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Problèmes de contrôle et équations hyperboliques non-linéaires.

THÈSE

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Ce que j'aime dans les mathématiques appliquées, c'est qu'elles ont pour ambition de donner du monde des systèmes une représentation qui permette de comprendre et d'agir. Et, de toutes les représentations, la représentation mathématique, lorsqu'elle est possible, est celle qui est la plus souple et la meilleure.

Jacques-Louis Lions

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Introduction générale

1 Théorie du contrôle

La théorie du contrôle trouve ses origines dans des problèmes d'ingénierie concrets qui étaient au départ résolus par des méthodes empiriques. La première approche mathématique des problèmes de contrôle est sans doute à trouver dans l'article de J. C. Maxwell [85] datant de 1867. Il y décrit mathématiquement les propriétés de stabilisation du régulateur de Watt qui était employé entre autres dans les machines à vapeur.

1.1 Description mathématique de deux problèmes.

Considérons un système dynamique général :

$$\begin{cases} \dot{X}(t) = F(X(t), U(t)), \\ X(0) = X_0, \end{cases} \tag{1}$$

où $X \in \mathcal{X}$ désigne l'état du système et $U \in \mathcal{U}$ le contrôle. Un aspect clé de l'analyse mathématique consistera à bien choisir l'espace des états $\mathcal{X}$ et l'espace des contrôles $\mathcal{U}$. Dans la pratique, le système dynamique (1) sera une équation différentielle ordinaire ou une équation aux dérivées partielles d'évolution. L'idée est de modéliser un système physique représenté par l'état X, sur lequel on a une influence via le contrôle U. La question est de savoir comment utiliser le contrôle pour obtenir un comportement souhaité du système. Dans cette optique, on va introduire deux problèmes classiques (parmi d'autres) de la théorie du contrôle.

1. Le problème le plus naturel est celui de la contrôlabilité exacte. Il s'agit d'amener le système dans un état désiré en un temps fixé et depuis un état initial quelconque. Plus précisément, pour deux états $X_0 \in \mathcal{X}$ et $X_1 \in \mathcal{X}$ et un temps $T > 0$, il faut trouver un contrôle $t \in [0, T] \mapsto U(t) \in \mathcal{U}$ tel que la solution de (1) satisfait $X(T) = X_1$.

2. Supposons maintenant l'on ait un état $(X_e, U_e) \in \mathcal{X} \times \mathcal{U}$ tel que $F(X_e, U_e) = 0$. C'est à dire qu'on a un état d'équilibre du système. Ce sont des solutions naturelles importantes de (1). Cependant on ne peut dans la pratique observer que les états d'équilibre stables du système. C'est le cas par exemple du pendule simple pour lequel l'état d'équilibre haut n'est jamais observé dans la nature, car il s'agit d'un équilibre instable. Pour compenser ce défaut, on pourrait chercher un contrôle $t \in [0, T] \mapsto U(t) \in \mathcal{U}$ tel que la solution de (1) satisfait $X(T) = X_e$ et que $U(T) = U_e$. Cependant, cette méthode souffre d'un défaut, elle

est très sensible aux perturbations et aux incertitudes : sur le modèle, sur le contrôle ou sur la connaissance de l'état initial. Pour des questions de robustesse on lui préférera donc un contrôle en boucle fermée. Plus précisément le problème de la stabilisation asymptotique demande de trouver une fonction $\mathbb{U} : \mathcal{U} \to \mathcal{X}$, satisfaisant $\mathbb{U}(X_e) = U_e$ et telle que, pour le système dit en boucle fermée :

$$\begin{cases} \dot{X}(t) = F(X(t), \mathbb{U}(X(t))) \\ X(0) = X_0, \end{cases} \tag{2}$$

l'état X_e soit asymptotiquement stable. Rappelons que cela signifie qu'on a les deux propriétés suivantes.

- Pour tout $\epsilon > 0$, il existe $\nu > 0$ tel que si X_0 est un état satisfaisant $||X_0 - X_e|| \leq \nu$ et si X est une solution maximale de (2), alors elle est globale en temps et satisfait :

$$\forall t \geq 0, \qquad ||X(t) - X_e|| \leq \epsilon.$$

- Pour tout état initial X_0, une solution maximale de (2) est globale en temps et satisfait,

$$||X(t) - X_e|| \underset{t \to +\infty}{\to} 0.$$

On a énoncé ici les propriétés globales de stabilisation asymptotique et de contrôlabilité exacte . Des variantes dites locales existent lorsque les données initiales (et finales pour la contrôlabilité) ne sont prises que dans de petits ouverts.

Dans les cas où le système dynamique (1) est régi par une équation différentielle ordinaire, la théorie est désormais mûre et des techniques robustes existent (voir par exemple [96], [34]).

1.2 Contrôle et équations aux dérivées partielles.

Concernant l'étude des problèmes de contrôlabilité exacte et de stabilisation asymptotique dans le cadre des équations aux dérivées partielles, les contrôles agiront le plus souvent via les données au bord et sur une partie de celui-ci, ou éventuellement comme un terme source localisé dans un sous domaine. Plusieurs méthodes générales existent pour les équations linéaires. Parmi les méthodes directes on citera la méthode de contrôlabilité par extension de Russell [92] dont une variante non-linéaire sera partiellement utilisée dans le chapitre 2. Une autre méthode très générale est la méthode HUM de Lions et Russell qui procède par dualité. L'élément essentiel en est l'équivalence entre la propriété de contrôlabilité sur une certaine E.D.P. et l'existence d'une inégalité d'observabilité sur le système adjoint. On remplace ainsi un problème non standard consistant à trouver une fonction, par un problème standard consistant à établir une certaine inégalité, ce nouveau problème n'étant pas facile pour autant. On pourra consulter [78] pour plus de détails. Il faut noter par contre que de nombreuses méthodes (dépendant de l'E.D.P. considérée) peuvent être utilisées pour montrer l'inégalité d'observabilité : par exemple les inégalités de Carleman, la méthode des multiplicateurs, l'analyse microlocale...

Pour contrôler/stabiliser une équation aux dérivées partielles non-linéaire, une première idée est de s'appuyer sur les méthodes linéaires. Si l'équation linéarisée est contrôlable/stabilisable, on peut espérer obtenir des propriétés locale de contrôle/stabilisation sur l'équation non-linéaire par exemple via une méthode de point fixe. Cependant, il est important de remarquer que cette condition est seulement suffisante. En effet, il peut arriver dans certains cas que le linéarisé ne soit pas, en général, contrôlable ou même que la linéarisation ne soit pas envisageable alors que le système non-linéaire, lui, l'est. Les problème abordés dans ce manuscrit appartiennent tous à cette catégorie. Différentes méthodes existent pour utiliser la non-linéarité (voir en particulier le livre [34]). Nous allons décrire, pour le système (1), la méthode du retour que J.-M. Coron a introduite dans [33] et qui est celle que l'on utilisera dans la suite.

La méthode consiste à trouver une trajectoire particulière $(\bar{X}, \bar{U})$ de (1) (autre que la trajectoire stationnaire (X_e, U_e)) telle que l'on ait

$$\bar{X}(T) = \bar{X}(0) = X_e \text{ et } \bar{U}(T) = \bar{U}(0) = U_e,$$

mais aussi que le système linéarisé le long de cette solution :

$$\begin{cases} \dot{X}(t) = \partial_X F(\bar{X}(t), \bar{U}(t))X(t) + \partial_U F(\bar{X}(t), \bar{U}(t))U(t), \\ X(0) = X_0, \end{cases} \tag{3}$$

soit contrôlable. On peut alors généralement en déduire un résultat de contrôlabilité locale au voisinage de X_e. Dans le cas où la linéarisation est problématique (ce qui sera le cas dans les chapitres 2 et 3), on demandera plutôt que certaines propriétés assurant la contrôlabilité soient satisfaites près de $(\bar{X}, \bar{U})$. Intuitivement cela signifie que même si les états initiaux et finaux sont petits, il n'y a pas de raison pour que la taille des états intermédiaires soient forcément du même ordre. Cette méthode a été utilisée avec succès pour résoudre des problèmes de contrôlabilité exacte et de stabilisation asymptotique dans de nombreux cas : l'équation d'Euler incompressible en deux dimensions ([37], [38], [59]) et en dimension 3 [60], l'équation de Camassa-Holm sur le tore [58], le système d'Euler isentropique [57], l'équation de Navier-Stokes [40], l'équation de Burgers ([69], [20], [72]), l'équation de Saint-Venant [36], le système de Vlasov-Poisson [61], l'équation de Schrödinger ([13], [14]), l'équation de Korteweg-de Vries [21] et des systèmes hyperboliques [39]. Pour tout ce qui concerne cette partie on pourra consulter [35].

2 Résultats principaux de la thèse.

Nous allons maintenant présenter les principaux résultats obtenus durant la thèse.

2.1 Équation de Camassa-Holm

Dans le chapitre 1, on s'intéresse au problème mixte non homogène, à la contrôlabilité exacte et à la stabilisation asymptotique de l'équation de Camassa-Holm sur un intervalle. Dans la pratique on travaillera sur $(0, 1)$ mais les résultats s'étendent clairement à un intervalle borné quelconque. Avec κ un nombre réel l'équation Camassa-Holm s'écrit :

$$\partial_t v - \partial^3_{txx} v + 2\kappa \partial_x v + 3v\partial_x v = 2\partial_x v \partial^2_{xx} v + v\partial^3_{xxx} v \text{ avec } (t, x) \in (0, T) \times (0, 1). \tag{4}$$

Cette équation décrit des ondes unidimensionnelles se propageant à la surface de l'eau sous l'action de la gravité lorsque la profondeur d'eau est petite. La fonction $v(t, x)$ représente la vitesse de l'onde au point x et à l'instant t. Mais d'après Camassa et Holm [22] la fonction représente aussi l'élévation de la surface d'eau dans l'approximation d'eau peu profonde. L'équation (4) a été introduite pour la première fois par Fokas et Fuchssteiner [56] en tant que modèle bi-Hamiltonien. Ce n'est que plus tard qu'elle a été utilisée pour modéliser la propagation d'ondes à la surface de l'eau par Camassa et Holm [22], et le long d'une tige rigide cylindrique par Dai [48].

L'équation (4) a de nombreuses propriétés communes avec l'équation KdV : elles sont toutes les deux bi-Hamiltoniennes, complétement intégrables et admettent des solutions soliton ([22],

[29], [32], [56], [76]). Cependant, (4) modélise aussi un nouveau phénomène via l'apparition de singularités en temps fini : c'est le déferlement des vagues. Ainsi, dans $H^s(\mathbb{T})$ ($s > \frac{3}{2}$) les solutions développent des singularités en temps fini de façon générale ([26],[27], [28]).

Le problème de Cauchy associé à l'équation (4) posée sur le tore ou sur la droite a été beaucoup étudié ces dernières années ([17], [23] ,[30], [49], [50], [67], [79], [98]). Il y a par contre beaucoup moins de résultats concernant le problème mixte. Le cas homogène, c'est à dire avec $v = 0$ aux bords, a été traité grâce à des arguments de symétrie dans [53] et de manière plus générale dans [54]. Et un cas particulier du problème non-homogène sur la droite entière est considéré dans [101] : la "condition au bord" consistant à demander qu'il existe une constante C telle que l'on ait :

$$\forall t \geq 0, \quad v(t,x) \underset{|x|\to+\infty}{\to} C.$$

Du point de vue de la théorie du contrôle, de nombreux résultats existent pour l'équation de KdV ([6], [24], [25], [41], [42], [63], [64], [90], [91], [93], [94], [100]). En comparaison les premiers résultats concernant (4) ont été donnés par O. Glass dans [58]. Il y prouve en utilisant un contrôle distribué sur un intervalle arbitraire du cercle :

- la contrôlabilité exacte dans $H^s(\mathbb{T})$ pour $s > \frac{3}{2}$ en temps arbitraire,
- la stabilisation asymptotique de l'état $v = -\kappa$ dans $H^2(\mathbb{T})$ via une loi de retour stationnaire à valeurs dans $H^{-1}(\mathbb{T})$.

Pour l'équation (4) posée sur un intervalle, après avoir montré que l'on a une bonne notion de solution faible pour le problème mixte, le résultat de Glass de contrôlabilité exacte sur le tore via un contrôle distribué sur un intervalle ω assure que l'on a contrôlabilité exacte sur un intervalle en utilisant des contrôles agissant sur les bords : il suffit d'identifier $(0,1)$ à $\mathbb{T} \setminus \omega$. Par contre on ne peut pas faire de même pour trouver une loi de retour stationnaire stabilisant $-\kappa$.

Avant d'expliciter nos conditions au bord pour (1.1), remarquons que l'équation (1.1) peut être reformulée en :

$$\begin{cases} \partial_t y + v\partial_x y = -2y\partial_x v, \\ y - \kappa = (1 - \partial_{xx}^2)v. \end{cases} \tag{5}$$

Sous cette forme, l'équation de Camassa-Holm a une analogie au moins formelle avec la formulation de l'équation d'Euler utilisant la vorticité ω :

$$\text{en dimension 2 : } \begin{cases} \partial_t \omega + (U.\nabla)\omega = 0, \\ \text{div } U = 0, \\ \text{rot } U = \omega, \end{cases} \tag{6}$$

$$\text{et encore plus en dimension 3 : } \begin{cases} \partial_t \omega + (U.\nabla)\omega = (\omega.\nabla)U, \\ \text{div } U = 0, \\ \text{rot } U = \omega. \end{cases} \tag{7}$$

On a un couplage entre une équation de transport et une équation stationnaire elliptique. Le problème mixte pour cette équation a été étudié dans [99] pour le cas 2D et [74] pour le cas 3D, et le problème de la stabilisation asymptotique dans [37] et [59] pour le cas 2D. On a cherché à adapter leurs méthodes à notre cas, sachant qu'on a un terme d'étirement dans l'équation de transport à l'instar du cas d'Euler 3D, mais que la géométrie du problème est plus simple. En particulier, la condition au bord de l'équation de transport d'Euler 3D due à Khazikov est la donnée de la composante normale de la vorticité sur la partie du bord où du fluide rentre ($\{U(t,x).n(t,x) < 0\}$), alors que dans notre cas (resp. pour Euler 2D) on a une équation de transport scalaire et la donnée au bord est y (resp. ω pour Euler 2D) sur la partie du bord ou du fluide rentre : $\{(t,0) \mid v_l(t) > 0\} \cup \{(t,1) \mid v_r(t) < 0\}$ (resp. $\{U(t,x).n(t,x) < 0\}$).

Introduisons maintenant des définitions nécessaires à l'énoncé des résultats. On choisit T un nombre strictement positif. Dans toute cette partie on notera $\Omega_T = [0,T] \times [0,1]$. Soient v_l et v_r deux fonctions de $\mathcal{C}^0([0,T],\mathbb{R})$ et $y_0 \in L^\infty(0,1)$. Posons

$$\Gamma_l = \{t \in [0,T] \mid v_l(t) > 0\} \text{ et } \Gamma_r = \{t \in [0,T] \mid v_r(t) < 0\}.$$

Nous allons faire l'hypothèse que les ensembles

$$P_l = \{t \in [0,T] \mid v_l(t) = 0\} \text{ et } P_r = \{t \in [0,T] \mid v_r(t) = 0\}, \tag{8}$$

ont un nombre fini de composantes connexes. Nous choisissons enfin $y_l \in L^\infty(\Gamma_l)$ et aussi $y_r \in L^\infty(\Gamma_r)$. Les fonctions v_l, v_r, y_l et y_r seront les données aux bords et y_0 sera la donnée initiale. Nous introduisons maintenant le relèvement $\mathcal{A}$ de v_l et v_r qui est défini par :

$$\begin{cases} (1-\partial^2_{xx})\mathcal{A}(t,x) = 0, \ \forall (t,x) \in \Omega_T, \\ \mathcal{A}(t,0) = v_l(t), \quad \mathcal{A}(t,1) = v_r(t), \quad \forall t \in [0,T]. \end{cases} \tag{9}$$

En notant $v = u + \mathcal{A}$, on peut réécrire (5) comme suit :

$$\begin{cases} y(t,x) - \kappa = (1-\partial^2_{xx})u(t,x), \ , \\ u(t,0) = u(t,1) = 0, \ dt \ p.p., \end{cases} \tag{10}$$

$$\begin{cases} \partial_t y + (u+\mathcal{A})\partial_x y = -2y\partial_x(u+\mathcal{A}), \\ y(0,.) = y_0, \ y(.,0)_{|\Gamma_l} = y_l \text{ et } y(.,1)_{|\Gamma_r} = y_r. \end{cases} \tag{11}$$

Le sens faible de l'équation (10) ne pose pas de problème, nous allons donc juste définir les solutions faibles de l'équation (11). On va d'abord commencer par introduire l'espace des fonctions test.

$$Adm(\Omega_T) = \{\psi \in \mathcal{C}^1(\Omega_T) \mid \psi(t,x) = 0 \text{ sur } [0,T]\backslash\Gamma_l \times \{0\} \cup [0,T]\backslash\Gamma_r \times \{0\} \cup \{T\} \times [0,1]\}. \tag{12}$$

Definition 1. *Lorsque* $u \in L^\infty((0,T);\mathrm{Lip}([0,1]))$, *une fonction* $y \in L^\infty(\Omega_T)$ *est une solution faible de* (1.12) *si* $\forall \psi \in Adm(\Omega_T)$ *:*

$$\iint_{\Omega_T} y(\partial_t\psi + (u+\mathcal{A})\partial_x\psi - \partial_x(u+\mathcal{A})\psi)dtdx = -\int_0^1 y_0(x)\psi(0,x)dx \\ + \int_0^T (\psi(t,1)v_r(t)y_r(t) - \psi(t,0)v_l(t)y_l(t))dt.$$

Il est évident que $\mathcal{C}^1_0(\Omega_T) \subset Adm(\Omega_T)$ donc une solution faible de (1.12) est aussi une solution de (1.12) au sens des distributions. Et à partir de là, on voit qu'une solution faible suffisamment régulière est bien une solution forte.

Ceci nous permet de prouver le théorème d'existence suivant.

Théorème 1. *Soit $\tilde{T} > 0$, $v_l, v_r \in \mathcal{C}^0([0,\tilde{T}])$ tels que les ensembles P_l et P_r n'ont qu'un nombre fini de composantes connexes. Soient $y_0 \in L^\infty(0,1)$, $y_l \in L^\infty(\Gamma_l)$ et $y_r \in L^\infty(\Gamma_r)$. Il existe un temps $T > 0$, et un couple de fonctions (u,y) solution faible de (10)-(11) avec $u \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H^1_0(0,1)\right)$ et $y \in L^\infty(\Omega_T)$. De plus, toute solution u appartient à $\mathcal{C}^0([0,T];W^{2,p}(0,1)) \cap C^1([0,1];W^{1,p}_0(0,1))$, $\forall p < +\infty$. De plus, le temps d'existence d'une solution maximale est plus grand que $\min(\tilde{T},T^*)$, où :*

$$T^* = \max_{\beta>0}\left(\frac{\ln(1+\beta/C_0)}{2(C_1+(2+\sinh(1))(C_0+|\kappa|+\beta))}\right), \tag{13}$$

$$C_0 = \max\left(||y_0||_{L^\infty(0,1)}, ||y_l||_{L^\infty(\Gamma_l)}, ||y_r||_{L^\infty(\Gamma_l)}\right), \tag{14}$$

$$C_1 = \frac{1}{\tanh(1)}(||v_r||_{L^\infty(0,T)} + ||v_l||_{L^\infty(0,T)}). \tag{15}$$

On prouvera également le résultat d'unicité fort-faible suivant :

Théorème 2. *Soit $(u,y) \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H^1_0(0,1)\right) \times L^\infty([0,T];\mathrm{Lip}([0,1]))$ une solution faible de (10) et (11). Elle est unique dans $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \times L^\infty(\Omega_T)$.*

Ayant maintenant une bonne théorie du problème mixte non-homogène, on s'intéresse au problème de la stabilisation asymptotique de (4) par une loi de retour stationnaire agissant au bord. Soient $A_l > 2\sinh(1)$, $A_r > A_l\cosh(1) + \sinh(2)$, $M > 0$ et $T > 0$. On choisit pour (5) la loi de retour :

$$y \in \mathcal{C}^0([0,1]) \mapsto \begin{cases} v_l(y) = A_l||y||_{\mathcal{C}^0([0,1])} - \kappa, \\ v_r(y) = A_r||y||_{\mathcal{C}^0([0,1])} - \kappa, \\ \dot{y}_l(t) + My_l(t) = 0. \end{cases} \tag{16}$$

On peut alors montrer le résultat suivant.

Théorème 3. *Quelque soit $y_0 \in \mathcal{C}^0([0,1])$ il existe $(y,v) \in \mathcal{C}^0(\Omega_T) \times \mathcal{C}^0([0,T],\mathcal{C}^2([0,1]))$ une solution faible de (5) and (16) satisfaisant*

$$\forall x \in [0,1], \quad y(0,x) = y_0(x). \tag{17}$$

De plus toute solution maximale de (5), (16) et (17) est globale en temps, et si on prend

$$c = \min(A_l - 2\sinh(1), \frac{A_r - A_l\cosh(1) - \sinh(2)}{\sinh(1)}) \text{ et } \tau = \frac{1}{M}\ln(\frac{2c||y_0||_{\mathcal{C}^0([0,1])}}{M}),$$

on a :

$$\forall t \geq \tau \qquad ||y(t,.)||_{\mathcal{C}^0([0,1])} \leq \frac{M}{2c}\frac{1}{1+M(t-\tau)}.$$

On a stabilisé $y = 0$ dans $\mathcal{C}^0([0,1])$, en considérant la définition du relèvement $\mathcal{A}$, cela correspond pour l'équation originale (4) à stabiliser $v = u + \mathcal{A} = -\kappa$ dans $\mathcal{C}^2([0,1])$. Ce résultat est résolument dans la lignée des résultats de stabilisation de Coron [37] et Glass [59] pour l'équation d'Euler incompressible et de Glass [58] pour l'équation de Camassa-Holm sur le tore avec contrôle interne : il utilise une forme de la méthode du retour adapté au problème de la stabilisation asymptotique. On exposera les idées essentielles de la preuve dans la partie 3 de cette introduction.

2.2 Lois de conservation scalaires et solutions entropiques.

Dans les chapitres 2 et 3, on étudie respectivement les problèmes de contrôlabilité exacte et de stabilisation asymptotique pour le système suivant :

$$\begin{aligned} &\partial_t u + \partial_x f(u) = g(t), \\ &u(0,x) = u_0(x), \\ &u(t,0) = u_l(t), \\ &u(t,1) = u_r(t). \end{aligned} \qquad (t,x) \in (0,T) \times (0,1), \tag{18}$$

La fonction de flux $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ est convexe, $u_0 \in \mathrm{BV}(0,1)$ est la donnée initiale, et enfin u_l, u_r et g sont les contrôles. Si on imagine que l'équation (18) modélise le champ de vitesse d'un fluide dans un tuyau, on peut penser au terme g comme à une force agissant directement sur ce tuyau rigide.

Quelques généralités sur les lois de conservation.

Les lois de conservation scalaires telles que :

$$\partial_t u + \partial_x(f(u)) = 0, \tag{19}$$

sont utilisées par exemple pour modéliser le trafic routier ou les réseaux de distribution de gaz. Cependant leur étude est aussi intéressante en tant que première étape en direction de l'analyse des systèmes de lois de conservation. Ce type de système apparaît naturellement dans la modélisation de très nombreux problèmes physiques : la dynamique des gaz, l'électromagnétisme, la magnéto-hydrodynamique, la propagation des ondes en eau peu profonde, les phénomènes de combustion... On pourra consulter [45] ou [95] pour plus de détails.

Pour de telles équations, on peut montrer que pour une donnée initiale régulière (par exemple $\mathcal{C}^1$), il existe une solution régulière en temps court. Cependant, ces solutions développent généralement des singularités en temps fini. Ainsi, pour l'équation de Burgers :

$$\partial_t u + \partial_x(\frac{u^2}{2}) = 0, \tag{20}$$

les seules données initiales engendrant une solution régulière globale en temps sont les fonctions croissantes de $\mathcal{C}^1(\mathbb{R})$. On peut même montrer par la méthode des caractéristiques que le temps maximal d'existence d'une solution régulière pour une donnée initiale u_0 dans $\mathcal{C}^1(\mathbb{R})$ vaut :

$$\sup_{x\in\mathbb{R}} -\frac{1}{u_0'(x)}.$$

Ceci montre la nécessité d'introduire une notion de solution faible pour espérer obtenir un théorème d'existence globale. Grâce à la forme particulière de l'équation, on obtient après une intégration par partie en x qu'une solution forte u de l'équation (19) satisfait pour toute fonction test ϕ dans $\mathcal{C}_c^1(\mathbb{R}^2)$:

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} u(t,x)\partial_t\phi(t,x) + f(u(t,x))\partial_x\phi(t,x)dxdt = \int_{-\infty}^{+\infty} u_0(x)\phi(0,x)dx. \tag{21}$$

Cette égalité a un sens à un niveau de régularité de u beaucoup plus faible que précédemment, par exemple $u \in L^\infty_{loc}$ suffit. Il faut noter que la possibilité d'intégrer par parties dépend entièrement de la formulation particulière des lois de conservation : la dérivation suivant x porte sur tout le terme $f(u)$. Par contre, l'équation linéarisée n'a pas la même forme, et il sera difficile de définir le produit $f'(u)\partial_x u$. A ce niveau de régularité, on ne pourra donc plus s'appuyer sur des techniques linéaires.

On peut aussi constater que si on se donne une fonction $c \in \mathcal{C}^1(\mathbb{R}^+)$ et deux fonctions $\mathcal{C}^1$, u^+ et u^- telles que l'on ait :

$$\forall t > 0, \text{ et } \forall x > c(t), \quad \partial_t u^+(t,x) + \partial_x(f(u^+))(t,x) = 0, \tag{22}$$

$$\forall t > 0, \text{ et } \forall x < c(t), \quad \partial_t u^-(t,x) + \partial_x(f(u^-))(t,x) = 0, \tag{23}$$

alors la fonction u égale à u^- sur $\{x < c(t)\}$ et à u^+ sur $\{x < c(t)\}$ satisfait (21) si et seulement si la condition dite de Rankine-Hugoniot est vérifiée :

$$\forall t > 0, \; f(u^+(t,c(t)^+)) - f(u^-(t,c(t)^-)) = \dot{c}(t)(u^+(t,c(t)^+) - u^-(t,c(t)^-)). \tag{24}$$

Cependant on ne peut se contenter d'adopter l'égalité (21) comme définition d'une solution faible car dans ce cas il serait impossible d'obtenir un résultat d'unicité, comme le montre l'exemple suivant.

On va à nouveau considérer l'équation de Burgers (20). Prenons a et b deux nombres réels tels que $a < b$, on définit u_0 par :

$$u_0(x) = \begin{cases} a & \text{si } x < 0, \\ b & \text{si } x > 0. \end{cases} \tag{25}$$

On peut alors voir que les fonctions u et v définies par :

$$u(t,x) = \begin{cases} a & \text{si } x < \frac{b^2-a^2}{b-a}t, \\ b & \text{si } x < \frac{b^2-a^2}{b-a}t, \end{cases} \tag{26}$$

$$v(t,x) = \begin{cases} a & \text{si } x < at, \\ b & \text{si } x > bt, \\ \frac{x}{t} & \text{autrement }, \end{cases} \tag{27}$$

sont, toutes les deux, des solutions faibles de (20) pour la donnée initiale u_0.

Un premier résultat important pour la résolution de ce problème a été celui d'Oleinik [87] en 1956. Elle a montré que pour un flux $f \in \mathcal{C}^2(\mathbb{R})$ tel que l'on ait :

$$\forall z \in \mathbb{R}, \quad f''(z) > 0,$$

et pour une fonction u_0 dans $L^\infty(\mathbb{R})$, il existe une unique fonction u dans $L^\infty(\mathbb{R}^2)$ (qu'on appellera la solution entropique) satisfaisant (21) et l'inégalité dite d'Oleinik :

$$\forall x \in \mathbb{R}, \; \forall t > 0, \; \forall a > 0, \quad \frac{u(t,x+a) - u(t,x)}{a} \leq \frac{E}{t}, \tag{28}$$

où E est une constante dépendant des quantités $\inf(f'')$ et $\sup(f')$ prises sur $[-||u_0||_{L^\infty}, ||u_0||_{L^\infty}]$.

En 1970, Kruzkov généralise le résultat d'Oleinik aux lois de conservation scalaires multidimensionnelles et avec un flux f de classe $\mathcal{C}^1$ qui n'est plus nécessairement convexe :

$$\partial_t u + \operatorname{div}(f(t,x,u)) = g(t,x,u), \text{ pour } t > 0, \; x \in \mathbb{R}^n, \; u : \mathbb{R}^n \mapsto \mathbb{R}.$$

Il introduit au passage une définition équivalente d'une solution entropique via une inégalité intégrale. Dans le cas de l'équation (19) cette définition est la suivante :

$$\forall k \in \mathbb{R},\ \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^2) \geq 0,$$
$$\int_0^{+\infty}\int_{\mathbb{R}} |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\phi_x dxdt + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx \geq 0. \quad (29)$$

Le problème aux limites, quant à lui, a été étudié par Leroux [80] pour le cas unidimensionnel avec données initiales et aux bords dans BV, par Bardos, Leroux et Nédélec [15] pour le cas multidimensionnel avec des données C^2 et nettement plus tard par Otto [88] pour des données initiales et aux bords L^∞. Le point clé est qu'on ne peut pas espérer obtenir une égalité de la solution avec les données aux bords au sens de Dirichlet. A la place, on a de nouveau une condition d'entropie que nous allons maintenant décrire.

Commençons par introduire la notation suivante :

$$\forall \alpha, \beta \in \mathbb{R} \qquad I(\alpha,\beta) = [\min(\alpha,\beta), \max(\alpha,\beta)]. \quad (30)$$

On va s'intéresser à l'équation suivante :

$$\begin{cases} \partial_t u + \partial_x(f(u)) = g(t) & \text{sur } (0,+\infty)\times(0,1), \\ u(0,.) = u_0 & \text{sur } (0,1), \\ \operatorname{sgn}(u(t,1^-) - u_r(t))(f(u(t,1^-)) - f(k)) \geq 0 & \forall k \in I(u_r(t), u(t,1^-)),\ dt \text{ p.p.}, \\ \operatorname{sgn}(u(t,0^+) - u_l(t))(f(u(t,0^+)) - f(k)) \leq 0 & \forall k \in I(u_l(t), u(t,0^+)),\ dt \text{ p.p.}, \end{cases} \quad (31)$$

où les deux dernières équations remplacent $u(t,1^-) = u_r(t)$ et $u(t,0^+) = u_l(t)$. Alors d'après [80] et [15] on dit qu'une fonction $u \in L^\infty((0,+\infty), \mathrm{BV}((0,1))$ est une solution entropique de (31) lorsque quelques soient le nombre $k \in \mathbb{R}$ et la fonction positive $\phi \in \mathcal{C}_c^1(\mathbb{R}^2)$ on a :

$$\int_0^{+\infty}\int_0^1 |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\phi_x + \operatorname{sgn}(u-k)g(t)\phi dxdt + \int_0^1 |u_0(x)-k|\phi(0,x)dx$$
$$+ \int_0^{+\infty} \operatorname{sgn}(u_r(t)-k)(f(k) - f(u(t,1^-)))\phi(t,1) - \operatorname{sgn}(u_l(t)-k)(f(k)-f(u(t,0^+))\phi(t,0)dt \geq 0. \quad (32)$$

Pour une classe d'équivalence $\bar{u}$ à la fois dans l'espace $L^\infty((0,+\infty);\mathrm{BV}(0,1))$ et aussi dans $\mathrm{Lip}([0,+\infty)\,;L^1(0,1))$ on peut trouver une fonction u mesurable sur $(0,T)\times(0,1)$, représentant cette classe et qui satisfait :

$$\forall t \geq 0, \quad u(t,.) \in \mathrm{BV}(0,1).$$

Les traces de $\bar{u}$ aux points $x=0$ et $x=1$ sont alors les limites de ce représentant en 0^+ et en 1^- pour tout temps. Mais alors les conditions aux bords de (3.14) sont vérifiées uniquement presque partout pour ce représentant privilegié. Ceci va rendre l'analyse de l'influence des conditions aux bords sur la solution plus difficile, comme on le verra au chapitre 3.

Nous allons maintenant nous intéresser à l'étude du système (18) du point de vue de la théorie du contrôle.

Résultats de contrôle.

Dans le cadre des solutions fortes des lois de conservation, il y a eu de nombreux travaux consacrés aux problèmes de la contrôlabilité exacte et de la stabilisation asymptotique parmi ceux-ci nous citerons seulement les articles [8], [9], [65], [66], [39] et le livre [77].

Dans le contexte des solutions entropiques, il n'y a que peu de résultats sur le problème de la contrôlabilité exacte et il semble qu'il n'y en ait aucun sur le problème de la stabilisation asymptotique par boucle fermée. On va maintenant détailler quelques résultats connus. Dans l'article [4], Fabio Ancona et Andrea Marson décrivent exactement l'ensemble atteignable pour l'équation scalaire suivante :

$$\begin{cases} \partial_t u + \partial_x(f(u)) = 0, \qquad t > 0, \quad x > 0, \\ u(0,x) = 0, \quad x > 0, \qquad u(t,0) = c(t), \quad t > 0. \end{cases} \tag{33}$$

avec $f : \mathbb{R} \mapsto \mathbb{R}$ strictement convexe et un contrôle au bord c. Un état $w \in L^\infty(0,+\infty)$ est atteignable en temps T si et seulement si les conditions suivantes sont satisfaites :

$$\begin{gathered} w(x) \neq 0 \Rightarrow f'(w(x)) \geq \frac{x}{T}, \\ \big(w(x^-) \neq 0 \text{ et pour tout } y > x,\ w(y) = 0\big) \Rightarrow f'(w(x^-)) > \frac{x}{T}, \\ \limsup_{h \to 0} \frac{w(x+h) - w(x)}{h} \leq \frac{f'(w(x))}{x f''(w(x))}, \end{gathered} \tag{34}$$

pour tout $x > 0$. Les deux premières conditions sont liées à la vitesse finie de propagation pour les solutions de (33) et la troisième est une variation de l'inégalité d'Oleinik (28) en présence d'un bord.
Thierry Horsin a apporté dans [69] des conditions suffisantes (liées à (34)) pour qu'un état soit atteignable par une solution de l'équation de Burgers (20) sur un intervalle compact, avec une donnée initiale quelconque et où les contrôles sont les données aux bords.
Il existe également des résultats de Bressan et Coclite [17], Ancona et Coclite [3], Ancona et Marson [5] et Glass [57] sur la contrôlabilité et la non-contrôlabilité des systèmes de lois de conservation dans le contexte des solutions entropiques. Dans tous les résultats évoqués, il y a de nombreux états naturels qui ne peuvent être atteints et ce quelque soit le temps fixé en utilisant juste des contrôles au bord. Ainsi l'état constant 0 ne peut être atteint par les solutions de (20) en temps quelconque pour la majorité des conditions initiales.

