show that $(t \mapsto S_t^\epsilon u_0)_\epsilon$ is equi-Lipschitz in $L^1(\mathbb{R})$. We can use a classical diagonal argument to build a limit $(S_t)_{t\geq 0}$ up to a subsequence of $\epsilon \to 0$.

To conclude let us try to motivate the fact that the family $(S_t)_{t\geq 0}$ satisfies the entropy inequalities that we described at the end of Subsection 1.2.3.

We pick a convex regular function $E : \mathbb{R} \to \mathbb{R}$ and its flux Q satisfying $Q' = E'f'$. We consider a test function ϕ regular and compactly supported in $\mathbb{R}_*^+ \times \mathbb{R}$. And we pick real numbers λ, t_0, t_1, a and b such that

$$0 < t_0 < t_1, \quad a < b, \quad \lambda > 0, \quad \lambda = \frac{b-a}{t_1 - t_0}, \quad \mathrm{supp}(\phi) \subset (t_0, t_1) \times (a, b).$$

Now given two real numbers u_l and u_r we define u by

$$\forall (t,x) \in (t_0, t_1) \times (a, b), \quad u(t,x) := \begin{cases} u_l & \text{if } x - a < \lambda\,(t - t_0), \\ u_r & \text{if } x - a > \lambda\,(t - t_0). \end{cases} \tag{5.12}$$

Then we have

$$\int_{t_0}^{t_1}\int_a^b \partial_t\phi(t,x)E(u(t,x)) + \partial_x Q(u(t,x))\partial_x\phi(t,x)\,dxdt$$
$$= (Q(u_l) - Q(u_r) - \lambda(E(u_l) - E(u_r)))\int_{t_0}^{t_1}\phi(t, a + \lambda(t - t_0))dt.$$

On the other hand we can also compute

$$\lambda = \int_0^1 f'(u_r + \theta(u_l - u_r))\,d\theta,$$
$$E(u_l) - E(u_r) = (u_l - u_r)\int_0^1 E'(u_r + \theta(u_l - u_r))\,d\theta,$$
$$Q(u_l) - Q(u_r) = (u_l - u_r)\int_0^1 (f'E')(u_r + \theta(u_l - u_r))\,d\theta.$$

Now since both E' and f' are non-decreasing we have for any (θ, η) in $[0, 1]^2$

$$(f'(u_r + \theta(u_l - u_r)) - f'(u_r + \eta(u_l - u_r)))\,(E'(u_r + \theta(u_l - u_r)) - E'(u_r + \eta(u_l - u_r))) \geq 0.$$

But then integrating on $[0, 1]^2$ we get

$$\int_0^1\int_0^1 (f'(u_r+\theta(u_l - u_r)) - f'(u_r+\eta(u_l - u_r)))\,(E'(u_r+\theta(u_l - u_r)) - E'(u_r+\eta(u_l - u_r)))\,d\eta d\theta \geq 0.$$

Expanding and reorganizing provides us with

$$\int_0^1 f'(u_r+\theta(u_l - u_r))E'(u_r+\theta(u_l - u_r))d\theta - \int_0^1 f'(u_r+\theta(u_l - u_r))d\theta\int_0^1 E'(u_r+\theta(u_l - u_r))d\theta \geq 0.$$

This clearly shows that when $u_l > u_r$ we have the Kruzkov inequality, while in the other case we get a lower bound in $-O(|u_r - u_l|)$ thus $-O(\epsilon)$ given the construction. This is enough to show that when passing to the limit in the wave front tracking approximation we do satisfy Kruzkov inequality.

To appreciate the potential complexity of a wave front approximation let us refer to Figure 5.2.

5.2 A particular problem in the scalar case

In [114], we were interested in solving the exact controllability problem for the system

$$\begin{cases} \partial_t u + \partial_x f(u) = g(t), \\ u(0, x) = u_0(x), \\ u(t, 0) = u_l(t), \\ u(t, L) = u_r(t), \end{cases} \qquad (t, x) \in (0, T) \times (0, L). \tag{5.13}$$

The key point compared to what we described in Subsection 1.3.3 is that besides the boundary conditions, we have an additional control g acting uniformly in space but depending

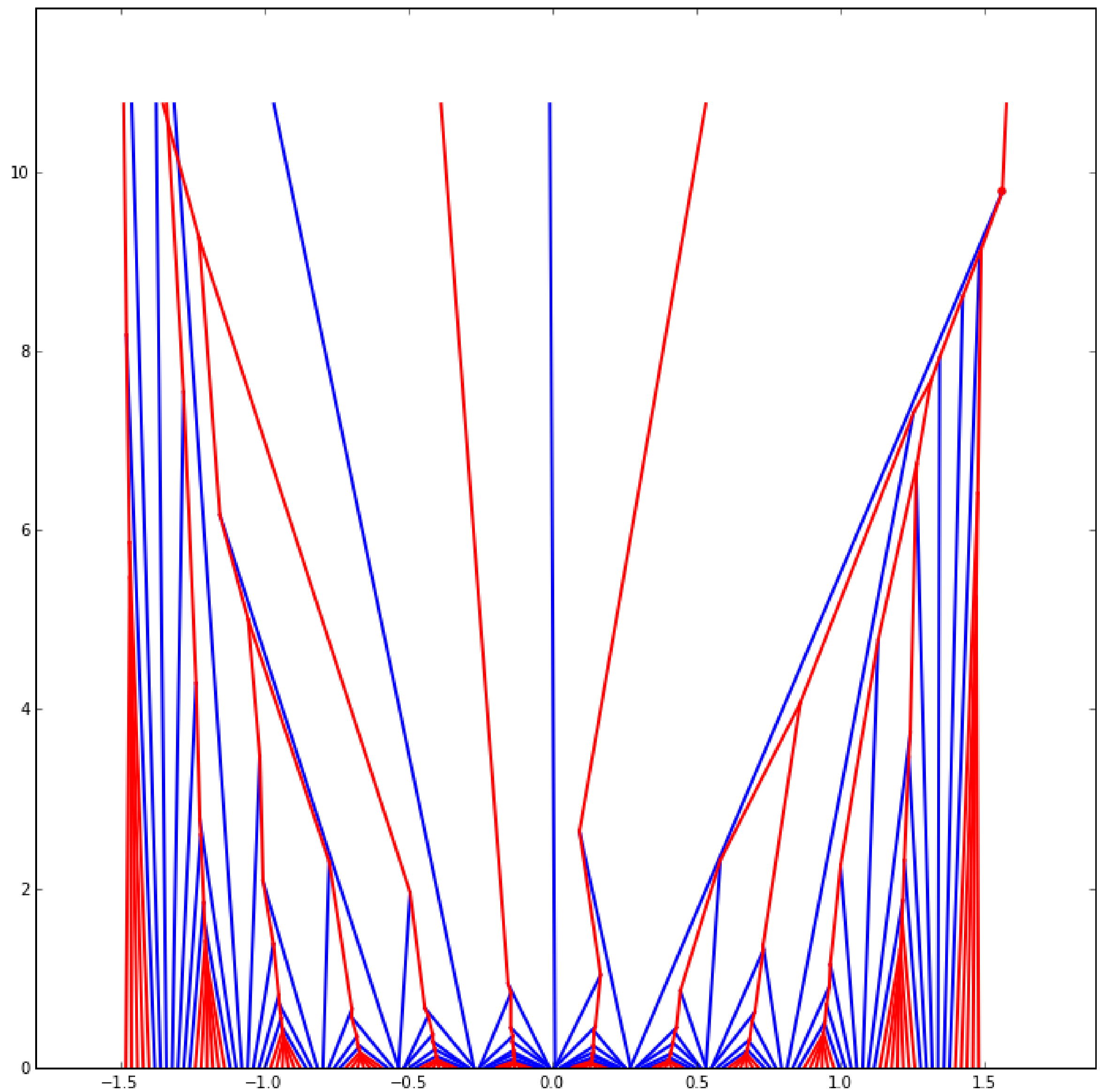

FIGURE 5.2 – Example of a wave front tracking approximation in the x-t plane for Burgers equation : red are shocks and blue are rarefactions

only on t. The idea was to see if this additional control allowed to have more reachable states than what was described in [10].

The physical motivation behind such control is to be found, for instance, when considering the transverse motion of a tank. The dynamic of such a system can be described by

$$\begin{cases} \partial_t H + \partial_x(Hv) = 0, \\ \partial_t(Hv) + \partial_x(gH^2 + \frac{Hv^2}{2}) = -u(t), \\ \ddot{D}(t) = g\,u(t). \end{cases} \tag{5.14}$$

Here H is the height of the fluid, v is the horizontal velocity of the fluid, D is the displacement of the tank and u is the force applied to the tank. From the viewpoint of control theory our control is u and the goal is to move the tank from a given point and with the liquid at rest to another given point with the liquid once again at rest. The exact controllability problem and the asymptotic stabilization by feedback control were already investigated in [44, 47] in the framework of regular solutions. However the framework of entropy solutions would hopefully provide more efficient motion planning and more robustness with regard to perturbations and errors.

Coming back to (5.13), if f besides being convex, satisfies

$$\frac{f'(M)}{\sup\limits_{z\in[0,M]} f''(z)} \to +\infty \text{ as } M \to +\infty \quad \text{or} \quad \frac{f'(M)}{\sup\limits_{z\in[M,0]} f''(z)} \to -\infty \text{ as } M \to -\infty, \tag{5.15}$$

then we can prove the following theorem.

Theorem 18: Perrollaz [114]

Let $u_1 \in \mathrm{BV}(0, L)$ satisfy :

$$\sup_{\substack{0<h<L \\ 0<x<L-h}} \frac{f'(u_1(x+h)) - f'(u_1(x))}{h} < +\infty. \tag{5.16}$$

Then for any positive time T and any u_0 in $\mathrm{BV}(0, L)$ there exist two functions g and u respectively in $\mathcal{C}^1([0,T])$ and $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$, such that

- u is an entropy solution of (5.13) on $(0,T) \times (0,L)$ (since we use the extension method of Russell we leave the boundary conditions alone)
- for almost all point x in $(0, L)$, we have $u(0, x) = u_0(x)$,
- for almost all point x in $(0, L)$, we have $u(T, x) = u_1(x)$.

Note that (5.16) is showing the added value of the third control g. Indeed, without it the condition for a state u_1 to be reachable in time T is much more complicated. Besides an inequality of Oleinik type involving T, u_1 must also satisfies

$$T \geq \max\left(\sup_{x\in(0,L)} \frac{x}{(f'(u_1(x)))^+}, \sup_{x\in(0,L)} \frac{L-x}{(f'(u_1(x)))^-} \right). \tag{5.17}$$

The way to understand the action of the control g is the following. We can still apply the front tracking algorithm, however we have functions of t instead of pure constants separating the singularities. We are thus curving the straight lines that were the trajectories of the singularities. To be more precise if we consider the solution of

$$\partial_t u(t,x) + \partial_x f(u)(t,x) = g(t), \tag{5.18}$$

corresponding to the Riemann initial data

$$\forall x \in \mathbb{R}, \quad u_0(x) := \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0. \end{cases} \tag{5.19}$$

then the semigroup $(S_t^\epsilon)_{t\geq 0}$ acts in the following way.

- If $u_l > u_r$ we define

$$\forall t \geq 0,\ v_l(t) := u_l + \int_0^t g(s)\,ds,\ v_r(t) := u_r + \int_0^t g(s)\,ds,\ \gamma(t) := \int_0^t \frac{f(v_r(s)) - f(v_l(s))}{v_r(s) - v_l(s)}ds, \tag{5.20}$$

  from then

$$\forall (t,x) \in \mathbb{R}_*^+ \times \mathbb{R}, \quad (S_t^\epsilon u_0)(x) = \begin{cases} v_l(t) & \text{if } x < \gamma(t), \\ v_r(t) & \text{if } x > \gamma(t). \end{cases} \tag{5.21}$$

- If however $u_l < u_r$, we saw in Subsection 1.2.3 that the *good* solution was a rarefaction wave that was not piecewise constant. We will therefore use the small parameter ϵ to discretize this solution. To this end, we introduce

$$n := \left\lfloor \frac{u_r - u_l}{\epsilon} \right\rfloor + 1, \tag{5.22}$$

$$\forall i \in \{0,\dots,n\}, \quad \forall t \geq 0, \quad v_i(t) := \left(1 - \frac{i}{n}\right) u_l + \frac{i}{n} u_r + \int_0^t g(s)\,ds, \tag{5.23}$$

$$\forall j \in \{1,\dots,n\}, \quad \gamma_j(t) := \int_0^t \frac{f(v_j(s)) - f(v_{j-1}(s))}{v_j(s) - v_{j-1}(s)}ds. \tag{5.24}$$

  We can now define

$$\forall (t,x) \in \mathbb{R}_*^+ \times \mathbb{R}, \quad (S_t^\epsilon u_0)(x) = \begin{cases} v_0(t) & \text{if } x < \gamma_1(t), \\ v_i(t) & \text{if } \gamma_i(t) < x < \gamma_{i+1}(t) \text{ and } 1 \leq i \leq n-1, \\ v_n(t) & \text{if } \gamma_n(t) < x. \end{cases} \tag{5.25}$$

Of course, by curving the trajectories of the singularities it is easier to make them leave the domain $(0, L)$. When using the extension method of Russell that means it is easier to replace the initial data by something from outside.

Finally, note that the result of this part were actually improved in [11].

5.3 A stabilization problem in the case of systems

In the case of systems of conservation laws, the situation is much more complicated. The reason is that the solution with a Riemann initial data is already much more complicated. However using the front tracking algorithm, we were able to prove the following asymptotic stabilization result using a feedback control.

We consider the following setting. Let Ω be an open subset of $\mathbb{R}^2$ with $0 \in \Omega$. And consider $f : \Omega \mapsto \mathbb{R}^2$ a smooth function (supposed to satisfy the strict hyperbolicity condition described below) and consider the following system of two conservation laws

$$\partial_t u + \partial_x(f(u)) = 0 \quad \text{for } (t, x) \in (0, \infty) \times (0, L). \tag{5.26}$$

In System (5.26), the solution $u = u(t, x) = (u_1, u_2)^T$ has 2 components and the space variable x belongs to the finite interval $(0, L)$. System (5.26) is completed with boundary conditions of the form

$$u(t, 0) = Ku(t, L), \tag{5.27}$$

where K is a 2×2 (real) matrix. Here, for sake of simplicity, we assume that the boundary condition is linear but other non-linear forms could be considered.

Clearly $u \equiv 0$ is an equilibrium of (5.26)-(5.27). We are interested in the exponential stability of this equilibrium in the BV space for entropy solutions. We recall that the space BV is natural for solutions of non-linear hyperbolic systems of conservation laws, and is in particular the space considered in the celebrated paper by Glimm [79].

Hypothesis. We assume the flux function f to satisfy the strict hyperbolicity conditions, that is,

$$\forall u \in \Omega, \quad \text{the matrix } A(u) = Df(u) \text{ has 2 real distinct eigenvalues } \lambda_1(u) < \lambda_2(u). \tag{5.28}$$

Furthermore, we make the assumption that both velocities are positive, so that we finally get :

$$0 < \lambda_1(u) < \lambda_2(u) \qquad \text{for all } u \in \Omega. \tag{5.29}$$

Note that the case of two strictly negative velocities is obviously equivalent by the change of variable $x \to L - x$.

In order to discuss the condition that we impose on K, we further introduce the left and right eigenvectors of $A(u) = Df(u)$. For each $k = 1, 2$, we define $r_k(u)$ as a right eigenvector of $A(u)$ corresponding to the eigenvalue $\lambda_k(u)$:

$$A(u)r_k(u) = \lambda_k(u)r_k(u),\ r_k(u) \neq 0, \qquad k = 1, 2,\ u \in \Omega, \tag{5.30}$$

and the functions r_k are regular. We also introduce correspondingly left eigenvectors $\ell_k(u)$ of $A(u)$

$$\ell_k(u)A(u) = \lambda_k(u)\ell_k(u), \qquad \text{with } \ell_k(u) \cdot r_{k'}(u) = \begin{cases} 1 \text{ if } k = k', \\ 0 \text{ if } k \neq k'. \end{cases} \tag{5.31}$$

We further impose that the hyperbolic system (5.26) is genuinely non-linear in the sense of Lax [96], i.e.

$$D\lambda_k(u) \cdot r_k(u) \neq 0 \text{ for all } u \in \Omega.$$

Changing the sign of $r_k(u)$ and $\ell_k(u)$ if necessary, we can therefore assume

$$D\lambda_k(u) \cdot r_k(u) > 0 \text{ for all } u \in \Omega. \tag{5.32}$$

As the total variation $\mathrm{TV}_{[0,L]}$ is only a semi-norm on $BV(0, L)$ (it vanishes for constant maps), it is convenient to define the following norm on $BV(0, L)$ as

$$\|u\|_{BV(0,L)} := \mathrm{TV}_{[0,L]}(u) + \int_0^L |u(x)|dx, \qquad u \in BV(0, L). \tag{5.33}$$

It is useful to recall that a function $u \in BV(0, L)$ has at most countably many discontinuities and has at each point left and right limits. In particular one can define $u(0^+)$ and $u(L^-)$ without ambiguity.