Par contre, si on utilise, en plus des données aux bords, un nouveau contrôle $g(t)$ tel qu'il est présenté dans (18), Marianne Chapouly a prouvé dans [20] que pour l'équation de Burgers et avec des solutions régulières on peut atteindre n'importe quel état régulier depuis n'importe quelle donnée initiale régulière et en temps arbitraire. Il faut noter que dans le contexte des solutions classiques, les contrôles doivent aussi empêcher l'explosion des solutions, ce qui n'est pas un problème dans le cadre des solutions entropiques.

Contrôlabilité exacte.

En ce qui concerne la contrôlabilité exacte, les contrôles aux bords peuvent être considérés comme des indéterminées. En effet pour deux fonctions $u_0, u_1 \in \mathrm{BV}(0,1)$ et un temps strictement positif T, si l'on arrive à trouver deux fonctions :

$$g \in \mathcal{C}^0([0,T]) \text{ et } u \in L^\infty((0,T); \mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,T]; L^1(0,1)),$$

telles que l'on ait :

$$\forall k \in \mathbb{R},\ \forall \phi \in \mathcal{C}_c^1((0,T)\times(0,1)) \geq 0, \tag{35}$$

$$\int_0^{+\infty}\int_0^1 |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\phi_x + \operatorname{sgn}(u-k)g(t)\phi dxdt \geq 0, \tag{36}$$

$$u(0,.0) = u_0, \qquad u(T,.) = u_1, \tag{37}$$

alors u est en fait l'unique solution entropique de (31) avec les données aux bord $u_l(t) = u(t,0^+)$ et $u_r(t) = u(t,1^-)$. On va donc se concentrer sur l'équation sous-déterminée :

$$\begin{cases} \partial_t u + \partial_x(f(u)) = g(t) \quad \text{sur } (0,T)\times(0,1), \\ u(0,.) = u_0 \quad \text{sur } (0,1). \end{cases} \tag{38}$$

Comme une solution entropique satisfait localement une inégalité telle que (28), on ne peut espérer atteindre n'importe quelle fonction dans BV(0,1). On va donc fournir quelques conditions suffisantes pour qu'une fonction de BV(0,1) soit atteignable en temps T. Dans le chapitre 2 nous montrerons les résultats suivant :

Théorème 4. *Soit* $u_1 \in \mathrm{BV}(0,1)$ *telle que :*

$$\sup_{\substack{0<h<1\\0<x<1-h}} \frac{u_1(x+h)-u_1(x)}{h} < +\infty, \tag{39}$$

et supposons que la fonction flux f *soit* $\mathcal{C}^2$*, convexe et satisfasse au moins une des deux conditions suivantes :*

$$\frac{f'(M)}{\sup_{z\in[0,M]} f''(z)} \underset{M\to+\infty}{\to} +\infty \quad ou \quad \frac{f'(M)}{\sup_{z\in[M,0]} f''(z)} \underset{M\to-\infty}{\to} -\infty. \tag{40}$$

Alors quelque soient le temps $T>0$ *et la donnée initiale* $u_0 \in \mathrm{BV}(0,1)$*, on peut trouver deux fonctions* g *et* u *appartenant respectivement aux espaces* $\mathcal{C}^1([0,T])$ *et* $L^\infty((0,T);\mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,T];L^1(0,1))$*, telles que* u *soit une solution entropique de* (38) *sur* $(0,T)\times(0,1)$ *et qu'on ait également :*

$$u(0,.) = u_0 \quad et \quad u(T,.) = u_1 \quad sur\ (0,1).$$

En comparaison avec les résultat précédemment cités on peut noter que :

- les inégalités (39) et (28) sont relativement semblables, mais (2.13) est nettement moins restrictive puisque le supremum peut être arbitrairement grand et ce indépendamment de t et f,
- les deux premières conditions de (34) sont ici remplacées par (40) qui ne concerne que le flux. De ce fait on peut atteindre beaucoup plus d'états grâce au contrôle g. De plus, on peut le faire en temps arbitrairement petit.

On peut améliorer le résultat en permettant une dégénérescence de la condition (39) près d'un bord. On explicitera juste le cas où la dégénérescence se produit en 0. Pour ce faire introduisons la fonction K définie par :

$$\forall x \in (0,1), \qquad K(x) = \left(\sup_{\substack{x\leq y<1\\0<h<1-y}} \frac{u_1(y+h)-u_1(y)}{h} \right)_+ . \tag{41}$$

On peut alors obtenir le résultat suivant :

Théorème 5. *Soit* $u_1 \in \mathrm{BV}(0,1)$ *satisfaisant à la fois :*

$$K(x) = O\left(\frac{1}{x^p}\right) \quad et \quad \left(u_1(0) - \inf_{0<y\leq x} u_1(y)\right) = O(x^{2p}) \quad lorsque\ x \to 0^+. \tag{42}$$

Définissons ensuite :

$$\forall M > 0,\ I_M = [\inf_{x\in(0,1)} u_1(x),\ \sup_{x\in(0,1)} u_1(x) + M], \tag{43}$$

et supposons que pour un certain $q > 0$*, tel que* $p(2q+1) \leq 1$*, la fonction flux* f *soit* $\mathcal{C}^2$*, convexe et vérifie les deux conditions suivantes :*

$$\frac{M^q}{\sup_{z\in I_M} f''(z)} \xrightarrow[M\to+\infty]{} +\infty \qquad et \qquad \frac{|h|^q}{|f'(u_1(0)+h)|} = O(1)\ en\ 0\ et\ en\ +\infty. \tag{44}$$

Alors pour n'importe quels temps $T > 0$*, et donnée initiale* $u_0 \in \mathrm{BV}(0,1)$*, il existe deux fonctions* g *et* u *appartenant respectivement aux espaces* $\mathcal{C}^1([0,T])$ *et* $L^\infty((0,T);\mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,T];L^1(0,1))$ *telles que l'on ait les propriétés suivantes :*

- *u est une solution entropique de* (38) *sur* $(0,T)\times(0,1)$ *avec* $u(0,.) = u_0$ *sur* $(0,1)$*,*
- *au temps final* T *on a à la fois* $u(T,.) = u_1$ *et* $g(T) = 0$*.*

On finit maintenant cette partie avec la condition la plus générale des trois présentées.

Théorème 6. *Nous supposons que* f *est* $\mathcal{C}^2$*, convexe et que* $f'(z)$ *tend vers* $+\infty$ *avec* z*. Soient* $u_1 \in \mathrm{BV}(0,1)$ *et* $\bar{T} > 0$*. Introduisons la notation :*

$$\forall x \in (0,1),\ \tau(x) = \min\left(\bar{T}, \frac{1}{2K(x)} \frac{1}{\sup_{z\in I_M} f''(z)}\right). \tag{45}$$

Supposons qu'il existe une fonction $\bar{g} \in \mathcal{C}^1([0,\bar{T}])$ *telle que :*

$$\liminf_{\beta\to0^+} \sup_{\frac{3\beta}{2}\leq\alpha<1} \left(\alpha - \int_{\bar{T}-\tau(\alpha-\beta)}^{\bar{T}} f'\left(\inf_{\alpha-\beta\leq x\leq\alpha} u_1(x) - \int_s^{\bar{T}} \bar{g}(r)dr\right) ds\right) \leq 0. \tag{46}$$

Alors pour tout temps $T > \bar{T}$ *et pour toute donnée initiale* $u_0 \in \mathrm{BV}(0,1)$*, on peut trouver deux fonctions* $g \in \mathcal{C}^1([0,T])$ *et* $u \in L^\infty((0,T);\mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,T];L^1(0,1))$ *telles que :*

u est une solution entropique de (38) *sur* $(0,T)\times(0,1)$*,*

$$u(0,.) = u_0 \quad et \quad u(T,.) = u_1 \quad sur\ (0,1).$$

Dans le chapitre 2, nous montrerons d'abord le théorème 6 puis nous en déduirons les deux précédents. Les conditions de ceux-ci sont moins générales mais ont l'avantage d'être plus lisibles. On pourra regarder la partie 3 de cette introduction pour des détails sur la stratégie de la démonstration.

Stabilisation asymptotique. Dans le chapitre 3 nous nous intéresserons au problème de la stabilisation asymptotique des états constants de (31) par une loi de retour stationnaire. Les fonctions g, u_l et u_r ne dépendront plus du temps mais de l'état du système : $u(t,.)$.

Pour un certain $\bar{u} \in \mathbb{R}$, il est clair que la fonction u définie par :

$$\forall (t,x) \in \mathbb{R}+ \times (0,1), \quad u(t,x) = \bar{u},$$

est une solution entropique de (31) avec des données initiales et aux bords constantes égales à $\bar{u}$. Dans ce qui suit nous allons introduire deux lois de retour (suivant que $f'(\bar{u}) \neq 0$ ou $f'(\bar{u}) = 0$), telles que la solution constante précédente soit asymptotiquement stable pour le système en boucle fermée. Nous allons commencer par le premier cas.
Si $f'(\bar{u}) \neq 0$, on utilise la loi de retour stationnaire donnée par :

$$\forall W \in L^1(0,1), \quad \mathcal{G}_1(W) = \frac{f'(\bar{u})}{2} ||W - \bar{u}||_{L^1(0,1)}, \tag{47}$$

$$\forall W \in L^1(0,1), \quad u_l(W) = u_r(W) = \bar{u}. \tag{48}$$

Dans l'équation (31), on va donc substituer : $g(t)$ par $\mathcal{G}_1(u(t,.))$, $u_l(t)$ par $u_l(u(t,.))$ et enfin $u_r(t)$ par $u_r(u(t,.))$ afin d'obtenir un système en boucle fermée où la seule donnée du problème est la donnée initiale u_0.

Dans toute cette partie, on va supposer que la fonction flux f est C^1**, strictement convexe**. On aura besoin de distinguer deux comportements différents pour f.

Definition 2. • *On dira que f est de type I s'il existe u^* tel que :*

$$f'(u^*) = 0. \tag{49}$$

L'équation de Burgers a par exemple un flux de type I.

• *On dira que f est de type II autrement, et dans ce cas on aura forcément :*

$$\forall z \in \mathbb{R}, \quad f'(z) > 0, \tag{50}$$

ou

$$\forall z \in \mathbb{R}, \quad f'(z) < 0. \tag{51}$$

Le flux $f(z) = e^z$ est de type II.

Si le flux f est de type I, on peut en déduire en utilisant également sa stricte convexité :

$$\lim_{z \to +\infty} f(z) = \lim_{z \to -\infty} f(z) = +\infty. \tag{52}$$

Si de plus $f'(\bar{u}) \neq 0$, on aura forcément $\hat{u} \neq \bar{u}$ tel que $f(\bar{u}) = f(\hat{u})$. On peut alors reformuler les conditions aux bords (3.14) comme suit (on traite le cas $f'(\bar{u}) > 0$) :

$$u(t,1^-) \in [u^*; +\infty) \quad dt \ p.p., \tag{53}$$

$$u(t,0^+) \in (-\infty, \hat{u}] \cup \{\bar{u}\} \quad dt \ p.p.. \tag{54}$$

L'idée est que soit la limite au bord est égale à la donnée au bord, soit la solution du problème de Riemann entre ces deux valeurs n'admet que des ondes qui quittent le domaine. On pourra consulter [51] pour plus de renseignements sur les conditions aux bords que l'on peut choisir pour les lois scalaires mais également pour les systèmes.

On peut alors démontrer le résultat suivant.

Théorème 7. *Quelque soit* $u_0 \in \mathrm{BV}(0,1)$*, le système en boucle fermée* (3.14) *où* u_l*,* u_r *et* g *sont données par les lois de retour* (3.17) *et* (3.16) *a une unique solution entropique* u*. Elle est globale en temps, appartient à l'espace* $L^\infty((0,+\infty);\mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,+\infty);L^1(0,1))$ *et dépend continûment de la donnée initiale. De plus si le flux* f *est de type I on a les propriétés suivantes.*

- *Il existe deux constantes* C_1 *et* C_2 *qui dépendent uniquement de* $\bar{u}$ *et telles que* u *satisfait :*

$$\forall t \geq 0, \qquad ||u(t,.) - \bar{u}||_{L^1(0,1)} \leq C_1 e^{-\frac{f'(\bar{u})}{2}t} ||u_0 - \bar{u}||_{L^1(0,1)}, \tag{55}$$

$$\forall t \geq 0, \qquad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq C_2 e^{-\frac{f'(\bar{u})}{2}t} ||u_0 - \bar{u}||_{L^\infty(0,1)}. \tag{56}$$

- *Il existe un temps* T *qui dépend seulement de* $\bar{u}$ *tel que* u *est régulière sur* $(T,+\infty) \times [0,1]$*.*

Si par contre le flux f *est de type II on a seulement les propriétés suivantes :*

- *il existe une constante* C_3 *qui dépend de* $\bar{u}$ *et de* $||u_0 - \bar{u}||_{L^\infty(0,1)}$ *telle que* u *satisfait :*

$$\forall t \geq 0, \quad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq C_3 e^{-\frac{f'(\bar{u})}{2}t} ||u_0 - \bar{u}||_{L^\infty(0,1)}. \tag{57}$$

- *Il existe un temps* T' *qui dépend de* $\bar{u}$ *et de* $||u_0 - \bar{u}||_{L^\infty(0,1)}$ *tel que* u *est régulière sur* $(T',+\infty) \times [0,1]$*.*

On a donc établi un résultat de stabilisabilité asymptotique comme annoncé. Ajoutons aussi que :

- dans le chapitre 3, on donnera des constantes explicites pour C_1, C_2, C_3, T et T',
- si cette loi de retour utilisant la norme L^1 permet d'assurer une stabilisation au sens de la norme L^∞, l'utilisation d'une loi de retour utilisant la norme L^∞ pourrait se révéler problématique. En raison de son absence de régularité en temps (déjà en boucle ouverte) mais aussi à cause de l'impossibilité de passer à la limite dans $||.||_{L^\infty(0,1)}$ lorsqu'on a juste convergence ponctuelle d'une suite de fonctions.

Nous allons maintenant considérer le deuxième cas, à savoir que $f'(\bar{u}) = 0$. Introduisons la fonction auxiliaire A :

$$\mathrm{A}(z) = \begin{cases} \frac{f(\bar{u}+z)-f(\bar{u})}{2} & \text{si } 0 \leq z \leq 1, \\ \frac{f'(\bar{u}+1)}{2}(z-1) + \frac{f(\bar{u}+1)-f(\bar{u})}{2} & \text{si } z \geq 1. \end{cases} \tag{58}$$

Nous allons de nouveau utiliser la loi de retour stationnaire :

$$\forall W \in L^1(0,1), \quad u_l(W) = \bar{u} = u_r(W),$$

pour les termes de bord. Par contre pour le terme source on va plutôt utiliser :

$$\forall W \in L^1(0,1), \quad \mathcal{G}_2(W) = \mathrm{A}(||W - \bar{u}||_{L^1(0,1)}), \tag{59}$$

Comme précédemment on échangera $g(t)$ et $\mathcal{G}_2(u(t,.))$ dans (3.14). On a alors le résultat suivant.

Théorème 8. *Le système en boucle fermée* (3.14) *où* u_l*,* u_r *et* g *sont fournis par les lois de retour* (3.17) *et* (3.28) *a les propriétés suivantes.*

- *Quelque soit* $u_0 \in \mathrm{BV}(0,1)$, *il existe une unique solution entropique* u. *Celle-ci est globale en temps, et appartient à l'espace*

$$L^\infty((0,+\infty);\mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,+\infty)\,;L^1(0,1)).$$

De plus elle dépend continûment de la donnée initiale.

- *La solution* u *satisfait également :*

$$||u(t,.)-\bar{u}||_{L^\infty(0,1)} \underset{t\to+\infty}{\to} 0. \tag{60}$$

- *Si on a l'hypothèse supplémentaire :*

$$\alpha = \inf_{z\in\mathbb{R}} f''(z) > 0,$$

alors il existe une fonction uniformément lipschitzienne R *telle que :*

$$R(0) = \frac{f'(1+\bar{u})}{2\alpha}\sqrt{\frac{2e}{e-1}} + \mathrm{A}^{-1}\left(\frac{e(f'(1+\bar{u}))^2}{4\alpha(e-1)}\right), \tag{61}$$

$$\forall t \geq 0, \qquad ||u(t,.)-\bar{u}||_{L^\infty(0,1)} \leq R(||u_0-\bar{u}||_{L^\infty(0,1)}). \tag{62}$$

La dernière propriété est plus faible que la stabilité, on ne peut donc pas dire qu'on stabilisation asymptotique de $\bar{u}$. Cependant en prenant $c>0$ et en ajustant la fonction A de la façon suivante :

$$\mathrm{A}(z) = \begin{cases} \frac{f(\bar{u}+z)-f(\bar{u})}{2} & \text{si } 0 \leq z \leq c, \\ \frac{f'(\bar{u}+c)}{2}(z-c) + \frac{f(\bar{u}+c)-f(\bar{u})}{2} & \text{si } z \geq c, \end{cases} \tag{63}$$

on peut voir que $\frac{f'(\bar{u}+c)}{2}$ tend vers 0 avec c et donc que $R(0)$ peut être arbitrairement petit.

Nous allons maintenant décrire dans les grandes lignes les méthodes qui nous permettront de prouver ces résultats.

3 Méthodologie.

Avant d'exposer les stratégies de contrôle spécifiques permettant d'obtenir les résultats précédemment présentés, nous allons commencer par présenter les idées communes aux trois chapitres et pour cela, faire des rappels sur les équations de transport linéaires. Le lecteur pourra consulter [43] pour plus détails sur celles-ci.

Considérons l'équation :

$$\partial_t Y(t,x) + a(t,x)\partial_x Y(t,x) = b(t,x)Y(t,x) + f(t,x), \qquad (t,x) \in \mathbb{R}^+ \times (0,1), \tag{64}$$

où Y, a, b et f sont des fonctions définies sur $\mathbb{R}^+ \times [0,1]$ à valeurs réelles, que nous supposerons régulières pour l'instant. La méthode des caractéristiques consiste à introduire le flot ϕ du champ a comme suit. Pour tout $(t,x) \in \mathbb{R}^+ \times (0,1)$ on a deux nombres positifs $e(t,x)$ et $h(t,x)$ telle que la fonction $s \mapsto \phi(s,t,x)$ est la solution maximale de l'équation différentielle suivante :

$$\begin{cases} \partial_s \phi(s,t,x) = a(s,\phi(s,t,x)), & s \in (e(t,x),h(t,x)) \\ \phi(t,t,x) = x. \end{cases} \tag{65}$$

Les fonctions $e(t,x)$ et $h(t,x)$ sont respectivement les temps d'entrée et de sortie de la caractéristique (la courbe $s \mapsto \phi(s,t,x)$) passant par le point (t,x). En effet, on a les propriétés suivantes :

$$e(t,x) > 0 \text{ implique que } \phi(e(t,x),t,x) \in \{0,1\}, \tag{66}$$

$$h(t,x) < +\infty \text{ implique que } \phi(h(t,x),t,x) \in \{0,1\}. \tag{67}$$

On peut alors constater que si Y est solution forte de (64) alors pour tout $(t,x) \in \mathbb{R}^+ \times (0,1)$:

$$\forall s \in (e(t,x),h(t,x)), \qquad \frac{\mathrm{d}}{\mathrm{d}s} Y(s,\phi(s,t,x)) = b(s,\phi(s,t,x))Y(s,\phi(s,t,x)) + f(s,\phi(s,t,x)). \tag{68}$$

Pour obtenir la valeur de Y au point (t,x) il suffit donc de remonter la caractéristique passant par ce point jusqu'à ce que celle-ci touche $x=1$, $x=0$ ou $t=0$ puis utiliser une donnée au bord ou la donnée initiale et l'équation différentielle (68). On pourra trier les points de $\mathbb{R}^+ \times (0,1)$ suivant que la caractéristique remonte jusqu'à $x=0$, $x=1$ ou $t=0$. Afin de pouvoir décrire la régularité de la solution Y, il faudra aussi identifier un ensemble problématique P qui fait la transition entre des zones où les caractéristiques se comportent différemment. Tout cela aboutit à la définition de la partition de $\mathbb{R}^+ \times (0,1)$ suivante.

$$\begin{aligned} P = \{(t,x) \in \mathbb{R}^+ \times (0,1) \mid \exists s \in [e(t,x),h(t,x)] \text{ tel que } \phi(s,t,x) \in \{0,1\} \text{ et } a(s,\phi(s,t,x)) = 0\} \\ \cup \ \{(s,\phi(s,0,0)) \mid \forall s \in [0,T]\} \ \cup \ \{(s,\phi(s,0,1)) \mid \forall s \in [0,T]\}, \end{aligned}$$

$$\begin{aligned} I &= \{(t,x) \in \mathbb{R}^+ \times (0,1) \setminus P \mid e(t,x) = 0\}, \\ L &= \{(t,x) \in \mathbb{R}^+ \times (0,1) \setminus P \mid \phi(e(t,x),t,x) = 0\}, \\ R &= \{(t,x) \in \mathbb{R}^+ \times (0,1) \setminus P \mid \phi(e(t,x),t,x) = 1\}. \end{aligned}$$

Les résultats sur les solutions faibles de (64) sont réunis dans la partie 1.4 du chapitre 1.

Du point de vue de la contrôlabilité, si les données aux bords sont des contrôles, il est clair qu'on ne pourra influencer la solution Y que sur l'ensemble $L \cup R$, il y aura donc des conditions géométriques sur a pour obtenir la contrôlabilité exacte. Dans les résultats énoncés dans les parties 2.1 et 2.2 de cette introduction, on se ramène à contrôler des équations de transport telles que (64) avec la nuance qu'il y a désormais un couplage entre la solution Y et la fonction a. C'est grâce à ce couplage qu'on obtiendra de bonnes propriétés de contrôlabilité. De manière plus précise, on utilisera la méthode du retour de Jean-Michel Coron (voir [33], [34]) : dans un premier temps on utilise les contrôles pour qu'après un certain T toutes les caractéristiques trouvent leur origine sur le même coté : $x=0$ ou $x=1$ (voir la figure 1). Un tel régime est stable vis à vis de petites perturbations, donc on a localement de bonnes propriétés de contrôlabilité. Nous allons maintenant décrire les spécificités des différents problèmes.

Équation de Camassa-Holm

Pour l'équation de Camassa-Holm étudiée dans le chapitre 1, on a $f=0$, $b=-2\partial_x a$. De plus, les fonctions a et Y sont couplées via l'équation stationnaire suivante :

$$\begin{cases} (1-\partial^2_{xx})a = Y - \kappa, \\ a(t,1) = d_r(t) \qquad a(t,0) = d_l(t). \end{cases} \tag{69}$$

Les données au bord d_l et d_r étant également des contrôles, on va agir sur la géométrie du flot de a via ces contrôles, puis on contrôlera les valeurs de Y via les données au bord de (64). Cette

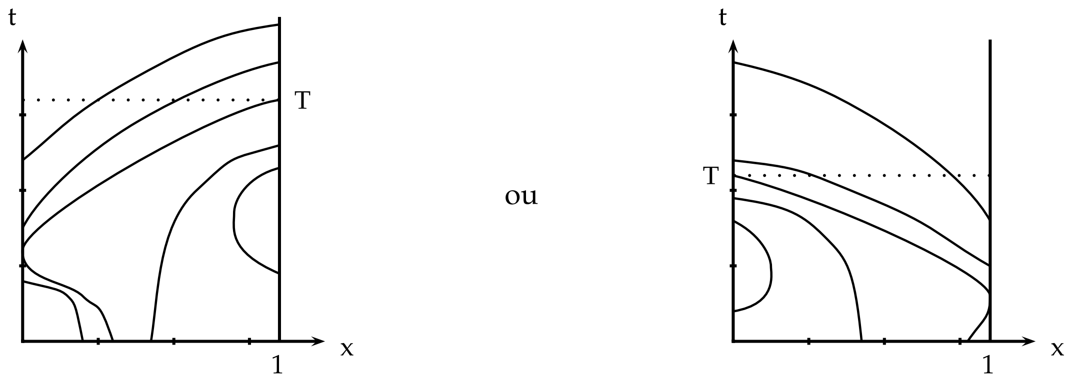


FIGURE 1 –

stratégie de contrôle pour un système couplé constitué d'une équation de transport et d'une équation elliptique stationnaire a déjà été utilisée par J.-M. Coron dans [34] pour stabiliser les états constants de l'équation d'Euler bidimensionnelle.

Pour établir le résultat d'existence en temps petit pour le problème mixte aux limites, on va utiliser un théorème de point fixe. En effet, on a deux équations couplées : l'équation de transport (64) et l'équation stationnaire (69). On va les résoudre successivement pour obtenir un certain opérateur donc tout point fixe sera une solution simultanée des deux équations. Plus précisément pour une fonction a dans $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H^1_0(0,1)\right)$ on regarde la solution Y de l'équation :

$$\partial_t Y(t,x) + a(t,x)\partial_x Y(t,x) = -2\partial_x a(t,x) Y(t,x), \qquad (t,x) \in \mathbb{R}^+ \times (0,1), \tag{70}$$

à laquelle on a rajouté des conditions aux bord et initiales telles que précisées dans la partie 1.4 du chapitre 1. On construit ensuite la fonction $\tilde{a}$ solution de :

$$(1-\partial^2_{xx})\tilde{a} = Y - \kappa, \tag{71}$$

avec les conditions de Dirichlet classiques aux bords. Grâce à de bonnes estimées linéaires (voir 1.4 chapitre 1) sur ces deux équations l'opérateur qui, à la fonction a, associe la fonction $\tilde{a}$ est continue pour la norme $||.||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$. De plus, l'espace $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H^1_0(0,1)\right)$ s'injecte de manière compacte dans $L^\infty\left((0,T);\mathrm{Lip}([0,1])\right)$. Ceci permet d'appliquer le théorème du point fixe de Schauder.

Pour établir le résultat de stabilisation asymptotique de l'état $Y=0$, $a=-\kappa$, on utilise la loi de retour :

$$d_r(Y) = A_r||Y||_{\mathcal{C}^0([0,1])}, \tag{72}$$

$$d_l(Y) = A_l||Y||_{\mathcal{C}^0([0,1])}, \tag{73}$$

$$\dot{Y}_l + MY_l = 0. \tag{74}$$

Les constantes A_l et A_r sont choisies de façon à utiliser les propriétés provenant de l'équation elliptique (69) :

$$\begin{cases} a(t,x) \geq (A_l - 2\sinh(1))||Y(t,.)||_{\mathcal{C}^0[0,1])}, \\ \partial_x a(t,x) \geq \frac{A_r - 2\cosh(1)A_l - \sinh(2)}{\sinh(1)}||Y(t,.)||_{\mathcal{C}^0[0,1])}. \end{cases} \tag{75}$$

La première inégalité garantit que le flot de a correspond bien à la situation décrite par la partie gauche de la figure 1 dès l'instant initial. La seconde inégalité implique quant à elle que l'on a décroissance de $|Y|$ le long des caractéristiques de a. Combiné à la décroissance de Y_l impliquée par (72) quand M est positive, cela permet d'obtenir :

$$||Y(t,.)||_{\mathcal{C}^0([0,1])} \underset{t\to+\infty}{\to} 0. \tag{76}$$

Pour montrer l'existence d'une solution de l'équation de Camassa-Holm en boucle fermée munie de la loi de retour (72), on procède de la même manière que pour le problème mixte aux limites. A ceci près que le gain de régularité est plus faible car les données aux bords sont maintenant des inconnues. La compacité sera obtenue dans $C^0([0,1])$ par le théorème d'Ascoli grâce à la propagation du module de continuité de la donnée initiale et des données aux bords par l'équation de transport (70).

Lois de conservation scalaires.
Pour une loi de conservation scalaire dotée d'un flux f telle que dans les chapitres 2 et 3, le couplage entre a et Y prendra la forme suivante :

$$\forall (t,x) \in \mathbb{R}^+ \times (0,1), \qquad a(t,x) = f'(Y(t,x)). \tag{77}$$

Sachant qu'on a $b=0$, $f(t,x)=g(t)$, où g est un contrôle supplémentaire, on arrive à l'équation :

$$\partial_t Y + \partial_x(f(Y)) = g(t). \tag{78}$$

La stratégie de contrôle consistera à agir à la fois sur la géométrie des caractéristiques de $f'(Y(t,x))$ et sur l'état Y. Cependant, et contrairement à ce qui s'est produit pour l'équation de Camassa-Holm, le couplage ne permet ici aucun gain de régularité en espace. Ceci oblige de travailler avec $Y(t,.) \in \mathrm{BV}(0,1)$, or à ce niveau de régularité, la construction du flot de $f'(Y(t,x))$ est beaucoup plus délicate et l'existence du flot n'est pas garantie. On pourra regarder [43] pour plus d'information à ce sujet. En pratique la double action des contrôles sur la géométrie et sur l'état sera mise en place via l'algorithme de suivi de fronts pour le problème de la contrôlabilité exacte dans le chapitre 2 et par la théorie des caractéristiques généralisées de Dafermos [44] dans le chapitre 3.

Contrôlabilité exacte.
Classiquement l'algorithme de suivi de fronts introduit par Dafermos [44] et développé également par Bressan [16] consiste à trouver des approximations constantes par morceaux sur des domaines polygonaux en utilisant les relations de Rankine-Hugoniot et les conditions d'entropie de Lax. On passe ensuite à la limite par compacité dans L^1 via des estimations uniformes sur la variation totale. On introduira dans le chapitre 2 une petite modification de l'algorithme permettant une analyse géométrique plus facile de l'action du contrôle g sur les solutions de (78). Le point de départ pour l'algorithme de suivi de front est la résolution approchée du problème de Riemann. Pour une donnée initiale Y_0 telle que :

$$\forall x \in \mathbb{R}, \quad Y_0(x) = \begin{cases} Y^- & \text{si } x < 0, \\ Y^+ & \text{si } x > 0, \end{cases} \tag{79}$$

on résout approximativement l'équation (78) de la façon suivante. Si $Y^- \geq Y^+$, on introduit la courbe :

$$\gamma(t) = \int_0^t \frac{f(Y^- + \int_0^s g(r)dr) - f(Y^+ + \int_0^s g(r)dr)}{Y^- - Y^+} ds. \tag{80}$$

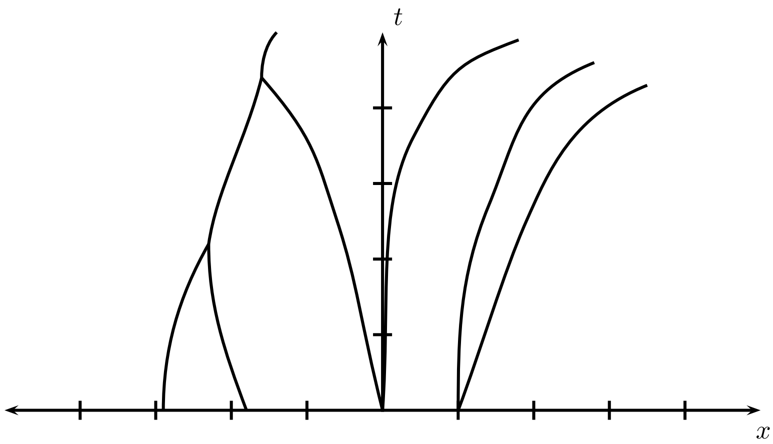


FIGURE 2 –

Ceci permet de définir la solution exacte :

$$Y(t,x) = \begin{cases} Y^- + \int_0^t g(s)ds & \text{si } x < \gamma(t), \\ Y^+ + \int_0^t g(s)ds & \text{si } x > \gamma(t). \end{cases}$$

Par contre, si $Y^- < Y^+$, on choisit un entier p (qui sera amené à tendre vers $+\infty$). Puis on introduit :

$$Y^l = \frac{p-l}{p} Y^- + \frac{l}{p} Y^+, \quad \text{pour } 0 \leq l \leq p, \tag{81}$$

$$\text{et pour } 1 \leq l \leq p, \quad \gamma_l(t) = \int_0^t \frac{f(Y^l + \int_0^s g(r)dr) - f(Y^{l-1} + \int_0^s g(r)dr)}{Y^l - Y^{l-1}} ds. \tag{82}$$

Ceci nous permet alors de définir une solution approchée de (78) :

$$Y(t,x) = \begin{cases} Y^0 + \int_0^t g(r)dr & \text{si } x < \gamma_1(t), \\ Y^l + \int_0^t g(r)dr & \text{si } \gamma_{l-1}(t) < x < \gamma_l(t) \text{ et } 1 \leq l \leq p-1, \\ Y^p + \int_0^t g(r)dr & \text{si } \gamma_p(t) < x. \end{cases} \tag{83}$$

On peut montrer que lorsque le nombre p tend vers $+\infty$, Y tend vers une solution d'entropie de l'équation (78). De cette façon, pour une donnée initiale constante par morceaux, on sait propager les discontinuités et de ce fait résoudre l'équation de manière approchée avec une fonction Y telle que $Y(t,.)$ est constante par morceaux pour tout temps t (voir figure 2). L'algorithme de suivi de front consiste alors à approcher la donnée initiale par des fonctions constantes par morceaux pour lesquelles on résout le problème de plus en plus finement.

Pour obtenir le résultat de contrôlabilité exacte, on procède alors en trois étapes en utilisant la méthode du retour de J.-M. Coron et la méthode d'extension de Russell.

- La première étape va consister à passer d'une donnée initiale Y_0 quelconque dans $BV(0,1)$ à un état final constant en un temps T_1 arbitrairement petit. Pour cela on va prolonger la donnée initiale à $\mathbb{R}$ par des constantes sur $(-\infty, 0)$ et $(1, +\infty)$. On va ensuite utiliser le contrôle g qui est lié d'après (80) et (82) à l'accélération des points de discontinuité, pour

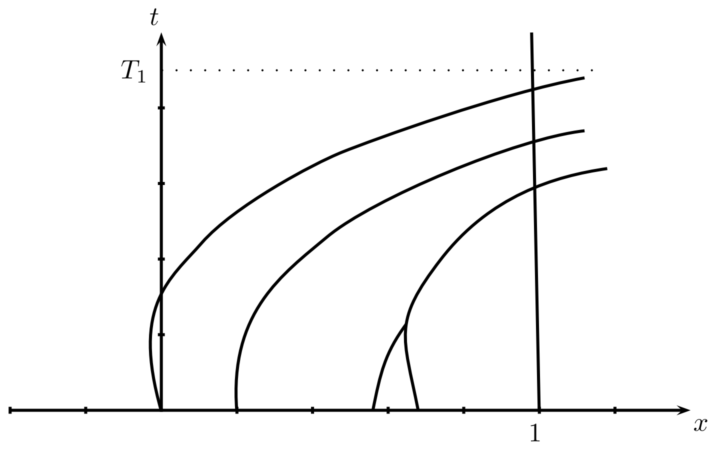


FIGURE 3 –

que la discontinuité partant de $x = 0$ à l'instant initial se trouve en un point à droite de $x = 1$ au temps T_1. Comme il s'agit du point de discontinuité le plus à gauche, la restriction de la solution $Y(T_1, .)$ à l'intervalle $(0, 1)$ sera constante (voir figure 3).

- La deuxième étape consiste à passer d'un état constant quelconque et à un autre état constant donné. Cette étape est triviale puisque pour des états constants en espace, l'équation (78) est réduite à une ODE.

- La dernière étape consiste à résoudre un problème rétrograde pour l'équation (78), on veut partir au temps final de l'état à atteindre et revenir à un instant antérieur à un état constant sur $(0, 1)$. Si Y_T est l'état à atteindre dans $BV(0, 1)$, on le prolonge sur $(1, +\infty)$ par la constante $Y_T(1^-)$. On utilise ensuite l'algorithme de suivi de fronts de manière rétrograde. L'objectif étant que le front de discontinuité partant de $x = 1$ au temps final T, doit passer à gauche de 0 à un instant antérieur. Mais on a cette fois-ci un problème : on ne peut pas résoudre de manière entropique la "collision" rétrograde de deux fronts de discontinuité. Il faut donc caractériser les états finaux pour lesquelles on arrive à faire sortir tous les fronts de discontinuité de $(0, +\infty)$ avant qu'ils ne se rencontrent (voire figure 4).

Au final, on peut se faire succéder les solutions obtenues aux trois étapes pour obtenir une solution entropique passant d'un état initial quelconque à un instant final vérifiant certaines conditions liées à la résolution du problème rétrograde. Cette stratégie est déjà celle adoptée dans l'article de Glass [57]. À ceci près que dans notre cas il n'y a qu'une seule famille d'onde, et que le contrôle g nous permet de modifier la dynamique des fronts de discontinuité.

Stabilisation asymptotique.
La théorie des caractéristiques généralisées de Dafermos [44] consiste à étudier les solutions faibles de l'équation (65) au sens de Filippov [55], avec $a(t, x) = f'(Y(t, x))$ et sachant que Y est BV en espace. A ce niveau de régularité on garde l'existence mais on perd l'unicité des solutions de (65). Cependant, grâce aux conditions d'entropie sur les solutions des lois de conservation de (18), on peut prouver l'existence de certaines caractéristiques privilégiées possédant de bonnes propriétés de régularité et qui permettent une analyse à posteriori des solutions entropiques de lois de conservation.

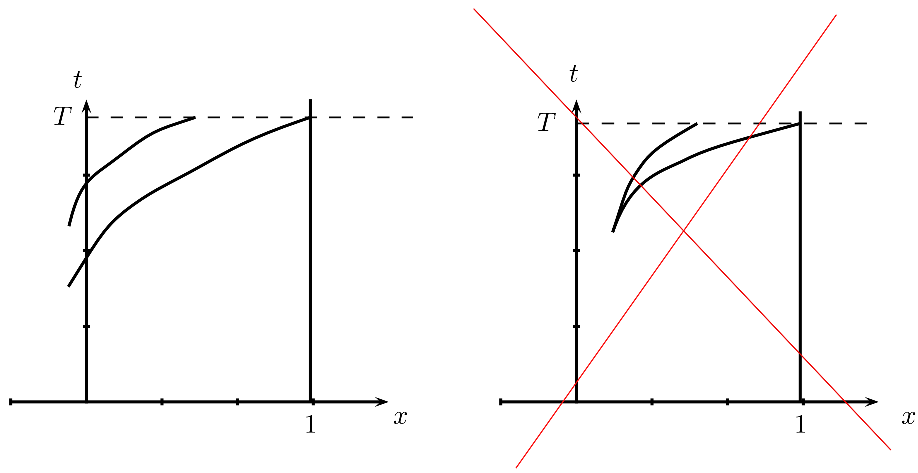


FIGURE 4 –

Pour stabiliser un état constant stationnaire $\bar{Y}$ de l'équation (78), on introduit une loi de retour sur les contrôles via :

$$g(Y(t,.)) = \frac{f'(\bar{u})}{2}||Y(t,.) - \bar{Y}||_{L^1(0,1)}, \tag{84}$$

et on choisit des données aux bords égales à $\bar{Y}$ quelque soit l'état $Y(t,.)$. Il faut d'abord montrer l'existence d'une solution du système en boucle fermée. Cela sera accompli via le théorème de point fixe Banach sur $g(Y(t,.))$ grâce aux bonnes propriétés de contraction du semi-groupe associé à une loi de conservation scalaire. L'analyse à posteriori des solutions est ensuite effectuée grâce aux caractéristiques généralisées. Il s'agit de courbes lipschitziennes γ vérifiant l'équation du flot (65) au sens suivant :

$$\dot{\gamma}(t) \in [f'(Y(t,\gamma(t)^+)), f'(Y(t,\gamma(t)^-))], \qquad dt\ p.p.. \tag{85}$$

La théorie de Filippov [55] assure qu'il existe toujours au moins une telle caractéristique passant par un point donné lorsque $Y(t,.) \in \mathrm{BV}$. Par contre il n'y a plus du tout de résultat d'unicité. Dafermos a montré qu'en fait, si Y est une solution entropique de (78) avec un flux f convexe, il y a unicité vers le futur de la caractéristique passant par (t,x) mais pas vers le passé. De plus, parmi les caractéristiques généralisées rétrogrades, certaines sont régulières et vérifient l'ODE (68) au sens classique. Ceci nous permettra de relier les valeurs de Y à différents points. La difficulté essentielle est alors de montrer que lorsqu'une telle caractéristique rencontre le bord $x = 0$ ou $x = 1$, l'ODE (68) fournit bien une valeur conforme à ce que laisse espérer les conditions aux bords associées à l'équation (78). Une fois ce type de résultat démontré, on verra qu'après un certain temps suffisamment long, le flot rétrograde de la solution entropique est tel que décrit dans la figure 1. Une estimation de ce temps, permettra alors d'obtenir des estimations uniformes assurant la stabilisation asymptotique.

4 Perspectives

Les travaux effectués durant cette thèse soulèvent de nouvelles questions que nous allons maintenant évoquer.

En ce qui concerne les résultats du chapitre 1, deux questions naturelles se posent dans le prolongement du travail présenté. Tout d'abord les conditions sur le bord proposées permettent-elles d'obtenir l'unicité de la solution faible dans l'espace $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right)\cap\mathrm{Lip}\left([0,T];H_0^1(0,1)\right)$ et pas seulement le résultat d'unicité forte-faible démontrée ici. Un deuxième point concerne le résultat de stabilisation asymptotique : peut-on trouver une loi qui stabilise les états stationnaires $u=\kappa$ comme dans le théorème 3 ou d'autres états mais cette fois-ci dans des espaces moins réguliers. Par exemple on pourrait espérer obtenir une stabilisation en norme $W^{2,p}$ avec des données initiales dans $W^{2,\infty}$ en s'appuyant sur les résultats de la partie 1.4 du chapitre 1. On pourrait également essayer d'adapter la méthode employée à d'autre équations de la même famille.

- Tout d'abord, l'équation de Degasperis-Procesi [52] :

$$u_t - u_{txx} + 2\kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{86}$$

  qui peut être mise sous la forme :

$$\begin{cases} y_t + uy_x = -3yu_x, \\ y - \frac{2\kappa}{3} = u - u_{xx}. \end{cases}$$

  Il semble raisonnable d'espérer pouvoir adapter les résultats obtenus pour l'équation de Camassa-Holm étant donnée la très forte ressemblance entre ces deux équations.

- On peut également s'intéresser à l'équation de Hunter-Saxton [71] qui est utilisée dans la modélisation des cristaux liquides.

$$(u_t + uu_x)_x = \frac{(u_x)^2}{2}.$$

  Celle-ci peut être mise sous la forme :

$$\begin{cases} y_t + uy_x = -y^2, \\ y = u_x, \end{cases}$$

  L'utilisation des méthodes employées sur Camassa-Holm semble ici nettement plus délicate. En effet on a un gain moindre de régularité de y vers u, et on a également une semi-linéarité explosive au second membre de l'équation de transport.

Il faut aussi mentionner que Alberto Bressan et Adrian Constantin ont introduit dans [18] une notion de solution plus faible que celle considérée dans notre étude. Ceci leur a permis de montrer l'existence et l'unicité d'une solution globale avec donnée initiale dans $H^1(\mathbb{R})$. Il pourrait être intéressant d'étudier les problèmes de contrôlabilité exacte et de stabilisation asymptotique dans ce cadre. Notons également que la même remarque peut être faite concernant l'équation d'Hunter-Saxton (voir l'article [19]). De plus à ce niveau de régularité, l'utilisation des caractéristiques généralisées devient pertinente, comme on peut le voir dans l'article de Dafermos [47], on peut donc peut être s'inspirer des méthodes du chapitre 3 pour le problème de la stabilisation asymptotique.

En ce qui concerne les résultats des chapitres 2 et 3, une premier problème naturel serait de remplacer le contrôle $g(t)$ dans (18) par des contrôles agissant non uniformément en espace. On peut essayer d'abord d'utiliser un contrôle distribué $1_{[a,b]}(x)g(t)$. Cependant, il semble probable qu'on ne puisse guère améliorer les résultats obtenus dans [69], [4]. Une manière plus judicieuse serait alors d'utiliser $V(x)g(t)$ où V est une fonction régulière et vérifiant par exemple $V(x) > 0$ sur $(0,1)$.

Des problèmes sans doute plus difficiles seraient d'adapter les méthodes utilisées pour les solutions entropiques des lois scalaires 1d à flux convexe à des cas plus généraux.

- Tout d'abord on peut s'intéresser aux solutions entropiques des lois scalaires 1d avec un flux non forcément convexe. L'algorithme de suivi de fronts employé dans le chapitre 2 se généralise à ce cas là, mais on génère plusieurs ondes en résolvant le problème de Riemann. De plus la correspondance entre l'état u et la vitesse $f'(u)$ devient nettement plus difficile à gérer. On pourrait également essayer d'utiliser la méthode de limite de viscosité évanescente telle qu'employée dans [62] et [75]. Les caractéristiques généralisées de Dafermos sont beaucoup moins efficaces dans cette situation et il faudrait dont trouver une autre méthode d'analyse à posteriori des solutions pour gérer les systèmes en boucle fermée.

- Un deuxième problème ouvert consiste à s'intéresser aux lois de conservation scalaires en dimension supérieure. Le problème mixte a déjà été étudié dans [15], mais il semble que les problèmes de contrôle n'ont pas encore été étudiés dans le cadre des solutions entropiques. Dans ce cadre là aussi l'algorithme de suivi de fronts se généralise [70], mais la géométrie du problème est forcément plus compliquée. Les méthodes par viscosité évanescentes sont là aussi dignes d'intérêt car on a des résultats de contrôlabilité exacte pour l'équation de Burgers visqueuse en 2-D [72].

- Une première étape pour généraliser les résultats du chapitre 2 aux systèmes hyperboliques de lois de conservation pourrait être de regarder les systèmes de Temple [97] pour lesquels les courbes de raréfactions et les courbes de chocs coïncident. C'est par exemple, ce qu'ont fait Ancona et Coclite dans [3] pour la contrôlabilité par le bord. Notons que le problème mixte pour les systèmes a été étudié dans [51], et l'algorithme de suivi de fronts adapté dans [1], [2].

- On peut également s'intéresser Euler-Poisson où l'équivalent du contrôle g des chapitres 2 et 3 apparaît naturellement. Le système s'écrit :

$$\begin{cases} \partial_t\rho + \partial_x m = 0, \quad \rho(0,.) = \rho_0, \quad \rho(t,0) = \rho_l(t), \quad \rho(t,L) = \rho_r(t), \\ \partial_t m + \partial_x(\dfrac{m^2}{\rho} + p(\rho)) = \rho\partial_x V - m, \quad m(0,.) = m_0, \ m(t,0) = m_l(t), \ m(t,L) = m_r(t), \\ -\partial^2_{xx} V = \rho, \quad V(t,0) = V_l(t), \quad V(t,L) = V_r(t), \end{cases} \tag{87}$$

et les contrôles sont les données aux bords ρ_l, ρ_r, m_l, m_r, V_l et V_r. Mais en prenant $g(t) = \frac{V_r(t)-V_l(t)}{L}$ on arrive à $V_x(t,.) = g(t) + A\rho(t,.)$ où A l'opérateur intégral défini par :

$$A\theta(x) = \int_0^x \theta(z)dz - \frac{1}{L}\int_0^L \int_0^y \theta(z)dzdy.$$

On se retrouve donc avec un système hyperbolique contrôlé par le bord et par un terme source agissant uniformément en espace. On pourra essayer d'utiliser la méthode d'extension comme dans le chapitre 2 pour résoudre le problème direct. La difficulté essentielle va consister à trouver à l'instar de ce qui est fait dans [57], de bonnes conditions suffisantes sur l'état final pour pouvoir résoudre le problème rétrograde mais également à construire une variante de l'algorithme de suivi de front pour résoudre le problème rétrograde tout en prenant en compte l'influence géométrique du contrôle g.

Enfin la formulation en vorticité de l'équation d'Euler en dimension 3 peut être vue comme une généralisation 3D de la formulation (5) de Camassa-Holm. En effet si U est la vitesse du fluide et ω sa vorticité, on peut écrire l'équation d'Euler comme une équation de transport avec un second membre :

$$\partial_t \omega + (U.\nabla)\omega = (\omega.\nabla)U, \tag{88}$$

couplée avec une équation stationnaire elliptique :

$$\begin{cases} \mathrm{rot} U = \omega, \\ \mathrm{div} U = 0. \end{cases} \tag{89}$$

Or rappelons que si pour l'équation d'Euler, le problème de la contrôlabilité exacte a été résolu en dimension 2 par J.-M. Coron [37] et en dimension 3 par O. Glass [60], le problème de la stabilisation asymptotique, lui, n'a été résolu qu'en dimension 2, par J.-M. Coron pour les domaines simplement connexes [38] et par O. Glass pour les domaines généraux.

Chapitre 1

Problème mixte et stabilisation asymptotique par une loi de retour stationnaire pour l'équation de Camassa-Holm sur un intervalle borné.

Abstract. We investigate the non-homogeneous initial boundary value problem for the Camassa-Holm equation on an interval. We provide a local in time existence theorem and a weak strong uniqueness result. Next we establish a result on the global asymptotic stabilization problem by means of a boundary feedback law.

1.1 Introduction

1.1.1 Origins of the equation and presentation of the problems

This article presents results concerning the initial boundary value problem and the possibility of asymptotic stabilization of the Camassa-Holm equation on a compact interval by means of a stationary feedback law acting on the boundary. The Camassa-Holm equation reads as follows (with κ a real constant):

$$\partial_t v - \partial^3_{txx} v + 2\kappa.\partial_x v + 3v.\partial_x v = 2\partial_x v.\partial^2_{xx} v + v.\partial^3_{xxx} v \text{ for } (t,x) \in [0,T] \times [0,1]. \quad (1.1)$$

The Camassa-Holm equation describes one-dimensional surface waves at a free surface of shallow water under the influence of gravity. Here $v(t,x)$ represents the fluid velocity at time t and position x. It is interesting to note that according to [22], it can equally represents the water elevation.

Equation (1.1) was first introduced by Fokas and Fuchssteiner [56] as a bi-Hamiltonian model, and was derived later as a water wave model by Camassa and Holm [22]. It turns out that this equation was also obtained as a model for propagating waves in cylindrical elastic rods, see Dai [48].

Equation (1.1) shares many features with the KdV equation, see [73]. It is bi-Hamiltonian, completely integrable, and admits soliton solutions see [22, 29, 32, 56, 76]. However, it can also model breaking waves, in fact in $H^s(\mathbb{T})$ ($s > \frac{3}{2}$) the solution generally develops singularity in finite time, see [26, 27, 28].

The Cauchy problem of (1.1) has been investigated in great details both on the torus and on the

real line, see [7, 23, 30, 49, 50, 67, 79, 98]. On the other hand, the study of the initial boundary value problem is much less complete, the homogeneous case was treated in [53] and in a more general setting in [54]. Finally a special case of the inhomogeneous case is considered in [101] (the boundary condition is that there is a constant C such that $\forall t \geq 0$ we have $v(t,x) \underset{|x|\to+\infty}{\to} C$).
The first part of this article will be devoted to the proofs of a local in time existence theorem and of a weak-strong uniqueness result for the initial boundary value problem of (1.1).
To explain our boundary formulation of (1.1), let us first remark that (1.1) is equivalent to the system:

$$\begin{cases} \partial_t y + v.\partial_x y = -2y.\partial_x v, \\ y - \kappa = (1-\partial^2_{xx})v. \end{cases} \tag{1.2}$$

This formulation of (1.1) and the vorticity formulation of the two dimensional Euler equation for incompressible perfect fluids (U is the speed and ω its vorticity) share similarities:

$$\begin{cases} \partial_t \omega + (U.\nabla)\omega = 0, \\ \text{div } U = 0, \\ \text{curl } U = \omega. \end{cases} \tag{1.3}$$

In both (1.2) and (1.3) there is a coupling between a transport equation and a stationary elliptic one. The initial boundary value problem for the two dimensional incompressible Euler equation was treated by Yudovitch in [99], where he showed that the problem is well-posed in a classical sense with strong solutions if one prescribes the initial velocity or vorticity, the normal velocity on the boundary and also the vorticity of the fluid on the parts of the boundary where fluid enters.
Similarly we will study the initial boundary value problem of (1.2) with v prescribed on the boundary, and y prescribed at time 0 and on the parts of the boundary where fluid enters.

Remark 1. *Note that* (1.2) *is even more similar to the vorticity formulation of the three dimensional incompressible Euler equation which reads:*

$$\begin{cases} \partial_t \omega + (U.\nabla)\omega = (\omega.\nabla)U \\ div\ U = 0 \\ curl\ U = \omega \end{cases} \tag{1.4}$$

because here we have a stretching term $(\omega.\nabla)U$ similar to the term $-2y\,\partial_x v$ in (1.2)*. Kazhikov has studied the local in time initial boundary value problem in three dimensions see [74]. However the Euler equation is much less understood in three dimensions. For example it is still unknown whether a singularity may appear in finite time, see [84]. Furthermore the asymptotic stabilization problem is still open for the three dimensional incompressible Euler equation which is not the case in two dimensions thanks to the papers of Coron [38] and Glass [59].*

In the second part of the article we will investigate equation (1.1) from the perspective of control theory. For a general control system

$$\begin{cases} \dot{x} = f(x,u), \\ x(t_0) = x_0, \end{cases} \tag{1.5}$$

(x being the state of the system and u the so called control), we can consider two classical problems among others in control theory.

1. First the exact controllability problem which asks, given two states x_0 and x_1 and a time T to find a certain function $u(t)$ such that the solution to (1.5) satisfies $x(T) = x_1$.
2. If $f(0,0) = 0$, the problem of asymptotic stabilization by a stationary feedback law asks to find a function $u(x)$, such that for any state x_0 a solution $x(t)$ to

$$\begin{cases} \dot{x}(t) = f(x(t), u(x(t))), \\ x(t_0) = x_0 \end{cases} \tag{1.6}$$

is global, satisfies $x(t) \underset{t\to+\infty}{\to} 0$ and also

$$\forall R > 0,\ \exists r > 0 \text{ such that } ||x_0|| \leq r \quad \Rightarrow \quad \forall t \in \mathbb{R},\ ||x(t)|| \leq R. \tag{1.7}$$

It may seem that if we have controllability, the asymptotic stabilization property is weaker. Indeed for any initial state x_0, we can find T and $u(t)$ such that the solution to (1.5) satisfies $x(T) = 0$ in this way we stabilize 0 in finite time. However this control suffers from a lack of robustness with respect to perturbation. Indeed with any error on the model, or on the initial state, the state at time T will only be approximately 0. This can be disastrous if $x = 0$ is unstable for the equation $\dot{x} = f(x, 0)$. This motivates the problem of asymptotic stabilization by a stationary feedback law which is clearly more robust. In fact in finite dimension, it automatically provides a Lyapunov function.

Concerning the Camassa-Holm equation, O. Glass provided in [58] the first results for the controllability and stabilization. More precisely he considered:

$$\partial_t v - \partial^3_{txx} v + 2\kappa.\partial_x v + 3v.\partial_x v = 2\partial_x v.\partial^2_{xx} v + v.\partial^3_{xxx} v + g(t,x)\mathbf{1}_\omega(x) \text{ for } (t,x) \in [0,T] \times \mathbb{T}, \tag{1.8}$$

where the control is the function g, and ω is a nonempty open subset of the torus $\mathbb{T}$. He proved that for any time $T > 0$ we have exact controllability in $H^s(\mathbb{T})$ ($s > \frac{3}{2}$), and also proposed a stationary feedback law $g : H^2(\mathbb{T}) \to H^{-1}(\omega)$ that stabilizes the state $v = -\kappa$ in $H^2(\mathbb{T})$. We will consider those problems, but in our case the control will be the boundary values of v and y. Since $[0,1]$ can be seen as $\mathbb{T} \setminus \omega$ the result of Glass on exact controllability by a distributed term on the torus implies a controllability result by boundary terms as soon as the initial boundary value problem makes sense, which will be the case by the end of the first part of this article (we also need enough regularity on the solution).

Therefore we will only investigate the asymptotic stabilization by a stationary feedback law acting on the boundary of (1.1). This time again we will consider the analogy with the asymptotic stabilization of the two dimensional Euler equation of incompressible fluids result by Coron [38] for a simply connected domain and Glass [59] for a general domain. It should be remarked that in three dimension the problem of asymptotic stabilization is still open. In both cases one of the main difficulty is that the linearized system around the equilibrium (which are $(y,v) = (0,-\kappa)$ for (1.2) and $(\omega, U) = (0,0)$ for (1.3)) is not stabilizable, so we will use the so called return method introduced by Coron in [33]. Since the evolution equation of (1.2) is on y, it will be much easier to work if we consider y and not v to be the state of the system.

1.1.2 Results

We begin with a general remark that will be used many times later.

Remark 2. *Changing $v(t,x)$ in $-v(t,1-x)$ and $y(t,x)$ in $-y(t,1-x)$ we change κ into $-\kappa$, therefore from now on we will suppose that $\kappa \leq 0$ (this choice is more convenient for the stabilization part).*

Let T be a positive number. In the following we take $\Omega_T = [0,T] \times [0,1]$. Let v_l and v_r be in $\mathcal{C}^0([0,T],\mathbb{R})$ and $y_0 \in L^\infty(0,1)$. We set

$$\Gamma_l = \{t \in [0,T] \mid v_l(t) > 0\} \text{ and } \Gamma_r = \{t \in [0,T] \mid v_r(t) < 0\}.$$

In the following, we will always suppose that the sets

$$P_l = \{t \in [0,T] \mid v_l(t) = 0\} \text{ and } P_r = \{t \in [0,T] \mid v_r(t) = 0\} \tag{1.9}$$

have a finite number of connected components. Finally let $y_l \in L^\infty(\Gamma_l)$ and $y_r \in L^\infty(\Gamma_r)$. The functions v_l, v_r, y_l and y_r will be the boundary values for the equation and y_0 is the initial data. Let now $\mathcal{A}$ be the auxiliary function which lifts the boundary values v_l and v_r and is defined by:

$$\begin{cases}(1-\partial_{xx}^2)\mathcal{A}(t,x) = 0, \ \forall (t,x) \in \Omega_T, \\ \mathcal{A}(t,0) = v_l(t), \quad \mathcal{A}(t,1) = v_r(t), \quad \forall t \in [0,T].\end{cases} \tag{1.10}$$

Setting $v = u + \mathcal{A}$, we can further rewrite the system (1.2) as:

$$\begin{cases}y(t,x) - \kappa = (1-\partial_{xx}^2)u(t,x), \ dx \ , \\ u(t,0) = u(t,1) = 0, \ dt \ a.e.,\end{cases} \tag{1.11}$$

$$\begin{cases}\partial_t y + (u+\mathcal{A}).\partial_x y = -2y.\partial_x(u+\mathcal{A}), \\ y(0,.) = y_0, \ y(.,0)_{|\Gamma_l} = y_l \text{ and } y(.,1)_{|\Gamma_r} = y_r.\end{cases} \tag{1.12}$$

The meaning of being a solution to (1.11)-(1.12) will be specified later but we can already say that we will have $u \in L^\infty((0,T);\mathrm{Lip}([0,1]))$ and $y \in L^\infty(\Omega_T)$. In the first part of this article, we will be interested in the initial boundary value problem on the interval for the system (1.11)-(1.12). We will first prove a local in time existence theorem:

Theorem 1. *For $\tilde{T} > 0$, we consider $v_l, v_r \in \mathcal{C}^0([0,\tilde{T}])$ such that the sets P_l and P_r have only a finite number of connected components. Let $y_0 \in L^\infty(0,1)$, $y_l \in L^\infty(\Gamma_l)$ and $y_r \in L^\infty(\Gamma_r)$. There exist $T > 0$, and (u,y) a weak solution of the system* (1.11)-(1.12) *with $u \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right)$ and $y \in L^\infty(\Omega_T)$. Moreover any such solution u is in fact in $\mathcal{C}^0([0,T];W^{2,p}(0,1)) \cap C^1([0,1];W_0^{1,p}(0,1))$, $\forall p < +\infty$. Furthermore the existence time of a maximal solution is larger than $\min(\tilde{T},T^*)$, with*

$$T^* = \max_{\beta>0}\left(\frac{\ln(1+\beta/C_0)}{2(C_1+(2+\sinh(1))(C_0+|\kappa|+\beta))}\right) \tag{1.13}$$

$$C_0 = \max\left(||y_0||_{L^\infty(0,1)}, ||y_l||_{L^\infty(\Gamma_l)}, ||y_r||_{L^\infty(\Gamma_l)}\right), \tag{1.14}$$

$$C_1 = \frac{1}{\tanh(1)}.(||v_r||_{L^\infty(0,T)} + ||v_l||_{L^\infty(0,T)}). \tag{1.15}$$

In a second step, we will show a weak-strong uniqueness property:

Theorem 2. *Let $(u,y) \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right) \times L^\infty([0,T];\mathrm{Lip}([0,1]))$ be a weak solution of* (1.11) *and* (1.12) *then it is unique in $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \times L^\infty(\Omega_T)$.*

In the second part of the paper, we will be interested in the asymptotic stabilization of the system (1.1) by a boundary feedback law. Let $A_l > 2.\sinh(1)$, $A_r > A_l.\cosh(1) + \sinh(2)$, $M > 0$ and $T > 0$. Our feedback law for (1.2) reads:

$$y \in \mathcal{C}^0([0,1]) \mapsto \begin{cases}v_l(y) = A_l.||y||_{\mathcal{C}^0([0,1])} - \kappa \\ v_r(y) = A_r.||y||_{\mathcal{C}^0([0,1])} - \kappa \\ \dot{y}_l(t) + M.y_l(t) = 0\end{cases}. \tag{1.16}$$

This allows us to get the following theorem:

Theorem 3. *For any* $y_0 \in \mathcal{C}^0([0,1])$ *there exists* $(y,v) \in \mathcal{C}^0(\Omega_T) \times \mathcal{C}^0([0,T],\mathcal{C}^2([0,1]))$ *a weak solution of* (1.2) *and* (1.16) *satisfying*

$$\forall x \in [0,1], \quad y(0,x) = y_0(x). \tag{1.17}$$

Furthermore any maximal solution of (1.2), (1.16) *and* (1.17) *is global, and if we let*

$$c = \min(A_l - 2.\sinh(1), \frac{A_r - A_l.\cosh(1) - \sinh(2)}{\sinh(1)}) \text{ and } \tau = \frac{1}{M}.\ln(\frac{2.c.||y_0||_{\mathcal{C}^0([0,1])}}{M})$$

then we have:

$$\forall t \geq \tau \qquad ||y(t,.)||_{\mathcal{C}^0([0,1])} \leq \frac{M}{2c}.\frac{1}{1+M(t-\tau)}.$$

1.2 Initial boundary value problem

We first define what we mean by a weak solution to (1.12). Our test functions will be in the space:

$$Adm(\Omega_T) = \{\psi \in \mathcal{C}^1(\Omega_T) \mid \psi(t,x) = 0 \text{ on } [0,T]\backslash\Gamma_l \times \{0\} \cup [0,T]\backslash\Gamma_r \times \{0\} \cup \{T\} \times [0,1]\}. \tag{1.18}$$

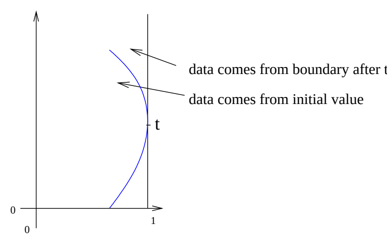
Definition 3. *When* $u \in L^\infty((0,T);\mathrm{Lip}([0,1]))$, *a function* $y \in L^\infty(\Omega_T)$ *is a weak solution to* (1.12) *if* $\forall \psi \in Adm(\Omega_T)$:

$$\begin{aligned}\iint_{\Omega_T} y(\partial_t\psi + (u+\mathcal{A})\partial_x\psi - \partial_x(u+\mathcal{A})\psi)dtdx = &-\int_0^1 y_0(x)\psi(0,x)dx \\ &+ \int_0^T (\psi(t,1)v_r(t)y_r(t) - \psi(t,0)v_l(t)y_l(t))dt.\end{aligned}$$

Remark 3. *It is obvious that* $\mathcal{C}_0^1(\Omega_T) \subset Adm(\Omega_T)$ *therefore a weak solution to* (1.12) *is also a solution to* (1.12) *in the distribution sense. And it is then clear that a regular weak solution is a classical solution.*

1.2.1 Strategy

In this part we will prove Theorems 1 and 2. Let us first explain the general strategy.
We want to solve (1.11) and (1.12). Equation (1.11) is a linear elliptic equation, and with u fixed (1.12) is a linear transport equation in y, with boundary data. Even when the flow is regular enough (and it will be in our case) to use the method of characteristics to solve the equation, singularity will generally appear, no matter how smooth the initial and boundary datas are, because of the boundary.



It is therefore useful to deal with weak solution of (1.12) belonging to $L^\infty(\Omega_T)$. This is done in the appendix.
Once we know how to deal with each equation separately and have appropriate linear estimates, we use a fixed point strategy. It is interesting to remark that Yudovitch dealt with the two dimensional incompressible Euler equation with non-homogeneous boundary conditions in a similar way. However with y only essentially bounded, we cannot easily estimate the difference of two couples (u_1, y_1) and (u_2, y_2), therefore we will rather use a compactness argument and a Schauder fixed point instead of a Banach fixed point. The auxiliary function $\mathcal{A}$ may be less regular in time than u and this is why we will be able to transfer the time regularity of y on u. We will only prove a weak-strong uniqueness property, for the same reason that prevented us from using a Banach fixed point theorem.
Therefore in Subsection 1.2.2 we will define precisely the fixed point operator $\mathcal{F}$ and study some of its properties. In Subsection 1.2.3 we will precise the domain on which we will apply Schauder's fixed point theorem, we will prove the continuity of $\mathcal{F}$ in Subsection 1.2.4 and also study the additional properties of a fixed point. Finally in Subsection 1.2.5 we will prove the weak-strong uniqueness property.

1.2.2 The operator $\mathcal{F}$

The operator $\mathcal{F}$ is obtained as follows. Given u in $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right)$ we will define y to be the solution of (1.12), and once we have y in $L^\infty(\Omega_T)$, we introduce $\tilde{u}$ solution of

$$(1-\partial_{xx}^2)\tilde{u} = y - \kappa. \tag{1.19}$$

Then $\mathcal{F}$ is defined as the operator associating $\tilde{u}$ to u.
Now let us describe the auxiliary function $\mathcal{A}$ once and for all.

Proposition 1. *The function $\mathcal{A}$ defined by* (1.10) *satisfies:*

$$\forall (t,x) \in \Omega_T \qquad \mathcal{A}(t,x) = \frac{1}{\sinh(1)}.(\sinh(x).v_r(t) + \sinh(1-x).v_l(t))$$

$$\mathcal{A} \in \mathcal{C}^0([0,T];\mathcal{C}^\infty([0,1])),$$

$$\textit{and hence } ||\mathcal{A}||_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \le \frac{\cosh(1)}{\sinh(1)}.(||v_r||_{L^\infty(0,T)} + ||v_l||_{L^\infty(0,T)}).$$

As in Subsection 1.4.1, for a function $u \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right)$ we consider ϕ the flow of $u+\mathcal{A}$. For $(t,x) \in \Omega_T$, $\phi(.,t,x)$ is defined on a set $[e(t,x), h(t,x)]$, here $e(t,x)$ is basically the entrance time in Ω_T of the characteristic curve going through (t,x).

Lemma 1. *The flow ϕ satisfies the following properties:*

1. *ϕ is $\mathcal{C}^1$ with the following partial derivatives*

$$\partial_1\phi(s,t,x) = (u+\mathcal{A})(s,\phi(s,t,x)),$$

$$\partial_2\phi(s,t,x) = -(u+\mathcal{A})(t,x).\exp\left(\int_t^s \partial_x(u+\mathcal{A})(r,\phi(r,t,x))dr\right),$$

$$\partial_3\phi(s,t,x) = \exp\left(\int_t^s \partial_x(u+\mathcal{A})(r,\phi(r,t,x))dr\right),$$

2. $\forall j \in \{1,2,3\},\ ||\partial_j \phi||_{\mathcal{C}^0} \leq (1+||u+\mathcal{A}||_{\mathcal{C}^0(\Omega_T)})e^{T.||\partial_x(u+\mathcal{A})||_{\mathcal{C}^0(\Omega_T)}}$,
3. *if* $e(t,x) > 0$ *then* $\phi(e(t,x),t,x) \in \{0,1\}$,
4. *if* $h(t,x) < T$ *then* $\phi(h(t,x),t,x) \in \{0,1\}$.

We introduce a partition of Ω_T, which allows us to distinguish the different influence zones in Ω_T.

Definition 4. *Let*

- $P = \big\{(t,x) \in \Omega_T \mid \exists s \in [e(t,x), h(t,x)]$ *for which* $(\phi(s,t,x) = 0$ *and* $v_l(s) = 0)$

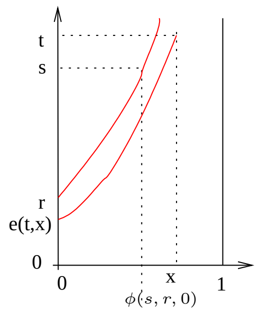
  or $(\phi(s,t,x) = 1$ *and* $v_r(s) = 0)\big\} \cup \{\phi(s,0,0) \mid s \leq h(0,0)\} \cup \{\phi(s,0,1) \mid s \leq h(0,1)\}$,
- $I = \{(t,x) \in \Omega_T \setminus P \mid e(t,x) = 0\}$,
- $L = \{(t,x) \in \Omega_T \setminus P \mid e(t,x) > 0$ *and* $\phi(e(t,x)t,x) = 0\}$,
- $R = \{(t,x) \in \Omega_T \setminus P \mid e(t,x) > 0$ *and* $\phi(e(t,x)t,x) = 1\}$.