Now we recall that entropy solutions are weak solutions of (5.26) in the sense of distributions, which satisfy moreover entropy conditions for the sake of uniqueness. A way to express these entropy conditions consists in introducing entropy/entropy flux couples for (5.26) as any couple of regular functions $(E, Q) : \Omega \to \mathbb{R}$ satisfying :

$$\forall U \in \Omega, \quad DE(U) \cdot Df(U) = DQ(U). \tag{5.34}$$

Of course $(E, Q) = (\pm \mathrm{Id}, \pm f)$ are entropy/entropy flux couples. Then we have the following definition (see [29, 54, 96]) :

Definition 7

A function $U \in L^\infty(0, T; BV(0, L)) \cap \mathrm{Lip}(0, T; L^1(0, L))$ is called an entropy solution of (5.26) when, for any entropy/entropy flux couple (E, Q), with E convex, one has in the sense of measures

$$\partial_t E(U) + \partial_x Q(U) \leq 0, \tag{5.35}$$

that is, for all $\varphi \in \mathcal{D}((0, T) \times (0, L))$ with $\varphi \geq 0$,

$$\int_{(0,T)\times(0,L)} \left(E(U(t, x))\varphi_t(t, x) + Q(U(t, x))\varphi_x(t, x)\right) dx\, dt \geq 0. \tag{5.36}$$

Our main result is the following one :

Theorem 19

Let the system (5.26) be strictly hyperbolic and genuinely non-linear in the sense of (5.32), and assume that the velocities are positive in the sense of (5.29).
If the matrix K satisfies

$$\inf_{\alpha \in (0,+\infty)} \Big(\max \big\{|\ell_1(0) \cdot Kr_1(0)| + \alpha|\ell_2(0) \cdot Kr_1(0)|,$$
$$\alpha^{-1}|\ell_1(0) \cdot Kr_2(0)| + |\ell_2(0) \cdot Kr_2(0)|\big\}\Big) < 1, \tag{5.37}$$

then there exist positive constants C, ν, $\varepsilon_0 > 0$, such that for every $u_0 \in BV(0, L)$ satisfying

$$\|u_0\|_{BV(0,L)} \leq \varepsilon_0, \tag{5.38}$$

there exists an entropy solution u of (5.26) in $L^\infty(0, \infty; BV(0, L))$ satisfying $u(0, \cdot) = u_0(\cdot)$ and (5.27) for almost all times, and such that

$$\|u(t)\|_{BV(0,L)} \leq C \exp(-\nu t)\|u_0\|_{BV(0,L)}, \qquad t \geq 0. \tag{5.39}$$

In other words, Theorem 19 states that (5.37) is a sufficient condition for the (local) exponential stability of $u \equiv 0$ with respect to the BV-norm for (5.26) with the boundary condition (5.27). The main idea of the proof it to prove an equivalent estimate to (5.39). To that end a modified version of the Glimm functional is introduced so that

- exponential weight are added on the strength of the singularities so that the propagation provides an exponential decay inside the domain,
- at each collision inside the domain, Glimm's analysis provides the instantaneous decay of the functional,
- at the boundary we cannot rely on Glimm's analysis to get the decay, but the condition on K is exactly the one that provides though at a linear level rather than at the quadratic level.

Let us try to be a bit more precise by giving a high level overview of the wave front tracking algorithm in our case.

The Riemann problem. The first step is still solving the Riemann problem. The analysis of Section 1.2.3 still shows that a function :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \quad u(t,x) := \begin{cases} u_l & \text{if } x < \lambda t, \\ u_r & \text{if } x > \lambda t, \end{cases}$$

is a weak solution of $\partial_t u + \partial_x f(u) = 0$ if and only if the Rankine-Hugoniot condition holds :

$$f(u_r) - f(u_s) = \lambda(u_r - u_l).$$

But this time we have an equality in $\mathbb{R}^2$ and λ is still a real number.

Now given a point in u in Ω one can show — using the Implicit Function theorem — that the set $\{v \in \Omega \,:\, \exists s \in \mathbb{R} \quad f(v) - f(u) = s(v-u)\}$ is actually the union of two regular curves $s \mapsto \Gamma_1(s)u$ and $s \mapsto \Gamma_2(s)u$ crossing in a transverse manner at $v = u$ — i.e. $s = 0$ — by the fact that

$$\Gamma_1'(0)u = r_1(u), \quad \Gamma_2'(0)u = r_2(u).$$

One can then use the inverse function theorem to show that the equation

$$v = \Gamma_2(s_2) \circ \Gamma_1(s_1)u,$$

is solvable in (s_1, s_2) when v and u are close to 0 in $\mathbb{R}^2$.

Basically, to solve the Riemann problem with initial data separating u_l and u_r we introduce s_1 and s_2 such That

$$u_r = \Gamma_2(s_2) \circ \Gamma_1(s_1)u_l.$$

If we introduce the intermediary state $w := \Gamma_1(s_1)u_l$, we solve the Riemann problem by patching

- a simple wave of the first family between u_l and w,
- a simple wave of the second family between w and u_r.

And the patching up is actually a superimposition thanks to the hypothesis of strict hyperbolicity which means that the two families of waves evolve with strictly distinct speeds. Actually, we have cheated a bit and positive s corresponds to a rarefaction (so we have to consider only a branch of the Rankine-Hugoniot locus) and only negative s to a shock in each family. We will call s the strength of the front. Finally, the total variation of the solution of the Riemann problem is comparable with $|s_1| + |s_2|$.

For more detail one should consult [29, Chapter 5].

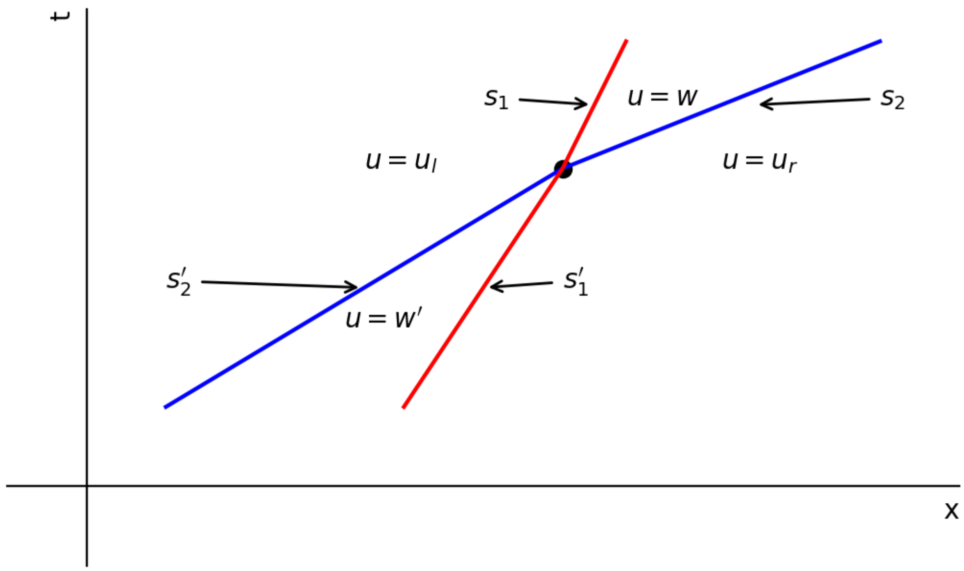

FIGURE 5.3 – Interaction of two waves.

Interactions analysis. Since we know how to solve — locally — the Riemann problem we can use the wave front tracking algorithm to produce approximations. To be complete, there is still a difficulty to show that we don't produce an infinite number of fronts in finite time. But since we are working with Ω in $\mathbb{R}^2$ the problem is not actually too bad and we refer to our paper [46] for the details. The difficulty lies in passing to the limit in the approximations.

When we have interaction between two waves, due to the propagation of the two families we can expect to have oncoming

- a wave of the second family on the left : $w' = \Gamma_2(s_2')u_l$,
- a wave of the first family on the right : $u_r = \Gamma_1(s_1')w'$.

But outgoing after the interaction, we have $u_r = \Gamma_2(s_2) \circ \Gamma_1(s_1)u_l$ and thus we end up with the relation :

$$\Gamma_2(s_2) \circ \Gamma_1(s_1)u_l = \Gamma_1(s_1') \circ \Gamma_2(s_2')u_l. \tag{5.40}$$

The key difference with the scalar case is that there is no reason why we should have

$$|s_1| + |s_2| \leq |s_1'| + |s_2'|,$$

and thus the total variation may actually increase.

Looking at (5.40) we see that both sides have the same first order approximation so what we get is actually :

$$|s_1 - s_1'| + |s_2 - s_2'| \leq C|s_1's_2'|.$$

This is an instance of what are called Glimm's estimates, after Glimm's original paper [79], see also [137] for an elegant derivation. Look at Figure 5.3 for a graphical representation of the interaction.

Glimm's functional. To have compactness of the sequence of wave front tracking approximations, it is necessary to keep the total variation under control. As we saw above, it is not non-increasing in time so we have to follow Glimm's original idea and introduce additional quadratic terms to the total variations.

Let us make this more precise. Suppose that at time t we have n discontinuities in the approximation. Ordering them by increasing x we introduce their positions $(p_1, \dots, p_n)$ and their strengths $(s^1_{i_1}, \dots, s^n_{i_n})$. We know that the total variation is equivalent to the quantity : $\sum_{k=1}^{n} |s^k_{i_k}|$. We now introduce A_t the set of approaching waves through :

$$A_t := \Big\{ (j,k) \in \{1, \dots, n\}^2 \,:\, p_j < p_k \text{ and } \Big((s^j_{i_j}, s^k_{i_k}) = (2,1) \text{ or } \big(i_j = i_k \text{ and } s^j_{i_j} < 0 \text{ or } s^k_{i_k} < 0 \big) \Big) \Big\}.$$

Basically only couple of rarefactions from the same family will be guaranteed to not interact. At this point, the value of the Glimm functional for the approximation at time t is given by the quantity :

$$\sum_{k=1}^{n} |s^k_{i_k}| + C \sum_{(j,k) \in A_t} |s^j_{i_j} s^k_{i_k}|.$$

The key point is that when there is no interaction, the value of the Glimm functional doesn't change. While at an interaction the increase in the first term is compensated by the disappearance of a couple of approaching fronts. Therefore the value of the Glimm functional is non-increasing, and clearly it controls the total variation.

Let us also mention that keeping the total variation under control besides providing compactness, is also necessary because we only know how to solve the Riemann problem locally.

Modification for our case. In our case, we have two modifications :

1. we want a functional controlling the total variation that decays exponentially in time, uniformly for all approximations,
2. a wave may interact with the boundary.

The way to deal with the first point it to introduce the modified Glimm functional taking the value :

$$\sum_{k=1}^{n} |s^k_{i_k}| e^{-\gamma p_k} + C \sum_{(j,k) \in A_t} |s^j_{i_j} s^k_{i_k}| e^{-\gamma (p_j + p_k)},$$

with γ a weight parameter to be adjusted.

At the boundary, using the same kind of analysis as in the previous paragraph, we see that a front of the family k touches the boundary we have :

$$\begin{cases} u_r = \Gamma_k(s') u_l, \\ K u_r = \Gamma_2(s_2) \circ \Gamma_1(s_1) K u_l. \end{cases}$$

Then we have :

$$|s_1 - s' \ell_1(K u_L) \cdot K r_k(u_L)| + |s_2 - s' \ell_2(K u_L) \cdot K r_k(u_L)| \le C_\delta |s'|^2, \tag{5.41}$$

and condition (5.37) let us concludes. Look at Figure 5.4 for a graphical representation.

Note that in the presence of a boundary, the definition of approaching waves is a bit more tricky since a front going through the boundary at the right will introduce two new fronts appearing on the left. This can be kept under control since the speed of all waves is bounded, thus we have a fixed window of time before two interacting fronts will interact again.

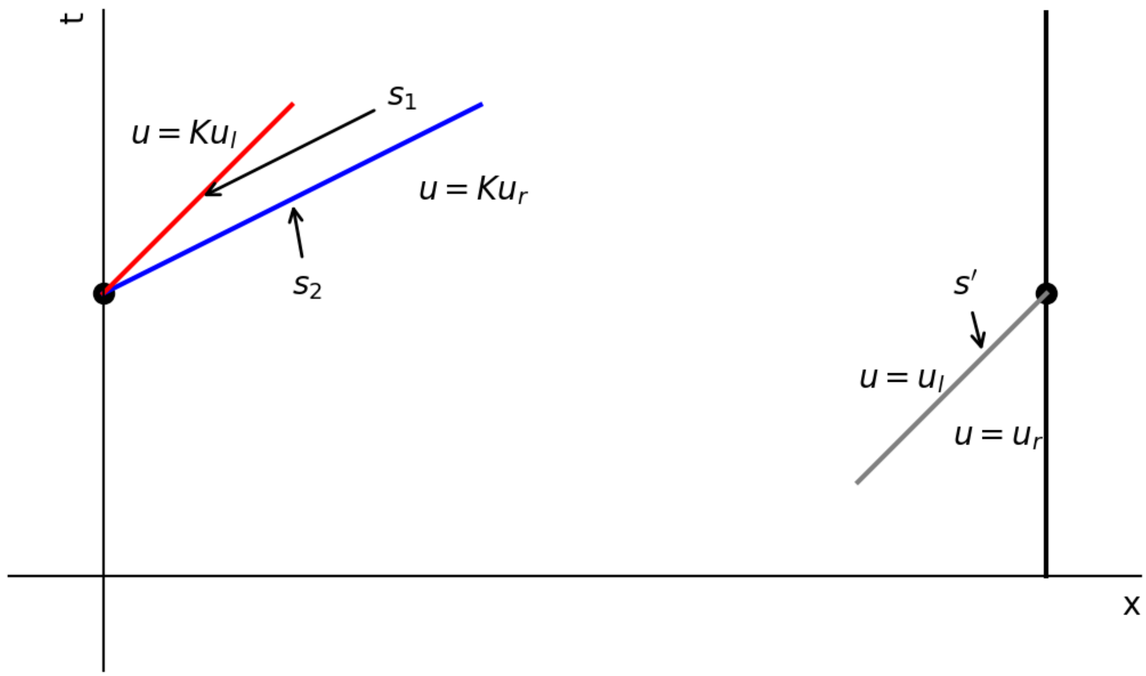

FIGURE 5.4 – Riemann problem at the boundary.

5.4 Discussion

Completing the stabilization result. In Theorem 19, there is no claim of uniqueness of the solution of (5.26) with initial condition $u(0,\cdot) = u_0(\cdot)$ and boundary law (5.27). Meaning that at this point, we only proved that those solutions we built using the wave front tracking algorithm are actually exponentially going to zero. One would therefore need to complete this result with a uniqueness result for the closed loop system or show that any solution can be generated by wave front tracking approximations.

The initial-boundary value problem for hyperbolic systems of conservation laws is a very delicate matter, even in the non-characteristic case (when no characteristic speed vanishes). We refer for instance to Amadori [3], Amadori-Colombo [4, 5], Colombo-Guerra [34], Donadello-Marson [63] and references therein. In particular, in the open loop case, it is possible to construct a Standard Riemann Semigroup as a limit of front-tracking approximations (which is also the construction method employed here). However, up to our knowledge, these results do not quite cover our feedback boundary condition (5.27), and no result of uniqueness in the same spirit as [29, Theorem 9.4] is available in our situation.

The case of opposite velocities is still open. The key difficulty is to show that the boundary condition we introduced does not lead to a resonance situation where an infinite number of fronts are generated in finite time. This would prevent the front tracking approximation of existing in large time and thus deal a crippling blow to the proof.

Finally, the case of $n \times n$ systems of conservation laws is also open. The dissipation estimates at the boundary are not the problem since the same methodology applies. But the necessity of introducing non-physical discontinuities to prevent an infinite number of fronts to be generated in finite time leads to having to review every step of the construction.

Proof assistant. There is no denying that when dealing with one dimensional homogeneous systems of conservation laws, the wave front tracking method is the most versatile and most used technique. The price to pay is that there is almost no abstraction level to hide the details from you. You are in control of all parameters and they are tangled together from the start to the end. This lead to some extremely complex proofs (in the computer science sense of having a lot of moving parts to juggle at all time). It might therefore be interesting to spend some time producing a computer verified proof, for instance using the lean theorem prover [59]. It has gained a lot of traction in the mathematical community, as can be seen by the contributions to the mathlib project (https ://leanprover-community.github.io/). In particular, it seems that most of the tools necessary (BV space, Implicit Function theorem,...) are already present in the library developed by the community. Such a verified proof would allow the community to iterate rapidly on different proof design and different strategies. This might lead us to find some abstractions allowing for a better encapsulation by Lemmas, Proposition...

Functional setting. A side effect of using the wave front tracking algorithm is to put the focus on BV estimate when tackling stabilization problem as shown in the result of the previous section or in [66]. This might not be so pertinent from a functional setting point of view in particular once the stabilization target is more complicated than a constant. By analogy with the results of Chapter 6, some investigation should probably be made to see if the functional appearing in the proof of the stability of the standard Riemann semigroup — see [29, Chapter 8] — could not be adapted to provide Lyapunov functionals when dealing with asymptotic stabilization problems.

Chapitre 6

Kato type Inequalities

This chapter is follows [117, 81, 64]

6.1 The gist of the method

Let us focus on a single transport equation to get a good picture of the method of this chapter. We consider two positive number L and λ and we pick a constant b in $\mathbb{R}$. Let us analyze the transport equation

$$\begin{cases} \partial_t u(t,x) + \lambda \partial_x u(t,x) = 0, \\ u(t,0) = b, & t > 0, \quad x \in (0,L), \\ u(0,x) = u_0(x), \end{cases} \tag{6.1}$$

where u_0 is regular enough and $b = u_0(0)$.

Using the characteristics method, it is straightforward to see that we have

$$\forall t > 0, \quad \forall x \in (0,L), \quad u(t,x) = \begin{cases} u_0(x - \lambda t) & \text{if } x > \lambda t \\ b & \text{otherwise,} \end{cases} \tag{6.2}$$

meaning that we actually have $u(L/\lambda, \cdot) = b$. We solved the exact controllability problem with target $x \mapsto b$ in time T/λ with a "feedback control".

What we want is to actually provide an alternative proof using Lyapunov functionals. To this end let us introduce the family of functional $(\mathcal{L}_\nu)_{\nu>0}$ defined on $L^2(0,L)$ and taking values in $\mathbb{R}$ by

$$\forall v \in L^2(0,L), \quad \mathcal{L}_\nu(v) := \int_0^L e^{-\nu x}(v(x) - b)^2 \, dx. \tag{6.3}$$

Now we want to understand the evolution of $t \mapsto \mathcal{L}_\nu(u(t,\cdot))$, we just provide a formal calculation since the goal is to gain some insights.

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_\nu(u(t,\cdot)) &= \int_0^L 2\partial_t u(t,x)(u(t,x)-b)e^{-\nu x}\,dx \\ &= -\lambda\int_0^L 2\partial_x u(t,x)(u(t,x)-b)e^{-\nu x}\,dx \\ &= -\lambda\Big[(u(t,x)-b)^2 e^{-\nu x}\Big]_0^L - \nu\lambda\int_0^L (u(t,x)-b)^2 e^{-\nu x}\,dx \\ &\le -\lambda\nu\mathcal{L}_\nu(u(t,\cdot)).\end{aligned}$$

A direct application to Gronwall's lemma thus asserts

$$\forall \nu > 0, \quad \forall t \ge 0, \quad \mathcal{L}_\nu(u(t,\cdot)) \le \mathcal{L}_\nu(u_0)e^{-\lambda\nu t}. \tag{6.4}$$

Now the trick is to compare the family $(\mathcal{L}_\nu)_{\nu>0}$ with the $\|\cdot\|_{L^2(0,L)}$ on which they are based. Clearly we have

$$\forall \nu > 0, \quad \forall v \in L^2(0,L), \quad e^{-\nu L}\|v-b\|^2_{L^2(0,L)} \le \mathcal{L}_\nu(v) \le \|v-b\|^2_{L^2(0,L)}. \tag{6.5}$$

Combining (6.5) and (6.4) we end up with :

$$\forall \nu > 0, \quad \forall t > 0, \quad \|u(t,\cdot)-b\|_{L^2(0,L)} \le e^{-\nu(\lambda t - L)/2}\|u_0-b\|_{L^2(0,L)}. \tag{6.6}$$

And clearly for $t > L/\lambda$, if we let ν tend to $+\infty$, we get back

$$\forall t > \frac{L}{\lambda}, \quad u(t,\cdot) = b. \tag{6.7}$$

Of course for this particular example, the Lyapunov based method was much more complicated than using the characteristics' method. However, we will see that we can adapt it to provide both

- robustness estimates when the evolution is perturbed,
- exact controllability results with feedback controls for conservation laws in a setting which was not easily accessible with the methods detailed in previous chapters.

6.2 Robustness analysis

In [117], we investigated the exact controllability problem using a feedback control — also called finite time stabilization problem — for the Saint-Venant equation :

$$\begin{cases}\partial_t H(t,x) + \partial_x(HV)(t,x) = 0, \\ \partial_t V(t,x) + \partial_x(V^2/2 + gH)(t,x) = 0\end{cases} \quad t > 0, \quad x \in (0,L). \tag{6.8}$$

This describes the evolution of the surface of water in a one-dimensional canal : $H(t,x)$ is the height of the water and $V(t,x)$ related to the speed of propagation, see [125] for the original article. The goal is to act on the boundary with sluice gates so that the canal gets back to an horizontal water level as fast as possible. Those sluice gates allow us to decide the flow rate at both extremities, the flow rate being $Q(t,x) := H(t,x)V(t,x)$. In the case where the target state is (H^*, V^*), the feedback controls that we investigated are actually

$$\forall t > 0, \quad \begin{cases} Q(t,0) = H(t,0)\left(u(t) - 2\sqrt{gH(t,0)} + V^* + 2\sqrt{g\,H^*}\right), \\ Q(t,L) = H(t,L)\left(v(t) + 2\sqrt{g\,H(t,L)} + V^* - 2\sqrt{g\,H^*}\right), \\ \frac{dv}{dt}(t) = -K\mathrm{sign}(v(t))|v(t)|^\gamma, \\ \frac{du}{dt}(t) = -K\mathrm{sign}(u(t))|u(t)|^\gamma, \end{cases} \tag{6.9}$$

with K a positive number and γ in $(0,1)$.

We were able to prove the following result.

Theorem 20: Perrollaz-Rosier [117]

Assume that $0 < V^* < \sqrt{g\,H^*}$ and pick any positive number c satisfying

$$2\,c < \sqrt{g\,H^*} - V^*.$$

The there exists a positive number δ such that for any initial data (H_0, V_0) Lipschitz on $[0,L]$ and satisfying

$$\max(\|H_0 - H^*\|_{W^{1,\infty}}, \|V_0 - V^*\|_{W^{1,\infty}}) < \delta,$$

there exists a unique classical solution (H,V) of (6.8) and (6.9) defined on $\mathbb{R}^+ \times (0,L)$ and a time T^* depending only on c, δ, K and γ such that

$$\forall t \geq T^*, \quad H(t,\cdot) = H^*, \quad V(t,\cdot) = V^*. \tag{6.10}$$

Note that [117] actually dealt with a network of canals.

Equation (6.8) assumes that the bottom of the canal is perfectly flat, to take into account a small slope of angle θ we use :

$$\begin{cases} \partial_t H(t,x) + \partial_x(H\,V)(t,x) = 0, \\ \partial_t V(t,x) + \partial_x(V^2/2 + g\,\cos(\theta)H)(t,x) = -g\sin(\theta) \end{cases} \quad t > 0, \quad x \in (0,L). \tag{6.11}$$

Since θ is very small, one is thus lead to investigate the dynamics of the solution when the boundary condition is given by (6.9).

In [81], we investigated an abstract version of the problem and considered the system

$$\begin{cases} \partial_t u^\epsilon + \lambda(u^\epsilon, v^\epsilon)\partial_x u^\epsilon = \epsilon f(u^\epsilon, v^\epsilon), \\ \partial_t v^\epsilon - \mu(u^\epsilon, v^\epsilon)\partial_x v^\epsilon = \epsilon g(u^\epsilon, v^\epsilon), \\ u^\epsilon(0,x) = u_0(x), \\ v^\epsilon(0,x) = v_0(x), \end{cases} \quad t > 0, \quad x \in (0,L), \tag{6.12}$$

In this form, the boundary control laws (6.9) translates as :

$$\begin{cases} u^\epsilon(t,0) = y_l(t), \\ v^\epsilon(t,L) = y_r(t), \\ y_l'(t) = -K\mathrm{sign}(y_l(t) - \bar{u})|y_l(t) - \bar{u}|^\gamma, \\ y_r'(t) = -K\mathrm{sign}(y_r(t) - \bar{v})|y_r(t) - \bar{v}|^\gamma, \end{cases} \tag{6.13}$$

where the target state for $\epsilon = 0$ is $(\bar{u}, \bar{v})$.

The first result is that the stationary solution of (6.12) exists : for ϵ sufficiently small we have $\bar{u}_\epsilon$ and $\bar{v}^\epsilon$ two regular functions satisfying

$$\begin{cases} \lambda(\bar{u}^\epsilon, \bar{v}^\epsilon)\partial_x \bar{u}^\epsilon = \epsilon f(\bar{u}^\epsilon, \bar{v}^\epsilon), \\ \mu(\bar{u}^\epsilon, \bar{v}^\epsilon)\partial_x \bar{v}^\epsilon = \epsilon g(\bar{u}^\epsilon, \bar{v}^\epsilon), \\ \bar{u}^\epsilon(0) = \bar{u}, \\ \bar{v}^\epsilon(L) = \bar{v}, \end{cases} \quad x \in (0, L). \tag{6.14}$$

Furthermore we have $\bar{u}^\epsilon \to \bar{u}$ and $\bar{v}^\epsilon \to \bar{v}$ when $\epsilon \to 0$.

The dynamics is then given by the following result.

Theorem 21: Gugat-Perrollaz-Rosier [81]

If $\lambda(\bar{u}, \bar{v}) > 0$ and $\mu(\bar{u}, \bar{v}) > 0$, there exist $\epsilon_0 > 0$ and $\delta > 0$ such that for any ϵ in $[0, \epsilon_0]$, any γ in $(0, 1)$, any positive K and any initial data (u_0, v_0) Lipschitz on $[0, L]$ and satisfying

$$\max(\|u_0 - \bar{u}\|_{W^{1,\infty}}, \|v_0 - \bar{v}\|_{W^{1,\infty}}) \leq \delta,$$

then the system (6.12)-(6.14) has a unique maximal regular solution. It is global in time and satisfies

$$\forall t > 0, \quad \|(u^\epsilon - \bar{u}^\epsilon, v^\epsilon - \bar{v}^\epsilon)\|^2_{L^2(0,L)} \leq M \min\left(1, \frac{e^{-C_\epsilon t}}{\epsilon^{1+\kappa}}\right) \|(u_0 - \bar{u}^\epsilon, v_0 - \bar{v}^\epsilon)\|^2_{L^2(0,L)}, \tag{6.15}$$

and

$$\forall t > 0, \quad \|(u^\epsilon - \bar{u}^\epsilon, v^\epsilon - \bar{v}^\epsilon)\|_{L^\infty(0,L)} \leq M \min\left(1, \frac{e^{-C_\epsilon t/3}}{\epsilon^{(1+\kappa)/3}}\right) \|(u_0 - \bar{u}^\epsilon, v_0 - \bar{v}^\epsilon)\|^{2/3}_{L^\infty(0,L)}, \tag{6.16}$$

with $\kappa := (c\delta^\gamma)/(KL\gamma)$, $M = M(\delta) > 0$ and $C_\epsilon = C_\epsilon(\delta) \underset{\delta \to 0^+}{\sim} -\frac{c}{L}\ln(\epsilon)$.

Compared to Theorem 20, we have lost the finite time stabilization to the target state. However we see that we still have exponential convergence to the perturbed stationary solution.

For the existence part, the proof is obtained using a Schauder fixed point theorem using the quasilinear form of (6.12). The compactness follows from a uniform control of the Lipschitz constant of the solutions by the Lipschitz constant of the initial data.

The asymptotic estimates (6.15) and (6.16) are obtained using the idea of the previous section and the family of functionals defined by

$$\forall \theta > 0, \quad \forall (a, b) \in [L^2(0, L)]^2, \quad \mathcal{L}_\theta(a, b) := \int_0^L [(a - \bar{u}^\epsilon)^2(x)e^{-\theta x} + (b - \bar{v}^\epsilon)^2(x)e^{-\theta(L-x)}]dx. \tag{6.17}$$

To be complete, we had to add some contributions from the boundary in the paper.

The key point is that, because of the perturbation, the quasilinear nature of the systems and the coupling on the speeds of propagation, there is cutoff on the parameter θ. Only below this cut off value do we have decreasing Lyapunov functionals. Of course this cutoff tends to $+\infty$ when ϵ goes to 0. For an example of such a cutoff, look at the viscous perturbation detailed in the last section of the present chapter.

6.3 Controllability to trajectories for entropy solutions

In this section, we will consider the conservation law :

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i} f_i(u)(t,x) = 0, t > 0, \quad x \in \Omega \subset \mathbb{R}^d, \tag{6.18}$$

where Ω is a regular domain.

Let us recall that a function u in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution to (6.18) when for any real number k and any non-negative test function ψ in $\mathcal{C}^1_c(\mathbb{R} \times \Omega)$ we have

$$\int_0^\infty \int_\Omega |u(t,x) - k| \partial_t \psi(t,x) + \text{sign}(u(t,x) - k) \left(\sum_{i=1}^{d} (f_i(u(t,x)) - f_i(k)) \partial_{x_i} \psi(t,x) \right) \, dx dt + \int_\Omega |u_0(x) - k| \psi(0,x) \, dx \geq 0. \tag{6.19}$$

From now on, we will suppose that the flux $(f_1, \ldots, f_d)$ are regular. Let us also fix a non-empty interval I of $\mathbb{R}$.

We will need two auxiliary functions L and c from $\mathbb{R}^d$ to $\mathbb{R}$ to state our result. Given a vector $w \in \mathbb{R}^d$, we define

$$L(w) := \sup_{x \in \Omega} \left(\sum_{i=1}^{d} x_i w_i \right) - \inf_{y \in \Omega} \left(\sum_{j=0}^{d} y_j \, w_j \right), \tag{6.20}$$

$$c(w) := \inf_{u \in I} \left(\sum_{i=1}^{d} w_i \, f_i'(u) \right). \tag{6.21}$$

Then we have the following result.

Theorem 22: Donadello-Perrollaz [64]

Let us suppose that there exists w in $\mathbb{R}^d$ such that $L(w) < +\infty$ and $c(w) > 0$. Consider v an entropy solution of (6.18) taking values in I and u_0 an initial data also taking values in I, then there exists en entropy solution u of (6.18) satisfying

$$\left\{ \forall x \in \mathbb{R}, \quad u(0,x) = u_0(x), \quad u(L(w)/c(w), x) = v(L(w)/c(w), x). \right. \tag{6.22}$$

Keeping Russell's extension method in mind (see Section 1.3.3), we have proven an exact controllability result for (6.18).

Idea of the proof. The key point in Kruzkov's proof of contraction of the semigroup is always the following Kato type inequality. Given two entropy solutions of (6.18) we have

$$\partial_t |u - v| + \text{div} \, \phi(u,v) \leq 0, \text{ on } (0,T) \times \Omega \tag{6.23}$$

where the inequality is understood in the distributional sense and we define ϕ by :

$$\forall(a,b)\in\mathbb{R}^2,\quad \phi(a,b):=\big(\mathrm{sign}(a-b)(f_1(a)-f_1(b)),\dots,\mathrm{sign}(a-b)(f_d(a)-f_d(b))\big)\in\mathbb{R}^d.$$

His idea when $\Omega=\mathbb{R}^d$ is to recognize that for each compact subset of $\mathbb{R}^2$, there exists a constant M such

$$\|\phi(u,v)\|\leq M|u-v|. \tag{6.24}$$

After fixing two positive numbers T and R, integrate inequality (6.23) on the trapezoid $\mathbb{T}:=\{(t,x)\in[0,T]\times\mathbb{R}^d\,:\,-R-M(T-t)\leq\|x\|\leq R+M(T-t)\}$. Using the Stokes formula, the spatial part of the boundary terms is found to have the good signs thanks to (6.24) and allow us to show that

$$t\mapsto\int_{B_{R+M(T-t)}}|u-v|(t,x)dx \text{ is non-increasing,}$$

where B_r is the ball centered at 0 of radius r.

Now when Ω has a boundary, the results from [6] allow us to integrate on Ω instead of $\mathbb{T}$ and still have contraction with respect not only to $|u_0-v_0|$ but also with respect to the boundary terms.