Remark 4. *The set P is constituted of the problematic points. Indeed those points belong to the characteristics tangent to the boundary, which are precisely the singular points of e and h.*

Proposition 2. *We have the following properties.*

1. *The sets P, I, L and R constitute a partition of Ω_T.*
2. *The set P is negligible and each spatial section of P is negligible for the 1d lebesgue measure.*
3. *The function e is $\mathcal{C}^1$ on $L \cup R \cup I$.*
4. *If $(t,x) \in L$ then $e(t,x) \in \Gamma_l$ and if $(t,x) \in R$ then $e(t,x) \in \Gamma_r$.*
5. *All those sets are invariant by the flow ϕ.*
6. *If $(t,x) \in L$ then $\forall \tilde{x} \in [0,x],\ (t,\tilde{x}) \in P \cup L$, if $(t,x) \in R$ then $\forall \tilde{x} \in [x,1],\ (t,\tilde{x}) \in P \cup R$ and if $(t,x) \in I$ and $(t,x+x') \in I$ then $\forall \tilde{x} \in [x,x+x'],\ (t,\tilde{x}) \in P \cup I$.*

Proof. The points 1, 4, 5, 6 are easy. The second point is true because for any $t \in [0,T]$ the set $\{(t,x) \mid x \in [0,1]\} \cap P\}$ is injected in the set of connected components of P_l and P_r, so it is countable and therefore 1d negligible. It implies that P itself is 2d negligible.
And the third point is shown in Proposition 9



□

For $u \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right)$, we define $y \in L^\infty(\Omega_T)$ by:

- if $(t,x) \in I$ $\quad y(t,x) = y_0(\phi(0,t,x)).\exp\left(-2\int_0^t \partial_x(u+\mathcal{A})(s,\phi(s,t,x))ds\right)$,
- if $(t,x) \in L$ $\quad y(t,x) = y_l(e(t,x)).\exp\left(-2\int_{e(t,x)}^t \partial_x(u+\mathcal{A})(s,\phi(s,t,x))ds\right)$,
- if $(t,x) \in R$ $\quad y(t,x) = y_r(e(t,x)).\exp\left(-2\int_{e(t,x)}^t \partial_x(u+\mathcal{A})(s,\phi(s,t,x))ds\right)$.

And we have:

1. the function y is the unique weak solution of (1.12) in the sense of definition 3, thanks to Theorem 6 and Proposition 13(which can be applied because $u \in \mathcal{C}^0(\Omega_T)$ and $\partial_x u \in \mathcal{C}^0(\Omega_T)$),
2. since $y \in L^\infty(\Omega_T)$ and satisfies (1.12), we immediately get $y \in W^{1,\infty}(0,T,H^{-1}(0,1))$,
3. the function y satisfies the estimates:
$$||y||_{L^\infty(\Omega_T)} \leq \max(||y_0||_{L^\infty}, ||y_l||_{L^\infty}, ||y_r||_{L^\infty}) \exp\left(2T\left(||\partial_x u||_{L^\infty(\Omega_T)} + ||\partial_x\mathcal{A}||_{L^\infty(\Omega_T)}\right)\right), \tag{1.20}$$
$$\begin{aligned}||\partial_t y||_{L^\infty((0,T),H^{-1})} \leq 3.\max&\left(||y_0||_{L^\infty(0,1)}, ||y_l||_{L^\infty(\Gamma_l)}, ||y_r||_{L^\infty(\Gamma_l)}\right)\\ &\times \exp\left(2T\left(||\partial_x u||_{L^\infty(\Omega_T)} + ||\partial_x\mathcal{A}||_{L^\infty(\Omega_T)}\right)\right)\\ &\times\left(||u||_{L^\infty((0,T);\mathrm{Lip}([0,1]))} + ||\mathcal{A}||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}\right),\end{aligned} \tag{1.21}$$
4. if $(t,x) \in I \cup L \cup R$ and if $(s,s') \in [e(t,x),h(t,x)]^2$, one has the following property:
$$y(s,\phi(s,t,x)) = y(s',\phi(s',t,x)).\exp\left(-2\int_{s'}^s \partial_x(u+\mathcal{A})(r,\phi(r,t,x))dr\right).$$

We can now focus on the elliptic equation (1.11).

Lemma 2. *There exists a unique $\tilde{u} \in L^\infty\left((0,T),H_0^1(0,1)\right)$ such that*
$$\forall t \in (0,T),\ y(t,.) - \kappa = (1-\partial_{xx}^2)\tilde{u}(t,.) \quad \text{in } \mathcal{D}'(0,1).$$
Furthermore we can see that $\tilde{u}$ belongs to $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left((0,T),H_0^1(0,1)\right)$ since $y \in L^\infty(\Omega_T) \cap \mathrm{Lip}([0,T];H^{-1}(0,1))$. Moreover we have the bounds:
$$||\tilde{u}||_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \leq (1+2\sinh(1)).(|\kappa| + ||y||_{L^\infty(\Omega_T)}), \tag{1.22}$$
$$||\partial_t\tilde{u}||_{L^\infty\left((0,T);H_0^1(0,1)\right)} \leq ||\partial_t y||_{L^\infty((0,T),H^{-1}(0,1))}. \tag{1.23}$$

Proof. In the first point, the constant comes from:
$$\tilde{u}(t,x) = \int_0^x \sinh(x-\tilde{x}).(\kappa - y(t,\tilde{x}))d\tilde{x} - \frac{\sinh(x)}{\sinh(1)}.\int_0^1 \sinh(\tilde{x}).(\kappa - y(t,\tilde{x}))d\tilde{x}. \tag{1.24}$$
The second point is classical □

Finally we can define $\mathcal{F}$ by:
$$\forall u \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right),$$
$$\mathcal{F}(u) = \tilde{u} \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right). \tag{1.25}$$

We now introduce a domain for the operator $\mathcal{F}$.

1.2.3 The domain

Let B_0 and B_1 be positive numbers, then we set:

$$\mathcal{C}_{B_0,B_1,T} = \{u \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H_0^1(0,1)\right) \mid \text{such that both} \\ ||u||_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \leq B_0 \text{ and } ||u||_{\mathrm{Lip}\left([0,T];H_0^1(0,1)\right)} \leq B_1\}. \quad (1.26)$$

Obviously $\mathcal{C}_{B_0,B_1,T}$ is convex. We will endow $\mathcal{C}_{B_0,B_1,T}$ with the norm $||.||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$.

Lemma 3. *There exist positive numbers B_0, B_1, T, such that $\mathcal{F}$ maps $\mathcal{C}_{B_0,B_1,T}$ into itself.*

Proof. Let us first introduce the two following constants depending only on the initial and boundary conditions

$$C_0 = \max\left(||y_0||_{L^\infty(0,1)}, ||y_l||_{L^\infty(\Gamma_l)}, ||y_r||_{L^\infty(\Gamma_r)}\right),$$
$$C_1 = \frac{\cosh(1)}{\sinh(1)}.(||v_r||_{L^\infty(0,T)} + ||v_l||_{L^\infty(0,T)}).$$

Estimates (1.20), (1.21), (1.22) and (1.23) on y and $\tilde{u}$ now read:

$$||y||_{L^\infty(\Omega_T)} \leq C_0.\exp\left(2T\left(||\partial_x u||_{L^\infty(\Omega_T)} + C_1\right)\right),$$
$$||\tilde{u}||_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \leq (1 + 2\sinh(1)).(|\kappa| + ||y||_{L^\infty(\Omega_T)}),$$
$$||\partial_t y||_{L^\infty((0,T);H^{-1}(0,1))} \leq 3.C_0.\exp\left(2T\left(||\partial_x u||_{L^\infty(\Omega_T)} + C_1\right)\right).\left(||u||_{L^\infty((0,T);\mathrm{Lip}([0,1]))} + C_1\right),$$
$$||\partial_t \tilde{u}||_{L^\infty\left((0,T);H_0^1(0,1)\right)} \leq ||\partial_t y||_{L^\infty((0,T);H^{-1}(0,1))}.$$

Combining those estimates we get:

$$||\tilde{u}||_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \leq (1 + 2\sinh(1)).\left(|\kappa| + C_0.\exp\left(2T\left(||\partial_x u||_{L^\infty(\Omega_T)} + C_1\right)\right)\right),$$
$$||\partial_t \tilde{u}||_{L^\infty\left((0,T);H_0^1(0,1)\right)} \leq 3.C_0.\exp\left(2T\left(||\partial_x u||_{L^\infty(\Omega_T)} + C_1\right)\right).\left(||u||_{L^\infty((0,T);\mathrm{Lip}([0,1]))} + C_1\right).$$

Now if $u \in \mathcal{C}_{B_0,B_1,T}$ we have

$$||\tilde{u}||_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \leq (1 + 2\sinh(1)).\left(|\kappa| + C_0.\exp\left(2T\left(B_0 + C_1\right)\right)\right),$$
$$||\partial_t \tilde{u}||_{L^\infty\left((0,T);H_0^1(0,1)\right)} \leq 3.C_0.\exp\left(2T\left(B_0 + C_1\right)\right).\left(B_0 + C_1\right).$$

Finally, to obtain $\tilde{u} \in \mathcal{C}_{B_0,B_1,T}$ it is sufficient that

$$(1 + 2\sinh(1)).\left(|\kappa| + C_0.\exp\left(2T\left(B_0 + C_1\right)\right)\right) \leq B_0$$
$$\text{and } B_0 + 3.C_0.\exp\left(2T\left(B_0 + C_1\right)\right).\left(B_0 + C_1\right) \leq B_1.$$

Once we have chosen T and B_0, it is easy to choose B_1 to satisfy the second inequality. For the first one we just choose B_0 sufficiently large and then T close to 0 . More precisely:

$$B_0 > (1 + 2\sinh(1)).(|\kappa| + C_0),$$

$$T \leq \frac{\ln\left(\frac{B_0}{1+2\sinh(1)} - |\kappa|\right) - \ln(C_0)}{2(B_0 + C_1)}.$$

It only to remains to maximize the bound of T to get the minimum existence, and with $\frac{B_0}{1+2\sinh(1)} = |\kappa| + C_0 + \beta$ we get the result announced. □

Let us now prove the compactness of the domain.

Proposition 3. *$\mathcal{C}_{B_0,B_1,T}$ is compact with respect to the norm $||.||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$.*

Proof. The fact that $\mathcal{C}_{B_0,B_1,T}$ is closed in $L^\infty((0,T);\mathrm{Lip}([0,1]))$ follows from the weak* compactness of the domain in $L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right)$ and in $\mathrm{Lip}\left([0,T];H_0^1(0,1)\right)$, and a classical use of a limit uniqueness.
We now show the relative compactness of $\mathcal{C}_{B_0,B_1,T}$ in $L^\infty((0,T);\mathrm{Lip}([0,1]))$. Let (u_n) be a sequence of $\mathcal{C}_{B_0,B_1,T}$. Since $H_0^1(0,1) \hookrightarrow \mathcal{C}^{\frac{1}{2}}([0,1])$ we can extract by Ascoli's theorem a subsequence $(u_{n'})$ converging in $L^\infty(\Omega_T)$. But since we have

$$\forall u \in L^\infty((0,T);W^{2,\infty}(0,1)),\ ||\partial_x u||_{L^\infty(\Omega_T)} \leq 2.\sqrt{||u||_{L^\infty(\Omega_T)}.||\partial^2_{xx}u||_{L^\infty(\Omega_T)}},$$

we can conclude that $(u_{n'})$ actually converges in $L^\infty((0,T);\mathrm{Lip}([0,1]))$. □

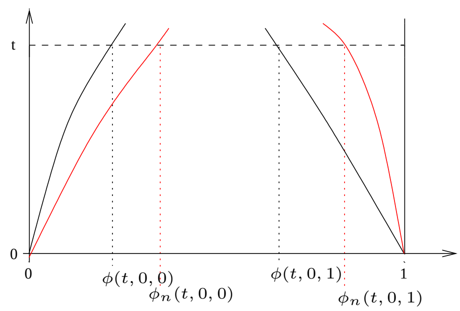
Before applying Schauder's fixed point theorem, it only remains to prove the continuity of the operator $\mathcal{F}$.

1.2.4 Continuity of $\mathcal{F}$ and properties of the fixed points

We begin with a result about the continuity of $\mathcal{F}$.

Proposition 4. *The operator $\mathcal{F} : \mathcal{C}_{B_0,B_1,T} \to \mathcal{C}_{B_0,B_1,T}$ is continuous with respect to the norm $||.||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$.*

Proof. Let us take a sequence (u_n) which tends to u with respect to $||.||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$. We call $\tilde{u}_n = \mathcal{F}(u_n)$ and $\tilde{u} = \mathcal{F}(u)$. Denote by ϕ_n the flow of $u_n + \mathcal{A}$ and ϕ the flow of $u + \mathcal{A}$. Thanks to Proposition 10, we have that $\phi_n \xrightarrow[n\to+\infty]{} \phi$ locally in $\mathcal{C}^1$. Let us show first that $||y_n(t,.) - y(t,.)||_{L^1(0,1)} \xrightarrow[n\to 0]{} 0 \quad dt$ a.e..
Let $t \in [0,T]$, having supposed that P_l and P_r have only a finite number of connected components (see (1.9)), we can assume, reducing t if necessary that v_l and v_r do not change sign on $[0,t]$. We will focus on the case where $v_l \geq 0$ and $v_r \leq 0$, the situation:



The characteristics of ϕ_n and ϕ may or may not cross before time t, but we are only interested in their relative positions at time t, which here correspond to $\phi(t,0,0) \leq \phi_n(t,0,0) \leq \phi(t,0,1) \leq \phi_n(t,0,1)$. The other cases are proved in the same way. We first point out that since $u_n \in \mathcal{C}_{B_0,B_1,T}$

we have a bound for (y_n) in $L^\infty(\Omega_T)$. Now

$$\begin{aligned}\int_0^1 |y(t,x)-y_n(t,x)|dx &= \int_0^{\phi(t,0,0)} |y(t,x)-y_n(t,x)|dx + \int_{\phi(t,0,0)}^{\phi_n(t,0,0)} |y(t,x)-y_n(t,x)|dx \\ &+ \int_{\phi_n(t,0,0)}^{\phi(t,0,1)} |y(t,x)-y_n(t,x)|dx + \int_{\phi(t,0,1)}^{\phi_n(t,0,1)} |y(t,x)-y_n(t,x)|dx \\ &+ \int_{\phi_n(t,0,1)}^{1} |y(t,x)-y_n(t,x)|dx \\ &= I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

Since $\phi_n(t,0,0) \xrightarrow[n\to+\infty]{} \phi(t,0,0)$ and $\phi_n(t,0,1) \xrightarrow[n\to+\infty]{} \phi(t,0,1)$ and thanks to the uniform bound on $||y_n||_{L^\infty(\Omega_T)}$ we see that both I_2 and I_4 tend to 0 when n goes to infinity.

For I_1 we have:

$$\begin{aligned}I_1 = \int_0^{\phi(t,0,0)} \Bigg| & y_l(e_n(t,x)).\exp\left(-2\int_{e_n(t,x)}^t \partial_x(u_n+\mathcal{A})(r,\phi_n(r,t,x))dr\right) \\ & - y_l(e(t,x)).\exp\left(-2\int_{e(t,x)}^t \partial_x(u+\mathcal{A})(r,\phi(r,t,x))dr\right)\Bigg| dx.\end{aligned}$$

But thanks to Proposition 8, if $(t,x)\notin P$ (defined by ϕ) we have $e_n(t,x) \xrightarrow[n\to+\infty]{} e(t,x)$. This implies that if y_l were continuous, since we have a uniform bound on $||u_n||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$ the dominated convergence theorem would provide:

$$I_1 = \int_0^{\phi(t,0,0)} |y(t,x)-y_n(t,x)|dx \xrightarrow[n\to+\infty]{} 0.$$

The same idea can be applied to I_3 and I_5.
Hence for y_l, y_r and y_0 continuous we have $||y_n(t,.)-y(t,.)||_{L^1(0,1)} \xrightarrow[n\to+\infty]{} 0.$
But now thanks to inequality (1.59), we have:

$$\begin{aligned}||y(t,.)||_{L^1(0,1)} \le \big(||y_0||_{L^1(0,1)} + ||y_l||_{L^1((0,t)\cap\Gamma_l)} + ||y_r||_{L^1((0,t)\cap\Gamma_r)}\big) \\ \times ||u+\mathcal{A}||_{L^\infty(\Omega_T)}.e^{3t.||\partial_x(u+\mathcal{A})||_{L^\infty(\Omega_T)}}, \quad (1.27)\end{aligned}$$

$$\begin{aligned}||y_n(t,.)||_{L^1(0,1)} \le \big(||y_0||_{L^1(0,1)} + ||y_l||_{L^1((0,t)\cap\Gamma_l)} + ||y_r||_{L^1((0,t)\cap\Gamma-r)}\big) \\ \times ||u_n+\mathcal{A}||_{L^\infty(\Omega_T)}.e^{3t.||\partial_x(u_n+\mathcal{A})||_{L^\infty(\Omega_T)}}. \quad (1.28)\end{aligned}$$

So by density of $\mathcal{C}^0$ in L^1, and with the uniform bound on $||u_n||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$, the general case follows,

$$||y_n(t,.)-y(t,.)||_{L^1(0,1)} \xrightarrow[n\to+\infty]{} 0.$$

Now only the restriction on t remains, we recall that until now we supposed that v_l and v_l did not change sign on $[0,t]$.
But if v_l and v_r do not change sign on $[0,t_1]$ and then on $[t_1,t]$, we have:

$$||y_n(t_1,.)-y(t_1,.)||_{L^1(0,1)} \xrightarrow[n\to+\infty]{} 0.$$

Let us call $\tilde{y}_n$ the solution of $\partial_t \tilde{y}_n + (u_n + \mathcal{A})\partial_x \tilde{y}_n = -2.\tilde{y}_n.\partial_x(u_n + \mathcal{A})$ on $[t_1, t] \times [0,1]$ with initial value $y(t_1, .)$ and boundary values y_l, y_r. Due to what precedes we have $||\tilde{y}_n(t,.) - y(t,.)||_{L^1(0,1)} \xrightarrow[n\to+\infty]{} 0$. Now we can conclude that:

$$\begin{aligned}
||y_n(t,.) - y(t,.)||_{L^1(0,1)} \leq & ||y_n(t,.) - \tilde{y}_n(t,.)||_{L^1(0,1)} + ||\tilde{y}_n(t,.) - y(t,.)||_{L^1(0,1)} \\
\leq & ||y_n(t_1,.) - \tilde{y}_n(t_1,.)||_{L^1(0,1)}.||u_n + \mathcal{A}||_{L^\infty(\Omega_T)}.e^{3(t-t_1)||\partial_x(u_n+\mathcal{A})||_{L^\infty(\Omega_T)}} \\
& + ||\tilde{y}_n(t,.) - y(t,.)||_{L^1(0,1)} \\
\leq & ||y_n(t_1,.) - y(t_1,.)||_{L^1(0,1)}.||u_n + \mathcal{A}||_{L^\infty(\Omega_T)}.e^{3(t-t_1)||\partial_x(u_n+\mathcal{A})||_{L^\infty(\Omega_T)}} \\
& + ||\tilde{y}_n(t,.) - y(t,.)||_{L^1(0,1)} \\
\xrightarrow[n\to+\infty]{} & 0.
\end{aligned}$$

Therefore the convergence in $L^1(0,1)$ propagates on each interval where v_l and v_r do not change sign, thanks to the hypothesis on P_r and P_l we have:

$$\forall t \in [0,T] \qquad ||y_n(t,.) - y(t,.)||_{L^1(0,1)} \xrightarrow[n\to+\infty]{} 0. \tag{1.29}$$

Combining this first convergence result with the uniform bound of $y_n - y$ in $L^\infty(\Omega_T)$ and using the dominated convergence theorem in the time variable we obtain:

$$y_n \to y \text{ in } L^1(\Omega_T).$$

In term of $\tilde{u}$ and $\tilde{u}_n$ it implies that

$$\tilde{u}_n \to \tilde{u} \text{ in } L^1(0,T,W^{2,1}(0,1)).$$

But we also have $\forall n \in \mathbb{N} \quad \mathcal{F}(u_n) \in \mathcal{C}_{B_0,B_1,T}$, and we know (see 3) that $\mathcal{C}_{B_0,B_1,T}$ is compact therefore $\tilde{u}_n \to \tilde{u}$ in $\mathcal{C}_{B_0,B_1,T}$ (as the unique accumulation point of the sequence). □

Now we can apply Schauder's fixed point theorem to $\mathcal{F}$ and we get a solution

$$u \in L^\infty\left((0,T);\mathcal{C}^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H^1_0(0,1)\right).$$

The additional regularity properties of any solution u, meaning

$$\forall p > +\infty \quad u \in \mathcal{C}^0([0,T],W^{2,p}(0,1)) \cap C^1([0,1],W^{1,p}_0(0,1)),$$

follow directly from the construction of $\mathcal{F}$ and from Proposition 14.
To obtain the minimum existence time announced we just have to realize that the only possible reduction of T occurred in subsection 1.2.3. This concludes the proof of Theorem 1.

1.2.5 Uniqueness

To conclude the part about the initial boundary value problem, we prove a weak-strong uniqueness property.

Theorem 4. *Let (y,u) and $(\tilde{y},\tilde{u})$ be two solutions of* (1.11) *and* (1.12) *for the same initial and boundary data, and such that $\tilde{y} \in L^\infty((0,T);\mathrm{Lip}([0,1]))$. Then $y = \tilde{y}$ and $u = \tilde{u}$.*

Proof. Define $Y = \tilde{y} - y$ and $U = \tilde{u} - u$. Then we have:

$$U \in \mathrm{Lip}\left([0,T]; H_0^1(0,1)\right),\ (1-\partial_{xx}^2)U(t,.) = Y(t,.)\ dt\ a.e.,$$

and $Y \in L^\infty(\Omega_T)$ is the unique weak solution of:

$$\partial_t Y + (u+\mathcal{A})\partial_x Y = -2.Y.\partial_x(u+\mathcal{A}) - \partial_x\tilde{y}.U - 2\tilde{y}.\partial_x U,$$

with $Y_0 = 0$, $Y_l = 0$, $Y_r = 0$. Using Theorem 6 and formula (1.53) we get with $b = -2.\partial_x(u+\mathcal{A})$ and $f = -U.\partial_x\tilde{y} - 2\tilde{y}.\partial_x U$:

$$\text{For } (t,x) \in P, \quad Y(t,x) = 0,$$
$$\text{For } (t,x) \in I, \quad Y(t,x) = \int_0^t f(r,\phi(r,t,x)).\exp\left(\int_r^t b(r',\phi(r',t,x))dr'\right)dr,$$
$$\text{For } (t,x) \in L, \quad Y(t,x) = \int_{e(t,x)}^t f(r,\phi(r,t,x)).\exp\left(\int_r^t b(r',\phi(r',t,x))dr'\right)dr,$$
$$\text{For } (t,x) \in R, \quad Y(t,x) = \int_{e(t,x)}^t f(r,\phi(r,t,x)).\exp\left(\int_r^t b(r',\phi(r',t,x))dr'\right)dr.$$

Now since $||U(t,.)||_{L^\infty(0,1)} \leq 5.||Y(t,.)||_{L^\infty(0,1)}$ and $\tilde{y}$, $\partial_x\tilde{y}$ bounded, we see that for some $C > 0$:

$$||f(t,.)||_{L^\infty(0,1)} \leq C.||Y(t,.)||_{L^\infty(0,1)}\ dt\ a.e.,$$

and since b is bounded, we get that for some $C' > 0$:

$$||Y(t,.)|| \leq C'.\int_0^t ||Y(s,.)||_{L^\infty(0,1)}ds \quad dt\ a.e.,$$

and we conclude using Gronwall's lemma. □

1.3 Stabilization

In this part we prove Theorem 3. Here again we suppose that $\kappa \leq 0$. We begin by reformulating (1.2) and we also give the corresponding statement to Theorem 3 for this new formulation. Rather than (1.2) we will work on:

$$\left\{\begin{array}{l} \partial_t y + (\check{u} + \check{\mathcal{A}} - \kappa).\partial_x y = -2y.\partial_x(\check{u} + \check{\mathcal{A}}) \\ (1-\partial_{xx}^2)\check{u} = y, \quad \check{u}(t,0) = \check{u}(t,1) = 0 \\ (1-\partial_{xx}^2)\check{\mathcal{A}} = 0, \quad \check{\mathcal{A}}(t,0) = v_l(t) + \kappa,\ \check{\mathcal{A}}(t,1) = v_r(t) + \kappa \end{array}\right. . \tag{1.30}$$

This system is equivalent to (1.2) with the change of unknown

$$v = \check{\mathcal{A}} + \check{u} - \kappa.$$

And our stationary feedback law still reads (1.16). One can check that Theorem 3 can be reformulated in terms of those new unknowns as:

Theorem 5. *Let $A_l > 2.\sinh(1)$, $A_r > A_l.\cosh(1) + \sinh(2)$, $M > 0$, $T > 0$. For any $y_0 \in \mathcal{C}^0([0,1])$ there exists $y \in \mathcal{C}^0(\Omega_T)$ such that if we define $\check{u}$ and $\check{\mathcal{A}}$ by:*

$$\forall (t,x) \in \Omega_T \quad (1-\partial_{xx}^2)\check{u}(t,x) = y(t,x), \quad \check{u}(t,0) = \check{u}(t,1) = 0,$$

$\forall (t,x) \in \Omega_T,\ (1-\partial^2_{xx})\check{\mathcal{A}}(t,x) = 0,\ \check{\mathcal{A}}(t,0) = A_l.||y(t,.)||_{\mathcal{C}^0([0,1])}$ *and* $\check{\mathcal{A}}(t,1) = A_r.||y(t,.)||_{\mathcal{C}^0([0,1])}$, *then y is the weak solution of*

$$\partial_t y + (\check{u} + \check{\mathcal{A}} - \kappa).\partial_x y = -2.y.\partial_x(\check{u} + \check{\mathcal{A}}). \tag{1.31}$$

This function y also satisfies:

$$\forall t \in [0,T] \qquad \partial_t y(t,0) + M.y(t,0) = 0,$$
$$\forall x \in [0,1] \qquad y(0,x) = y_0(x).$$

Besides, if y is a maximal solution of the closed loop system (1.16),(1.30) *then y is defined on* $[0,+\infty) \times [0,1]$.
And finally if we let $c = min(A_l - 2.\sinh(1), \frac{A_r - A_l.\cosh(1) - \sinh(2)}{\sinh(1)})$ and $\tau = \frac{1}{M}.\ln(\frac{2.c.||y_0||_{\mathcal{C}^0([0,1])}}{M})$, we have:

$$\forall t \geq \tau \qquad ||y(t,.)||_{\mathcal{C}^0([0,1])} \leq \frac{M}{2c}.\frac{1}{1+M(t-\tau)}. \tag{1.32}$$

We now prove Theorem 5.

1.3.1 Strategy

Let us first describe the main steps of the proof of Theorem 5. In terms of the new unknowns, the equilibrium state that we want to stabilize is $y = 0$, $\check{u} = \check{\mathcal{A}} = 0$. A first natural idea would be to look at the linearized system around the equilibrium state. Its stabilization would provide a local stabilization result on the nonlinear system. But the linearized system reads:

$$\begin{cases} \partial_t y - \kappa.\partial_x y = 0 \\ (1-\partial^2_{xx})\check{u} = y, \quad \check{u}(t,0) = \check{u}(t,1) = 0 \\ (1-\partial^2_{xx})\check{\mathcal{A}} = 0, \quad \check{\mathcal{A}}(t,0) = v_l(t) + \kappa,\ \check{\mathcal{A}}(t,1) = v_r(t) + \kappa \end{cases}. \tag{1.33}$$

In the case $\kappa = 0$, the state y is constant therefore the system is not stabilizable.
In this situation we will apply a rough version of the return method that J.-M. Coron introduced in [33]. We will try to use the control in order to put the system in a simpler dynamic where it is easier to stabilize.
When we look at the transport equation we see that the sign of $\check{u} + \check{\mathcal{A}} - \kappa$ controls the geometry of the characteristics, and the sign of $\partial_x(\check{u}+\check{\mathcal{A}})$ controls the growth of y along the characteristics. Therefore we would like our feedback law to provide $\check{u}+\check{\mathcal{A}} \geq 0$ (since $-\kappa \geq 0$) and $\partial_x(\check{u}+\check{\mathcal{A}}) \geq 0$. Considering the estimates ((1.36),(1.37)) on $\check{u}$ we can get from the elliptic equation of (1.30) we see that with

$$v_l(t) = A_l.||y(t,.)||_{\mathcal{C}^0([0,1])} - \kappa, \tag{1.34}$$
$$v_r(t) = A_r.||y(t,.)||_{\mathcal{C}^0([0,1])} - \kappa, \tag{1.35}$$

the function $\check{\mathcal{A}}$ will dominate $\check{u}$ and we will have the desired signs.

For the existence of a solution we cannot adapt our proof of existence for the initial boundary value problem completely. Our feedback law makes us lose some regularity in time because $\check{\mathcal{A}}$ is now an unknown and it has exactly the time regularity of $||y(t,.)||_{\mathcal{C}^0([0,1])}$. To compensate for this, we will work in the space of continuous functions for y. This is now possible because the flow will always point toward $x = 1$. Therefore we have to prescribe y_l, and we just need to make a continuous transition at $(t,x) = (0,0)$ and have y_l decreasing in time. This is guaranteed by $\partial_t y_l(t) + M.y_l(t) = 0$. In the next part we will prove the existence part of Theorem 5. The asymptotic properties will be proved in the last part.

1.3.2 Existence of a solution to the closed loop system

Once again, we use a fixed point strategy on an operator $\mathcal{S}$ we describe now. We begin by defining the domain of the operator.

Definition 5. *Let X be the space of $(g,N) \in \mathcal{C}^0([0,T]\times[0,1]) \times \mathcal{C}^0([0,T])$ satisfying:*

1. $\forall (t,x) \in [0,T]\times[0,1] \quad g(0,x) = y_0(x) \qquad g(t,0) = y_0(0).e^{-M.t}$,
2. $\forall t \in [0,T] \qquad ||g(t,.)||_{\mathcal{C}^0([0,1])} \leq N(t)$,
3. *N is non-increasing and* $N(0) \leq ||y_0||_{\mathcal{C}^0([0,1])}$.

Proposition 5. *The domain X is non empty, convex, bounded and closed with respect to the uniform topology.*

The proof is elementary and one notices that $(y_0(x).e^{-Mt}, ||y_0||_{\mathcal{C}^0([0,1])}.e^{-Mt}) \in X$. Now for $(y,N) \in X$ we define $\check{u}$ and $\check{\mathcal{A}}$ as the solutions of:

$$\forall (t,x) \in \Omega_T \qquad (1-\partial^2_{xx})\check{u}(t,x) = y(t,x) \text{ and } \check{u}(t,0) = \check{u}(t,1) = 0,$$

$$\forall (t,x) \in \Omega_T \qquad (1-\partial^2_{xx})\check{\mathcal{A}}(t,x) = 0, \quad \check{\mathcal{A}}(t,0) = A_l\, N(t) \text{ and } \check{\mathcal{A}}(t,1) = A_r\, N(t).$$

One has the following exact formulas:

$$\forall (t,x) \in \Omega_T \qquad \check{u}(t,x) = -\int_0^x \sinh(x-\tilde{x}).y(t,\tilde{x})d\tilde{x},$$

$$\forall (t,x) \in \Omega_T \qquad \check{\mathcal{A}}(t,x) = \frac{N(t)}{\sinh(1)}.(A_r.\sinh(x) + A_l.\sinh(1-x)).$$

Therefore we have the following inequalities:

$$\forall (t,x) \in [0,T]\times[0,1] \qquad |\check{u}(t,x)| \leq 2\sinh(1)||y(t,.)||_{\mathcal{C}^0([0,1])}, \tag{1.36}$$

$$|\partial_x \check{u}(t,x)| \leq 2\cosh(1)||y(t,.)||_{\mathcal{C}^0([0,1])}, \quad |\partial^2_{xx}\check{u}(t,x)| \leq (1+2\sinh(1))||y(t,.)||_{\mathcal{C}^0([0,1])}, \tag{1.37}$$

$$|\partial_x \check{\mathcal{A}}(t,x)| \geq \frac{A_r - 2\cosh(1)A_l}{\sinh(1)}.N(t), \quad |\check{\mathcal{A}}(t,x)| \geq A_l.N(t). \tag{1.38}$$

And in turn those provide:

$$\forall (t,x) \in [0,T]\times[0,1] \qquad (\check{u}+\check{\mathcal{A}})(t,x) \geq (A_l - 2.\sinh(1)).||y(t,.)||_{\mathcal{C}^0([0,1])}, \tag{1.39}$$

$$\forall (t,x) \in [0,T]\times[0,1] \qquad \partial_x(\check{u}+\check{\mathcal{A}})(t,x) \geq \frac{A_r - 2.\cosh(1).A_l - \sinh(2)}{\sinh(1)}.||y(t,.)||_{\mathcal{C}^0([0,1])}. \tag{1.40}$$

Now if ϕ is the flow of $\check{u}+\check{\mathcal{A}}-\kappa$, ϕ is $\mathcal{C}^1$ and since $\check{u}+\check{\mathcal{A}}-\kappa \geq 0$ (thanks to the inequalities above), $\phi(.,t,x)$ is nondecreasing. This allows us to define the entrance time and then the operator $\mathcal{S}$ as follows. Let $e(t,x) = min\{s \in [0,T] \mid \phi(s,t,x) = 0\}$ with the convention that $\min \emptyset = 0$. Now for $(t,x) \in [0,T]\times[0,1]$, $\mathcal{S}(y,N) = (\tilde{y},\tilde{N})$ with:

1. if $x \geq \phi(t,0,0) \quad \tilde{y}(t,x) = y_0(\phi(0,t,x)).\exp(-2\int_0^t \partial_x(\check{u}+\check{A})(s,\phi(s,t,x))ds)$,
2. if $x \leq \phi(t,0,0) \quad \tilde{y}(t,x) = y_0(0).e^{-M.e(t,x)}.\exp(-2.\int_{e(t,x)}^t \partial_x(\check{u}+\check{A})(s,\phi(s,t,x))ds)$,

3. $\tilde{N}(t) = ||\tilde{y}(t,.)||_{\mathcal{C}^0([0,1])}$.

From Theorem 6 we know that $\tilde{y}$ is the weak solution of:

$$\partial_t \tilde{y} + (\breve{u} + \breve{\mathcal{A}} - \kappa)\partial_x \tilde{y} = -2\tilde{y}\partial_x(\breve{u} + \breve{\mathcal{A}}) \qquad \tilde{y}(0,.) = y_0 \qquad \tilde{y}(t,0) = y_0(0)\, e^{-M\,t}. \tag{1.41}$$

Before applying Schauder's fixed point theorem to $\mathcal{S}$ we prove the following identities.