In our case, the idea is to modify the starting Kato type inequality in the following way. Consider w the vector from $\mathbb{R}^d$ from Theorem 22, given a positive parameter ν we define the weight function p_ν by

$$\forall x\in\mathbb{R}^d,\quad p_\nu(x):=e^{-\nu\langle w|x\rangle}, \tag{6.25}$$

where $\langle\cdot\mid\cdot\rangle$ denotes the classical scalar product.

Then we have the following new Kato inequality :

$$\begin{aligned}\partial_t(p_\nu|u-v|)+\mathrm{div}(p_\nu\phi(u,v))&=p_\nu(\partial_t|u-v|+\mathrm{div}\phi(u,v))\\&\quad+|u-v|\partial_t p_\nu+\langle\phi(u,v)\mid\nabla p_\nu\rangle.\end{aligned}$$

But clearly p_ν is non-negative and $\nabla p_\nu=-\nu\, w p_\nu$, so we end up with :

$$\partial_t(p_\nu|u-v|)+\mathrm{div}(p_\nu\phi(u,v))\leq-\nu p_\nu\langle w\mid\phi(u,v)\rangle.$$

Finally using $c(w)>0$ we see that :

$$\forall(a,b)\in I^2,\quad\langle w\mid\phi(a,b)\rangle=\sum_{k=1}^{d}w_k\mathrm{sign}(a-b)(f(a)-f(b))\geq|a-b|c(w).$$

And we obtain the final Kato type inequality

$$\partial_t(p_\nu|u-v|)+\mathrm{div}(p_\nu\phi(u,v))\leq-\nu c(w)p_\nu|u-v|.$$

Using the ideas of Kruzkov — as modified in [6] — and the right kind of boundary conditions, we can apply the Gronwall lemma to the family of Lyapunov functions :

$$\mathcal{L}_\nu(u,v):=\int_\Omega|u-v|(t,x)p_\nu(x)dx,$$

and finish the proof using the ideas from the first section of this chapter.

6.4 Discussion

Geometric hypothesis. The geometric condition of Section 6.3 is sufficient but clearly not necessary. In one dimension of space, it would not even allow us to get exact controllability with a stationary shock wave as a target. It is clearly important to adapt the method and probably the weight functions to this kind of situation. It would probably be necessary to have some kind of predictor providing some estimate on the localization of the singularity when one is close enough to the shock in the spirit of the result of Chapter 4.

General robustness. When looking at the analysis done in the first section of this Chapter on the transport equation, one might realize that the robustness results are very general. Rather than looking at a perturbation by a source term, let us consider the viscous perturbation of (6.1)

$$\begin{cases} \partial_t u^\epsilon(t,x) + \lambda \partial_x u^\epsilon(t,x) = \epsilon \partial_{xx}^2 u^\epsilon, \\ u^\epsilon(t,0) = b, \\ u^\epsilon(t,L) = b, \\ u^\epsilon(0,x) = u_0(x), \end{cases} \qquad t > 0, \quad x \in (0,L), \tag{6.26}$$

Using the family $(\mathcal{L}_\nu)_{\nu>0}$ defined in (6.3), we can compute :

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_\nu(u^\epsilon(t,\cdot)) &= \int_0^L 2\partial_t u^\epsilon(t,x)(u^\epsilon(t,x)-b)e^{-\nu x}\,dx \\ &= \int_0^L 2(\epsilon\partial_{xx}^2 u^\epsilon(t,x) - \lambda\partial_x u^\epsilon(t,x))(u^\epsilon(t,x)-b)e^{-\nu x}\,dx \\ &= 2\epsilon\int_0^L \partial_{xx}^2(u^\epsilon(t,x)-b)(u^\epsilon(t,x)-b)e^{-\nu x}dx \\ &\quad - 2\lambda\int_0^L \partial_x(u^\epsilon(t,x)-b)(u^\epsilon(t,x)-b)e^{-\nu x}dx \\ &= 2\epsilon\big[\partial_x(u^\epsilon(t,x)-b)(u^\epsilon(t,x)-b)e^{-\nu x}\big]_0^L \\ &\quad - 2\epsilon\int_0^L (\partial_x(u^\epsilon(t,x)-b))^2 e^{-\nu x}dx \\ &\quad + \nu\epsilon\int_0^L \partial_x(u^\epsilon(t,x)-b)^2 e^{-\nu x}dx \\ &\quad - \lambda\int_0^L \partial_x(u^\epsilon(t,x)-b)^2 e^{-\nu x}dx \\ &\leq -(\lambda-\nu\epsilon)\Big[(u^\epsilon(t,x)-b)^2 e^{-\nu x}\Big]_0^L - \nu(\lambda-\nu\epsilon)\int_0^L (u^\epsilon(t,x)-b)^2 e^{-\nu x}\,dx \\ &\leq -\nu(\lambda-\nu\epsilon)\mathcal{L}_\nu(u^\epsilon(t,\cdot)). \end{aligned}$$

So we see that for ν up to $\frac{\lambda}{\epsilon}$, $\mathcal{L}_\nu$ is a Lyapunov functional so we don't have finite stabilization — i.e. exact controllability — for the viscous perturbation. However passing to the L^2 norm as in the first section we get :

$$\|u^\epsilon(t,\cdot)-b\|_2^2 \leq e^{\nu(L-t(\lambda-\nu\epsilon))}\|u_0-b\|_2^2,$$

and optimizing for $0 < \nu < \lambda/\epsilon$ we end up with :

$$\|u^\epsilon(t,\cdot) - b\|_2 \leq e^{-\frac{(\lambda t - L)^2}{8\epsilon t}} \|u_0 - b\|_2.$$

We dealt with a perturbation of 2×2 system of conservation laws by a small source term in Section 6.2. The calculation we just detailed suggests that some results could probably obtained when considering viscous perturbations. With the added value that we could deal with regular solutions of the viscous problem.

A more challenging problem is to look at the case where the controls acting on the system are sampled in time. From an application perspective this is a very important question since any digital setup leads to this kind of situation. Of course, in the case of non-linear equations, using discontinuous boundary value data leads directly to working in the framework of entropy solutions so the method of Section 6.3 could be instrumental to solve this kind of problem.

Relative entropy method. There is classical method to prove weak-strong uniqueness results in a very general setting : the relative entropy method [54, Chapter 5 Section 2]. The idea is to prove some contraction in L^2 using Kato type inequalities. It should be possible to use the techniques from this Chapter to adapt the method and get exact controllability to regular trajectories in a very general setting. Let us describe the idea for a scalar one dimensional conservation law, keeping in mind that it does generalize. We consider therefore one classical solution u_1 and one weak solution u_2 of

$$\partial_t u + \partial_x f(u) = 0.$$

We suppose that we have one entropy entropy flux couple (E, Q) associated to f with E being uniformly convex. Note that the existence of an entropy is not general for systems, but that for many physical equations there is indeed one. Since u_1 is a classical solution we also have :

$$\partial_t E(u_1) + \partial_x Q(u_1) = 0.$$

We request as an additional hypothesis that u_2 satisfies the one corresponding entropy inequality

$$\partial_t E(u_2) + \partial_x Q(u_2) \leq 0.$$

Note that we manipulate all equalities and inequalities in the distributional sense. At this point we clearly have

$$\begin{cases} \partial_t(u_2 - u_1) + \partial_x(f(u_2) - f(u_1)) = 0, \\ \partial_t \left(E(u_2) - E(u_1)\right) + \partial_x \left(Q(u_2) - Q(u_1)\right) \leq 0. \end{cases}$$

But of course $(a, b) \mapsto E(b) - E(a)$ may take negative values and thus does not allow us to compare a and b. Keeping in mind the uniform convexity of E we introduce the function h through :

$$\forall (a, b) \in \mathbb{R}, \quad h(a, b) := E(a) - E(b) - E'(b)(a - b).$$

Clearly we have

$$\forall (a, b) \in \mathbb{R}^2, \quad h(a, b) \geq (a - b)^2 \inf_{z \in I(a,b)} E''(z).$$

The goal becomes to find a differential inequality on $h(u_2, u_1)$. But of course

$$\partial_t h(u_1, u_2) = \partial_t(E(u_2) - E(u_1)) + E'(u_1)\partial_t(u_2 - u_1) + \partial_t u_1 E''(u_1)(u_2 - u_1),$$

and the first two terms on the right hand side are easy to recognize. We obtain therefore

$$\partial_t h(u_2, u_1) \leq -\partial_x(Q(u_2) - Q(u_1)) - E'(u_1)\partial_x(f(u_2) - f(u_1)) + \partial_t u_1 E''(u_1)(u_2 - u_1).$$

Working on the second and third terms on the right hand side to exploit the analogy with the definition of h we get

$$\begin{aligned}\partial_t h(u_2, u_1) \leq -\partial_x(Q(u_2) - Q(u_1) - E'(u_1)(f(u_2) - f(u_1)))\\ + \partial_x u_1 E''(u_1)(f(u_2) - f(u_1)) - \partial_x u_1 f'(u_1)E''(u_1)(u_2 - u_1).\end{aligned}$$

This leads to defining two functions p and r on $\mathbb{R}^2$ by

$$\forall (a, b) \in \mathbb{R}^2, \quad \begin{cases} p(a, b) = Q(a) - Q(b) - E'(b)(f(a) - f(b)), \\ r(a, b) = E''(b)\,(f(a) - f(b) - f'(b)(a - b))\,. \end{cases}$$

And as a consequence

$$\partial_t h(u_2, u_1) + \partial_x p(u_2, u_1) \leq \partial_x u_1 r(u_2, u_1). \tag{6.27}$$

But for a given couple (a, b) the Taylor-Lagrange formula provides the existences c_1, c_2 and c_3 such that :

$$\begin{cases} p(a, b) = (a - b)^2 \frac{Q''(c_1) + E'(b)f''(c_2)}{2}, \\ r(a, b) = (a - b)^2 \frac{E''(b)f''(c_3)}{2}. \end{cases}$$

As a consequence there exists positive constants M_1 and M_2 — provided we have L^∞_{loc} bound on both u_1 and u_2 — such that

$$\begin{cases} |p(u_2, u_1)| \leq M_1\, h(u_2, u_1), \\ |r(u_2, u_1)| \leq M_2\, h(u_2, u_1). \end{cases} \tag{6.28}$$

Let us fix two positive numbers T and R. Integrating the Kato type inequality (6.27) on the trapezoid $\mathbb{T} := \{(t, x) \in [0, T] \times \mathbb{R} \,:\, -R - M_1(T - s) \leq x \leq R + M_1(T - s)\}$, we get :

$$\begin{aligned}\int_{\partial\mathbb{T}} h(u_2, u_1)dx - p(u_2, u_1)dt &= \int_{\mathbb{T}} (\partial_t h(u_2, u_1) + \partial_x p(u_2, u_1))\, dt \wedge dx \\ &\leq \int_{\mathbb{T}} \partial_x u_1 r(u_2, u_1) dt \wedge dx,\end{aligned}$$

which becomes

$$\begin{aligned}\int_{R+M_1T}^{-R-M_1T} h(u_2, u_1)(0, x)dx + \int_0^T -M_1 h(u_2, u_1)(t, R+M_1(T-t)) - p(u_2, u_1)(t, R+M_1(T-t))dt \\ + \int_{-R}^{R} h(u_2, u_1)(T, x)dx + \int_T^0 +M_1 h(u_2, u_1)(t, -R-M_1(T-t)) - p(u_2, u_1)(t, -R-M_1(T-t))dt \\ \leq \int_{\mathbb{T}} \partial_x u_1 r(u_2, u_1) dt \wedge dx.\end{aligned}$$

So in the end, keeping in mind the bounds (6.28), we see that the term corresponding to the sides $x = R + M_1(T-t)$ and $x = -R - M_1(T-t)$ are signed and we end up with

$$\int_{-R}^{R} h(u_2, u_1)(T, x)dx \leq \int_{-R-M_1T}^{R+M_1T} h(u_2, u_1)(0, x)dx + M_2 \int_0^T \int_{-R-M_1(T-t)}^{R+M_1(T-t)} h(u_2, u_1)(t, x)dxdt. \tag{6.29}$$

We can apply Gronwall to $t \mapsto \int_{-R-M_1(T-t)}^{R+M_1(T-t)} h(u_2, u_1)(t, x)dx$ and conclude.

Let us finish by mentioning that since Vasseur managed to adapt the method to deal with some discontinuous solutions [128], we could hope to follow his methods to improve our control/stabilization results.

Chapitre 7

Résumé en Français

On propose ici une description non exhaustive du contenu du manuscrit en français.

Ce manuscrit propose un guide de lecture des articles [2, 36, 37, 38, 39, 40, 45, 64, 81, 113, 114, 115, 117, 118, 116].

Le Chapitre 2 s'intéresse aux équations aux dérivées partielles suivantes :

$$\partial_t u + \partial_x \left(H(x, u) \right) = 0, \tag{7.1}$$

et

$$\partial_t U + H(x, \partial_x U) = 0. \tag{7.2}$$

On y montre que les problèmes sont bien posés au sens de Hadamard en temps grand sous une hypothèse plus général que les résultats fondamentaux de Kruzkov dans [95] et de Crandall-Lions dans [49]. Ces papiers s'intéressent cependant à un cadre beaucoup plus large. Dans une second étape, la correspondance formelle entre ces équations — via $u = \partial_x U$ — est prouvée rigoureusement au niveau des semigroupes.

Tous les chapitres qui suivent sont consacrés à des problèmes des contrôle dans le cadre de lois de conservation hyperboliques. Ils sont organisés suivant la méthodologie utilisée pour traiter les problèmes. Et mis à part la méthode de viscosité évanescente, toues les méthodes existantes sont abordées.

Dans le Chapitre 3, on montre comment il est possible d'utiliser la connexion avec les équations d'Hamilton-Jacobi, et plus important encore avec les problèmes de contrôle optimal sous-jacent à celles-ci, pour résoudre des problèmes d'identification de données initiales.

Dans le Chapitre 4, on prouve des résultats de stabilisation asymptotique par retour d'état stationnaire. A cette fin, on utilise la théorie des caractéristiques généralisées de Dafermos [51].

Dans le Chapitre 5, on commence par décrire l'algorithme de suivi de fronts. Celui-ci a été initialement introduit par Dafermos dans [53] pour traiter le cas des équations scalaires. Il a ensuite été étendu par Di Perna dans [62] pour traiter le cas des systèmes 2×2. Il a finalement été poussé dans son cadre le plus général par Bressan — par exemple dans [27, 31] — pour prouver l'existence et l'unicité du semigroupe standard de Riemann pour les systèmes $n \times n$.

Dans le cas scalaire, Holden et ses co-auteurs ont depuis longtemps utilisé cette méthode en particulier à des fins numériques, à partir du papier [85]. Dans la deuxième partie du chapitre, on utilisera cet algorithme pour résoudre un problème de contrôlabilité exacte sur une équation scalaire et un problème de stabilisation asymptotique par retour d'état stationnaire pour un système hyperbolique 2×2.

Finalement, dans le Chapitre 6, on montre comment il est possible d'utiliser des inégalités de type Kato pour obtenir à la fois des résultats de contrôlabilité exacte, des résultats de stabilisation asymptotique et des estimations de robustesse dans des contextes à priori inaccessibles aux méthodes précédentes.

Un objectif secondaire de ce manuscrit est de proposer une sorte de comparatif des différentes méthodes utilisées pour résoudre des problèmes de contrôle dans le cadre des solutions entropiques de lois de conservation. Ceci à l'exception notable de la méthode par viscosité évanescente. Le Chapitre 2 utilise, de fait des approximations visqueuses pour obtenir des résultats d'existence en temps grands pour les équations (7.1) et (7.2) et étudie ensuite le passage à la limite via la méthode de compacité par compensation. De même, plusieurs question ouvertes du Chapitre 6 ont aussi un lien avec cette méthode. Cependant, on conseille plutôt de consulter par exemple [75, 91, 106] pour explorer des résultats de contrôle via cette méthode.

Toutes les figures de ce manuscrit ont été réalisé grâce à python [93], numpy [83], matplotlib [89] et jupyter [94].

Comme la majorité des résultats de ce manuscrit traite de problèmes de contrôle dans le cadre des solutions entropiques de lois de conservation, on va fournir

- une introduction à trois problèmes standard la théorie du contrôle dans la prochaine section,
- une introduction aux lois de conservation hyperboliques dans celle qui suit.