Proposition 6. *1. The operator $\mathcal{S}$ maps X to X.*

2. The family $\mathcal{S}(X)$ is uniformly bounded and equicontinuous.

3. $\mathcal{S}$ is continuous w.r.t. the uniform topology.

Proof. 1. It will be useful to distinguish the cases where $y_0(0) = 0$ (case 1) and $y_0(0) \neq 0$ (case 2). First remark that $\tilde{y}$ being continuous, $\tilde{N}$ is continuous. Now in case 1, we have:

$$\forall (t,x) \in \Omega_T,\, x \leq \phi(t,0,0) \quad \Rightarrow \quad \tilde{y}(t,x) = 0,$$

and both the continuity on $\{(t,x) \in \Omega_T \mid x > \phi(t,0,0)\}$ and the continuity at the interface $\{(t,x) \in \Omega_T \mid x = \phi(t,0,0)\}$ are obvious.
In case 2, one must first remark that $\forall t \in [0,T],\ y(t,0) \neq 0$, so

$$\forall t \in [0,T],\ 0 < ||y(t,.)||_{\mathcal{C}^0([0,1])} \leq N(t.$$

This implies that every characteristic curve points to the right and so e corresponds to Definition 1.4.1. Therefore e is $\mathcal{C}^1$ on $\{(t,x) \in \Omega_T \mid x < \phi(t,0,0)\}$ and continuous at the interface $\{(t,x) \in \Omega_T \mid x = \phi(t,0,0)\}$, once again we see that $\tilde{y}$ is continuous in Ω_T, and so is $\tilde{N}$.
Now it is straightforward from its definition that

$$\forall (t,x) \in [0,T] \times [0,1], \quad \tilde{y}(0,x) = y_0(x), \quad \tilde{y}(t,0) = y_0(0).e^{-M.t}.$$

It only remains to see that $\tilde{N} = ||\tilde{y}(t,.)||_{\mathcal{C}^0([0,1])}$ is non-increasing. Since $\partial_x(\breve{u} + \breve{\mathcal{A}}) \geq 0$ (see (1.40)), we see from the definition of $\tilde{y}$ that $|\tilde{y}|$ does not increase along the characteristics, and since $|\tilde{y}(t,0)|$ is also non-increasing we can conclude.

2. Since X is already bounded and thanks to the first part of the proof, $\mathcal{S}(X)$ is bounded. The equicontinuity of the family $\{\tilde{N}\}$ being implied by the one of the family $\{\tilde{y}\}$, we will show that we have a common continuity modulus for all $\{\tilde{y}\}$. For now let us focus only on $\{(t,x) \in \Omega_T \mid x \leq \phi(t,0,0)\}$. On this set $\tilde{y}(t,x) = 0$ in case 1. In the second case, we need the following inequalities valid on Ω_T and which follow from the definition of $\breve{u}$ and $\breve{\mathcal{A}}$:

$$||\breve{u}||_{\mathcal{C}^0(\Omega_T)} \leq 2.\sinh(1).||y_0||_{\mathcal{C}^0([0,1])}, \tag{1.42}$$

$$||\partial_x \breve{u}||_{\mathcal{C}^0(\Omega_T)} \leq 2.\cosh(1).||y_0||_{\mathcal{C}^0([0,1])}, \tag{1.43}$$

$$||\partial^2_{xx} \breve{u}||_{\mathcal{C}^0(\Omega_T)} \leq (1 + 2.\sinh(1)).||y_0||_{\mathcal{C}^0([0,1])}, \tag{1.44}$$

$$||\breve{\mathcal{A}}||_{\mathcal{C}^0(\Omega_T)} = ||\partial^2_{xx} \breve{\mathcal{A}}||_{\mathcal{C}^0(\Omega_T)} \leq (A_r + A_l)||y_0||_{\mathcal{C}^0([0,1])}, \tag{1.45}$$

$$||\partial_x \breve{\mathcal{A}}||_{\mathcal{C}^0(\Omega_T)} \leq \frac{A_r + A_l}{\tanh(1)}.||y_0||_{\mathcal{C}^0([0,1])}. \tag{1.46}$$

And since ϕ is the flow of $\check{u}+\check{\mathcal{A}}-\kappa$ we also have:

$$\begin{aligned}
||\partial_1\phi||_{\mathcal{C}^0([0,1])} &\leq -\kappa + (2\sinh(1)+A_l+A_r)||y_0||_{\mathcal{C}^0([0,1])},\\
||\partial_2\phi||_{\mathcal{C}^0([0,1])} &\leq (-\kappa + (2\sinh(1)+A_l+A_r)||y_0||_{\mathcal{C}^0([0,1])})\\
&\quad \times \exp\left(2.T.\cosh(1).\left(2+\frac{A_r+A_l}{\sinh(1)}\right)||y_0||_{\mathcal{C}^0([0,1])}\right),\\
||\partial_3\phi||_{\mathcal{C}^0([0,1])} &\leq \exp\left(2.T.\cosh(1).\left(2+\frac{A_r+A_l}{\sinh(1)}\right)||y_0||_{\mathcal{C}^0([0,1])}\right).
\end{aligned}$$

Now since we have

$$\tilde{y}(t,x) = y_0(0).e^{-M.e(t,x)}.\exp(-2.\int_{e(t,x)}^t \partial_x(\check{u}+\check{\mathcal{A}})(r,\phi(r,t,x))dr),$$

we see that we only need a uniform bound on $||e||_{\mathcal{C}^1}$ to conclude about the equicontinuity on $\{(t,x)\in\Omega_T \mid x\leq\phi(t,0,0)\}$.
We have $0\leq e(t,x)\leq T$, and thanks to the definition of e, to (1.42), (1.45) and also thanks to:

$$||y(t,.)||_{\mathcal{C}^0([0,1])} \geq |y(t,0)| = |y_0(0)|.e^{-M.t} \geq |y_0(0).e^{-M.T}|,$$

we get:

$$|\partial_t e(t,x)| \leq \frac{(\kappa+(2\sinh(1)+A_l+A_r)||y_0||_{\mathcal{C}^0([0,1])})\exp(2T\cosh(1)(2+\frac{A_r+A_l}{\sinh(1)})||y_0||_{\mathcal{C}^0([0,1])})}{(A_l-2\sinh(1))e^{-MT}|y_0(0)|}.$$

In the same way:

$$|\partial_x e(t,x)| \leq \frac{\exp(2.T.\cosh(1).(2+\frac{A_r+A_l}{\sinh(1)})||y_0||_{\mathcal{C}^0([0,1])})}{(A_l-2\sinh(1)).e^{-M.T}.|y_0(0)|}.$$

In the end, we see that both in case 1 and case 2, the family $\{\tilde{y}\}$ is uniformly Lipschitz on the set:

$$\{(t,x)\in\Omega_T \mid x\leq\phi(t,0,0)\}.$$

Now on $\{(t,x)\in\Omega_T \mid x\geq\phi(t,0,0)\}$, we know

$$\tilde{y}(t,x) = y_0(\phi(0,t,x)).\exp(-2.\int_0^t \partial_x(\check{u}+\check{\mathcal{A}})(r,\phi(r,t,x))dr).$$

Clearly y_0 is continuous on $[0,1]$ therefore it is both bounded and uniformly continuous, the family of functions ϕ is uniformly Lipschitz and the family

$$\left\{\exp(-2.\int_0^t \partial_x(\check{u}+\check{\mathcal{A}})(r,\phi(r,t,x))dr)\right\},$$

is uniformly bounded and equicontinuous. We can conclude that the family $\{\tilde{y}\}$ is also equicontinuous on $\{(t,x)\in\Omega_T \mid x\geq\phi(t,0,0)\}$. Since we have continuity on the set $\{(t,x)\in\Omega_T \mid x=\phi(t,0,0)\}$, we can conclude that the family $\mathcal{S}(X)$ is uniformly bounded and equicontinuous on Ω_T, $\mathcal{S}(X)$ is therefore relatively compact in X.

3. It remains to prove that $\mathcal{S}$ is continuous w.r.t. to the uniform convergence.
Let (y_n) be a sequence in X converging uniformly to $y \in X$. We only have to show that $\tilde{y}_n$ converges uniformly to $\tilde{y}$, since it immediately implies that $\tilde{N}_n$ converges uniformly to $\tilde{N}$. First the uniform convergence of y_n and N_n implies the uniform convergence of $\check{u}_n$ and $\check{\mathcal{A}}_n$. Then by Gronwall's lemma, we also have $\phi_n \to \phi$ uniformly in $\mathcal{C}^1(\Omega_T)$. Using Proposition 8, we then obtain $e_n \to e$ uniformly in $\mathcal{C}^0(\Omega_T)$. Now we decompose Ω_T in three parts depending on n.

$$\begin{aligned} L_n &= \{(t,x) \in \Omega_T \mid x \leq \min(\phi_n(t,0,0), \phi(t,0,0))\}, \\ R_n &= \{(t,x) \in \Omega_T \mid x \geq \max(\phi_n(t,0,0), \phi(t,0,0))\}, \\ I_n &= \overline{\Omega_T \setminus (L_n \cup R_n)}. \end{aligned}$$

Let us point out first that when $n \to +\infty$:

$$\liminf L_n = \{(t,x) \in \Omega_T \mid x \leq \phi(t,0,0)\}, \quad \liminf R_n = \{(t,x) \in \Omega_T \mid x \geq \phi(t,0,0)\},$$
$$\text{and } \limsup I_n = \{(t,x) \in \Omega_T \mid x = \phi(t,0,0)\}.$$

- For $(t,x) \in L_n$ if $y_0(0) = 0$ then y_n and $\tilde{y}$ are equal to zero otherwise we have the formulas:

$$\tilde{y}(t,x) = y_0(0).e^{-M.e(t,x)}.\exp(-2\int_{e(t,x)}^{t} \partial_x(\check{u} + \check{\mathcal{A}})(r, \phi(r,t,x))dr),$$
$$\tilde{y}_n(t,x) = y_0(0).e^{-M.e_n(t,x)}.\exp(-2\int_{e_n(t,x)}^{t} \partial_x(\check{u}_n + \check{\mathcal{A}}_n)(r, \phi_n(r,t,x))dr),$$

  and the uniform convergence of $\tilde{y}_n$ follows from the uniform boundedness and convergence of $\partial_x \check{u}_n$, $\partial_x \check{\mathcal{A}}_n$, e_n and ϕ_n.
- For $(t,x) \in R_n$ the proof is similar.
- It remains only to prove the convergence in I_n. But the width of I_n tends to zero, and the family $\{\tilde{y}_n\}$ is equicontinuous. Therefore the uniform convergence of $\tilde{y}_n$ in I_n follows from those in L_n and R_n.

□

Now we can apply Schauder's fixed point theorem to $\mathcal{S}$ and get (y, N) fixed point of $\mathcal{S}$. It remains to show that it satisfies all of the properties of Theorem 5 except (1.32) which will be proven in the next subsection.
First we have $y(t,0) = \tilde{y}(t,0) = y_0(0).e^{-M.t}$ and it implies $\partial_t y(t,0) = -M.y(t,0)$.
But also $N(t) = \tilde{N}(t) = ||\tilde{y}(t,.)||_{\mathcal{C}^0([0,1])} = ||y(t,.)||_{\mathcal{C}^0([0,1])}$, therefore $||y(t,.)||_{\mathcal{C}^0([0,1])}$ is non-increasing and, thanks to Theorem 6, $y = \tilde{y}$ is a weak solution of

$$\begin{cases} (1-\partial_{xx}^2)\check{u} = y, \quad \check{u}(t,0) = \check{u}(t,1) = 0 \\ (1-\partial_{xx}^2)\check{\mathcal{A}} = 0, \quad \check{\mathcal{A}}(t,0) = A_l.||y(t,.)||_{\mathcal{C}^0([0,1])}, \ \check{\mathcal{A}}(t,1) = A_r.||y(t,.)||_{\mathcal{C}^0([0,1])} \\ \partial_t y + (\check{u} + \check{\mathcal{A}} - \kappa).\partial_x y = -2y.\partial_x(\check{u} + \check{\mathcal{A}}) \end{cases} . \tag{1.47}$$

Remark 5. • *Since* $(\check{u} + \check{\mathcal{A}} - \kappa)(t,1) = A_r.||y(t,.)||_{\mathcal{C}^0([0,1])} - \kappa \geq 0$ *we had all along* $\Gamma_r = \emptyset$.

- *Since* $(\check{u} + \check{\mathcal{A}} - \kappa)(t,0) = A_l.||y(t,.)||_{\mathcal{C}^0([0,1])} - \kappa$, *we see that a priori,* Γ_l *depends on* y. *But in fact if* $y_0(0) \neq 0$ *then* $\forall t, y(t,0) \neq 0$ *and* $\Gamma_l = \mathbb{R}+$. *And if* $y_0(0) = 0$ *then*

$\forall t,\ y_l(t) = y(t,0) = 0$ and it makes no difference in the weak formulation(1.56) if we enlarge Γ_l to $\mathbb{R}+$. Therefore the space of test functions is always:

$$Adm(\Omega_T) = \{\phi \in \mathcal{C}^1(\Omega_T) \mid \forall x \in [0,1]\ \phi(T,x) = 0,\ \forall t \in [0,T]\ \phi(t,1) = 0\}.$$

- *It must be noted that while we required $T < \infty$, we did not need T to be small.*

1.3.3 Stabilization and global existence

To finish the proof of Theorem 5 we have to prove the global existence of a maximal solution and estimate (1.32).

Proof. First we rewrite (1.39),(1.40) as:

$$\forall (t,x) \in \Omega_T \qquad \begin{aligned}&(\check{u} + \check{\mathcal{A}})(t,x) \geq c||y(t,.)||_{\mathcal{C}^0([0,1])},\\ &\partial_x(\check{u} + \check{\mathcal{A}})(t,x)) \geq c||y(t,.)||_{\mathcal{C}^0([0,1])}.\end{aligned}$$

But y is the solution of the transport equation (1.31) and it satisfies:

$$y(t,x) = y(s,\phi(s,t,x)).\exp(-2\int_s^t \partial_x(\check{u} + \check{\mathcal{A}})(r,\phi(r,t,x))dr).$$

Combining those facts, we get for $t \geq s$:

$$|y(t,x)| \leq |y(s,\phi(s,t,x))|.\exp(-2\int_s^t c.||y(r,.)||_{\mathcal{C}^0([0,1])}dr).$$

This implies that $|y|$ decreases along the characteristics (strictly for the times where $y(t,.) \not\equiv 0$). But we have also imposed $y(t,0) = y(s,0).e^{-M(t-s)}$, therefore $|y|$ also decreases along $x = 0$. This already shows, thanks to the existence theorem that a maximal solution of the closed loop system is global. To get a more precise statement, we consider all the characteristics between time t and s and we obtain:

$$\forall t \geq s \geq 0,\ ||y(t,.)||_{\mathcal{C}^0([0,1])} \leq ||y(s,.)||_{\mathcal{C}^0([0,1])} \max_{r\in[s,t]} \left(e^{-M(r-s)} \exp(-2c\int_r^t ||y(\alpha,.)||_{\mathcal{C}^0([0,1])}d\alpha)\right).$$

Now let us define the function g as follows:

$$g(r) = e^{-M(r-s)} \exp(-2c\int_r^t ||y(\alpha,.)||_{\mathcal{C}^0([0,1])}d\alpha).$$

We then have:

$$g'(r) = (2c||y(r,.)||_{\mathcal{C}^0([0,1])} - M)g(r),$$

and we know that as long as the quantity $||y(r,.)||_{\mathcal{C}^0([0,1])}$ is not equal to zero, it is decreasing. So if $||y_0||_{\mathcal{C}^0([0,1])} > \frac{M}{2c}$, for t small enough $||y(t,.)||_{\mathcal{C}^0([0,1])} \geq \frac{M}{2c}$ and we have:

$$||y(t,.)||_{\mathcal{C}^0([0,1])} \leq ||y_0||_{\mathcal{C}^0([0,1])}.e^{-M.t}$$

which implies $||y(\tau,.)||_{\mathcal{C}^0([0,1])} \leq \frac{M}{2c}$. This provides for $\tau \leq s \leq t$, the inequality (which was clear when $||y_0||_{\mathcal{C}^0([0,1])} \leq \frac{M}{2c}$)

$$||y(t,.)||_{\mathcal{C}^0([0,1])} \leq ||y(s,.)||_{\mathcal{C}^0([0,1])}.\exp(-2c\int_s^t ||y(r,.)||_{\mathcal{C}^0([0,1])}dr).$$

And we conclude with a classical comparison principle for ODES. ☐

Remark 6. • *For $\kappa \neq 0$ the result is easily improved.*
Indeed if $t \geq \tau - \frac{2\sinh(1)+A_l+A_r}{\kappa.c}$ *we have* $-\kappa + \breve{u} + \breve{\mathcal{A}} \geq -\frac{\kappa}{2}$.
And therefore $t \geq \tau - \frac{2\sinh(1)+A_l+A_r}{\kappa.c} - \frac{2}{\kappa} \Rightarrow ||y(t,.)||_{\mathcal{C}^0([0,1])} \leq |y_0(0)|.e^{-M(t+\frac{2}{\kappa})}$

- *In particular if $y_0(0) = 0$ we see that we stabilize the null state in finite time.*
- *Of course similar results hold for $\kappa \geq 0$ thanks to Remark 2.*

1.4 Initial boundary value problem for a linear transport equation

In this section we will consider the initial boundary value problem for the following linear transport equation:

$$\partial_t y + a(t,x).\partial_x y = b(t,x).y + f(t,x) \tag{1.48}$$

We will look at strong and weak solutions of (1.48) on $\Omega_T = [0,T] \times [0,1]$. It should be noted that the backward problem is transformed in a standard one by the change of variables: $t \to T - t$.

1.4.1 Properties of the flow

Let $a \in \mathcal{C}^0(\Omega_T)$ be uniformly Lipschitz in the second variable. We will denote the Lipschitz constant by

$$L = ||a||_{L^\infty((0,T),\mathrm{Lip}([0,1]))}.$$

Since we want to use the method of characteristics to solve (1.48) we need to study the flow of a.

Definition 6. *For $(t,x) \in \Omega_T$, let $\phi(.,t,x)$ be the $\mathcal{C}^1$ maximal solution to :*

$$\begin{cases} \partial_s \phi(s,t,x) = a(s,\phi(s,t,x)) \\ \phi(t,t,x) = x \end{cases}, \tag{1.49}$$

which is defined on a certain set $[e(t,x),h(t,x)]$ (which is closed because $[0,1]$ is compact) and with possibly $e(t,x)$ and/or $h(t,x) = t$.

Remark 7. *Obviously $e(t,x) > 0 \Rightarrow \phi(e(t,x),t,x) \in \{0,1\}$.*

Now we take into account the influence of the boundaries by introducing the sets:

$$P = \{(t,x) \in \Omega_T \mid \exists s \in [e(t,x),h(t,x)] \text{ such that } \phi(s,t,x) \in \{0,1\} \text{ and } a(s,\phi(s,t,x)) = 0\}$$
$$\cup\ \{(s,\phi(s,0,0)) \mid \forall s \in [0,T]\}\ \cup\ \{(s,\phi(s,0,1)) \mid \forall s \in [0,T]\},$$

$$I = \{(t,x) \in \Omega_T \setminus P \mid e(t,x) = 0\},$$
$$L = \{(t,x) \in \Omega_T \setminus P \mid \phi(e(t,x),t,x) = 0\},$$
$$R = \{(t,x) \in \Omega_T \setminus P \mid \phi(e(t,x),t,x) = 1\},$$
$$\Gamma_l = \{t \in [0,T] \mid a(t,0) > 0\},$$
$$\Gamma_r = \{t \in [0,T] \mid a(t,1) < 0\}.$$

Proposition 7. *The function ϕ is uniformly Lipschitz on its domain.*

Proof. This is easily deduced from the standard case by the use of a Lipschitzian extension of a. □

We can now study the regularity of e.

Proposition 8. *Let $(t,x) \in \Omega_T \setminus P$, $(a_n) \in \mathcal{C}^0(\Omega_T) \cap L^\infty((0,T);\mathrm{Lip}([0,1]))$ a sequence such that $||a_n - a||_{\mathcal{C}^0(\Omega_T)} \to 0$, $||a_n||_{L^\infty(0,1;\mathrm{Lip}([0,1]))}$ is bounded and $(t_n;x_n) \in \Omega_T$ such that $(t_n,x_n) \to (t,x)$. Then $e_n(t_n,x_n) \to e(t,x)$.*

Proof. Once again we will use a Lipschitzian extension operator Π and we set $\tilde{a}_n = \Pi(a_n)$ and $\tilde{a} = \Pi(a)$. Now let $\tilde{\phi}_n$ and $\tilde{\phi}$ be their respective flows. Using Gronwall's lemma we have:

$$|(\tilde{\phi}_n - \tilde{\phi})(s,t,x)| \leq T.||\tilde{a}_n - \tilde{a}||_{\mathcal{C}^0(\Omega_T)}.e^{T.||\tilde{a}||_{L^\infty((0,T);\mathrm{Lip}([0,1]))}}. \tag{1.50}$$

But we can see that:

$$e_n(t_n,x_n) = \min\{s \in [0,t_n] \mid \forall r \in [s,t_n] \quad \tilde{\phi}_n(r,t,x) \in [0,1]\}.$$

- If $(t,x) \in I$ since we have excluded the characteristics coming from $(0,0)$ and $(0,1)$ we have:

  $$\inf_{s\in[0,T]}(\mathrm{d}(\phi(s,t,x),[0,t]\times\{0\} \cup [0,t]\times\{1\})) > 0.$$

  So we can conclude from (1.50) that for n large enough $\phi_n(.,t,x)$ is defined back to 0 that is $e_n(t,x) = 0$. From now on $(t,x) \in L \cup R$.

- Now we can take s strictly lower and close enough to $e(t,x)$, $\tilde{\phi}(s,t,x) \notin [0,1]$, since $(t,x) \notin P \quad \Rightarrow \quad e(t,x) \in \Gamma_l \cup \Gamma_r$. But $\tilde{\phi}_n(s,t_n,x_n) \to \tilde{\phi}(s,t,x)$, therefore for n large enough $\tilde{\phi}_n(s,t_n,x_n) \notin [0,1]$ and $s < t_n$ and we can conclude that $\liminf e_n(t_n,x_n) \geq s$. But s is arbitrarily close to $e(t,x)$ and we get

  $$\liminf e_n(t_n,x_n) \geq e(t,x).$$

- If $e(t,x) = t$ then $\limsup e_n(t_n,x_n) \leq \limsup t_n = t$ and $e_n(t_n,x_n) \to e(t,x)$. Otherwise since $(t,x) \notin P$ then $\forall s \in]e(t,x),t[\quad \phi(s,t,x) \in]0,1[$. And now $\forall \epsilon > 0 \quad \exists \alpha > 0$ such that

  $$\forall s \in [e(t,x)+\epsilon, t-\epsilon], \quad \min(\phi(s,t,x), 1-\phi(s,t,x)) \geq \alpha.$$

  But for n large enough we have:

  $$||\phi_n - \phi||_{\mathcal{C}^0(\Omega_T)} \leq \frac{\alpha}{4},$$
  $$|\phi_n(s,t_n,x_n) - \phi_n(s,t,x)| \leq \frac{\alpha}{4},$$

  (the second estimate comes from the uniform bound on $||a_n||_{L^\infty((0,1);\mathrm{Lip}([0,1]))}$). But now, combining those two inequalities we see that for n large and for all s between $e(t,x)+\epsilon$ and $t-\epsilon$ we have:

  $$\min(\phi_n(s,t_n,x_n), 1-\phi_n(s,t_n,x_n)) \geq \frac{\alpha}{2}.$$

  This provides $\limsup e_n(t_n,x_n) \leq e(t,x)+\epsilon$, and since ϵ is arbitrarily small we obtain:

  $$\limsup e_n(t_n,x_n) \leq e(t,x).$$

□

This proposition has a few immediate corollaries.

Remark 8.
- *For $a_n = a$ it shows that e is continuous outside of P.*
- *If $P = \emptyset$, since Ω_T is compact the proposition implies that e_n converges uniformly toward e.*

Proposition 9. *If we assume that $\partial_x a \in \mathcal{C}^0(\Omega_T)$ then ϕ is $\mathcal{C}^1$ and e is $\mathcal{C}^1$ on $\Omega_T \setminus P$ with:*

$$\partial_t e(t,x) = \frac{a(t,x).\exp(\int_{e(t,x)}^s \partial_x a(r,\phi(r,t,x))dr)}{a(e(t,x),\phi(e(t,x),t,x))}, \quad \partial_x e(t,x) = -\frac{\exp(\int_{e(t,x)}^s \partial_x a(r,\phi(r,t,x))dr)}{a(e(t,x),\phi(e(t,x),t,x))}. \tag{1.51}$$

Proof. The regularity of ϕ is a classical result. If $(t,x) \in I$, $\quad e(t,x) = 0$ and it is obvious. For $(t,x) \in L$ we have $\phi(e(t,x),t,x) = 0$ and $e(t,x) \in \Gamma_l$ therefore $\partial_1\phi(e(t,x),t,x) > 0$ and the implicit function theorem let us conclude, we can proceed in the same way for R. The inclusion of the characteristics of $(0,0)$ and $(0,1)$ in P is needed here. □

Proposition 10. *Let (a_n) be a sequence of $\mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$ and $a \in \mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$ such that $||a_n - a||_{L^\infty((0,T);\mathrm{Lip}([0,1]))} \xrightarrow[n\to+\infty]{} 0$. If we call ϕ_n the flow of a_n and ϕ the flow of a then $\phi_n \xrightarrow[n\to+\infty]{} \phi$ locally in $\mathcal{C}^1$.*

Proof. Once again using a $\mathcal{C}^1$ extension operator on a_n and a we deduce the result from the classical standard case, which follows from applications of Gronwall's lemma. □

1.4.2 Strong solutions

Here we consider the case of data $a \in \mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$, $y_l \in \mathcal{C}^1_c(\Gamma_l)$, $y_r \in \mathcal{C}^1_c(\Gamma_r)$, $y_0 \in \mathcal{C}^1_c(0,1)$, $b \in \mathcal{C}^1(\Omega_T)$ and $f \in \mathcal{C}^1_c(\Omega_T \setminus P)$. We define the function y in the following way:

$$\text{for } (t,x) \in P \quad y(t,x) = 0, \tag{1.52}$$

$$\begin{aligned}
\text{for } (t,x) \in I \quad y(t,x) &= y_0(\phi(0,t,x)).\exp\left(\int_0^t b(r,\phi(r,t,x))dr\right) \\
&+ \int_0^t f(r,\phi(r,t,x)).\exp\left(\int_r^t b(r',\phi(r',t,x))dr'\right) dr, \\
\text{for } (t,x) \in L \quad y(t,x) &= y_l(e(t,x)).\exp\left(\int_{e(t,x)}^t b(r,\phi(r,t,x))dr\right) \\
&+ \int_{e(t,x)}^t f(r,\phi(r,t,x)).\exp\left(\int_r^t b(r',\phi(r',t,x))dr'\right) dr, \\
\text{for } (t,x) \in R \quad y(t,x) &= y_r(e(t,x)).\exp\left(\int_{e(t,x)}^t b(r,\phi(r,t,x))dr\right) \\
&+ \int_{e(t,x)}^t f(r,\phi(r,t,x)).\exp\left(\int_r^t b(r',\phi(r',t,x))dr'\right) dr.
\end{aligned} \tag{1.53}$$

Proposition 11. *We have* $y \in \mathcal{C}^1(\Omega_T)$, $\text{supp}(y) \subset \Omega_T \setminus P$ *and* y *is a strong solution of* (1.48) *with the additional conditions that for all* x *in* $[0,1]$ $y(0,x) = y_0(x)$, *for all* t *in* Γ_l $y(t,0) = y_l(t)$ *and for all* t *in* Γ_r $y(t,1) = y_r(t)$. *Besides we have the estimate:*

$$||y||_{\mathcal{C}^0(\Omega_T)} \leq \left(\max\left(||y_0||_{\mathcal{C}^0(0,1)}, ||y_l||_{\mathcal{C}^0(\Gamma_l)}, ||y_r||_{\mathcal{C}^0(\Gamma_r)}\right) + T.||f||_{\mathcal{C}^0(\Omega_T)}\right).e^{T.||b||_{\mathcal{C}^0(\Omega_T)}}. \quad (1.54)$$

Proof. First, y is equal to 0 in a neighborhood of P because we chose y_0, y_l, y_r, f to be null close to P and because of(1.53). Outside of this neighborhood, the regularity of y comes from the integral formulas (1.53) and from the regularity of y_0, y_l, y_r, f, b, ϕ and e (proved in proposition 9). The fact that y satisfies (1.48) is a straightforward calculation. ☐

Remark 9. *We have that:*

$$\forall (t,x) \in \Omega_T \text{ and } \forall s \in [e(t,x), h(t,x)] \quad y(t,x) = y(s,\phi(s,t,x)).\exp\left(\int_s^t b(r,\phi(r,t,x))dr\right)$$
$$+ \int_s^t f(r,\phi(r,t,x)).\exp\left(\int_r^t b(r',\phi(r',t,x))dr'\right)dr. \quad (1.55)$$

1.4.3 Weak solutions

In this section we will consider the case of data $a \in \mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$, $b, f \in L^\infty(\Omega_T)$, $y_0 \in L^\infty(0,1)$, $y_l \in L^\infty(\Gamma_l)$ and $y_r \in L^\infty(\Gamma_r)$. We introduce the space of test functions:

$$Adm(\Omega_T) = \{\phi \in \mathcal{C}^1(\Omega_T) |\ \forall x \in [0,1],\ \phi(T,x) = 0,$$
$$\forall t \in [0,T] \setminus \Gamma_l,\ \phi(t,0) = 0,\ \forall t \in [0,T] \setminus \Gamma_r,\ \phi(t,1) = 0\}.$$

Proposition 12. *For* $y \in \mathcal{C}^1(\Omega_T)$, y *is a strong solution of* (1.48), *if and only if it satisfies* $\forall \phi \in Adm(\Omega_T)$

$$\int_{\Omega_T} y.(\partial_t \phi + a.\partial_x \phi + (b + \partial_x a)\phi)dxdt = -\int_{\Omega_T} f(t,x).\phi(t,x)dtdx$$
$$-\int_0^1 \phi(0,x).y(0,x)dx + \int_0^T (a(t,1).\phi(t,1).y(t,1) - a(t,0).\phi(t,0).y(t,0))dt. \quad (1.56)$$

This legitimates the following definition of a weak solution.

Definition 7. *For* $a \in L^\infty(0,T,\text{Lip}(0,1))$, $b, f \in L^1(\Omega_T)$, $y_0 \in L^1(0,1)$, $y_l \in L^1(\Gamma_l)$ *and* $y_r \in L^1(\Gamma_r)$, *we say that* $y \in L^\infty(\Omega_T)$ *is a weak solution of* (1.48) *if it satisfies* (1.56).

Theorem 6. *Let* $a \in \mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$, $b, f \in L^\infty(\Omega_T)$, $y_0 \in L^\infty(0,1)$, $y_l \in L^\infty(\Gamma_l)$ *and* $y_r \in L^\infty(\Gamma_r)$. *We will also suppose that the sets*

$$P_l = \{t \in [0,T] \mid a(t,0) = 0\} \text{ and } P_r = \{t \in [0,T] \mid a(t,1) = 0\}$$

have at most a countable number of connected components. Then the function y *defined by the formula* (1.53), *is a weak solution of* (1.48) *and satisfies:*

$$||y||_{L^\infty(\Omega_T)} \leq \left(\max\left(||y_0||_{L^\infty(0,1)}, ||y_l||_{L^\infty(\Gamma_l)}, ||y_r||_{L^\infty(\Gamma_r)}\right) + T.||f||_{L^\infty(\Omega_T)}\right).e^{T.||b||_{L^\infty(\Omega_T)}}. \quad (1.57)$$

Proof. If we let $P_{\tilde{t}} = P \cap \{(t,x) \in \Omega_T \mid t = \tilde{t}\}$, we can see that each points of a $P_{\tilde{t}}$ corresponds to at least one connected component of $P_l \cup P_r$ (since only one characteristic curve goes through the whole connected component) therefore, $P_{\tilde{t}}$ is at most countable and thus 1d negligible, this implies that P is 2d negligible.
Now we have:

- $\mathcal{C}_c^1(\Omega_T \setminus P)$ is dense in $L^1(\Omega_T)$,
- $\mathcal{C}_c^1(0,1)$ is dense in $L^1(0,1)$,
- $\mathcal{C}_c^1(\Gamma_l)$ is dense in $L^1(\Gamma_l)$,
- $\mathcal{C}_c^1(\Gamma_r)$ is dense in $L^1(\Gamma_r)$.

And we can take, thanks to the hypothesis on b, f, y_0, y_l and y_r:

- $(b_n) \in \mathcal{C}^1(\Omega_T)$ such that $||b_n - b||_{L^1(\Omega_T)} \to 0$ and $||b_n||_{L^\infty(\Omega_T)}$ is bounded,
- $(f_n) \in \mathcal{C}_c^1(\Omega_T \setminus P)$ such that $||f_n - f||_{L^1(\Omega_T)} \to 0$ and $||f_n||_{L^\infty(\Omega_T)}$ is bounded,
- $(y_{0,n}) \in \mathcal{C}_c^1(0,1)$ such that $||y_{0,n} - y_0||_{L^1(0,1)} \to 0$ and $||y_{0,n}||_{L^\infty(0,1)}$ is bounded,
- $(y_{l,n}) \in \mathcal{C}_c^1(\Gamma_l)$ such that $||y_{l,n} - y_l||_{L^1(\Gamma_l)} \to 0$ and $||y_{l,n}||_{L^\infty(\Gamma_l)}$ is bounded,
- $(y_{r,n}) \in \mathcal{C}_c^1(\Gamma_r)$ such that $||y_{r,n} - y_l||_{L^1(\Gamma_r)} \to 0$ and $||y_{r,n}||_{L^\infty(\Gamma_r)}$ is bounded.

We call (y_n) the sequence of strong solutions to (1.48). Thanks to (1.54) we can extract so that:

$$\exists y \in L^\infty(\Omega_T) \text{ such that } y_n \text{ converges to } y \text{ for the weak-* topology of } L^\infty(\Omega_T).$$

Now we take the limit in (1.56) and conclude that y is a weak solution to (1.48).
We can also suppose (we just need to extract again) that we have pointwise convergence almost everywhere of:

$$b_n \to b, \quad f_n \to f, \quad y_{0,n} \to y_0, \quad y_{l,n} \to y_l, \quad y_{r,n} \to y_r.$$

Thanks to the dominated convergence theorem and to the limit uniqueness, we see that y satisfies (1.53) and (1.55) almost everywhere, and this provides (1.57). □

1.4.4 Uniqueness of the weak solution

We have proved the existence of a weak solution to (1.48) and we have the bound (1.57), therefore the initial boundary value problem will be well posed once we have shown the uniqueness of the weak solution.

Proposition 13. *Under the hypothesis of the theorem 6, there is only one weak solution to* (1.48).

Proof. By linearity we only need to prove the uniqueness for $f = 0$, $y_0 = 0$, $y_l = 0$, $y_r = 0$. Which is $\forall y \in L^\infty(\Omega_T)$:

$$\left(\forall \phi \in Adm(\Omega_T) \quad \int_{\Omega_T} y.(\partial_t \phi + a.\partial_x \phi + (b + \partial_x a).\phi) dx dt = 0 \right) \Rightarrow y = 0 \quad a.e.$$

Let y be such as above, we take:

- $y_n \in \mathcal{C}_c^1(\Omega_T \setminus P)$ such that $||y_n - y||_{L^2(\Omega_T)} \to 0$ and $||y_n||_{L^\infty(\Omega_T)}$ is bounded,

- $d_n \in \mathcal{C}^1(\Omega_T)$ such that $||d_n - (b + \partial_x a)||_{L^2(\Omega_T)} \to 0$ and $||d_n||_{L^\infty(\Omega_T)}$ is bounded.

We want $\phi_n \in Adm(\Omega_T)$ to be a strong solution of $\partial_t \phi_n + a.\partial_x \phi_n + d_n.\phi_n = y_n$, but the boundary conditions for functions in $Adm(\Omega_T)$ makes it a backward problem. Indeed for ϕ_n to be a test function we must have $\forall x \in [0,1],\ \phi_n(T,x) = 0,\ \forall t \in [0,T] \setminus \Gamma_l,\ \phi_n(t,0) = 0$ and $\forall t \in [0,T] \setminus \Gamma_r,\ \phi_n(t,1) = 0$. As we said previously the change of variables $t \to T - t$ transforms a backward problem in a regular forward one, which we can solve thanks to section 1.4.2. We just need to realize that the change of variables $t \to T - t$ sends the old P on the new P, the old $[0,T] \setminus \Gamma_l$ on the new $\Gamma_l \cup P_l$ and the old $[0,T] \setminus \Gamma_r$ on the new $\Gamma_l \cup P_r$.

$$\text{And therefore: } \forall n \in \mathbb{N},\ \int_{\Omega_T} y.(y_n + \phi_n(b + \partial_x a - d_n))dxdt = 0.$$

Now thanks to the hypothesis on y_n and d_n, and to (1.54), when $n \to +\infty$ we get

$$\int_{\Omega_T} |y(t,x)|^2 dxdt = 0.$$

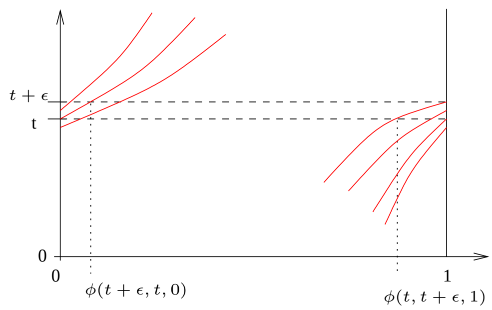
□

1.4.5 Additional properties of y

Until now weak solutions had only the L^∞ regularity but in fact we have more.

Lemma 4. *If a and $\partial_x a$ are continuous and if the sets $P_l = \{t \in [0,T] \mid a(t,0) = 0\}$ and $P_r = \{t \in [0,T] \mid a(t,1) = 0\}$ have a finite number of connected components, and if b and f are in $L^\infty(\Omega_T)$ then $\forall p < +\infty$ we have $||y||_{L^p(0,1)} \in \mathcal{C}^0([0,T])$.*

Proof. Let $t \in [0,T]$ and $\epsilon \geq 0$. Reducing ϵ if necessary we can suppose that $a(s,0)$ and $a(s,1)$ have a constant sign on $[t, t+\epsilon]$. Hence we will prove the result in the case $a(t,0) \geq 0$ and $a(t,1) \geq 0$ (the other cases being similar). This implies $h(t,0) \geq t + \epsilon$ and $e(t+\epsilon, 1) \leq t$:



Now we have:

$$||y(t+\epsilon,.)||^p_{L^p(0,1)} = \int_0^{\phi(t+\epsilon,t,0)} |y(t+\epsilon,x)|^p dx + \int_{\phi(t+\epsilon,t,0)}^1 |y(t+\epsilon,x)|^p dx$$

since $\phi(t+\epsilon,t,0) \underset{\epsilon \to 0}{\longrightarrow} 0$ and $y \in L^\infty(\Omega_T)$ the first integral tends to 0. Then, if $x \in [\phi(t+\epsilon,t,0), 1]$ we recall that thanks to (1.55) and after performing the change of variables $\tilde{x} = \phi(t, t+\epsilon, x)$ one has:

$$\int_{\phi(t+\epsilon,t,0)}^{1} |y(t+\epsilon,x)|^p dx = \int_0^{\phi(t,t+\epsilon,1)} \Big| y(t,\tilde{x}).\exp\left(\int_t^{t+\epsilon} b(s,\phi(s,t,\tilde{x}))ds\right)$$
$$+ \int_t^{t+\epsilon} f(t,\phi(r,t,\tilde{x})).\exp\left(\int_r^{t+\epsilon} b(r',\phi(r',t,\tilde{x}))dr'\right)dr\Big|^p \times \exp\left(\int_t^{t+\epsilon} \partial_x a(s,\phi(s,t,\tilde{x}))ds\right) d\tilde{x}. \tag{1.58}$$

And finally since $\phi(t,t+\epsilon,1) \underset{\epsilon\to 0^+}{\to} 1$, $f,b,y \in L^\infty(\Omega_T)$ and $\partial_x a \in \mathcal{C}^0(\Omega_T)$ we get

$$\int_{\phi(t+\epsilon,t,0)}^{1} |y(t+\epsilon,x)|^p dx \xrightarrow[\epsilon\to 0^+]{} \int_0^1 |y(t,x)|^p dx.$$

The other geometries of the characteristics are treated in the same way. And the argument is clearly reversible in time so we also have the case $\epsilon \leq 0$. □

Now we can get some additional regularity for y.

Proposition 14. *If a and $\partial_x a$ are continuous, if the sets P_l and P_r defined by:*

$$P_l = \{t \in [0,T] \mid a(t,0) = 0\}, \qquad P_r = \{t \in [0,T] \mid a(t,1) = 0\},$$

have a finite number of connected components, if y_0, y_l y_r are essentially bounded and if b and f are in $L^\infty(\Omega_T)$ then $\forall p < +\infty$ we have $y \in \mathcal{C}^0([0,T], L^p(0,1))$.

Proof. We take $t = 0$ and $\epsilon > 0$. Reducing ϵ if necessary, we can suppose that $a(s,0)$ and $a(s,1)$ have a constant sign on $[t, t+\epsilon]$. We will prove the result in the case $a(t,0) \geq 0$ and $a(t,1) \leq 0$ (the others can be treated in the same way). This implies:

$$h(0,1), h(0,0) \geq \epsilon.$$

Let $\gamma > 0$, since $y_0 \in L^\infty(0,1)$ we have a function $\tilde{y}_0 \in \mathcal{C}^0([0,1])$ such that $||y_0 - \tilde{y}_0||_{L^p(0,1)} \leq \gamma$. We now consider $\tilde{y}$ the weak solution of (1.48) with boundary value y_l and y_r and initial value $\tilde{y}_0$. By linearity it is clear that $y - \tilde{y}$ is solution to (1.48) with boundary value 0 and initial value $y_0 - \tilde{y}_0$. Therefore the previous lemma asserts that $t \mapsto ||y(t,.) - \tilde{y}(t,.)||_{L^p(0,1)}$ is continuous and we see that for t sufficiently small

$$||y(t,.) - \tilde{y}(t,.)||_{L^p(0,1)} \leq 2\gamma.$$

Now since $\tilde{y}$ satisfies (1.55), since $b, f, \tilde{y} \in L^\infty(\Omega_T)$ and more importantly since $\tilde{y}_0$ continuous, we obtain:

$$\tilde{y}(\epsilon,x) \underset{\epsilon\to 0^+}{\to} \tilde{y}_0(x) \text{ for any } x \text{ in } (0,1).$$

This in turn provides:

$$\int_{\phi(\epsilon,0,0)}^{\phi(\epsilon,0,1)} |\tilde{y}(\epsilon,x) - \tilde{y}_0(x)|^p dx \underset{\epsilon\to 0^+}{\to} 0.$$

And finally we conclude that for ϵ sufficiently small we have:

$$||\tilde{y}(\epsilon,.) - \tilde{y}_0(.)||_{L^p(0,1)} \leq \gamma,$$

which implies that for ϵ small enough:

$$||y(\epsilon,.) - y_0(.)||_{L^p(0,1)} \leq 4\gamma.$$

We can both translate and reverse the argument in time. □

To finish this part we will prove an inequality about the continuity property of the linear operator providing y in term of f, y_0, y_l and y_r.

Proposition 15. *If a and $\partial_x a$ are continuous and if the sets P_l and P_r defined by:*

$$P_l = \{t \in [0,T] \mid a(t,0) = 0\}, \qquad P_r = \{t \in [0,T] \mid a(t,1) = 0\},$$

have a finite number of connected components then we have the inequality:

$$\forall t \in [0,T] \quad ||y(t,.)||_{L^1(0,1)} \leq (||f||_{L^1((0,t)\times(0,1))} + ||y_0||_{L^1(0,1)} + ||y_l||_{L^1((0,t)\cap\Gamma_l)} + ||y_r||_{L^1((0,t)\cap\Gamma_r)}) \times ||a||_{L^\infty(\Omega_T)}) e^{t(||\partial_x a||_{L^\infty(\Omega_T)} + ||b||_{L^\infty(\Omega_T)})}. \quad (1.59)$$

Proof. Let us first suppose that $a(s,0), a(s,1) \geq 0$ on $[0,T]$, this implies $h(0,0) \geq t$ and $e(t,1) = 0$, therefore we can write:

$$\begin{aligned}
||y(t,.)||_{L^1(0,1)} \leq & \int_0^{\phi(t,0,0)} \left| y_l(e(t,x)).\exp\left(\int_{e(t,x)}^t b(r,\phi(r,t,x))dr\right)\right| dx \\
&+ \int_0^{\phi(t,0,0)} \left| \int_{e(t,x)}^t f(t,\phi(r,t,x)).\exp\left(\int_s^t b(s,\phi(s,t,x))ds\right) dr\right| dx \\
&+ \int_{\phi(t,0,0)}^1 \left| y_0(\phi(0,t,x)).\exp\left(\int_0^t b(r,\phi(r,t,x))dr\right)\right| dx \\
&+ \int_{\phi(t,0,0)}^1 \left| \int_0^t f(t,\phi(r,t,x)).\exp\left(\int_s^t b(s,\phi(s,t,x))ds\right) dr\right| dx \\
=& I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now we will treat each I_k separately. In I_1 we perform the change of variables: $s = e(t,x)$ (or equivalently $x = \phi(t,s,0)$) and we get:

$$I_1 = \int_0^t |y_l(s)|.a(s,0).\exp\left(\int_s^t b(r,\phi(r,s,0)) + \partial_x a(r,\phi(r,s,0))dr\right) ds$$

Therefore we have $I_1 \leq ||y_l||_{L^1(0,t)}.||a||_{L^\infty(\Omega_T)}.e^{t.(||\partial_x a||_{L^\infty(\Omega_T)} + ||b||_{L^\infty(\Omega_T)})}$.
For the second integral we have:

$$\begin{aligned}
I_2 =& \int_0^{\phi(t,0,0)} \left| \int_{e(t,x)}^t f(t,\phi(r,t,x)).\exp\left(\int_s^t b(s,\phi(s,t,x))ds\right) dr\right| dx \\
\leq & \int_0^t \int_{\phi(s,0,0)}^{\phi(t,0,0)} |f(t,\phi(r,t,x))|.\exp\left(\int_s^t b(s,\phi(s,t,x))ds\right) dxdr.
\end{aligned}$$

This time we perform the change of variables: $\tilde{x} = \phi(r,t,x)$. And we get:

$$I_2 \leq e^{t(||\partial_x a||_{L^\infty(\Omega_T)} + ||b||_{L^\infty(\Omega_T)})} \times \int_0^t \int_0^{\phi(r,0,0)} |f(t,\tilde{x})| d\tilde{x} dr.$$

In the same way we obtain:

$$I_3 \leq e^{t(||b||_{L^\infty(\Omega_T)} + ||\partial_x a||_{L^\infty(\Omega_T)})} \times \int_0^{\phi(0,t,1)} |y_0(\tilde{x})| d\tilde{x}.$$

And finally for I_4 we use $\tilde{x} = \phi(r,t,x)$ to obtain:

$$I_4 \leq e^{t(||b||_{L^\infty(\Omega_T)}+||\partial_x a||_{L^\infty(\Omega_T)})} \int_0^t \int_{\phi(r,0,0)}^{\phi(r,t,1)} |f(t,\tilde{x})| d\tilde{x} dr.$$

Combining the inequalities on I_1, I_2, I_3 and I_4 we get (1.59). However we supposed that $a(s,0)$ and $a(s,1)$ did not change signs between on $[0,T]$. Therefore if either $a(s,0)$ or $a(s,1)$ change sign at time t_1 we only have the desired estimates separately on $[t_0,t_1]$ and on $[t_1,t_2]$ where on each interval, $a(s,0)$ and $a(s,1)$ do not change sign. More precisely if $t \in [t_1,t_2]$ we have:

$$\begin{aligned}||y(t_1,.)||_{L^1(0,1)} \leq (&||f||_{L^1((t_0,t_1)\times(0,1))} + ||y(t_0,.)||_{L^1(0,1)} + ||y_l||_{L^1((t_0,t_1)\cap\Gamma_l)} + ||y_r||_{L^1((t_0,t_1)\cap\Gamma_r)}) \\ &\times ||a||_{L^\infty(\Omega_T)}).e^{(t_1-t_0)(||\partial_x a||_{L^\infty(\Omega_T)}+||b||_{L^\infty(\Omega_T)})} \\ ||y(t,.)||_{L^1(0,1)} \leq (&||f||_{L^1((t_1,t)\times(0,1))} + ||y(t_1,.)||_{L^1(0,1)} + ||y_l||_{L^1((t_1,t)\cap\Gamma_l)} + ||y_r||_{L^1((t_1,t)\cap\Gamma_r)}) \\ &\times ||a||_{L^\infty(\Omega_T)}).e^{(t-t_1)(||\partial_x a||_{L^\infty(\Omega_T)}+||b||_{L^\infty(\Omega_T)})}\end{aligned}$$

And now we can substitute $||y(t_1,.)||_{L^1(0,1)}$ in the right side of (1.4.5) with the right side of (1.4.5), which provides (1.59) on the whole interval $[t_0,t_2]$. Finally since we know that $a(s,0)$ and $a(s,1)$ change sign only a finite number of time, the previous argument allows us to extend (1.59) to $[0,T]$. □

Remark 10. *The previous estimate and the well posedness in $L^\infty(\Omega_T)$ of the initial boundary value problem* (1.12) *for data y_0, y_l, y_r and f in L^∞ show that the same problem is well posed in $\mathcal{C}([0,T];L^1(0,1))$ with data in L^1. And then since the equation is linear and because we have both the well-posedness in $L^\infty(\Omega_T)$ with essentially bounded data, and also the well-posedness in $\mathcal{C}^0([0,T];L^1(0,1))$ with summable data we can interpolate the two results and get well posedness in $\mathcal{C}^0([0,T];L^p(0,1))$ with data in L^p.*

Chapitre 2

Contrôlabilité exacte des solutions entropiques de lois de conservation scalaires avec un contrôle additionnel.

Abstract. In this paper, we study the exact controllability problem for nonlinear scalar conservation laws on a compact interval, with a regular convex flux and in the framework of entropy solutions. With the boundary data and a source term depending only on the time as controls, we provide sufficient conditions for a state to be reachable in arbitrary small time. To do so we introduce a slightly modified wave-front tracking algorithm.

2.1 Introduction

This paper is concerned with the exact controllability problem of a nonlinear scalar conservation law with a source term, on a bounded interval and in the framework of entropy solutions:

$$\begin{aligned} &\partial_t u + \partial_x f(u) = g(t), \\ &u(0,x) = u_0(x), \\ &u(t,0) = u_l(t), \\ &u(t,L) = u_r(t), \end{aligned} \qquad (t,x) \in (0,T)\times(0,L), \tag{2.1}$$

where f is assumed to be a $\mathcal{C}^2$ strictly convex function.

Scalar conservation laws are used for instance to model traffic flow or gas networks, but their importance also consists in being a first step in the understanding of systems of conservation laws. Those systems of equations model a huge number of physical phenomena: gas dynamics, electromagnetism, magneto-hydrodynamics, shallow water theory, combustion theory... see [45, Chapter2].

In this paper, we study (3.1) from the point of view of control theory and we regard the boundary data u_l, u_r and the source term g as controls. We will provide sufficient conditions on a state u_1 in $\mathrm{BV}(0,L)$ so that for any time T and any u_0 in $\mathrm{BV}(0,L)$ there exist u_l and u_r in $\mathrm{BV}(0,T)$ and g in $\mathcal{C}^1([0,T])$ such that $u(T,.) = u_1$.

For equations such as (3.1), the Cauchy problem on the whole line is well posed in small time in the framework of classical solutions and with a classical initial value. However those solutions generally blow up in finite time: shock waves appear. Hence to get global in time results, a weaker notion of solution is called for. In [87] Oleinik proved that given a flux $f \in \mathcal{C}^2$ such that $f'' > 0$ and any $u_0 \in L^\infty(\mathbb{R})$ there exists one and only one weak solution to:

$$u_t + (f(u))_x = 0, \qquad x \in \mathbb{R} \text{ and } t > 0, \tag{2.2}$$
$$u(0,.) = u_0, \tag{2.3}$$

satisfying the additional condition:

$$\frac{u(t,x+a) - u(t,x)}{a} < \frac{E}{t} \qquad \text{for } x \in \mathbb{R}, \quad t > 0, \text{ and } a > 0, \tag{2.4}$$

and where E depends only on the quantities $\inf(f'')$ and $\sup(f')$ taken on $[-||u_0||_{L^\infty}, ||u_0||_{L^\infty}]$ and not on u_0. Later in [82], Kruzkov extended this global result to the multidimensional problem, with a C^1 flux $f : \mathbb{R} \to \mathbb{R}^n$ not necessarily convex and with a different entropy condition:

$$u_t + \operatorname{div}(f(u)) = 0, \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^n. \tag{2.5}$$

This time the weak entropy solution is defined as satisfying the following integral inequality:

$$\text{for all real numbers } k \text{ and all positive functions } \phi \text{ in } \mathcal{C}^1_c(\mathbb{R}^2), \tag{2.6}$$
$$\iint_{\mathbb{R}^2} |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\nabla\phi dtdx + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx \geq 0. \tag{2.7}$$

The initial boundary value problem for equation (3.1) is also well posed as shown by Leroux in [80] for the one dimensional case with BV data, by Bardos Leroux and Nedelec in [15] for the multidimensional case with C^2 data and later by Otto in [88] (see also [86]) for L^∞ data. However the meaning of the boundary condition is quite intricate and the Dirichlet condition may not be fulfilled pointwise a.e. in time. In the following, we will use the fact that the restriction of a weak entropy solution of (3.1) on the whole line is the weak entropy solution to the IBVP on an interval with boundary data given by its trace at the boundary points (which exists the solution being in BV).

In the framework of entropy solutions, only a few controllability results exist for equation (3.1). In [4], Ancona and Marson characterized exactly the reachable states of

$$\begin{cases} \partial_t u + \partial_x(f(u)) = 0, & t > 0, \quad x > 0, \\ u(0,x) = 0, \quad x > 0, & u(t,0) = c(t), \quad t > 0. \end{cases} \tag{2.8}$$

where f is strictly convex and with a boundary control c. A state $w \in L^\infty(0,+\infty)$ is reachable in time T if and only if the following conditions hold:

$$\begin{gathered} w(x) \neq 0 \Rightarrow f'(w(x)) \geq \frac{x}{T}, \\ \big(w(x^-) \neq 0 \text{ and for every } y \text{ greater than } x,\ w(y) = 0\big) \Rightarrow f'(w(x^-)) > \frac{x}{T}, \\ \limsup_{h\to 0} \frac{w(x+h) - w(x)}{h} \leq \frac{f'(w(x))}{xf''(w(x))}, \end{gathered} \tag{2.9}$$

for every $x > 0$. The first two conditions are related to the propagation speed of (2.8) and the third is analogous to (2.4) but in the presence of a boundary.
In [69] Horsin provided sufficient conditions (related to (2.9)) on a state to be reachable for the Burgers equation posed on a compact interval and with a general initial data and where the

controls are the two boundary values.
There are also some results on the controllability and non-controllability of systems of conservation laws in the context of entropy solutions by Bressan and Coclite [17], Ancona and Coclite [3], Ancona and Marson [5] and by Glass [57]. In all those cases, some very reasonable looking states cannot be reached in any time using only boundary control. For example in the case of Burgers' equation on a compact interval, the constant state 0 cannot be reached from most initial states in any given time.

However with an additional control $g(t)$ as in (3.1) and with $f(z) = \frac{z^2}{2}$ (Burgers equation), Chapouly showed in [20] that in the framework of classical solutions, any state is reachable from any initial data and in any time (note that in this context, the controls also had to prevent the blow up of the solution, which will not be a concern for entropy solutions). This is the kind of improvement we want to obtain in the framework of entropy weak solutions, and for more general convex fluxes.

In the case of the Burgers equation, our additional control g can be seen as a pressure field, but (3.1) is also a toy model for the Euler-Poisson system:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \qquad \rho(0,.) = \rho_0, \qquad \rho(t,0) = \rho_l(t), \quad \rho(t,L) = \rho_r(t), \\ \partial_t m + \partial_x(\dfrac{m^2}{\rho} + p(\rho)) = \rho \partial_x V - m, \quad m(0,.) = m_0, \ m(t,0) = m_l(t), \ m(t,L) = m_r(t), \\ -\partial^2_{xx} V = \rho, \qquad V(t,0) = V_l(t), \quad V(t,L) = V_r(t), \end{cases} \tag{2.10}$$

where the controls are ρ_l, ρ_r, m_l, m_r, V_l and V_r. Indeed once we take $g(t) = \frac{V_r(t) - V_l(t)}{L}$ we get $V_x = g(t) + A\rho(t)$ with A a linear integral operator. So we have to deal with an hyperbolic system controlled by the boundary data and an additional source term depending only on the time variable.

2.2 Statement of the results.

In what follows f is a $\mathcal{C}^2$ strictly convex function and L and T are positive numbers. We consider the equation:

$$\partial_t u + \partial_x(f(u)) = g(t), \qquad \text{for } (t,x) \in (0,T) \times (0,L), \tag{2.11}$$

where g is a $\mathcal{C}^1$ function that we can specify, that is, a control.

We begin by recalling the definition of an entropy solution for a scalar conservation law.

Definition 8. *A couple of $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ functions (η, q) is a convex entropy-flux pair for f if:*

$$\eta \text{ is convex and } \forall z \in \mathbb{R}, \ \eta'(z) f'(z) = q'(z).$$

Now we say that $u \in L^\infty((0,T)\times(0,L))$ is an entropy solution of (2.11) if for all non-negative functions ϕ in $\mathcal{C}^1_c((0,T) \times (0,L))$ and all convex entropy-flux pairs (η, q) we have:

$$\int_0^T \int_0^L \eta(u(t,x)) \partial_t \phi(t,x) + q(u(t,x)) \partial_x \phi(t,x) + \eta'(u(t,x)) \phi(t,x) g(t) dx dt \geq 0. \tag{2.12}$$

It will be useful to consider only the class representatives of BV functions that are right-continuous, which is possible since the discontinuity points of such a function are countable, we will do so in all the paper. We now provide our first controllability result concerning (2.11).

Theorem 7. *Let* $u_1 \in \mathrm{BV}(0,L)$ *satisfy:*

$$\sup_{\substack{0<h<L \\ 0<x<L-h}} \frac{u_1(x+h)-u_1(x)}{h} < +\infty, \tag{2.13}$$

and suppose that f *satisfy one of the following conditions:*

$$\frac{f'(M)}{\sup\limits_{z\in[0,M]} f''(z)} \to +\infty \text{ as } M \to +\infty \quad \text{or} \quad \frac{f'(M)}{\sup\limits_{z\in[M,0]} f''(z)} \to -\infty \text{ as } M \to -\infty. \tag{2.14}$$

Then for any positive time T *and any* u_0 *in* $\mathrm{BV}(0,L)$ *there exist two functions* g *and* u *respectively in* $\mathcal{C}^1([0,T])$ *and* $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$*, such that* u *is an entropy solution of* (2.11) *on* $(0,T)\times(0,L)$ *and*

$$u(0,.) = u_0 \quad \text{and} \quad u(T,.) = u_1 \quad \text{in } (0,L).$$

Remark 11.
- *Estimates* (2.13) *and* (2.4) *are of similar nature but* (2.13) *is much less restrictive since this supremum can be arbitrarily large.*
- *The first two conditions of* (2.9) *are replaced here by* (2.14) *which concerns only the flux. Therefore many more states are reachable with the additional control* g*. Furthermore they are reachable in arbitrarily small time.*

We now provide some results in the case where the semi-Lipschitz condition (2.13) degenerates near one boundary point. Indeed we can see that in the third condition of (2.9), the right-hand side can blow up as $x \to 0^+$, which is not the case for (2.13). Since the transformation:

$$X = L - x, \quad F(z) = f(-z), \quad v(t,X) = -u(t,x), \tag{2.15}$$

transforms an entropy solution u of (2.11) in an entropy solution v of $\partial_t v + \partial_X F(v) = -g$ with $F(z) = f(-z)$ also a convex function, and exchanges the boundary points, we will only consider the case where the degeneracy takes place at 0.

Now to quantify this degeneracy, we introduce the following function K:

$$\forall x \in (0,L), \qquad K(x) = \left(\sup_{\substack{x\le y<L \\ 0<h<L-y}} \frac{u_1(y+h)-u_1(y)}{h} \right)_+ . \tag{2.16}$$

From now on, we will always suppose that K is finite at each point of $(0,L)$. It is obviously non-increasing and non-negative therefore it may only blow up at 0. In the case of such a blow-up we have the following sufficient condition for controllability.

Theorem 8. *Let* $u_1 \in \mathrm{BV}(0,L)$ *satisfy both*

$$K(x) = O\left(\frac{1}{x^p}\right) \quad \text{and} \quad \left(u_1(0) - \inf_{0<y\le x} u_1(y)\right) = O(x^{2p}) \quad \text{as } x \to 0^+. \tag{2.17}$$

Let us define:

$$\forall M > 0, \ I_M = [\inf_{x\in(0,L)} u_1(x), \ \sup_{x\in(0,L)} u_1(x) + M], \tag{2.18}$$

and suppose that for a certain $q > 0$, such that $p(2q+1) \leq 1$, the flux f satisfies both:

$$\frac{M^q}{\sup\limits_{z \in I_M} f''(z)} \xrightarrow[M \to +\infty]{} 0 \qquad \text{and} \qquad \frac{|h|^q}{|f'(u_1(0)+h)|} = O(1) \text{ at } 0 \text{ and at } +\infty. \tag{2.19}$$

Then for any T positive and any u_0 in $\mathrm{BV}(0,L)$ there exist two functions g and u respectively in $\mathcal{C}^1([0,T])$ and $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$ such that the following holds:

- *u is an entropy solution of (2.11) on $(0,T) \times (0,L)$ with $u(0,.) = u_0$,*
- *at the final time T we have both $u(T,.) = u_1$ and $g(T) = 0$.*

Remark 12.

- *This contains the fluxes of shape $f(z) = |z|^{q+1}$ with q less than $\frac{1}{2p} - \frac{1}{2}$.*
- *The fact that at time T, g is C^1 and equal to zero is restrictive. The compatibility condition $p(2q+1) \leq 1$ could be improved by removing either hypotheses.*
- *Note that in comparison to conditions (2.9), the compatibility condition $p(2q+1) \leq 1$ is a new phenomenon.*
- *Using the theory of generalized characteristics of Dafermos [44], it is easy to show that an entropy solution u of (2.11) satisfies the following necessary condition. For $0 < x < y < L$ let us take:*

$$V = \max(|f'(||u(t,.)||_{L^\infty(0,L)} + ||g||_{L^1(0,t)})|, |f'(-||u(t,.)||_{L^\infty(0,L)} - ||g||_{L^1(0,t)})|),$$

$$\text{then we have:} \quad u(t,y) - u(t,x) \leq \frac{V}{\inf(f''(z))} \frac{1}{\min(L-y, x, Vt)}(y-x). \tag{2.20}$$

So we see that the semi-Lipschitz condition (2.13) may a priori blow up at both endpoints.

We conclude this part with the most general result on controllability properties for equation (2.11).

Theorem 9. *Suppose that $f'(z)$ tends to infinity as z does. Let u_1 be in $\mathrm{BV}(0,L)$ and let $\bar{T}$ be a positive number. We introduce the following notations:*

$$\forall x \in (0,L), \ \tau(x) = \min\left(\bar{T}, \frac{1}{2K(x)} \frac{1}{\sup\limits_{z \in I_M} f''(z)}\right). \tag{2.21}$$

Suppose that there exists a non-positive function $\bar{g} \in \mathcal{C}^1([0,\bar{T}])$ such that:

$$\liminf_{\beta \to 0^+} \sup_{\frac{3\beta}{2} \leq \alpha < L} \left(\alpha - \int_{\bar{T}-\tau(\alpha-\beta)}^{\bar{T}} f'\left(\inf_{\alpha-\beta \leq x \leq \alpha} u_1(x) - \int_s^{\bar{T}} \bar{g}(r)dr\right) ds\right) \leq 0. \tag{2.22}$$

Then for any time T larger that $\bar{T}$ and any function u_0 in $\mathrm{BV}(0,L)$ there exist two functions g in $\mathcal{C}^1([0,T])$ and u in $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$ such that:

u is an entropy solution of (2.11) on $(0,T) \times (0,L)$,

$u(0,.) = u_0$ and $u(T,.) = u_1$ in $(0,L)$.

Before proving the results above let us make a few general comments on the problem and on the method which we will use. The linearization of equation (2.11) is problematic because of the lack of regularity and also because the linearized equation is no longer in conservative form. Therefore we will rather construct approximate solutions using a wave-front tracking algorithm and then use a classical compactness argument to get a trajectory solving the exact controllability problem. It should be noted that another approach would be to control the viscous equation and then let the viscosity tend to zero while keeping uniformly bounded controls, as in [62] or [75].

A first obvious remark is that when both the initial and final states u_0 and u_1 are constant functions on $(0, L)$, the exact controllability problem for (2.11) is reduced to finding g in $\mathcal{C}^1([0,T])$ such that $\int_0^T g(s)ds = u_1 - u_0$ which is trivial for any choice of T, $g(0)$ and $g(T)$. Now we follow the strategy of the return method J.-M. Coron introduced in [33] (see also [34]): rather than keeping the control small and use a linearization argument, we use large controls and the nonlinearity to perturb the system. More precisely more precisely we proceed in two steps, in the first we begin with a general initial value and end with a chosen constant one, in the second with begin with a constant initial value and end with a more general one

The rest of the paper is organized as follows. In the next section, we will prove that for any initial condition u_0 in $\mathrm{BV}(0, L)$ and any positive T, we can find g in $C^1([0,T])$ and an entropy solution of (2.11) such that both $u(0, .) = u_0$ and $u(T, .)$ is constant on $(0, L)$. In Section 4 we will prove the remaining part of Theorem 9: given $\bar{T}$ positive, u_1 in $\mathrm{BV}(0, L)$ and a flux f satisfying the hypotheses of the theorem we can construct g and an entropy solution u of (2.11) such that $u(\bar{T}, .) = u_1$ and $u(0, .)$ is constant on $(0, L)$. In Section 5 we show how we deduce Theorem 7 from Theorem 9. And in Section 6 we prove Theorem 8 using Theorem 9. Finally we collect useful results on our wave-front tracking algorithm in an appendix.

2.3 Control toward a constant.

The aim of this section is to prove the following result dealing with the exact controllability problem from a general initial data toward a final constant state in arbitrarily small time.

Proposition 16. *Let u_0 be in $\mathrm{BV}(0, L)$, and T be a positive number. There exist g and u respectively in $\mathcal{C}^1([0,T])$ and $L^\infty((0,T); \mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T]; L^1(0,L))$ such that:*

- *u is an entropy solution of (2.11),*
- *$u(0, .) = u_0$ on $(0, L)$,*
- *$u(T, .)$ is constant on $(0, L)$.*

Proof. Take g non-negative in $\mathcal{C}^1([0,T])$ such that the following condition is satisfied:

$$\int_0^T f'\left(\int_0^t g(s)ds - ||u_0||_{L^\infty(0,1)}\right) dt \geq L, \tag{2.23}$$

and define:

$$c(t) = \int_0^t f'\left(\int_0^r g(s)ds - ||u_0||_{L^\infty(0,1)}\right) dr. \tag{2.24}$$

We first recall a classical lemma.

Lemma 5. *If (u_n) is a family of functions defined on $[t_1, t_2] \times (a, b)$ and C a constant independent of n such that:*

$$\forall t \in [t_1, t_2], \qquad ||u_n(t, .)||_{\mathrm{BV}((a,b))} \le C, \tag{2.25}$$

$$\forall t, s \in [t_1, t_2], \qquad \int_a^b |u_n(t, z) - u_n(s, z)|dz \le C|t - s|. \tag{2.26}$$

Then we can extract $(u_{\psi(n)})_{n\ge 0}$ and get u satisfying (2.25), (2.26) and such that:

$$\left.\begin{array}{r}||u_{\psi(n)} - u||_{L^1_{loc}((0,T)\times(a,b))} \to 0 \\ \forall t \in [0, T], \quad ||u_{\psi(n)}(t, .) - u(t, .)||_{L^1_{loc}(a,b)} \to 0\end{array}\right. \quad \text{when } n \to +\infty. \tag{2.27}$$

Proof. See [16] Chapter 2 Section 4. □

Now we will to construct u by approximation.

Lemma 6. *Suppose that we have a sequence $(u_n)_{n\ge 1}$ satisfying the following properties:*

1. *the family $(u_n)_{n\ge 1}$ is bounded in $L^\infty((0, T); \mathrm{BV}(\mathbb{R})) \cap \mathrm{Lip}([0, T], L^1_{loc}(\mathbb{R}))$,*
2. *we have $||u_n(0, .) - u_0||_{L^1(0,L)} \to 0$ when $n \to +\infty$,*
3. *for every entropy-flux pair (η, q), we have:*

$$\limsup_{n\to+\infty} \int_0^T\int_0^L \eta(u_n(t, x))\partial_t\phi(t, x) + q(u_n(t, x))\partial_x\phi(t, x) + \eta'(u_n(t, x))\phi(t, x)g(t)dtdx \le 0, \tag{2.28}$$

4. *the functions $u_n(t, x) - \int_0^t g(s)ds$ are constant on the set $\{(t, x) \in [0, T] \times \mathbb{R} \mid x \le c(t)\}$.*

Then there exists u as in Proposition 16.