7.1 Sur la théorie du contrôle

Au tournant du 17ème siècle Galilée décrivait la nécessité d'introduire des mathématiques dans les sciences naturelles par sa citation célèbre : « La philosophie est écrite dans ce grand livre qui s'étend chaque jour devant nos yeux : l'univers. Mais on ne peut le comprendre si nous n'apprenons d'abord son langage et si nous ne comprenons les symboles avec lesquels il est écrit ». On peut considérer grossièrement les efforts de Fermat et Descartes dans « La géométrie » utilisant des mathématiques pour décrire des courbes générales en utilisant des moyens algébriques comme la première étape de ce programme. La seconde étape étant alors constituée des travaux de Newton utilisant les méthodes mathématiques pour prédire des phénomènes physiques.

Il est remarquable de constater qu'une troisième étape naturelle du programme consiste à introduire des mathématiques pour comprendre comme on peut agir pour influencer la nature d'une façon voulue. Cette étape a seulement démarré avec le travail de Maxwell dans l'article "On governors" de 1868 [108].

Il y a étudié la stabilité du régulateur de Watt et à cette fin introduit le critère de linéarisation. A se suite une littérature gigantesque s'est construite que l'on dénotera sous le terme de théorie du contrôle.

7.1.1 Description formel de quelques problèmes

Commençons par introduire des problèmes sous une forme plutôt abstraite. Pour ce faire on considère un système dynamique générique.

$$\begin{cases} \dot{X}(t) = F(X(t), U(t)), \\ X(0) = X_0, \end{cases} \tag{7.3}$$

ici $X \in \mathcal{X}$ est l'état du système tandis que $U \in \mathcal{U}$ est le contrôle. Un aspect de l'analyse mathématique consiste à bien sélectionner les espaces $\mathcal{X}$ et $\mathcal{U}$.

Plus concrètement le système (7.3) serait une équation différentielle ordinaire ou une équation aux dérivées partielles d'évolution. On modélise ainsi un système physique par X sur lequel on ne peut agir qu'à travers U. La question est bien sûr de trouver un moyen efficace de sélectionner le contrôle pour que le système adopte le comportement désiré.

A cette fin introduisons trois problèmes classiques (de nombreux autres existent).

1. Le plus direct est celui dit de contrôlabilité exacte. Il consiste à piloter le système, depuis un état initial, vers un état cible, en un temps donné. Plus précisément, étant donnés deux états $X_0 \in \mathcal{X}$, $X_1 \in \mathcal{X}$ et un temps $T > 0$, existe-t-il une fonction $t \in [0, T] \mapsto U(t) \in \mathcal{U}$ telle que la solution de (7.3) satisfait $X(T) = X_1$?
2. Une approche alternative consiste à piloter le système en cherchant à garder un certain coût minimal, il s'agit du problème de contrôle optimal. De manière plus précise, introduisons un temps $T > 0$, un coût d'exploitation $R : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ et une fonction de pénalisation $P : \mathcal{X} \to \mathbb{R}$. A une stratégie particulière de contrôle $U : [0, T] \to \mathcal{U}$ on peut associer sont coût totale par la formule

   $$C(U) := \int_0^T R(X(t), U(t))dt + P(X(T)), \tag{7.4}$$

   où X est la solution associée à U par (7.3).

   L'objectif est alors de déterminer un minimiseur U_m de la fonctionnelle C parmi toutes les stratégies de contrôle admissibles.
3. Décrivons pour finir le problème de stabilisation asymptotique par retour d'état stationnaire. A cette fin, supposons donné un couple $(X_e, U_e) \in \mathcal{X} \times \mathcal{U}$ tel que $F(X_e, U_e) = 0$. On a donc un équilibre du système. Il s'agit bien sûr de solutions structurellement importants pour le système (7.3). Bien sûr lorsque l'évolution se fait sans contrôle seuls les états stables sont observés. Par exemple, lorsque on pense au pendule, seule la position basse est rencontrée dans la nature. Pour compenser ceci on pourrait chercher un contrôle $t \in [0, T] \mapsto U(t) \in \mathcal{U}$ telle que la solution de (7.3) satisfasse $X(T) = X_e$ et $U(T) = U_e$. Mais il se trouve que cette stratégie est horriblement fragile. Elle est de fait très sensible aux perturbations et approximations. On lui préfère donc le problème de stabilisation par retour d'état stationnaire. On cherche une fonction $\mathbb{U} : \mathcal{U} \to \mathcal{X}$ satisfaisant $\mathbb{U}(X_e) = U_e$ et telle que pour le système en boucle fermée :

   $$\begin{cases} \dot{X}(t) = F(X(t), \mathbb{U}(X(t))) \\ X(0) = X_0, \end{cases} \tag{7.5}$$

   le point d'équilibre X_e est asymptotiquement stable.

   En termes plus précis, cela revient à demander à ce que les propriétés suivantes soient satisfaites.

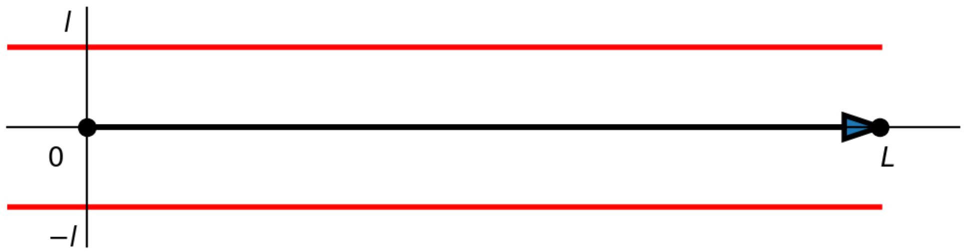

FIGURE 7.1 – Les bords de la route sont en rouge.

- Pour tout $\epsilon > 0$, il existe $\nu > 0$ tel que si X_0 est un état satisfaisant $\|X_0 - X_e\| \leq \nu$ et si X est une solution maximale de (7.5), alors elle est globale en temps et vérifie

$$\forall t \geq 0, \qquad \|X(t) - X_e\| \leq \epsilon.$$

- Pour tout état initial X_0, une solution maximale de (7.5) est globale en temps et vérifie

$$\|X(t) - X_e\| \underset{t \to +\infty}{\longrightarrow} 0.$$

7.1.2 Un exemple tangible

On va maintenant étudier un système simplifié pour illustrer ces notions. On considère le problème d'une voiture évoluant sur une ligne parfaitement droite. Le véhicule sera considéré ponctuel et avançant à une vitesse constante Le seul contrôle sera la direction de la voiture.

L'état du système sera encodé par un couple (x, y). Et être sur la route signifiera vérifier $-l < y < l$, où $2l$ est la largeur de la route. A ce stade, on a $\mathcal{X} := \mathbb{R} \times [-l, l]$. Par simplicité, on considérera que la direction est fournie par une fonction $t \mapsto u(t) \in \mathcal{U} := \mathbb{R}$. L'évolution est alors régie par

$$\begin{cases} \dot{x}(t) = \nu \cos(u(t)), \\ \dot{y}(t) = \nu \sin(u(t)), \end{cases} \tag{7.6}$$

où ν est la vitesse de déplacement.

Un exemple de contrôlabilité exacte pourrait être de partir de $(0, 0)$ et de chercher à rejoindre un autre point de l'axe des x par exemple $(L, 0)$ (où $L > 0$). La stratégie $u : t \mapsto 0$ résout évidemment le problème en temps $T_m := \frac{L}{\nu}$. Notons que pour $T < T_m$, il n'y aurait pas de solution et que pour $T > T_m$ il y en aurait une infinité. On pourra consulter la Figure 7.1 pour une représentation géométrique.

Un exemple de contrôle optimal pourrait être de conduire en restant le plus proche possible du centre de la route tout en finissant le plus près possible du point $(L, 0)$. On chercherait alors à minimiser la quantité

$$C(U) := \int_0^T y(t)^2 dt + y(T)^2 + (x(T) - L)^2.$$

pour une stratégie $t \mapsto U(t)$ donnée. Lorsque $T = T_m$, la solution du problème de contrôlabilité exacte est encore solution de ce nouveau problème.

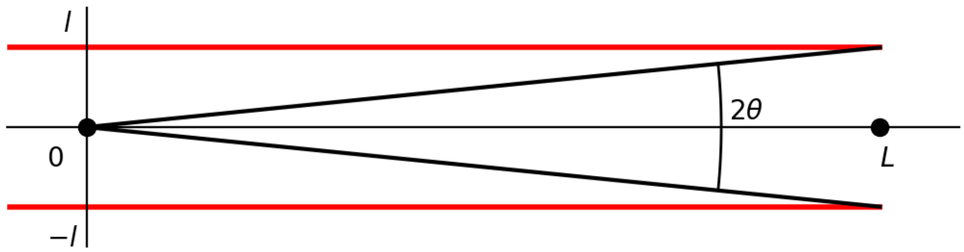

FIGURE 7.2 – Ici θ est égal à 3 minutes d'arc

Pour comprendre l'intérêt de la stabilisation par retour d'état stationnaire, réalisons tout d'abord que la stratégie considérée jusqu'à présent revient à bloquer son volant au démarrage connaissant la position de départ et celle d'arrivée, puis on avance pour une durée déterminée, tout en gardant les yeux fermés.

Il est évident que personne ne se risquerait à utiliser cette stratégie en réalité. Ainsi pour une route de deux mètres de large et un kilomètre de long, il faudrait sélectionner la position du volant à trois minutes d'arc près. La Figure 7.2 explicite le calcul. Ce n'est évidemment pas envisageable pour un être humain, et cela couterait inutilement cher dans le cas d'un système électro-mécanique.

Une stratégie alternative, réellement utilisée, consisterait à tourner son volant dans la direction opposée au côté de la route que l'on occupe à chaque instant. Mathématiquement cela signifierait par exemple choisir $\mathbb{U}((x,y)) := -y/l$ avec une évolution donnée par

$$\begin{cases} \dot{x}(t) = v\cos\left(-\frac{y(t)}{l}\right), \\ \dot{y}(t) = v\sin\left(-\frac{y(t)}{l}\right). \end{cases}$$

Pour analyser cette dynamique, notons juste que si $y(t) \in [-l, l]$ alors

$$\frac{\mathrm{d}\left(y^2\right)}{\mathrm{d}t}(t) = 2vy(t)\sin(-\frac{y(t)}{l}) \leq 0,$$

et donc la quantité $y(t)^2$ est décroissante. Dans le pire des cas, on a la garantie que l'on reste sur la route. Il est en fait aisé en utilisant le lemme de Gronwall de raffiner l'argument pour prouver que si $y(0) \in [-l, l]$, alors on a

$$\forall t \geq 0, \quad y(t)^2 \leq y(0)^2 e^{-\frac{2\,v\,t\,\cos(1)}{l}},$$

ce qui montre la stabilité exponentielle de $y = 0$. Techniquement, on a montré que $\mathcal{L}(y) := y^2$ est une fonctionnelle de Lyapunov. On pourra consulter [45, Chapter 12, Section 1] pour une introduction à cet outil.

Pour comprendre l'intérêt de cette stratégie, considérons que le contrôle est perturbée par une fonction parasite $t \mapsto p(t) \in [-\epsilon, \epsilon]$ où ϵ est un petit paramètre. La dynamique pour la stratégie de contrôlabilité exacte présentée plus haut — mais généralisée ici à une position initiale (x_0, y_0) — est donnée par

$$\begin{cases} \dot{x}(t) = v\cos\left(p(t) + \arctan\left(\frac{-y_0}{L-x_0}\right)\right), \\ \dot{y}(t) = v\sin\left(p(t) + \arctan\left(\frac{-y_0}{L-x_0}\right)\right) \end{cases} \tag{BOUCLE OUVERTE}$$

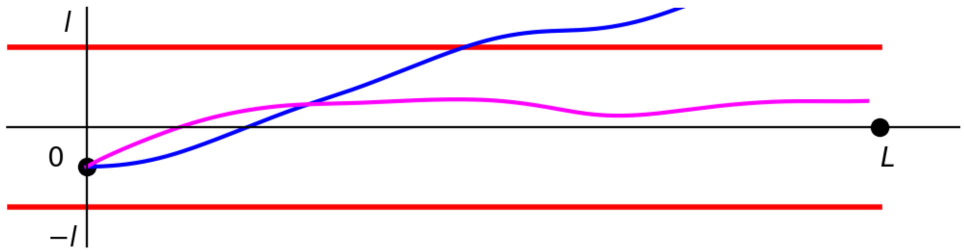

FIGURE 7.3 – Dynamique de (BOUCLE FERMEE) en magenta et de (BOUCLE OUVERTE)en bleu

pour la stratégie par retour d'état stationnaire on a la dynamique

$$\begin{cases} \dot{x}(t) = \nu \cos\left(p(t) - \frac{y(t)}{l}\right), \\ \dot{y}(t) = \nu \sin\left(p(t) - \frac{y(t)}{l}\right). \end{cases} \tag{BOUCLE FERMEE}$$

Ajouter $\arctan\left(\frac{-y_0}{L-x_0}\right)$ dans l'équation (BOUCLE OUVERTE) correspond exactement à emprunter la ligne droite reliant les positions initiale et finale.

Plutôt que de fournir une analyse mathématiques, examinons des simulations numérique où la perturbation est limitée à 5 degré — $\epsilon = \pi/36$ ci-dessus. On montre le résultat d'une telle simulation dans la Figure 7.3. On a juste utilisé un schéma d'Euler explicite pour les équations (BOUCLE OUVERTE) et (BOUCLE FERMEE). En ce qui concerne la fonction p, on a tronqué la formule

$$\forall t \in [0,T], \quad p(t) := \sum_{k=0}^{\infty} \frac{\epsilon\, u_k}{2^{k+1}} \cos\left(\frac{k\,\nu\, t}{L}\right),$$

où $(u_k)_{k\geq 0}$ est une famille de variables aléatoires indépendantes et uniformément distribuées sur $[-1,1]$. Le script générant cette simulation se trouve dans la Section A.1 de l'Annexe. Pour faire rentrer la figure dans une page, on a choisi une route de longueur $L = 10$, tout en gardant la largeur de $l = 1$, ce qui rend bien sûr la perte de contrôle de la stratégie en boucle ouverte encore plus terrifiante.

La raison pour la différence de robustesse entre les deux stratégies de la Figure 7.3 vient de ce que les erreurs s'accumulent pour la boucle ouverte, alors qu'il y a un système d'auto-correction pour la boucle fermée.

L'utilisation de mécanismes de contrôle en boucle fermée est beaucoup plus ancien que leur analyse mathématique. On consultera [109] pour des exemples remontant à l'antiquité.

Pour les cas où l'on a affaire à une équation différentielle ordinaire des techniques confirmées existent. On consultera [131] et [45] pour en apprendre plus.

Dans ce manuscrit on traite d'équations aux dérivées partielles et plus précisément de lois de conservation hyperboliques. Et les méthodes de dimension finie se généralisent très mal à cette classe d'équations. Découvrons les maintenant.

7.2 Lois de conservation hyperboliques, solutions entropiques

7.2.1 Origines des lois de conservation

Par simplicité, on considère une évolution en dimension un d'espace seulement. On se donne donc une fonction de densité $(t,x) \mapsto \rho(t,x)$ qui soit L^1_{loc}. La quantité de matière contenu dans un intervalle $[x_0, x_1]$ à un temps t est alors donnée par $\int_{x_0}^{x_1} \rho(t,x)dx$. On se donne également une fonction flux $(t,x) \mapsto F(t,x)$, aussi dans L^1_{loc}.

La dynamique de conservation consiste à dire que la quantité de matière contenu dans un intervalle (a,b) à un temps t_2 est égale à la quantité dans le même intervalle au temps t_1 modifié par ce qui a été échangé à travers la frontière. En termes précis, on demande à ce que pour presque tout $x_0 < x_1$ et presque tout $t_1 < t_2$ on ait

$$\int_{x_0}^{x_1} \rho(t_2,x)dx = \int_{t_1}^{t_2} \rho(s,x_0)ds - \int_{t_1}^{t_2} F(s,x_1)ds + \int_{x_0}^{x_1} F(t_1,x)dx, \tag{7.7}$$

voir Figure 7.4 pour une visualisation.