Proof. Using the first property above and the standard compactness result of lemma 5, we can extract a subsequence $(u_{\phi(n)})_{n\ge 1}$ that converges in $L^1_{loc}(\mathbb{R}^2)$ toward a function u such that:

$$u \in L^\infty((0, T); \mathrm{BV}(\mathbb{R})) \cap \mathrm{Lip}([0, T], L^1_{loc}(\mathbb{R})).$$

Furthermore we can also suppose that for every t in $[0, T]$ we have $||u_{\phi(n)}(t,) - u(t, .)||_{L^1(0,L)} \to 0$. The third property satisfied by $(u_n)_{n\ge 1}$ implies that u is an entropy solution of (2.11), the second that $u_0 = u(0, .)$ on $(0, L)$ and the last property together with (2.23) and (2.24) implies that $u(T, .)$ is constant on $(0, L)$. □

It only remains to construct such a family. We will do so using a wave-front tracking algorithm. Compared to the classical wave-front tracking algorithm (see [45] chapter 14 or [16] chapter 6) the discontinuities travel along piecewise $\mathcal{C}^1$ curves and not polygonal lines. Note that while we use this modification to deal with a source term $g(t)$, the same ideas might be used with $g(t, u)$.
To be more precise take

$$G_1(t) = \int_0^t g(s)ds \tag{2.29}$$

and introduce the following notion of wave-front tracking approximation.

Definition 9. *If ϵ is a positive number and u_ϵ a function defined on $[0,T]\times(0,L)$ we say that u_ϵ is an ϵ-approximate front tracking solution of (2.11) if:*

- *as a function of two variables $u_\epsilon(t,x)-G_1(t)$ is locally constant except on a finite number of curves $x=x_\alpha(t)$ which are $\mathcal{C}^1$ where the discontinuities are located and which we will call discontinuity fronts,*
- *for each curve x_α we have for a.e. t:*

$$\dot{x}_\alpha(t)=\frac{f(u_\epsilon(t,x_\alpha(t)^+))-f(u_\epsilon(t,x_\alpha(t)^-))}{u(t,x_\alpha(t)^+)-u(t,x_\alpha(t)^-)}, \tag{2.30}$$

- *for each curve x_α and a.e. time t we have*

$$u_\epsilon(t,x_\alpha(t)^+)\leq u_\epsilon(t,x_\alpha(t)^-)+\epsilon. \tag{2.31}$$

We have the following key property of the ϵ-approximate front tracking solution.

Lemma 7. *If u_ϵ is an ϵ-approximate front tracking solution of (2.11) and (η,q) is an entropy-flux pair then we have for every positive function ϕ in $\mathcal{C}^1_c((0,T)\times(0,L))$:*

$$\int_0^T\int_0^L(\eta(u_\epsilon(t,x))\partial_t\phi(t,x)+q(u_\epsilon(t,x))\partial_x\phi(t,x)+\eta'(u_\epsilon(t,x))\phi(t,x)g(t)dtdx \\ \geq -C||\phi||_{\mathcal{C}^0((0,T)\times(0,L))}\epsilon. \tag{2.32}$$

Where the constant C depends only on f, η, $||g||_{L^1(0,T)}$ and $||u_\epsilon||_{L^\infty((0,T),\mathrm{BV}(0,L))}$.

Proof. See the Appendix. □

Now to construct such a wave-front tracking approximation we proceed as follows. Let n be a positive integer, we define:

$$u_n^k=u_0(\frac{2k-1}{2n}L),\quad \text{for } 1\leq k\leq n,\qquad u_n^0=u_0(0),\quad u_n^{n+1}=u_0(L). \tag{2.33}$$

We take $u_n(0,x)$ on $\mathbb{R}$ equal to:

- u_n^k for $\frac{k-1}{n}L<x<\frac{k}{n}L$ and $1\leq k\leq n$,
- u_n^0 for $x<0$,
- u_n^{n+1} for $x>L$.

Now at each discontinuity point of $u^n(0,.)$, we approximately solve the Riemann problem as follows. We suppose that the discontinuity is at $x=0$ and that the left and right state are respectively v^- and v^+. Then

- if $v^->v^+$ the discontinuity is a shock and defining

$$\gamma(t)=\int_0^t\frac{f(v^-+\int_0^s g(r)dr)-f(v^++\int_0^s g(r)dr)}{v^--v^+}ds, \tag{2.34}$$

  we take:

$$v(t,x)=\begin{cases}v^-+\int_0^t g(r)dr & \text{if } x<\gamma(t)\\ v^++\int_0^t g(r)dr & \text{if } x>\gamma(t)\end{cases}. \tag{2.35}$$

- if $v^- < v^+$ take $p = \lceil n(v^+ - v^-) \rceil + 1$ and define

$$\text{for } 0 \leq l \leq p, \quad v^l = \frac{p-l}{p} v^- + \frac{l}{p} v^+, \tag{2.36}$$

$$\text{and for} 1 \leq l \leq p, \quad \gamma_l(t) = \int_0^t \frac{f(v^l + \int_0^s g(r)dr) - f(v^{l-1} + \int_0^s g(r)dr)}{v^l - v^{l-1}} ds. \tag{2.37}$$

Finally we define:

$$v(t,x) = \begin{cases} v^0 + \int_0^t g(r)dr & \text{if } x < \gamma_1(t) \\ v^l + \int_0^t g(r)dr & \text{if } \gamma_{l-1}(t) < x < \gamma_l(t) \text{ and } 1 \leq l \leq p-1 \\ v^p + \int_0^t g(r)dr & \text{if } \gamma_p(t) < x. \end{cases} \tag{2.38}$$

Now there is a small time during which all the discontinuity fronts created at time $t = 0$ do not intersect. And when two or more fronts interact at a time $t > 0$ we use the same procedure. It should be noted that only one front leaves the interaction point. In order to see that we begin with the following simple lemma.

Lemma 8. *let u^1, u^2, u^3 be three real numbers such that:*

$$\frac{f(u^1) - f(u^2)}{u^1 - u^2} > \frac{f(u^2) - f(u^3)}{u^2 - u^3}, \tag{2.39}$$

then $u^3 < u^1$

Proof. Straightforward from the convexity of f. □

Now if m fronts separating $m+1$ states $u^1, ..., u^{m+1}$ are interacting at time τ we have, thanks to the order of their respective speed:

$$\frac{f(u^{i-1}(\tau)) - f(u^i(\tau))}{u^{i-1}(\tau) - u^i(\tau)} > \frac{f(u^i(\tau)) - f(u^{i+1}(\tau))}{u^i(\tau) - u^{i+1}(\tau)}, \qquad \forall 1 \leq i \leq m. \tag{2.40}$$

Now using the lemma we get if m is even $u^1 > u^3 > u^5 > \cdots > u^{m+1}$ and the resulting front is a shock, if m is odd we have $u^1 > u^3 > u^5 > \cdots > u^m$ and since $u^{m+1} \leq u^m + \frac{1}{n}$ we can conclude that $u^{m+1} \leq u^1 + \frac{1}{n}$ and we have either a shock or a single rarefaction front.

Since the number of discontinuity fronts decreases at each interaction, this scheme allows us to define u_n on $\mathbb{R}+ \times \mathbb{R}$ and produce a $\frac{1}{n}$-approximate wave-front tracking solution. Furthermore since all the states separated by the discontinuity fronts are translated by the same value $\int_0^t g(s)ds$, the quantity $\mathrm{TotVar}(u^n(t,.))$ does not increase with time and the quantity $||u^n(t,.)||_{L^\infty(\mathbb{R})}$ is bounded by $||u_0||_{L^\infty(0,L)} + ||g||_{L^1(0,T)}$. Since u_n is a $\frac{1}{n}$-approximate wave-front tracking solution we can apply Lemma 7. And we see that the third property of Lemma 6 is satisfied. Obviously the second property is satisfied. And we also have the following estimate:

$$||u_n||_{L^\infty(0,T;\mathrm{BV}(0,L))} \leq ||u_0||_{\mathrm{BV}(0,L)} + ||g||_{L^1(0,T)}. \tag{2.41}$$

Furthermore since the speed of the discontinuity fronts is bounded by $f'(||u_n||_{L^\infty((0,T)\times(0,L)})$ we have:

$$||u_n(t+h,.) - u_n(t,.)||_{L^1(_{loc}(\mathbb{R})} \leq |h| \left(||g||_{L^\infty(0,T)} + 2||u_n||_{L^\infty((0,T)\times\mathbb{R})}.f'(||u_n||_{L^\infty((0,T)\times\mathbb{R})}) \right), \tag{2.42}$$

and we see that the first property of Lemma 6 is satisfied.

Finally for any n larger than 0 the leftmost discontinuity front $\gamma(t)$ satisfies the following:

$$\dot{\gamma}(t) = \frac{f(u_n^0 + \int_0^t g(r)dr) - f(u_n^k + \int_0^t g(r)dr)}{u_n^0 - u_n^k} \geq f'\left(-||u_0||_{L^\infty(0,L)} + \int_0^t g(r)dr\right) = \dot{c}(t), \tag{2.43}$$

where k may depend on t. Since $c(0) \leq \gamma(0)$ we end up with $\gamma(t) \geq c(t)$ for all positive time t. And using (2.23) and (2.24) we see that the fourth property of Lemma 6 is satisfied by $(u_n)_{n\geq 1}$. □

2.4 Proof of Theorem 9.

We now prove the following result which deals with the exact controllability from a constant state toward a state u_1 belonging to $\mathrm{BV}(0,L)$ and satisfying the hypotheses of Theorem 9. This concludes the proof of Theorem 9.

We recall that the function τ is defined in (2.21), and for a given number M the interval I_M in (2.18).

Proposition 17. *Consider $T > 0$, $M > 0$ and u_1 in $\mathrm{BV}(0,L)$. Suppose that there exists g in $\mathcal{C}^1([0,T])$ a non-positive function satisfying:*

$$||g||_{L^1(0,T)} \leq M, \qquad g(T) = 0, \tag{2.44}$$

$$\liminf_{\beta\to 0^+} \sup_{\alpha\in[\frac{3\beta}{2},L]} \left(\alpha - \int_{T-\tau(\alpha-\beta)}^T f'\left(\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x)) - \int_s^{\bar{T}} g(r)dr\right)ds\right) \leq 0. \tag{2.45}$$

Then there exists u in $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$ an entropy solution of (2.11), such that:

$$u(T,.) = u_1 \quad \text{and} \quad u(0,.) \text{ constant on } (0,L).$$

From now on we let G_2 be the function defined by:

$$G_2(s) = -\int_s^T g(r)dr. \tag{2.46}$$

We begin with a lemma dealing with two discontinuity fronts of a wave-front tracking approximation:

Lemma 9. *For $\alpha \in (0,L)$ and $0 < \beta < min(\alpha, L-\alpha)$ consider γ_+ and γ_- as follows:*

$$\gamma_\pm(T) = \alpha \pm \frac{\beta}{2}, \qquad \dot{\gamma}_\pm(t) = \frac{f(u_1(\alpha\pm\beta) + G_2(t)) - f(u_1(\alpha) + G_2(t))}{u_1(\alpha\pm\beta) - u_1(\alpha)}, \tag{2.47}$$

we have the two following properties:

$$\forall t \in [T-\tau(\alpha-\beta), T], \qquad \gamma_-(t) \leq \gamma_+(t), \tag{2.48}$$

$$\gamma_-(T-\tau(\alpha-\beta)) \leq \alpha - \frac{\beta}{2} - \int_{T-\tau(\alpha-\beta)}^T f'(\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x)) + G_2(s))ds. \tag{2.49}$$

Proof. Both properties are consequences of the convexity of f. The first one follows from:

$$\begin{aligned}
\gamma_+(t)-\gamma_-(t) = \beta - \int_t^T & \frac{f(u_1(\alpha+\beta)+G_2(s))-f(u_1(\alpha)+G_2(s))}{u_1(\alpha+\beta)-u_1(\alpha)} \\
& - \frac{f(u_1(\alpha-\beta)+G_2(s))-f(u_1(\alpha)+G_2(s))}{u_1(\alpha-\beta)-u_1(\alpha)} ds \\
&\geq \beta - \int_t^T f'(\max(u_1(\alpha),u_1(\alpha+\beta))+G_2(s)) - f'(\min(u_1(\alpha-\beta),u_1(\alpha))+G_2(s))ds \\
&\geq \beta - \int_t^T f'(u_1(\alpha)+K(\alpha)\beta+G_2(s)) - f'(u_1(\alpha)-K(\alpha-\beta)\beta+G_2(s))ds \\
&\geq \beta - \int_t^T \sup_{z\in I_M}(f''(z))(K(\alpha)+K(\alpha-\beta))\beta \\
&\geq \beta(1-(T-t)\sup_{z\in I_M}(f''(z))(K(\alpha)+K(\alpha-\beta))) \\
&\geq 0 \qquad \text{when } T-t\leq \tau(\alpha-\beta).
\end{aligned}$$

And the second one comes from:

$$\begin{aligned}
\gamma_-(t) &= \alpha - \frac{\beta}{2} - \int_t^T \frac{f(u_1(\alpha-\beta)+G_2(s))-f(u_1(\alpha)+G_2(s))}{u_1(\alpha-\beta)-u_1(\alpha)} ds \\
&\leq \alpha - \frac{\beta}{2} - \int_t^T f'[\min(u_1(\alpha),u_1(\alpha-\beta))+G_2(s)]ds \\
&\leq \alpha - \frac{\beta}{2} - \int_t^T f'[\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x))+G_2(s)]ds.
\end{aligned}$$

□

We prove Proposition 17 by constructing appropriate wave-front tracking approximations.

Proof. Thanks to (2.22) we can take (β_n) and (δ_n) two decreasing sequences such that $\beta_n \to 0$, $\delta_n \to 0$ and

$$\sup_{\alpha\in[\frac{3\beta_n}{2},L]} \left(\alpha - \int_{T-\tau(\alpha-\beta_n)}^T f'\left(\inf_{x\in[\alpha-\beta_n,\alpha]}(u_1(x)) + G_2(s)\right) ds \right) \leq \delta_n. \tag{2.50}$$

We can also suppose that:

$$K(\delta_n)\beta_n \to 0. \tag{2.51}$$

Now we construct $u_n \in L^\infty((0,T);\mathrm{BV}(\delta_n,L))$ as follows. Let $p = \lceil \frac{L-\delta_n}{\beta_n} \rceil$. For $k\in\{1,..,p\}$, we take:

$$x_k = \delta_n + k\beta_n, \tag{2.52}$$

$$v_k(t) = u_1(\delta_n+(k+\frac{1}{2})\beta_n) + G_2(t). \tag{2.53}$$

For $k\in\{1,..,p-1\}$ we define the curve γ_k by:

$$\gamma_k(T) = x_k, \tag{2.54}$$

$$\dot\gamma_k(t) = \frac{f(v_k(t))-f(v_{k-1}(t))}{v_k(t)-v_{k-1}(t)}. \tag{2.55}$$

Thanks to (2.48), (2.49) and (2.50) we see that the curves γ_k do not cross each other inside $[0,T]\times[\delta_n,L]$ therefore we can define u_n as follows:

$$u_n(t,x)=\begin{cases}v_1(t) \text{ if } x\leq\gamma_1(t),\\ v_k(t) \text{ if } \gamma_{k-1}(t)\leq x\leq\gamma_k(t) \text{ and } 2\leq k\leq p-1,\\ v_p(t) \text{ if } \gamma_{p-1}(t)\leq x.\end{cases} \tag{2.56}$$

Furthermore thanks to (2.21) and (2.49) we see that:

$$\forall x\in[\delta_n,L],\qquad u_n(0,x)=u_1(\delta_n+(p-\frac{1}{2})\beta_n)+G_2(0). \tag{2.57}$$

We also have the estimates:

$$||u_n||_{L^\infty((0,T);\mathrm{BV}(\delta_n,L))}\leq||u_1||_{\mathrm{BV}(0,L)}+M, \tag{2.58}$$

$$||u_n||_{\mathrm{Lip}([0,T];L^1(\delta_n,L))}\leq\max(L(||u_1||_{L^\infty(0,L)}+M),||g||_{\mathcal{C}^0([0,T])}+||u_1||_{\mathrm{BV}(0,L)}f'(||u_1||_{L^\infty(0,L)}+M)). \tag{2.59}$$

Finally thanks to (2.16), (2.51) and (2.32), for every convex entropy-flux couple (η,q) we get a constant C independent of n such that $\forall\phi\in\mathcal{C}^1_c((0,T)\times(\delta_n,L))$ non-negative we have:

$$\int_0^T\int_{\delta_n}^L(\eta(u_n(t,x))\partial_t\phi(t,x)+q(u_n(t,x))\partial_x\phi(t,x)+\eta'(u_n(t,x))g(t)\phi(t,x))dtdx \geq -C.||\phi||_{\mathcal{C}^0((0,T)\times(0,L))}\beta_nK(\delta_n). \tag{2.60}$$

Using lemma 5, we extract a subsequence from (u_n) and get a solution to equation (2.11). Since $\delta_n\to0$ the limit u is defined on $(0,T)\times(0,L)$ and is an entropy solution of (2.11). And taking the limit $n\to+\infty$ in (2.57) we see that $u(0,.)$ is constant on $(0,L)$. □

2.5 Proof of Theorem 7.

We now show that under the hypotheses of Theorem 7, condition (2.22) is satisfied. We know that:

$$\forall\alpha\in(0,L),\quad K(\alpha)\leq K<+\infty, \tag{2.61}$$

$$\frac{f'(M)}{\sup\limits_{z\in[0,M]}f''(z)}\underset{M\to+\infty}{\to}+\infty. \tag{2.62}$$

Recalling the definitions of I_M in (2.18) and τ in (2.21), it is clear that

$$\forall\alpha\in(0,L)\quad\tau(\alpha)\geq\tau_M:=\frac{1}{2K}\frac{1}{\sup\limits_{z\in I_M}f''(z)}.$$

Therefore with g non-positive and such that G_2 satisfies:

$$G_2(T)=G_2'(T)=0, \tag{2.63}$$

$$\forall t\in[0,T-\frac{\tau_M}{2}]\qquad G_2(t)=M, \tag{2.64}$$

we have when M is large enough so that $f'(\inf_{x\in(0,L)} u_1(x)+M)\geq 0$:

$$\sup_{\alpha\in[\frac{3\beta}{2},L]}\left(\alpha-\int_{T-\tau(\alpha-\beta)}^{T} f'(\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x))+G_2(s))ds\right)$$
$$\leq L-\frac{\tau_M}{2}\left(f'(\inf_{x\in(0,L)}(u_1(x)))+f'(\inf_{x\in(0,L)}(u_1(x))+M)\right). \quad (2.65)$$

But now we can get:

$$\begin{aligned}\liminf_{\beta\to 0^+}\sup_{\alpha\in[\frac{3\beta}{2},L]}&\left(\alpha-\int_{T-\tau(\alpha-\beta)}^{T} f'(\min_{x\in[\alpha-\beta,\alpha]}(u_1(x))+G_2(s))ds\right)\\ &\leq L-\frac{\tau_M}{2}\left(f'(\inf_{x\in(0,L)}(u_1(x)))+f'(\inf_{x\in(0,L)}(u_1(x))+M)\right)\\ &\leq \frac{L}{4K}\left(1-\frac{f'(\inf_{x\in(0,L)}(u_1(x)))+f'(\inf_{x\in(0,L)}(u_1(x))+M)}{\sup_{z\in I_M} f''(z)}\right).\end{aligned}$$

And thanks to (2.62) this expression is non-positive for M large enough.

2.6 Proof of Theorem 8.

In this part we prove that under the hypotheses of Theorem 8 condition (2.22) is satisfied and therefore that we have exact controllability in arbitrarily small time.
We recall that $u_1\in \mathrm{BV}(0,L)$, $M>0$ and $0<\epsilon<T$, where ϵ is the amount of time needed to get controllability and therefore is as small as we want. Indeed if we can control in time T_1 we can obviously control in any time $T_2\geq T_1$ since in our strategy we can spend an arbitrary amount of time between the two intermediate states constant spaces.
We use once again the functions K, τ and the interval I_M defined in (2.16) (2.21) and (2.18), and also the following:

$$U_i(\alpha)=\inf_{z\in(0,\alpha]}(u_1(z)) \quad \text{and } \alpha_c(\epsilon)=\sup\{\alpha\in(0,L)\mid \tau(\alpha)<\epsilon\}. \quad (2.66)$$

It is clear that K and U_i are non-increasing, that τ is non-increasing in α but non-decreasing in M and that α_c is non-decreasing in ϵ but non-decreasing in M. We can suppose that $\tau(\alpha)\underset{\alpha\to 0^+}{\to} 0$. It implies that $\alpha_c(\epsilon)\underset{\epsilon\to 0^+}{\to} 0$.

The assumptions made in Theorem 8 can be reformulated as follows, for any $H>0$ there exist $C>0$ and $\bar{U}_i>0$ such that the following holds:

$$\frac{M^q}{\sup_{z\in[0,M]} f''(z)}\underset{M\to+\infty}{\to}+\infty, \qquad \forall h>0,\ f'(U_i(0^+)+h)\geq\frac{h^q}{C}$$
$$\text{and}\quad \forall h\in(0,H]\ f'(U_i(0^+)-h)\geq -Ch^q, \quad (2.67)$$

$$\forall\alpha\in(0,L),\qquad K(\alpha)\leq\frac{\bar{K}}{\alpha^p}\ \text{and}\quad U_i(0^+)-U_i(\alpha)\leq \bar{U}_i\alpha^{2p}, \quad (2.68)$$

and finally we have the compatibility condition:

$$p(2q+1) \leq 1. \tag{2.69}$$

Now we define:

$$\tilde{\tau}(\alpha) = \frac{\alpha^p}{2\bar{K}} \frac{1}{\sup\limits_{z \in I_M} f''(z)}, \tag{2.70}$$

$$\mathcal{F}(\alpha,\beta) = \alpha - \int_{T-\tau(\alpha-\beta)}^{T} \left(U_i(0^+) - \bar{U}_i \alpha^{2p} + G_2(s)\right) ds. \tag{2.71}$$

Finally we take g such that:

$$G_2(t) = -\int_t^T g(s)ds \text{ is decreasing}, \tag{2.72}$$

$$\forall s \in [T-\epsilon, T] \quad G_2(s) = \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2, \tag{2.73}$$

$$\forall s \in [0,T] \quad G_2(s) \leq M. \tag{2.74}$$

We want to prove that (2.22) holds, but since f' is monotone and using the bound on U_i in (2.68) it is sufficient to prove:

$$\liminf_{\beta \to 0^+} \sup_{\frac{3\beta}{2} \leq \alpha \leq L} \mathcal{F}(\alpha,\beta) \leq 0, \tag{2.75}$$

for M large enough and ϵ given by:

$$\epsilon = \frac{1}{\sqrt{f'\left(U_i(0^+) - \bar{U}_i L^{2p} + \frac{M}{8}\right) . \sup\limits_{z \in I_M} f''(z)}}. \tag{2.76}$$

Note that with this choice $\epsilon \to 0$ when $M \to +\infty$.

We will get upper bounds of $\mathcal{F}$ in two different ways.

Lemma 10. *There exists M_0 such that for all (α,β) such that $\alpha \geq \alpha_c(\epsilon) + \beta$ and for any $M \geq M_0$ we have $\mathcal{F}(\alpha,\beta) \leq 0$.*

Proof. If $\alpha - \beta \geq \alpha_c(\epsilon)$ we have $\tau(\alpha-\beta) = \epsilon$ and therefore:

$$\mathcal{F}(\alpha,\beta) = \alpha - \int_{T-\epsilon}^{T} \left(U_i(0^+) - \bar{U}_i \alpha^{2p} + G_2(s)\right) ds \tag{2.77}$$

$$\leq L - \frac{\epsilon}{2}\left(f'\left(U_i(0^+) - \bar{U}_i L^{2p}\right) + f'\left(U_i(0^+) - \bar{U}_i L^{2p} + \frac{M}{8}\right)\right) \tag{2.78}$$

$$\leq L - \frac{\epsilon}{2} f'\left(U_i(0^+) - \bar{U}_i L^{2p}\right) - \left(\frac{f'\left(U_i(0^+) - \bar{U}_i L^{2p} + \frac{M}{8}\right)}{\sup\limits_{z \in I_M} f''(z)}\right)^{\frac{1}{2}}. \tag{2.79}$$

And using the definition of ϵ and (2.67) we can conclude. □

It remains to bound $\mathcal{F}$ on $\frac{\beta}{2} \leq \alpha - \beta < \alpha_c(\epsilon)$, we begin by a few observations.

Lemma 11. *The following properties hold:*

- *when* $M \to +\infty$ *we have* $\alpha_c(\epsilon) \to 0$,
- *for* $\alpha \leq \alpha_c(\epsilon) + \beta$ *we have* $\tilde{\tau}(\alpha - \beta) \leq \tau(\alpha - \beta)$
- *for* $\alpha \leq \alpha_c(\epsilon) + \beta$ *we have:*

$$\alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0 \Rightarrow \mathcal{F}(\alpha, \beta) \leq 0. \tag{2.80}$$

Proof. The first part of the lemma comes from the definitions of α_c and ϵ through the following calculations:

$$\begin{aligned}
\alpha_c(\epsilon) &= \sup\{\alpha \in (0, L) \mid \tau(\alpha) < \epsilon\} \\
&= \sup\Big\{\alpha \in (0, L) \mid \frac{1}{2K(\alpha)} \frac{1}{\sup\limits_{z \in I_M} f''(z)} < \epsilon\Big\} \\
&= \sup\Big\{\alpha \in (0, L) \mid K(\alpha) > \frac{1}{2\epsilon \sup\limits_{z \in I_M} f''(z)}\Big\} \\
&= \sup\Big\{\alpha \in (0, L) \mid K(\alpha) > \frac{1}{2} \frac{f'\left(U_i(0^+) - \bar{U}_i L^{2p} + \frac{M}{8}\right)}{\sup\limits_{z \in I_M} f''(z)}\Big\}.
\end{aligned}$$

The second part is an immediate consequence of the definitions of $\tilde{\tau}$, τ, α_c and of the first part of (2.68).
And for the third part we have, thanks to the monotonicity of f':

$$\begin{aligned}
\alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} & f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0 \\
&\Rightarrow f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{\tilde{\tau}(\alpha-\beta)}{\epsilon}\right)^2\right) \geq 0 \\
&\Rightarrow \int_{T-\tilde{(}\alpha-\beta)}^{T-\tilde{\tau}(\alpha-\beta)} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \geq 0.
\end{aligned} \tag{2.81}$$

Hence:

$$\mathcal{F}(\alpha, \beta) \leq \alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0. \tag{2.82}$$

□

Lemma 12. *There exists* M_1 *such that for any* $M \geq M_1$ *and if* $\frac{3\beta}{2} \leq \alpha \leq \alpha_c(\epsilon) + \beta$ *we have:*

$$\alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0. \tag{2.83}$$

Proof. Let

$$\begin{aligned}Q &:= \alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds\\ &\leq \alpha - \frac{\tilde{\tau}(\alpha-\beta)}{2}\left(f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p}\right) + f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{\tilde{\tau}(\alpha-\beta)}{2\epsilon}\right)^2\right)\right).\end{aligned}$$

Using the definition of $\tilde{\tau}$ and (2.67) we get:

$$\begin{aligned}Q &\leq \alpha - \frac{(\alpha-\beta)^p}{4\bar{K}\sup\limits_{z\in I_M} f''(z)}\left(-C\bar{U}_i\alpha^{2pq} + \frac{1}{C}\left(\frac{M(\alpha-\beta)^{2p}}{8\bar{K}^2}\frac{f'\left(U_i(0^+) - \bar{U}_iL^{2p} + \frac{M}{8}\right)}{\sup\limits_{z\in I_M} f''(z)} - \bar{U}_i\alpha^{2p}\right)^q\right)\\ &\leq \alpha - \alpha^{p(2q+1)}\frac{\left(1-\frac{\beta}{\alpha}\right)^p}{4\bar{K}\sup\limits_{z\in I_M} f''(z)}\left(-C\bar{U}_i + \frac{1}{C}\left(\frac{M\left(1-\frac{\beta}{\alpha}\right)^{2p}}{8\bar{K}^2}\frac{f'\left(U_i(0^+) - \bar{U}_iL^{2p} + \frac{M}{8}\right)}{\sup\limits_{z\in I_M} f''(z)} - \bar{U}_i\right)^q\right).\end{aligned}$$

Now using (2.69) and the fact that $1-\frac{\beta}{\alpha} \geq \frac{1}{3}$ we see that, for any $A > 0$ and for M large enough (independently of α and β) we have:

$$\frac{\left(1-\frac{\beta}{\alpha}\right)^p}{4\bar{K}\sup\limits_{z\in I_M} f''(z)}\left(-C\bar{U}_i + \frac{1}{C}\left(\frac{M\left(1-\frac{\beta}{\alpha}\right)^{2p}}{8\bar{K}^2}\frac{f'\left(U_i(0^+) - \bar{U}_iL^{2p} + \frac{M}{8}\right)}{\sup\limits_{z\in I_M} f''(z)} - \bar{U}_i\right)^q\right) \geq A.$$

Finally thanks to (2.69) we see that for any $A > 1$, $\alpha - A\alpha^{p(2q+1)} \leq 0$ and since α_c tends to 0 when M goes to infinity and β is arbitrarily small we have the lemma. □

2.7 Wave-front tracking approximations.

Here we provide two results useful for the wave-front tracking algorithm. Let $T > 0$, $a, b \in \mathbb{R}$ such that $a < b$. We consider g a continuous function on $[0,T]$ and recall that $G_1(t) = \int_0^t g(s)ds$. And finally we suppose that f is a $\mathcal{C}^2$ convex function defined on $\mathbb{R}$.
We will be interested in the entropic solutions of the equation:

$$\partial_t u(t,x) + \partial_x(f(u(t,x))) = g(t) \text{ on } [0,T]\times(a,b). \tag{2.84}$$

Note that while here we deal only with $g(t)$, the same idea could be used to deal with a source term $g(t,u)$ though of course one would need some additional informations on g to get the existence in large time. We recall that approximate wave-front tracking approximations were defined in Definition 9, and we will now prove Lemma 7.

Proof of Lemma 7. We evaluate the left hand side of (2.32) using the fact that $v = u - G_1$ is piecewise constant.
More precisely we apply Green's theorem to the vector field $X = (\phi\eta(v+G_1), q(v+G_1)\phi)$ on the parts where it is regular. We know that η, q and G_1 are regular. Furthermore we know that v is piecewise constant therefore regular except on the curves x_α.
Now consider D a connected component of the open subset of $(0,T)\times(a,b)$ constituted of the

points (t,x) on a neighborhood of which, v is constant. Thanks to the definition of approximate front tracking solutions we know that the boundary of D is constituted of

$D=\{(t,x)\in(t_1,t_2)\times(a,b)|x_{\alpha_1}(t)<x<x_{\alpha_2}(t)\}$ such that no other curve x_α lies in D, and that we have the following alternatives:

1. either $t_1=0$ or $x_{\alpha_1}(t_1)=x_{\alpha_2}(t_1)$ or $x_1(t_1)=a$ or $x_2(t_1)=b$,
2. either $t_2=T$ or $x_{\alpha_1}(t_2)=x_{\alpha_2}(t_2)$ or $x_1(t_2)=a$ or $x_2(t_2)=b$,

When we apply Green's theorem to X on D we get the following:

$$\begin{aligned}\iint_D \operatorname{div}(X)dxdt &= \int_{\partial D} X.n\ ds\\ &= \int_{t_1}^{t_2}\left(X(t,x_{\alpha_2}(t)).(-\dot{x}_{\alpha_2}(t),1)+X(t,x_{\alpha_1}(t)).(\dot{x}_{\alpha_1}(t),-1)\right)dt\\ &+\int_{x_{\alpha_1}(t_2)}^{x_{\alpha_2}(t_2)} q(u(t_2,x))\phi(t_2,x)dx-\int_{x_{\alpha_1}(t_1)}^{x_{\alpha_2}(t_1)} q(u(t_1,x))\phi(t_1,x)dx.\end{aligned}$$

Now since either $x_{\alpha_1}(t_2)=x_{\alpha_2}(t_2)$ or $\phi(T,.)=0$ we get:

$$\int_{x_{\alpha_1}(t_2)}^{x_{\alpha_2}(t_2)} q(u(t_2,x))\phi(t_2,x)dx=0.$$

And with the same kind of reasoning we also have

$$\int_{x_{\alpha_1}(t_1)}^{x_{\alpha_2}(t_1)} q(u(t_1,x))\phi(t_1,x)dx=0.$$

On the other hand we have:

$$\iint_D \operatorname{div}(X)dxdt=\iint_D \eta(u(t,x))\partial_t\phi(t,x)+q(u(t,x))\partial_x\phi(t,x)+\eta'(u(t,x))\phi(t,x)g(t)dtdx. \tag{2.85}$$

In the end we obtain:

$$\begin{aligned}&\iint_D \eta(u(t,x))\partial_t\phi(t,x)+q(u(t,x))\partial_x\phi(t,x)+\eta'(u(t,x))\phi(t,x)g(t)dtdx=\\ &\qquad\int_{t_1}^{t_2}\left(q(u(t,x_{\alpha_2}(t)^-))-\dot{x}_{\alpha_2}(t)\eta(u(t,x_{\alpha_2}(t)^-))\right)\phi(t,x_{\alpha_2}(t))\\ &\qquad\qquad-\left(q(u(t,x_{\alpha_1}(t)^+))-\dot{x}_{\alpha_1}(t)\eta(u(t,x_{\alpha_1}(t)^+))\right)\phi(t,x_{\alpha_1}(t))dt.\end{aligned} \tag{2.86}$$

Furthermore by adding arbitrary fronts to the family $\{x_\alpha\}$ on the parts of the domain where v is constant (since there is no jump those artificial fronts automatically satisfy (2.30) and (2.31)), we can obtain a partition of $(a,b)\times(0,T)$ by sets such as D. As a consequence we get:

$$\begin{aligned}&\int_a^b\int_0^T \eta(u(t,x))\partial_t\phi(t,x)+q(u(t,x))\partial_x\phi(t,x)+\eta'(u(t,x))\phi(t,x)g(t)dtdx=\\ &\sum_\alpha\int_0^T\left(\dot{x}_\alpha(t)(\eta(u(t,x_\alpha(t)^+))-\eta(u(t,x_\alpha(t)^-))+q(u(t,x_\alpha(t)^-))-q(u(t,x_\alpha(t)^+))\right)\phi(t,x_\alpha(t))dt.\end{aligned} \tag{2.87}$$

Now we replace $\dot{x}_\alpha$ by its value in terms of u and we use the following lemma which we will prove later:

Lemma 13. *Let z^- and z^+ be real numbers such that $z^+ \leq z^- + \epsilon$ then we have:*

$$\frac{f(z^+) - f(z^-)}{z^+ - z^-}\left(\eta(z^+) - \eta(z^-)\right) - \left(q(z^+) - q(z^-)\right) \geq -C\epsilon|z^+ - z^-|, \tag{2.88}$$

with $C = (||f''||_{\mathcal{C}^0([z^-,z^+])}||\eta'||_{\mathcal{C}^0([z^-,z^+])} + ||f'||_{\mathcal{C}^0([z^-,z^+])}||\eta''||_{\mathcal{C}^0([z^-,z^+])})$.