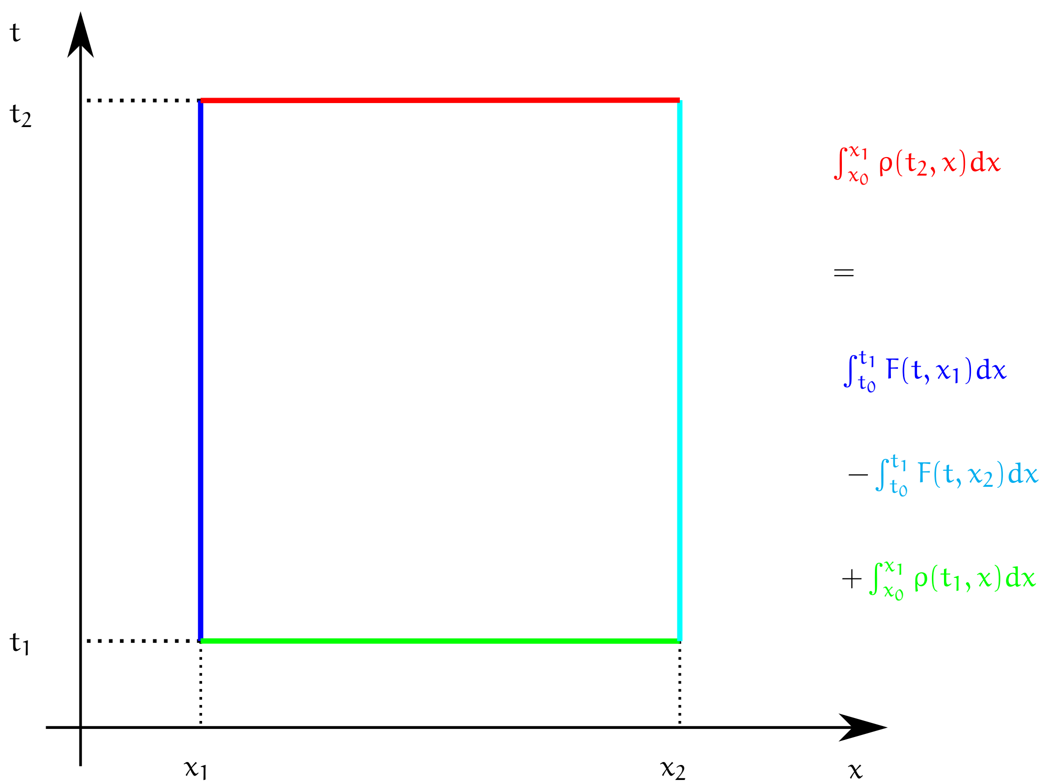

FIGURE 7.4 – Représentation de l'équation (7.7)

Si les fonctions densité et flux sont plus régulières, i.e. $\mathcal{C}^1$, il est clair — en utilisant le théorème fondamental du calcul différentiel —que l'équation (7.7) est en fait équivalent à

$$\partial_t \rho(t,x) + \partial_x F(t,x) = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{7.8}$$

Pour fermer le système et obtenir quelque chose de déterministe, on a besoin d'une loi constitutive reliant le flux et la densité. De nombreuses possibilités existent dans la nature et peuvent mener à des équations très différentes.

- Si on choisit $F(t,x) := \frac{\rho(t,x)^2}{2}$, l'équation (7.8) est l'équation de Burgers' inspirée de l'équation de la dynamique des gaz, voir (7.9) ci-dessous.
- Pour $F(t,x) := -\kappa\partial_x\rho(t,x)$, l'équation (7.8) est maintenant l'équation de la chaleur en dimension de Fourier [74].
- Comme exemple final, si on prend $F(t,x) := \nu\,\rho(t,x)\left(1-\frac{\rho(t,x)}{\rho_m}\right)$, on obtient l'équation de Lighthill-Whitham [104] et Richards [121] qui modélise l'évolution du trafic routier.

Dans ce manuscrit on se restreint aux équations hyperboliques, c'est à dire qu'on ne considère que des relations du type $F(t,x) := f(x,\rho(t,x))$.

Pour des raisons de simplicité, on s'est contenté de regarder des équations scalaires en dimension un d'espace, mais on peut avoir des quantités vectorielles en dimensions multiples. Pour le cadre de modélisation le plus général, on consultera [54, Chapter 1]. Finalement, mentionnons les équations les plus fondamentales du domaine, il s'agit des équations d'Euler de la dynamique des gaz en dimension d (compressible mais isentropique) :

$$\begin{cases} \partial_t\rho(t,x_1,\dots,x_d) + \sum_{k=1}^d \partial_{x_k} m_k(t,x_1,\dots,x_d) = 0, \\ \partial_t m_1(t,x_1,\dots,x_d) + \partial_{x_1} P(\rho)(t,x_1,\dots,x_d) + \sum_{k=1}^d \partial_{x_k}\left(\frac{m_1 m_k}{\rho}\right)(t,x_1,\dots,x_d) = 0, \\ \dots, \\ \partial_t m_d(t,x_1,\dots,x_d) + \partial_{x_d} P(\rho)(t,x_1,\dots,x_d) + \sum_{k=1}^d \partial_{x_k}\left(\frac{m_d m_k}{\rho}\right)(t,x_1,\dots,x_d) = 0, \end{cases} \tag{7.9}$$

ici P est la loi de pression, ρ est la densité de gaz, m_k la quantité de mouvement dans la direction x_k d'espace.

7.2.2 Paradis perdu

Par souci de simplicité, on considérera l'équation de Burgers' dans cette sous section. Étant donné une donnée initiale u_0 dans $\mathcal{C}_c^\infty(\mathbb{R})$ on chercher une fonction régulière $(t,x) \mapsto u(t,x)$ telle que :

$$\begin{cases} \partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0, \\ u(0,x) = u_0(x), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}. \tag{7.10}$$

On va procéder par analyse-synthèse.

Analyse. On suppose donnée une solution u de (7.10) qui est dans $\mathcal{C}^1([0,T]\times\mathbb{R})$, pour un temps positif T. On applique la règle de composition des dérivées pour obtenir

$$\forall t \in [0,T], \quad \forall x \in \mathbb{R}, \qquad \partial_t u(t,x) + u(t,x)\,\partial_x u(t,x) = 0, \text{ i.e. } (\partial_t + u(t,x)\partial_x)\,u(t,x) = 0.$$

La dérivée directionnelle de u selon le champs de vecteurs $(t,x) \mapsto (1, u(t,x))$ est donc nulle. On en déduit alors que la fonction u est constante le long des courbes intégrales de ce champs de vecteurs. En considérant la première composante du champs de vecteur, il est clair que les courbes intégrales sont à chercher sous le forme $t \mapsto (t, p(t))$ où p est solution de

$$\forall t \in [0,T], \quad \dot{p}(t) = u(t, p(t)).$$

Mais maintenant le terme de droite est constant par construction et les courbes intégrales sont donc forcément des droites

En résumer on a obtenu qu'une solution régulière u satisfait forcément

$$\forall (t,x) \in [0,T] \times \mathbb{R}, \quad u(t,x) = u_0(x - t\,u(t,x)). \tag{7.11}$$

Synthèse. Étant donnée une fonction u_0 dans $\mathcal{C}_c^\infty(\mathbb{R})$, on considère un temps T tel que $T\,\|u_0'\|_\infty < 1$. On introduit alors une fonction auxiliaire $\mathcal{A}$ définie par

$$\forall (t,x,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \quad \mathcal{A}(t,x,v) := v - u_0(x - t\,v). \tag{7.12}$$

Il est clair que

$$\forall x \in \mathbb{R}, \quad \mathcal{A}(0,x,u_0(x)) = 0. \tag{7.13}$$

Mais on a aussi

$$\forall (t,x,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \quad \mathcal{A}(t,x,v) = 1 - t\,u_0'(x - tv) \geq 1 - T\|u_0'\|_\infty > 0. \tag{7.14}$$

Un argument de connexité et le théorème des fonctions assurent alors l'existence et l'unicité d'une fonction u de $\mathcal{C}^\infty([0,T] \times \mathbb{R})$ satisfaisant

$$\forall (t,x,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \quad \mathcal{A}(t,x,u(t,x)) = 0. \tag{7.15}$$

En dérivant (7.15) on obtient finalement

$$\begin{cases} \partial_t u(t,x) = -\frac{u_0'(x - t\,u(t,x))u(t,x)}{1 + t\,u_0'(x - t\,u(t,x))}, \\ \partial_x u(t,x) = \frac{u_0'(x - t\,u(t,x))}{1 + t\,u_0'(x - t\,u(t,x))}, \end{cases}$$

delà u satisfait clairement la première équation de (7.10). En choisissant $t = 0$ dans (7.15), on obtient aussi la seconde équation de (7.10).

On a donc finalement prouvé le théorème suivant.

Theorem 23

Pour tout donnée initiale u_0 de $\mathcal{C}_c^\infty(\mathbb{R})$, et pour tout temps $T < 1/\|u_0'\|_\infty$, il existe une unique solution régulière de (7.10) définie sur $[0,T] \times \mathbb{R}$.

Singularités. Mettons de côté le cas inintéressant où $u_0 \equiv 0$ pour lequel $u \equiv 0$. Grâce au résultat précédent, on peut considérer une solution maximale de (7.10), u, que l'on considère définie jusqu'en T^*. On peut alors appliquer rigoureusement le raisonnement présenté dans la partie Analyse ci-dessus et obtenir

$$\forall (t,x) \in \mathbb{R}^2, \quad 0 \leq t < T^* \implies u_0(x) = u(t, x + t\,u_0(x)). \tag{7.16}$$

Mais il existe forcément (x_1, x_2) dans $\mathbb{R}^2$ tels que

$$x_1 < x_2 \text{ et } u_0(x_1) > u_0(x_2),$$

car la seule fonction croissante à support compact est $x \mapsto 0$. Mais alors pour $t_p := -(x_2 - x_1)/(u_0(x_2) - u_0(x_1))$, on obtient

$$t_p > 0 \text{ et } x_1 + t_p\,u_0(x_1) = x_2 + t_p\,u_0(x_2).$$

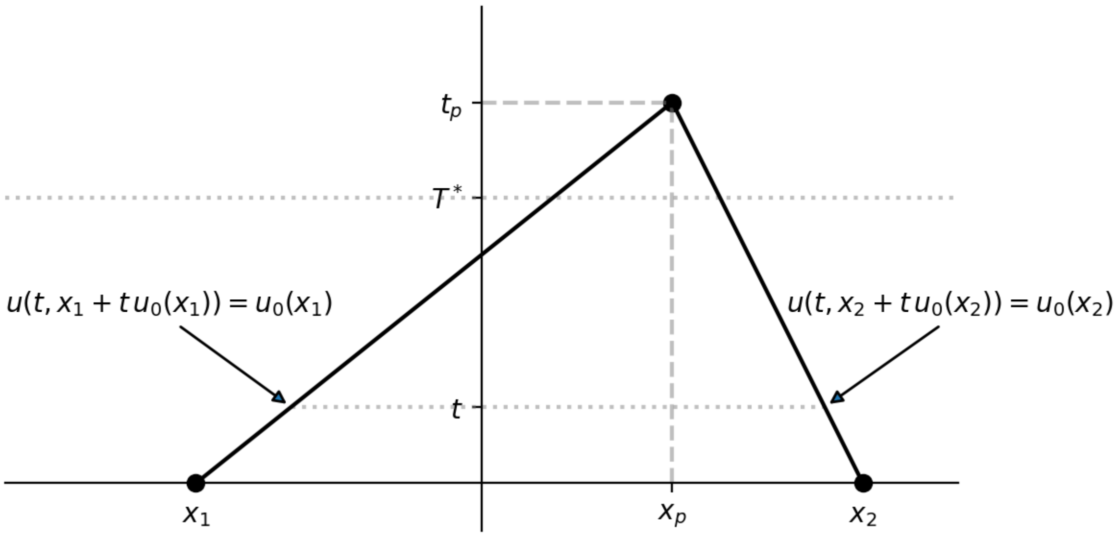

FIGURE 7.5 – Singularité pour l'équation de Burgers par croisement de courbes intégrales

En notant x_p la quantité de droite, on voit que pour $t_p < T^*$, on aurait $u_0(x_1) = u(t_p, x_p) = u_0(x_2)$ ce qui est contradictoire, et donc on en conclut $T^* \leq t_p$. On consultera la Figure 7.5 pour une représentation graphique de cette construction. Combinée avec l'égalité (7.11), on a donc obtenu le résultat complémentaire suivant.

Theorem 24

Pour tout donnée initiale u_0 dans $\mathcal{C}_c^\infty(\mathbb{R})$, la solution maximale de (7.10) explose en temps fini, mais on a malgré tout

$$\forall (t, x) \in \mathbb{R}^2, \quad 0 \leq t < T^* \implies |u(t, x)| \leq \|u_0\|_\infty.$$

A ce stade, il doit être clair que la formulation (7.8) est en cause alors que l'originale (7.7) garde tout son sens.

Intuition géométrique et méthode des caractéristiques. Essayons de donner maintenant un peu d'intuition géométrique en ce qui concerne le raisonnement utilisé dans la phase d'Analyse ci-dessus.

Introduire les courbes intégrales le long du champs de vecteur $(t, x) \mapsto (t, u(t, x))$ puis regarder l'évolution de u sur celles-ci est une technique appelée la *méthode des caractéristiques*. Le fait que une solution régulière de (7.10) soit constante le long d'une telle courbe intégrale — qui se retrouve dès lors à être une droite — peut être visualisée de la façon suivante.

En terme formel, si on introduit les graphes

$$\forall t \geq 0, \quad G_t := \{(x, y) \in \mathbb{R}^2 \, : \, y = u(t, x)\}, \tag{7.17}$$

ainsi que les transformations de $\mathbb{R}^2$

$$\forall t \geq 0, \quad T_t(x, y) := (x + t\, y, y), \tag{7.18}$$

on a alors $G_t = T_t(G_0)$ tant que la solution u est régulière. De manière concrète les points d'ordonnées y doivent avancer à vitesse y dans la direction x. On regardera la Figure 7.6 pour visualiser le graphe G_0 et les transformations $(T_t)_{t\geq 0}$. On regardera la Figure 7.7 pour observer l'évolution de u en utilisant ces transformations. Notons au passage que la pente augmente dans la partie du graphe où la fonction est décroissante. Jusq'au moment où on se retrouve avec une pente parfaitement verticale, ce qui signifie qu'on a explosion de la dérivée $x \mapsto \partial_x u(t,x)$ et une discontinuité de $x \mapsto u(t,x)$. Au passage, si on applique T_t à G_o pour un temps t plus grand que le temps d'explosion, on obtient un ensemble qui n'est pas le graphe d'une fonction. On pourra consulter la Section A.2 dans l'Annexe pour le code python générant ces figures.

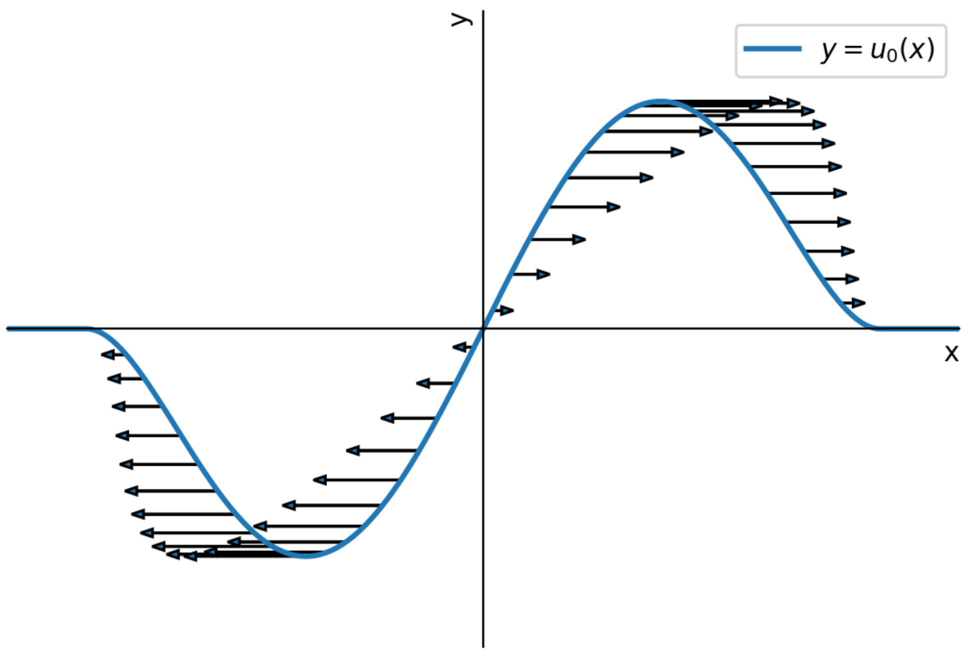

FIGURE 7.6 – La transformation T_t et G_0.

7.2.3 Solutions entropiques et non entropiques

Dans cette section, on va décrire comment obtenir une solution globale en temps de

$$\begin{cases} \partial_t u(t,x) + \partial_x \left(f(u(t,x))\right) = 0, \\ u(0,x) = u_0(x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \tag{7.19}$$

où la fonction f sera suffisamment régulière (i.e. $\mathcal{C}^1$).

Condition de Rankine-Hugoniot. Comme on l'a vu dans la section précédente pour obtenir une solution globale en temps de (7.19), il est obligatoire de considérer des solutions potentiellement discontinues. Il nous faut alors une formulation de l'équation dans un sens plus faible, par exemple (7.7).

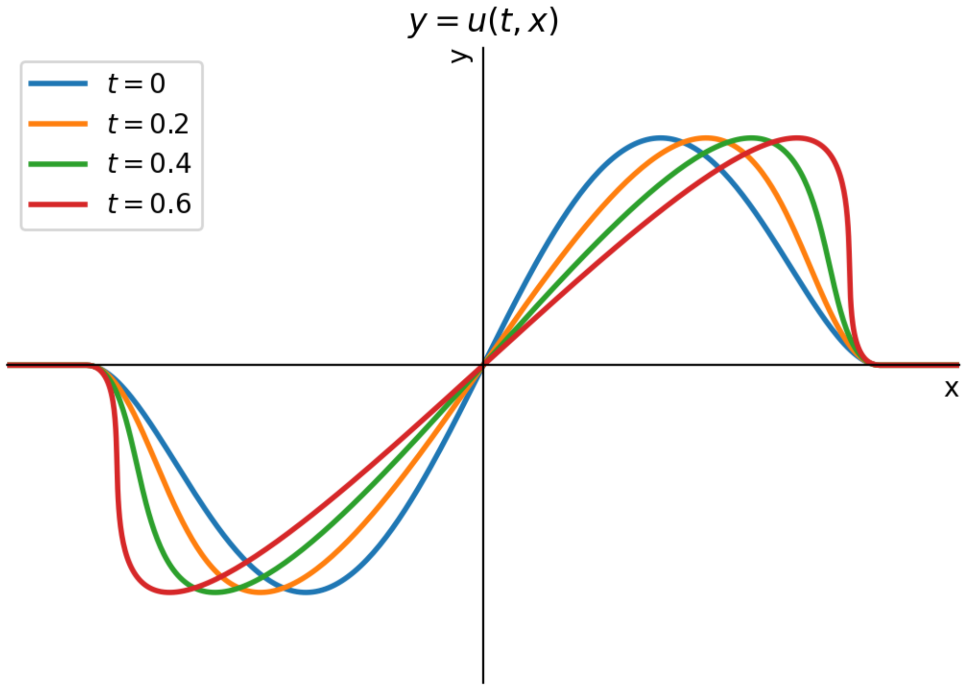

FIGURE 7.7 – Les graphes G_t pour plusieurs valeurs de t.

Il est évident que les fonctions constantes en t et x sont des solutions de (7.19). Le niveau suivant en terme de complexité est donc de considérer une donnée de Riemman (d'après l'article [122] de Riemann lui-même). Précisément, étant donnés trois réels x_c, u_l et u_r on s'intéresse à la donnée initiale définie par

$$\forall x \in \mathbb{R}, \quad u_0(x) := \begin{cases} u_l & \text{if } x < x_c, \\ u_r & \text{if } x > x_c, \end{cases} \tag{7.20}$$

et on cherche une solution $(t, x) \mapsto u(t, x)$ satisfaisant (7.19) au sens de (7.7). C'est à dire que pour tous réels x_0, x_1, t_0 et t_1 on doit avoir

$$\begin{cases} x_0 \leq x_1 \\ 0 \leq t_0 < t_1 \end{cases} \implies \int_{x_0}^{x_1} u(t_1, x)\, dx + \int_{t_0}^{t_1} f(u(t, x_1))\, dt - \int_{x_0}^{x_1} u(t_0, x)\, dx - \int_{t_0}^{t_1} f(u(t, x_0))\, dt = 0. \tag{7.21}$$

Il sera commode de réaliser que cette égalité s'interprète comme le fait que l'intégrale suivant le rectangle dont les sommets sont (x_0, t_0), (x_1, t_0), (x_1, t_1) et (x_0, t_1) de la forme différentielle $u(t, x)\, dx - f(u(t, x))\, dt$ est nulle.

Il est facile de voir que si u satisfait (7.21) alors pour tout réel a la fonction $(t, x) \mapsto u(t, x - a)$ satisfait aussi l'équation (7.21), et bien sûr la donnée initiale est translatée de la même façon. On peut donc se ramener dans (7.20) à $x_c = 0$.

De la même façon, un calcul élémentaire montre que pour un réel positif α, si u satisfait (7.21), alors c'est aussi le cas de $(t, x) \mapsto u(\alpha t, \alpha x)$ (qui est aussi définie sur $\mathbb{R}^+ \times \mathbb{R}$). Bien sûr la donnée initiale correspondante est donnée par $x \mapsto u_0(\alpha x)$.

On se ramène donc à chercher la solution u avec l'ansatz suivant

$$\forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u_l & \text{if } x < \gamma\, t \\ u_r & \text{if } x > \gamma\, t \end{cases} \tag{7.22}$$

où γ est justement à déterminer.

A ce stade, injecter $x_0 = \min(\gamma\, t_0, \gamma t_1)$ et $x_1 = \max(\gamma\, t_0, \gamma\, t_1)$ dans (7.21) fournit

$$\gamma\,(t_1 - t_0)\,(u_l - u_r) = (t_1 - t_0)(f(u_l) - f(u_r)),$$

(mais il faut distinguer les cas $\gamma > 0$ et $\gamma < 0$), on consultera la Figure 7.8 pour une représentation graphique de ce calcul.

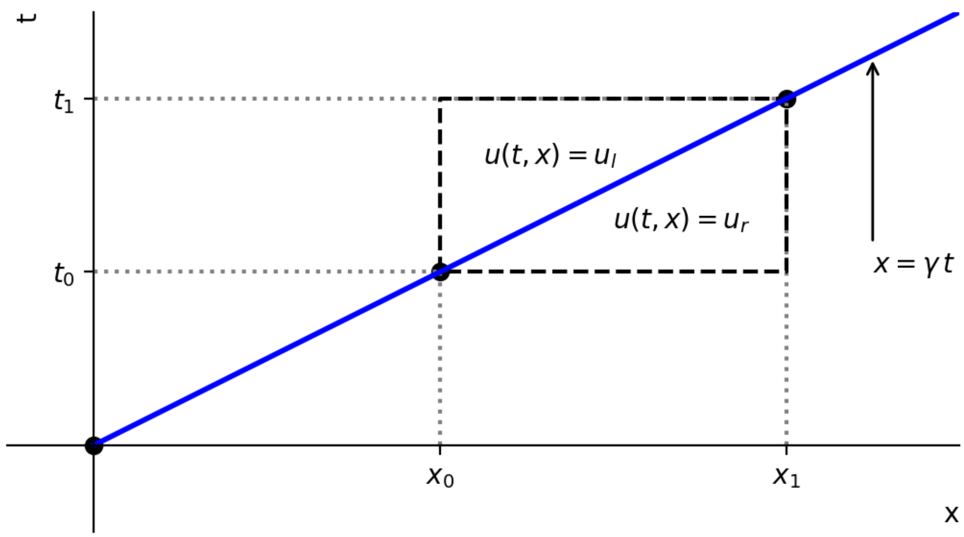

FIGURE 7.8 – Calcul pour la condition de Rankine-Hugoniot

On a donc prouvé qu'une discontinuité est admissible — i.e. pour que u définie par (7.22) satisfasse (7.21) — si la condition de Rankine-Hugoniot est satisfaite

$$\gamma = \frac{f(u_r) - f(u_l)}{u_r - u_l}. \tag{7.23}$$

Il s'avère que cette condition est également suffisante. En effet, en suivant le calcul sur la Figure 7.9, on peut décomposer les quantités apparaissant dans (7.21) en utilisant la relation de Chasles et le fait que les arrêtes intérieures sont parcourues dans les deux sens et donc se neutralisent. Plus précisément, il est clair que l'intégrale de la forme différentielle $\omega := u(t,x)\,dx - f(u(t,x))\,dt$ le long du rectangle noir est égale à la somme des intégrales le long des triangles bleu, vert, magenta et rouge. L'analyse ci-dessus montre que l'intégrale le long du rectangle bleu est nulle lorsque la condition de Rankine-Hugoniot est satisfaite. Et comme u est constante le long des rectangles rouge, vert et magenta, les intégrales correspondantes sont également nulles.

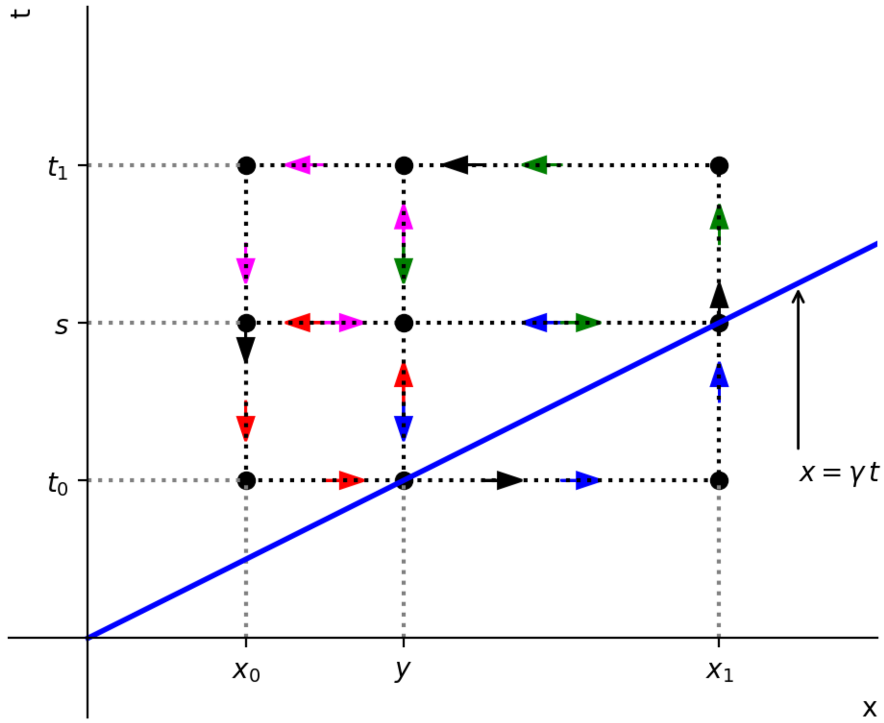

FIGURE 7.9 – Suffisance de la condition de Rankine-Hugoniot

Condition entropique. A ce stade on sait comment propager tout discontinuité pour obtenir une solution de (7.19).

On peut en fait construire un semigroupe $(S_t^{pc})_{t\geq 0}$ agissant sur l'espace des fonctions constantes par morceaux allant de $\mathbb{R}$ dans $\mathbb{R}$ et résolvant l'équation (7.19) au sens où on vérifie (7.21). Lorsque deux discontinuités se rencontrent, on a une nouvelle donnée de Riemann et on sait grâce à la condition de Rankine-Hugoniot comme cette singularité fusionnée se propage. On se retrouve de fait avec la situation de la Figure 7.10.

On voudrait maintenant naturellement étendre ce semigroupe à un espace de fonctions plus gros. On doit donc étudier les propriétés de continuité du semigroupe.

On considère pour cela trois réels u_l, u_m et u_r satisfaisant

$$\gamma_l := \frac{f(u_l) - f(u_m)}{u_l - u_m} > \frac{f(u_r) - f(u_m)}{u_r - u_m} =: \gamma_r. \tag{7.24}$$

Pour un réel positif ϵ on introduit la donnée initiale

$$u_\epsilon(x) := \begin{cases} u_l & \text{si } x < 0 \\ u_m & \text{si } 0 < x < \epsilon \\ u_r & \text{si } \epsilon < x. \end{cases} \tag{7.25}$$

En utilisant les résultats précédents on voit facilement que

$$(S_t^{pc} u_\epsilon)(x) := \begin{cases} u_l & \text{si } x < \gamma_l t \\ u_m & \text{si } \gamma_l t < x < \epsilon + \gamma_r t \\ u_r & \text{si } \epsilon + \gamma_r t < x. \end{cases} \tag{7.26}$$

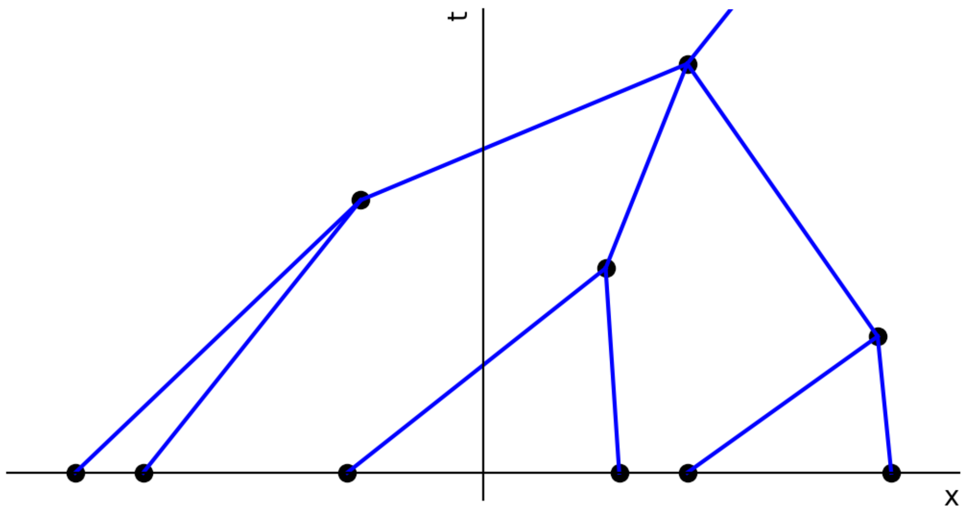

FIGURE 7.10 – Propagation des discontinuités pour des solutions constantes par morceaux

(consulter Figure 7.11 pour visualiser ces données).

En passant à la limite $\epsilon \to 0^+$ on voit

$$\lim_{\epsilon \to 0^+} (S_t^{pc} u_\epsilon)(x) := \begin{cases} u_l & \text{si } x < \gamma_l t \\ u_m & \text{si } \gamma_l t < x < \gamma_r t \\ u_r & \text{si } \gamma_r t < x, \end{cases} \tag{7.27}$$

ce qui est clairement différent de

$$(S_t^{pc} \lim_{\epsilon \to 0^+} u_\epsilon)(x) := \begin{cases} u_l & \text{si } x < \gamma_m t \\ u_r & \text{si } \gamma_m t < x, \end{cases} \tag{7.28}$$

pour $\gamma_m := (f(u_r) - f(u_l))/(u_r - u_l)$. Remarquons quand même que les deux formules fournissent des solutions de (7.19) au sens de (7.21) pour une donnée initiale définie par (7.20). D'un côté, le semigroupe $(S_t^{pc})_{t \geq 0}$ n'est pas continu et ne peut donc pas être étendu, de l'autre côté il y a plusieurs solutions de (7.19) au sens où (7.21) est satisfaite.

La façon de s'en sortir est de déclarer admissibles les seules discontinuités satisfaisant en plus de Rankine-Hugoniot la condition dite entropique

$$\forall u \in [\min(u_l, u_r), \max(u_l, u_r)], \quad \frac{f(u_l) - f(u)}{u_l - u} \geq \frac{f(u) - f(u_r)}{u - u_r}. \tag{7.29}$$

Pour les discontinuités ne satisfaisant pas cette condition, il y a en fait une façon presque régulière de déterminer une solution en utilisant la méthodes caractéristiques de la sous Section 7.2.2. Considérons par exemple le cas de l'équation de Burgers' 7.10 pour une donnée initiale de Riemann satisfaisant $u_l < u_r$ et $x_c = 0$. Un calcul élémentaire montre que

$$\forall x \in \mathbb{R}, \quad \forall t > 0, \quad u(t, x) := \begin{cases} u_l & \text{si } x < t\, u_l \\ \frac{x}{t} & \text{si } t\, u_l \leq x \leq t\, u_r \\ u_r & \text{si } t\, u_r < x. \end{cases} \tag{7.30}$$

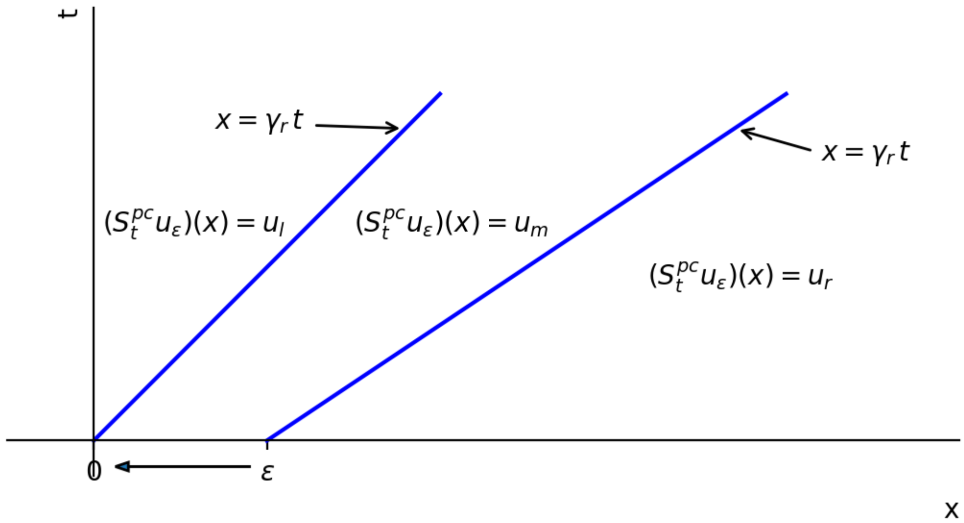

FIGURE 7.11 – Donnée initiale montrant le problème de $(S_t^{pc})_{t\geq 0}$

définit une solution régulière, on l'appelle une onde de raréfaction. Dans la Figure 7.12 et la Figure 7.13 on a représenté l'évolution de l'onde de raréfaction suivant l'interprétation géométrique du dernier paragraphe de la précédente sous section. On regardera la Figure 7.14 pour visualiser les courbes caractéristiques associées à la solution.

On peut en principe utiliser les discontinuités admissibles et les ondes de raréfaction pour construire un semigroupe continu de solutions faibles. En pratique, cela conduit à l'algorithme de suivi de front que l'on aborde dans le Chapitre 5. On va maintenant considérer une façon équivalente de sélectionner les bonnes solutions suivant les idées de Kruzkov dans [95]. On utilisera ce type de technique dans le Chapitre 2 et le Chapitre 6.

Solutions entropiques Le point de départ de l'analyse consiste à réaliser qu'il est fréquent dans la phase de modélisation d'avoir négliger des effets de viscosité. En regardant (7.7), cela signifie que $F(t,x)$ en plus de dépendre de $\rho(t,x)$ dépend aussi de $\partial_x \rho(t,x)$. On peut cependant supposer que ce terme additionnel est petit et additif ce qui mène à considérer (7.19) comme une limite de

$$\begin{cases} \partial_t u^\epsilon(t,x) + \partial_x \left(f(u^\epsilon(t,x))\right) = \epsilon\, \partial_{xx}^2 u^\epsilon(t,x), \\ u^\epsilon(0,x) = u_0(x), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \tag{7.31}$$

lorsque ϵ tend vers 0^+.

On étudiera rigoureusement ce type d'approximation et cette limite — très singulière — dans le Chapitre 2. Contentons nous pour l'instant de dire que pour $\epsilon > 0$ empêche la formation de singularité dans les solutions de (7.31) et qu'on s'attend de plus à avoir une convergence ponctuelle de la suite d'approximations visqueuses.

Cela implique qu'on peut travailler avec les solutions régulières de (7.31). L'idée est ensuite de considérer une fonction E dans $\mathcal{C}^2(\mathbb{R},\mathbb{R})$ — qu'on qualifiera d'entropie — et de multiplier l'équation (7.31) par $E'(u^\epsilon(t,x))$. Simultanément, on introduit Q — qu'on qualifie de flux

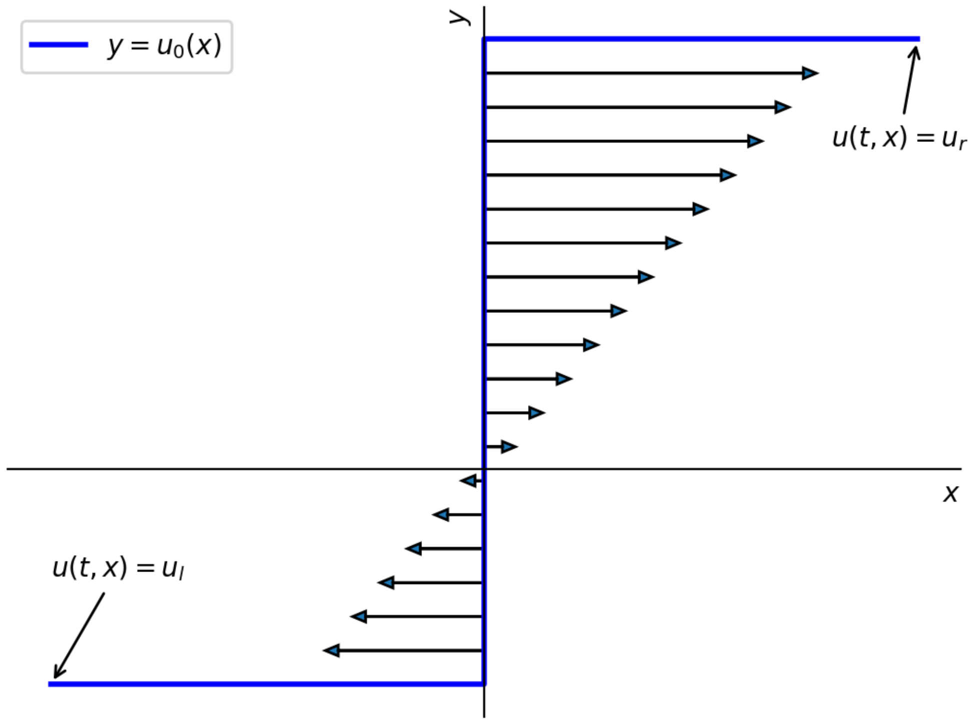

Figure 7.12 – Visualisation de l'onde de raréfaction sur le graphe de u_0

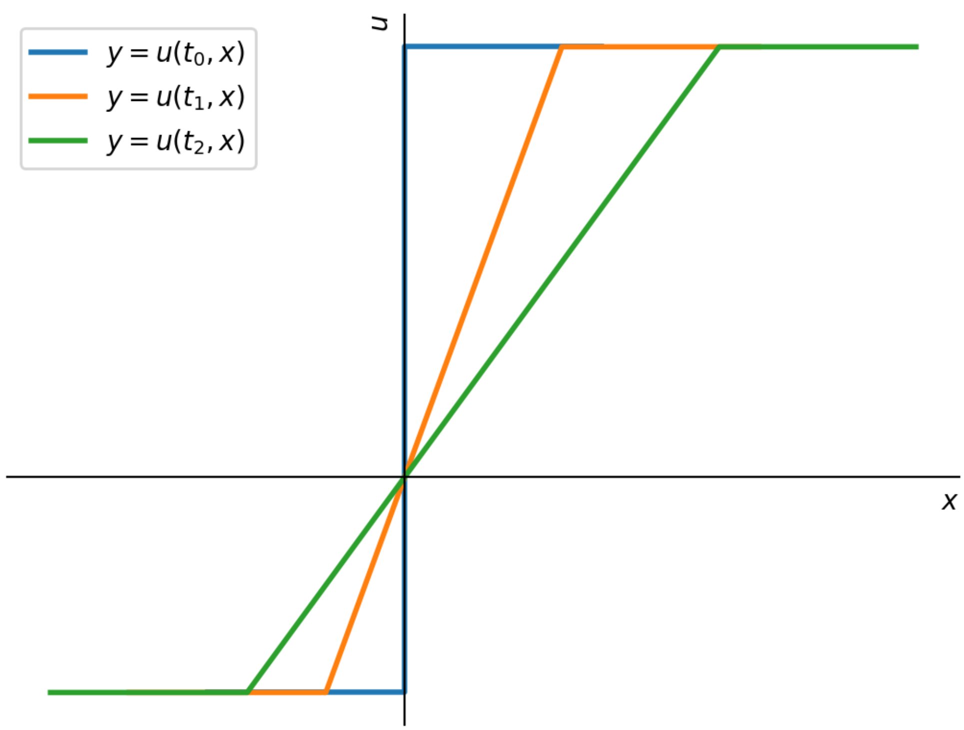

Figure 7.13 – Évolution de l'onde de raréfaction pour $0 = t_0 < t_1 < t_2$

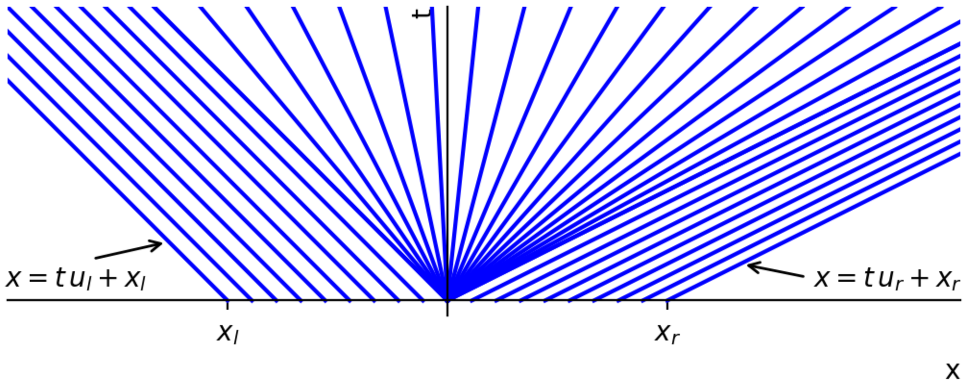

FIGURE 7.14 – Courbes caractéristiques pour une onde de raréfaction

d'entropie — la fonction associée à E par

$$\forall z \in \mathbb{R}, \quad Q(z) := \int_0^z f'(w)\, E'(w)\, dw. \tag{7.32}$$

On réalise alors que u^ϵ satisfait

$$\partial_t E(u^\epsilon)(t,x) + \partial_x Q(u^\epsilon)(t,x) = \epsilon \left[\partial^2_{xx} E(u^\epsilon)(t,x) - E''(u^\epsilon(t,x))(\partial_x u(t,x))^2\right]. \tag{7.33}$$

Si E est une fonction convexe, le dernier terme a alors un signe. Si on multiplie (7.33) par une fonction test positive ϕ in $C_c^\infty(\mathbb{R}^2)$ et qu'on intègre sur le demi plan supérieur en espace temps, on obtient après quelques intégrations par parties

$$\int_0^\infty \int_{-\infty}^\infty \partial_t\phi(t,x)E(u^\epsilon(t,x)) + \partial_x\phi(t,x)Q(u^\epsilon(t,x)) + \epsilon\partial^2_{xx}\phi(t,x)E(u^\epsilon(t,x))\, dxdt$$
$$+ \int_{-\infty}^\infty E(u_0(x))\phi(0,x)\, dx = \epsilon \int_0^\infty \int_{-\infty}^\infty E''(u^\epsilon(t,x))(\partial_x u^\epsilon(t,x))^2 \phi(t,x)\, dx\, dt \geq 0. \tag{7.34}$$

Mais si on a convergence presque partout de $(u^\epsilon)_{\epsilon>0}$ vers une fonction u — et une borne L^∞ uniforme — on peut appliquer le théorème de convergence dominée sur la partie de gauche pour obtenir

$$\int_0^\infty \int_{-\infty}^\infty \partial_t\phi(t,x)E(u(t,x)) + \partial_x\phi(t,x)Q(u(t,x))\, dxdt + \int_{-\infty}^\infty E(u_0(x))\phi(0,x)\, dx \geq 0. \tag{7.35}$$

On adoptera cette formulation comme définition d'une solution entropique de (7.19), qui doit être satisfaite quelque soit la fonction convexe et $\mathcal{C}^2$ E le flux entropique associé Q et la fonction test ϕ dans $\mathcal{C}_c^\infty(\mathbb{R}^2)$ et positive.

L'article de Kruzkov [95] prouve — dans un cadre beaucoup plus général et accompagné d'autres résultats — que cette notion de solution satisfait les conditions entropiques obtenues précédemment. De plus, il en déduit l'existence d'un semigroupe $(S_t)_{t\geq 0}$ qui est contractant dans $L^1(\mathbb{R})$.

Il est critique de remarquer que contrairement aux solutions régulières de (7.19), les solutions entropiques ne sont pas réversibles en temps. C'est la source d'une majorité des difficultés rencontrées dans le manuscrit. En effet, le changement de variables $t \mapsto T - t$ dans le côté gauche de (7.35) changerait le signe de l'équation. On discutera de ce phénomène de manière plus détaillée dans 3.

Annexe A

Python scripts

A.1 Open loop vs closed loop

We provide here the python script used to generate Figure 1.3 in Chapter 1, Section 1.1.

```
#imports
import numpy as np
import matplotlib.pyplot as plt

# setting up the data
T = 10
N = 1000
DT = T / N
x0, y0 = 0, -0.5
l = 1
L = 10
v = 1.
eps = 5 / (2 * np.pi)

#creating the base of the figure
fig, rep = plt.subplots()
rep.spines.top.set_visible(False)
rep.spines.right.set_visible(False)
rep.spines.bottom.set_position("zero")
rep.spines.left.set_position("zero")
rep.set_xlim(-1, L+1)
rep.set_ylim(-l - 0.5, l + 0.5)
rep.set_xticks([])
rep.set_yticks([])
rep.set_aspect("equal")
rep.text(-0.3, l + 0.2, "$l$")
rep.text(-0.5, -l - 0.5, "$-l$")
rep.text(-0.5, -0.5, "$0$")
rep.text(10, -0.5, "$L$")
rep.plot([-1, L], [1, 1], lw=2, color="red")
rep.plot([-1, L], [-1, -1], lw=2, color="red")
rep.scatter([x0, L], [y0, 0], color="black")

#generating the perturbation function
```

```
np.random.seed(15)
coefs = np.random.uniform(low=-1., high=1., size=(15,))
def perturbation(ts):
    resultat = np.zeros_like(ts)
    for (i,coef) in enumerate(coefs):
        resultat += coef * np.cos(i * ts) / (2 ** (i+1))
    return eps * resultat


#simulating the ODEs
xs = [x0]
ys = [y0]
for i in range(N):
    xs.append(xs[-1] + v * np.cos(perturbation(i * DT)) * DT)
    ys.append(ys[-1] + v * np.sin(perturbation(i * DT)) * DT)

Xs = [x0]
Ys = [y0]
for i in range(N):
    Xs.append(Xs[-1] + v * np.cos(perturbation(i * DT) - Ys[-1] / l) * DT)
    Ys.append(Ys[-1] + v * np.sin(perturbation(i * DT) - Ys[-1] / l) * DT)

#plotting the trajectories
rep.plot(xs, ys, color="blue")
rep.plot(Xs, Ys, color="magenta")
fig.savefig("route_perturbee.png", dpi=200, bbox_inches="tight")
```

A.2 Characteristics's method

Script generating Figure 1.6 in Chapter 1, Section 1.2.

```
import numpy as np
import matplotlib.pyplot as plt
L = 1.2
H = 0.8
l = 1.
def u0(xs):
    return 2 * np.where((xs < l) & (-l < xs), xs * (1. - xs) ** 2 * (1 + xs)**2, 0)

xs = np.linspace(-L, L, 300)
y0 = u0(xs)

fig, rep = plt.subplots()
rep.spines.top.set_visible(False)
rep.spines.right.set_visible(False)
rep.spines.bottom.set_position(("data", 0))
rep.spines.left.set_position(("data", 0))
rep.set_xlim(-L, L)
rep.set_ylim(-H, H)
rep.set_xlabel("x", loc="right")
rep.set_ylabel("y", loc="top")
rep.set_aspect("equal")
```

```
rep.set_xticks([], [])
rep.set_yticks([], [])
rep.plot(xs, y0, lw=2, label="$y=u_0(x)$")

ps = np.linspace(-l * 0.9, l * 0.9, 40)
qs = u0(ps)
for p, q in zip(ps, qs):
    rep.arrow(p, q, q / 2, 0, head_width=0.02 )
rep.legend()
fig.savefig("caracteristique_transform.png", dpi=200, bbox_inches="tight")
```

Script generating Figure 1.7 in Chapter 1, Section 1.2.

```
import numpy as np
import matplotlib.pyplot as plt
L = 1.2
H = 0.8
l = 1.
ts = (0, 0.2, 0.4, 0.6)

def u0(xs):
    return 2 * np.where((xs < l) & (-l < xs), xs * (1. - xs) ** 2 * (1 + xs)**2, 0)

xs = np.linspace(-L, L, 300)
y0 = u0(xs)

fig, rep = plt.subplots()
rep.set_title("$y=u(t, x)$")
rep.spines.top.set_visible(False)
rep.spines.right.set_visible(False)
rep.spines.bottom.set_position(("data", 0))
rep.spines.left.set_position(("data", 0))
rep.set_xlim(-L, L)
rep.set_ylim(-H, H)
rep.set_xlabel("x", loc="right")
rep.set_ylabel("y", loc="top")
rep.set_aspect("equal")
rep.set_xticks([], [])
rep.set_yticks([], [])
for t in ts:
    rep.plot(xs + t * y0, y0, lw=2, label=f"$t={t}$")

rep.legend()
fig.savefig("caracteristique_evolution.png", dpi=200, bbox_inches="tight")
```

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