Thus we get:

$$\begin{aligned}\int_a^b\int_0^T &\eta(u(t,x))\partial_t\phi(t,x) + q(u(t,x))\partial_x\phi(t,x) + \eta'(u(t,x))\phi(t,x)g(t)dtdx\\ &\geq \int_0^T \sum_\alpha -C\epsilon|u(t,x_\alpha(t)^+) - u(t,x_\alpha(t)^-)|||\phi||_{\mathcal{C}^0((0,T)\times(a,b))}\\ &\geq \int_0^T -C\epsilon||\phi||_{\mathcal{C}^0((0,T)\times(a,b))}\mathrm{TotVar}(u(t,.))dt\\ &\geq -\tilde{C}\epsilon||\phi||_{\mathcal{C}^0((0,T)\times(a,b))}||u||_{L^\infty((0,T);\mathrm{BV}(a,b))}.\end{aligned} \tag{2.89}$$

Where the constant $\tilde{C}$ is given by:

$$\begin{aligned}\tilde{C} = T\,\big(&||f''||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}||\eta'||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}\\ &+ ||f'||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}||\eta''||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}\big).\end{aligned} \tag{2.90}$$

□

Proof of Lemma 13. We begin with a basic equality:

$$\begin{aligned}&\frac{f(z^+) - f(z^-)}{z^+ - z^-}\frac{\eta(z^+) - \eta(z^-)}{z^+ - z^-} - \frac{q(z^+) - q(z^-)}{z^+ - z^-}\\ &\quad= \int_{z^-}^{z^+} f'(z)\frac{dz}{z^+ - z^-}\int_{z^-}^{z^+}\eta'(z)\frac{dz}{z^+ - z^-} - \int_{z^-}^{z^+} q'(z)\frac{dz}{z^+ - z^-}\\ &\quad= \int_{z^-}^{z^+} f'(z)\frac{dz}{z^+ - z^-}\int_{z^-}^{z^+}\eta'(z)\frac{dz}{z^+ - z^-} - \int_{z^-}^{z^+} f'(z).\eta'(z)\frac{dz}{z^+ - z^-}.\end{aligned} \tag{2.91}$$

But thanks to the convexity of f and η we also have the following:

$$\int_{z^-}^{z^+}\int_{z^-}^{z^+}(f'(z) - f'(w))(\eta'(z) - \eta(w))\frac{dz}{z^+ - z^-}\frac{dw}{z^+ - z^-} \geq 0$$

$$\text{so } \int_{z^-}^{z^+}\int_{z^-}^{z^+} f'(z)\eta'(z) + f'(w)\eta'(w)\frac{dz}{z^+ - z^-}\frac{dw}{z^+ - z^-}$$

$$\geq \int_{z^-}^{z^+}\int_{z^-}^{z^+} f'(z)\eta'(w) + f'(w)\eta'(z)\frac{dz}{z^+ - z^-}\frac{dw}{z^+ - z^-}$$

$$\text{therefore } \int_{z^-}^{z^+} f'(z)\eta'(z)\frac{dz}{z^+ - z^-} \geq \int_{z^-}^{z^+} f'(z)\frac{dz}{z^+ - z^-}\int_{z^-}^{z^+}\eta'(z)\frac{dz}{z^+ - z^-}.$$

Now if $z^+ < z^-$ we get:

$$\frac{f(z^+) - f(z^-)}{z^+ - z^-}\big(\eta(z^+) - \eta(z^-)\big) - \big(q(z^+) - q(z^-)\big)$$
$$= (z^+ - z^-)\left(\frac{f(z^+) - f(z^-)}{z^+ - z^-}\frac{\eta(z^+) - \eta(z^-)}{z^+ - z^-} - \frac{q(z^+) - q(z^-)}{z^+ - z^-}\right) \geq 0. \quad (2.92)$$

On the other hand if we have $z^- \leq z^+ \leq z^- + \epsilon$ we can take $z_1, z_2, z_3 \in [z^-, z^- + \epsilon]$ such that:

$$\frac{f(z^+) - f(z^-)}{z^+ - z^-} = f'(z_1), \qquad \frac{\eta(z^+) - \eta(z^-)}{z^+ - z^-} = \eta'(z_2), \qquad \frac{q(z^+) - q(z^-)}{z^+ - z^-} = f'(z_3)\eta'(z_3). \quad (2.93)$$

And now we get:

$$\left|\frac{f(z^+) - f(z^-)}{z^+ - z^-}\frac{\eta(z^+) - \eta(z^-)}{z^+ - z^-} - \frac{q(z^+) - q(z^-)}{z^+ - z^-}\right| = \left|(f'(z_1) - f'(z_3))\eta'(z_2) + f'(z_3)(\eta'(z_2) - \eta'(z_3))\right|$$
$$\leq (z^+ - z^-)(||f''||_{\mathcal{C}^0([z^-,z^+])}||\eta'||_{\mathcal{C}^0([z^-,z^+])} + ||f'||_{\mathcal{C}^0([z^-,z^+])}||\eta''||_{\mathcal{C}^0([z^-,z^+])}). \quad (2.94)$$

□

Chapitre 3

Stabilisation asymptotique des solutions entropiques de lois de conservation scalaires par une loi de retour stationnaire.

Abstract In this paper, we study the problem of asymptotic stabilization by closed loop feedback for a scalar conservation law with a convex flux and in the context of entropy solutions. Besides the boundary data, we use an additional control which is a source term acting uniformly in space.

3.1 Introduction.

This paper is concerned with the asymptotic stabilization problem for a nonlinear scalar conservation law with a source term, on a bounded interval and in the framework of entropy solutions:

$$\begin{aligned} &\partial_t u + \partial_x f(u) = g(t),\\ &u(0,x) = u_0(x),\\ &u(t,0) = u_l(t),\\ &u(t,L) = u_r(t). \end{aligned} \qquad (t,x) \in (0,T) \times (0,1), \tag{3.1}$$

Here u is the sate and u_l, u_r and g are the controls. For any regular strictly convex flux f and any state $\bar{u} \in \mathbb{R}$ we will provide explicit stationnary feedback law for g, u_r and u_l such that the state $\bar{u}$ is asymptotically stable in the $L^1(0,1)$ norm and in the $L^\infty(0,1)$ norm.

3.1.1 Generalities and previous results.

Scalar conservation laws are used for instance to model traffic flow or gas networks, but their importance also consists in being a first step in the understanding of systems of conservation laws. Those systems of equations model a huge number of physical phenomena: gas dynamics, electromagnetism, magneto-hydrodynamics, shallow water theory, combustion theory... see [45, Chapter2].

For equations such as (3.1), the Cauchy problem on the whole line is well posed in small time in the framework of classical solutions and with a $\mathcal{C}^1$ initial value. However those solutions generally blow up in finite time: shock waves appear. Hence to get global in time results, a weaker notion of solution is called for. In [87] Oleinik proved that given a flux $f \in \mathcal{C}^2$ such that $f'' > 0$ and any $u_0 \in L^\infty(\mathbb{R})$ there exists a unique weak solution to:

$$u_t + (f(u))_x = 0, \qquad x \in \mathbb{R} \text{ and } t > 0, \tag{3.2}$$

$$u(0,.) = u_0, \tag{3.3}$$

satisfying the additional condition:

$$\frac{u(t,x+a) - u(t,x)}{a} \leq \frac{E}{t} \qquad \text{for } x \in \mathbb{R}, \quad t > 0, \text{ and } a > 0, \tag{3.4}$$

and where E depends only on the quantities $\inf(f'')$ and $\sup(f')$ taken on $[-||u_0||_{L^\infty}, ||u_0||_{L^\infty}]$ and not on u_0. Later in [82], Kruzkov extended this global result to the multidimensional problem, with a C^1 flux $f : \mathbb{R} \to \mathbb{R}^n$ not necessarily convex:

$$u_t + \operatorname{div}(f(u)) = 0, \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^n. \tag{3.5}$$

This time the weak entropy solution is defined as satisfying the following integral inequality:

$$\text{for all real numbers } k \text{ and all nonnegative functions } \phi \text{ in } \mathcal{C}^1_c(\mathbb{R}^2), \tag{3.6}$$

$$\iint_{\mathbb{R}^2} |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\nabla\phi dtdx + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx \geq 0. \tag{3.7}$$

The initial boundary value problem for equation (3.1) is also well posed as shown by Leroux in [80] for the one dimensional case with BV data, by Bardos, Leroux and Nédélec in [15] for the multidimensional case with C^2 data and later by Otto in [88] (see also [86]) for L^∞ data. However the meaning of the boundary condition is quite intricate and the Dirichlet condition may not be fulfilled pointwise a.e. in time. We will go into further details later.

Now for a general control system:

$$\begin{cases} \dot{X} = F(X,U), \\ X(t_0) = X_0, \end{cases} \tag{3.8}$$

(X being the state of the system and U the so called control), we can consider two classical problems (among others) in control theory.

1. First the exact controllability problem which asks, given two states X_0 and X_1 and a time T the possibility to find a certain function $U(t)$ such that the solution to (3.8) satisfies $X(T) = X_1$.

2. If $F(0,0) = 0$, the problem of asymptotic stabilization by a stationary feedback law asks to find a function $U(X)$, such that for any state X_0 a maximal solution $X(t)$ of the closed loop system:

$$\begin{cases} \dot{X}(t) = f(X(t), u(X(t))), \\ X(t_0) = X_0, \end{cases} \tag{3.9}$$

is global and satisfies additionally:

$$\forall R > 0, \ \exists r > 0 \text{ such that } ||X_0|| \leq r \quad \Rightarrow \quad \forall t \in \mathbb{R}, \ ||X(t)|| \leq R, \tag{3.10}$$

$$X(t) \underset{t \to +\infty}{\to} 0. \tag{3.11}$$

The asymptotic stabilization property might seem weaker than exact controllability: for any initial state X_0, we can find T and $U(t)$ such that the solution to (3.8) satisfies $X(T) = 0$ in this way we stabilize 0 in finite time. However this method suffers from a lack of robustness with respect to perturbation: with any error on the model, or on the initial state, the control may not act properly anymore. This motivates the problem of asymptotic stabilization by a stationary feedback law which is more robust. In fact in finite dimension, the asymptotic stabilization property automatically guarantees the existence of a Lyapunov function.

In the framework of entropy solutions, only a few results exist for the exact controllability problem see [4], [5], [3], [17], [57], [62], [69]. In all cases the control act only at the boundary points. Furthermore many of those results show that boundary controls are not sufficient to reach many states. However with an additional control $g(t)$ as in (3.1) and with $f(z) = \frac{z^2}{2}$ (Burgers equation), Chapouly showed in [20] that in the framework of classical solutions, any regular state is reachable from any regular initial data and in any time (note that in this context, the controls also had to prevent the blow up of the solution, which will not be a concern for entropy solutions). It was shown in [89] that the same kind of improvement occur also in the framework of entropy solutions.

The aim of this paper is to investigate the problem of asymptotic stabilization with this additional control on the right-hand side. To the author's knowledge it is the first result about asymptotic stabilization through a closed loop feedback law in the framework of entropy solutions. However it should be noted that in the framework of classical solutions the problem has been studied extensively see for example: [77], [8] or [9] among many others.

As for the physical significance of g, it can be seen as a pressure field in the case where the Burgers equation is considered as a one dimensional Euler incompressible equation. Equation (3.1) is also a toy model for the compressible Euler-Poisson system:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \qquad \rho(0,.) = \rho_0, \qquad \rho(t,0) = \rho_l(t), \quad \rho(t,L) = \rho_r(t), \\ \partial_t m + \partial_x(\dfrac{m^2}{\rho} + p(\rho)) = \rho \partial_x V - m,\ m(0,.) = m_0,\ m(t,0) = m_l(t),\ m(t,L) = m_r(t), \\ -\partial_{xx}^2 V = \rho, \qquad V(t,0) = V_l(t), \quad V(t,L) = V_r(t), \end{cases} \tag{3.12}$$

where the controls are ρ_l, ρ_r, m_l, m_r, V_l and V_r. Indeed once we take $g(t) = \frac{V_r(t) - V_l(t)}{L}$ we get $V_x = g(t) + A\rho(t)$ with A a linear integral operator. So we have to deal with a hyperbolic system controlled by the boundary data and an additional source term depending only on the time variable.

3.1.2 Results.

In this article, functions in BV($\mathbb{R}$) will be considered continuous from the left in order to prevent ambiguity on the representative of the L^1 equivalence class.

We will also suppose that the flux f is a $\mathcal{C}^1$ strictly convex function.

We will make use the following notation:

$$\forall \alpha, \beta \in \mathbb{R} \qquad I(\alpha, \beta) = [\min(\alpha, \beta), \max(\alpha, \beta)]. \tag{3.13}$$

We are interested in the following equation:

$$\begin{cases} \partial_t u + \partial_x(f(u)) = g(t) & \text{on } (0,+\infty)\times(0,1), \\ u(0,.) = u_0 & \text{on } (0,1), \\ \operatorname{sgn}(u(t,1^-) - u_r(t))(f(u(t,1^-)) - f(k)) \geq 0 & \forall k \in I(u_r(t), u(t,1^-)),\ dt \text{ a.e.}, \\ \operatorname{sgn}(u(t,0^+) - u_l(t))(f(u(t,0^+)) - f(k)) \leq 0 & \forall k \in I(u_l(t), u(t,0^+)),\ dt \text{ a.e.}. \end{cases} \tag{3.14}$$

We recall that following [80] and [15] a function u in $L^\infty((0,+\infty);\mathrm{BV}(0,1))$ is an entropy solution of (3.14) when it satisfies the following inequality for every k in $\mathbb{R}$ and every nonnegative function ϕ in $\mathcal{C}^1_c(\mathbb{R}^2)$:

$$\int_0^{+\infty}\!\!\int_0^1 |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\phi_x + \operatorname{sgn}(u-k)g(t)\phi\, dxdt + \int_0^1 |u_0(x)-k|\phi(0,x)dx$$
$$+\int_0^{+\infty} \operatorname{sgn}(u_r(t)-k)(f(k)-f(u(t,1^-)))\phi(t,1) - \operatorname{sgn}(u_l(t)-k)(f(k)-f(u(t,0^+))\phi(t,0)dt \geq 0. \tag{3.15}$$

Remark 13. *It should be noted that if u in $L^\infty((0,+\infty);\mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,+\infty)\,;L^1(0,1))$ the exists a unique representative u such that:*

$$u \in \mathrm{Lip}([0,+\infty)\,;L^1(0,1)) \quad \text{and } \forall t \geq 0, \quad u(t,.) \in \mathrm{BV}(0,1).$$

Thus the traces of u at $x=0$ and $x=1$ are taken as the limit of this representative for every time t and the boundary conditions in (3.14) hold almost everywhere and not necessarily everywhere. This will make the analysis more delicate as will be seen in Section 3.3.

Here the functions g, u_l and u_r will not depend on the time but on the state $u(t,.)$. Their value will be prescribed by a closed loop feedback law.

Consider $\bar{u} \in \mathbb{R}$. It is clear that if we define u by:

$$\forall (t,x) \in \mathbb{R}+\times(0,1), \quad u(t,x) = \bar{u},$$

then u is an entropy solution of (3.14) for initial and boundary data equal to $\bar{u}$. In the following we will provide two feedback laws and two corresponding results in the respective cases $f'(\bar{u}) = 0$ and $f'(\bar{u}) \neq 0$, such that the previous stationary solution is asymptotically stable.

If $f'(\bar{u}) \neq 0$, we will use the following stationary feedback laws:

$$\forall W \in L^1(0,1), \quad \mathcal{G}_1(W) = \frac{f'(\bar{u})}{2}||W-\bar{u}||_{L^1(0,1)}, \tag{3.16}$$

$$\forall W \in L^1(0,1), \quad u_l(W) = u_r(W) = \bar{u}. \tag{3.17}$$

In the system (3.14) we will exchange: $g(t)$ and $\mathcal{G}_1(u(t,.))$, $u_l(t)$ and $u_l(u(t,.))$ and finally $u_r(t)$ and $u_r(u(t,.))$ to obtain a closed loop system.

We will need to distinguish between two possible behavior of f as follows:

Definition 10. • *We say that f if of type I if there exists u^* such that:*

$$f'(u^*) = 0. \tag{3.18}$$

The Burgers equation has a flux of type I.

- *We say that f is of type II otherwise. In this case we have either*

$$\forall z \in \mathbb{R}, \quad f'(z) > 0, \tag{3.19}$$

or

$$\forall z \in \mathbb{R}, \quad f'(z) < 0. \tag{3.20}$$

The flux $f(z) = e^z$ is of type II.

Remark 14. *If the flux f is of type I, we can deduce since it also strictly convex:*

$$\lim_{z \to +\infty} f(z) = \lim_{z \to -\infty} f(z) = +\infty. \tag{3.21}$$

If $f'(\bar{u}) \neq 0$, guarantees the existence of $\hat{u} \neq \bar{u}$ such that $f(\bar{u}) = f(\hat{u})$. We can then reformulate the boundary condition of (3.14) *as follows (we describe the case $f'(\bar{u}) > 0$):*

$$u(t, 1^-) \in [u^*; +\infty) \quad dt \text{ a.e.}, \tag{3.22}$$

$$u(t, 0^+) \in (-\infty, \hat{u}] \cup \{\bar{u}\} \quad dt \text{ a.e.}. \tag{3.23}$$

We now have the following result.

Theorem 10. *For any u_0 in* $\mathrm{BV}(0,1)$*, the closed loop system* (3.14) *where u_l, u_r and g are given by the feedback laws* (3.17) *and* (3.16) *has a unique entropy solution u. It is global in time, belongs to the space $L^\infty((0,+\infty); \mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,+\infty) ; L^1(0,1))$ and continuously depends on the initial data. Furthermore if the flux f is of type I we have:*

- *There exist two positive constants C_1 and C_2 depending only on $\bar{u}$ such that u satisfies:*

$$\forall t \geq 0, \qquad ||u(t,.) - \bar{u}||_{L^1(0,1)} \leq C_1 e^{-\frac{f'(\bar{u})}{2} t} ||u_0 - \bar{u}||_{L^1(0,1)}. \tag{3.24}$$

$$\forall t \geq 0, \qquad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq C_2 e^{-\frac{f'(\bar{u})}{2} t} ||u_0 - \bar{u}||_{L^\infty(0,1)}. \tag{3.25}$$

- *There exists a certain time T depending only on $\bar{u}$ such that u is actually regular on $(T, +\infty) \times [0,1]$.*

On the other hand if the flux f is of type II we have:

- *There exists a positive constants C_3 depending on $\bar{u}$ and $||u_0 - \bar{u}||_{L^\infty(0,1)}$ such that u satisfies:*

$$\forall t \geq 0, \quad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq C_3 e^{-\frac{f'(\bar{u})}{2} t} ||u_0 - \bar{u}||_{L^\infty(0,1)}. \tag{3.26}$$

- *There exists a certain time T' depending on $\bar{u}$ and $||u_0 - \bar{u}||_{L^\infty(0,1)}$ such that u is actually regular on $(T', +\infty) \times [0,1]$.*

Remark 15.

- *In Section 3.4, we will provide explicit formulae for C_1, C_2, C_3, T and T'.*
- *It is interesting to see that a feedback using the L^1 norm actually provides a control in the L^∞ norm. On the other hand a feedback relying on the L^∞ norm may be problematic due to the impossibility of taking the limit in $||.||_{L^\infty(0,1)}$ with a pointwise convergence and also due to the lack of time regularity of $||u(t,.)||_{L^\infty(0,1)}$ for an etropy solution of the open loop system.*

Now let us suppose that $f'(\bar{u})=0$, we introduce the following auxiliary function A:

$$\mathrm{A}(z)=\begin{cases}\frac{f(\bar{u}+z)-f(\bar{u})}{2} & \text{if } 0\le z\le 1,\\ \frac{f'(\bar{u}+1)}{2}(z-1)+\frac{f(\bar{u}+1)-f(\bar{u})}{2} & \text{if } z\ge 1.\end{cases} \tag{3.27}$$

We will use (3.17) for the boundary terms but the stationary feedback law for the source term will now be:

$$\forall W\in L^1(0,1),\quad \mathcal{G}_2(W)=\mathrm{A}(||W-\bar{u}||_{L^1(0,1)}), \tag{3.28}$$

and as previously we will exchange $g(t)$ and $\mathcal{G}_2(u(t,.))$ in (3.14). This allows us to prove the following.

Theorem 11. *The closed loop system* (3.14) *where* u_l, u_r *and* g *are provided by the feedback laws* (3.17) *and* (3.28) *has the following properties.*

- *For any* u_0 *in* $\mathrm{BV}(0,1)$ *there exists a unique entropy solution* u. *It is global in time, belongs to the space*
$$L^\infty((0,+\infty);\mathrm{BV}(0,1))\cap \mathrm{Lip}([0,+\infty)\,;L^1(0,1)).$$
and depends continuously on the initial data.
- *The solution satisfies:*
$$||u(t,.)-\bar{u}||_{L^\infty(0,1)}\underset{t\to+\infty}{\to}0. \tag{3.29}$$
- *If additionally*
$$\alpha=\inf_{z\in\mathbb{R}}f''(z)>0,$$
then we have a globally Lipschitz function R *such that:*
$$R(0)=\frac{f'(1+\bar{u})}{2\alpha}\sqrt{\frac{2e}{e-1}}+\mathrm{A}^{-1}\left(\frac{e(f'(1+\bar{u}))^2}{4\alpha(e-1)}\right), \tag{3.30}$$
$$\forall t\ge 0,\qquad ||u(t,.)-\bar{u}||_{L^\infty(0,1)}\le R(||u_0-\bar{u}||_{L^\infty(0,1)}). \tag{3.31}$$

Remark 16. *The last property is weaker than stability, thus we do not have asymptotic stability of* $\bar{u}$. *However taking* c *positive and adjusting* A *as follows:*

$$\mathrm{A}(z)=\begin{cases}\frac{f(\bar{u}+z)-f(\bar{u})}{2} & \text{if } 0\le z\le c,\\ \frac{f'(\bar{u}+c)}{2}(z-c)+\frac{f(\bar{u}+c)-f(\bar{u})}{2} & \text{if } z\ge c,\end{cases} \tag{3.32}$$

we can see that $\frac{f'(\bar{u}+c)}{2}$ *tends to* 0 *with* c *and therefore* $R(0)$ *can be as small as we want.*

The feedback laws (3.16) and (3.28) act in two steps. In the first step the control g uniformly increases the state $u(t,.)$ and therefore the characteristic speed $f'(u(t,.))$ (in the case where $f'(\bar{u})\ge 0$) to eventually reach a point where the speed is everywhere positive on $(0,1)$ (the same speed profile as the target state $\bar{u}$). It should be noted that we may potentially increase $||u(t,.)-\bar{u}||$ during this part. Once such a speed profile is reached the feedback loop increases the speed $f'(u)$ more than the state u and we have stabilization toward $\bar{u}$. This is the same strategy as the return method of J.-M. Coron [33], [34].

The paper will be organized as follows. In Section 3.2, we will prove using a Banach fixed point theorem that the the closed loop systems of both Theorem 10 and 11 has a unique maximal entropy solution which furthermore is global in time (a Lax-Friedrichs scheme with a discrete $||u(t,.) - \bar{u}||_{L^1(0,1)}$ would also have provided existence). In Section 3.3, we will adapt the result of [44] and describe the influence of the boundary conditions on the generalized characteristics touching the boundary points. In Section 3.4 we prove Theorem 10 in the case $f'(\bar{u}) > 0$ the other case being directly deduced from it using the transformation:

$$\begin{aligned} X &= 1 - x, \\ F(z) &= f(-z), \\ U(t, X) &= -u(t, 1 - x). \end{aligned}$$

Finally in Section 3.5 we will prove Theorem 11.

3.2 Cauchy problem for the closed loop system.

In this section, we will prove the following result which will imply the first part of Theorem 10 and Theorem 11 about existence uniqueness and continuous dependence on the initial data for the closed loop systems.

Proposition 18. *For $\bar{u}$ in $\mathbb{R}$, u_0 in $\mathrm{BV}(0,1)$ and g a $\mathcal{C}^1$ function on $\mathbb{R}$ which is globally Lipschitz (with constant L_G) and satisfies $g(0) = 0$, there exists a unique entropy solution of:*

$$\begin{cases} \partial_t u + \partial_x(f(u)) = g(||u(t,.) - \bar{u}||_{L^1(0,1)}) & \text{on } (0,+\infty) \times (0,1), \\ u(0,.) = u_0 & \text{on } (0,1), \\ \mathrm{sgn}(u(t,1^-) - \bar{u})(f(u(t,1^-)) - f(k)) \geq 0 & \forall k \in I(\bar{u}, u(t,1^-)),\ dt\ a.e., \\ \mathrm{sgn}(u(t,0^+) - \bar{u})(f(u(t,0^+)) - f(k)) \leq 0 & \forall k \in I(\bar{u}, u(t,0^+)),\ dt\ a.e.. \end{cases} \tag{3.33}$$

Furthermore two solutions u and v of (3.33) *for two initial data u_0 and v_0 satisfy:*

$$\forall t \geq 0, \qquad ||u(t,.) - v(t,.)||_{L^1(0,1)} \leq ||u_0 - v_0||_{L^1(0,1)} e^{L_G t}.$$

It is crucial that the boundary data is independent of u so that we can use a fixed point theorem on the source term of the equation. This proposition implies the first parts of Theorem 10 and Theorem 11 because both take the form of (3.33).

Thanks to [80], [81] and [15], we know that for any u_0 in $\mathrm{BV}(0,1)$ and any function h in $\mathcal{C}^0(\mathbb{R}^+)$ there exists a unique entropy solution v in $L^\infty_{loc}((0,+\infty);\mathrm{BV}(0,1)) \cap \mathrm{Lip}_{loc}([0,+\infty);L^1(0,1))$ to

$$\begin{cases} \partial_t v + \partial_x(f(v)) = h(t), & \text{on } (0,+\infty) \times (0,1), \\ v(0,.) = u_0, & \text{on } (0,1), \\ \mathrm{sgn}(v(t,1^-) - \bar{u})(f(v(t,1^-)) - f(k)) \geq 0, & \forall k \in I(\bar{u}, v(t,1^-)),\ dt\ a.e., \\ \mathrm{sgn}(v(t,0^+) - \bar{u})(f(v(t,0^+)) - f(k)) \leq 0, & \forall k \in I(\bar{u}, v(t,0^+)),\ dt\ a.e.. \end{cases} \tag{3.34}$$

We now have the following key estimate, which is a classical result of Kruzkov [82] when there is no boundary.

Lemma 14. *If v and $\tilde{v}$ are entropy solutions of* (3.34) *with respective source terms h and $\tilde{h}$ and respective initial data u_0 and $\tilde{u}_0$ then we have:*

$$\forall T \geq 0, \quad ||v(T,.) - \tilde{v}(T,.)||_{L^1(0,1)} \leq ||u_0 - \tilde{u}_0||_{L^1(0,1)} + \int_0^T |h(s) - \tilde{h}(s)| ds. \tag{3.35}$$

Proof. Following the method of Kruzkov[82], we take ψ nonnegative function in $\mathcal{C}_c^1(\mathbb{R}^4)$. Since for any $(\tilde{t},\tilde{x})$ in $\mathbb{R}^2$, $\psi(.,\tilde{t},.,\tilde{x})$ is in $\mathcal{C}_c^1(\mathbb{R}^2)$ we get when substituting k and $\tilde{v}(\tilde{t},\tilde{x})$ in (3.15):

$$\int_0^{+\infty}\int_0^1 |v(t,x)-\tilde{v}(\tilde{t},\tilde{x})|\psi_t(t,\tilde{t},x,\tilde{x}) + \operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))(f(v(t,x))-f(\tilde{v}(\tilde{t},\tilde{x})))\psi_x(t,\tilde{t},x,\tilde{x})$$
$$+\operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))h(t)\psi(t,\tilde{t},x,\tilde{x})dxdt + \int_0^1 |u_0(x)-\tilde{v}(\tilde{t},\tilde{x})|\psi(t,\tilde{t},x,\tilde{x})dx$$
$$+\int_0^{+\infty} \operatorname{sgn}(\bar{u}-\tilde{v}(\tilde{t},\tilde{x}))(f(\tilde{v}(\tilde{t},\tilde{x}))-f(v(t,1^-)))\psi(t,\tilde{t},1,\tilde{x})$$
$$-\operatorname{sgn}(\bar{u}-\tilde{v}(\tilde{t},\tilde{x}))(f(\tilde{v}(\tilde{t},\tilde{x}))-f(v(t,0^+))\psi(t,\tilde{t},0,\tilde{x})dt \geq 0.$$

Integrating the above inequality in the variables $\tilde{t}$ and $\tilde{x}$ and using the compactness of the support of the left-hand side, we get:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 |v(t,x)-\tilde{v}(\tilde{t},\tilde{x})|\psi_t(t,\tilde{t},x,\tilde{x}) + \operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))h(t)\psi(t,\tilde{t},x,\tilde{x})$$
$$+\operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))(f(v(t,x))-f(\tilde{v}(\tilde{t},\tilde{x})))\psi_x(t,\tilde{t},x,\tilde{x})dxd\tilde{x}dtd\tilde{t}$$
$$+\int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}(\bar{u}-\tilde{v}(\tilde{t},\tilde{x}))(f(\tilde{v}(\tilde{t},\tilde{x}))-f(v(t,1^-)))\psi(t,\tilde{t},1,\tilde{x})$$
$$-\operatorname{sgn}(\bar{u}-\tilde{v}(\tilde{t},\tilde{x}))(f(\tilde{v}(\tilde{t},\tilde{x}))-f(v(t,0^+))\psi(t,\tilde{t},0,\tilde{x})dtd\tilde{x}d\tilde{t}$$
$$+\int_0^{+\infty}\int_0^1\int_0^1 |u_0(x)-\tilde{v}(\tilde{t},\tilde{x})|\psi(0,\tilde{t},x,\tilde{x})dxd\tilde{x}d\tilde{t} \geq 0. \quad (3.36)$$

But now we can reverse the roles of the functions v and $\tilde{v}$. So we have:

$$\int_0^{+\infty}\int_0^1 |v(t,x)-\tilde{v}(\tilde{t},\tilde{x})|\psi_{\tilde{t}}(t,\tilde{t},x,\tilde{x}) + \operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))(f(v(t,x))-f(\tilde{v}(\tilde{t},\tilde{x})))\psi_{\tilde{x}}(t,\tilde{t},x,\tilde{x})$$
$$-\operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))\tilde{h}(t)\psi(t,\tilde{t},x,\tilde{x})d\tilde{x}d\tilde{t} + \int_0^1 |u_0(x)-\tilde{v}(\tilde{t},\tilde{x})|\psi(t,\tilde{t},x,\tilde{x})d\tilde{x}$$
$$+\int_0^{+\infty} \operatorname{sgn}(\bar{u}-v(t,x))(f(v(t,x))-f(\tilde{v}(\tilde{t},1^-)))\psi(t,\tilde{t},x,1)$$
$$-\operatorname{sgn}(\bar{u}-v(t,x))(f(v(t,x))-f(\tilde{v}(\tilde{t},0^+))\psi(t,\tilde{t},x,0)d\tilde{t} \geq 0.$$

and then we obtain:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 |v(t,x)-\tilde{v}(\tilde{t},\tilde{x})|\psi_{\tilde{t}}(t,\tilde{t},x,\tilde{x}) - \operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))\tilde{h}(\tilde{t})\psi(t,\tilde{t},x,\tilde{x})$$
$$+\operatorname{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))(f(v(t,x))-f(\tilde{v}(\tilde{t},\tilde{x})))\psi_{\tilde{x}}(t,\tilde{t},x,\tilde{x})dxd\tilde{x}dtd\tilde{t}$$
$$+\int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}(\bar{u}-v(t,x))(f(v(t,x))-f(\tilde{v}(\tilde{t},1^-)))\psi(t,\tilde{t},x,1)$$
$$-\operatorname{sgn}(\bar{u}-\tilde{v}(t,x))(f(v(t,x))-f(\tilde{v}(\tilde{t},0^+))\psi(t,\tilde{t},x,0)dtdxd\tilde{t}$$
$$+\int_0^{+\infty}\int_0^1\int_0^1 |\tilde{u}_0(\tilde{x})-v(t,x)|\psi(t,0,x,\tilde{x})d\tilde{x}dxdt \geq 0. \quad (3.37)$$

And finally adding (3.36) and (3.37) we get:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 |v(t,x)-\tilde v(\tilde t,\tilde x)|(\psi_t(t,\tilde t,x,\tilde x)+\psi_{\tilde t}(t,\tilde t,x,\tilde x))\\
&\qquad + \mathrm{sgn}(v(t,x)-\tilde v(\tilde t,\tilde x))(f(v(t,x))-f(\tilde v(\tilde t,\tilde x)))(\psi_x(t,\tilde t,x,\tilde x)+\psi_{\tilde x}(t,\tilde t,x,\tilde x))\\
&\qquad\qquad + \mathrm{sgn}(v(t,x)-\tilde v(\tilde t,\tilde x))(h(t)-\tilde h(\tilde t))\psi(t,\tilde t,x,\tilde x)dxd\tilde x dt d\tilde t\\
&+\int_0^{+\infty}\int_0^1\int_0^1 |\tilde u_0(\tilde x)-v(t,x)|\psi(t,0,x,\tilde x)d\tilde x dx dt + \int_0^{+\infty}\int_0^1\int_0^1 |u_0(x)-\tilde v(\tilde t,\tilde x)|\psi(0,\tilde t,x,\tilde x)dxd\tilde x d\tilde t\\
&\qquad + \int_0^{+\infty}\int_0^1\int_0^{+\infty} \mathrm{sgn}(\bar u - v(t,x))(f(v(t,x))-f(\tilde v(\tilde t,1^-)))\psi(t,\tilde t,x,1)\\
&\qquad - \mathrm{sgn}(\bar u-\tilde v(t,x))(f(v(t,x))-f(\tilde v(\tilde t,0^+))\psi(t,\tilde t,x,0)dtdxd\tilde t\\
&\qquad + \int_0^{+\infty}\int_0^1\int_0^{+\infty} \mathrm{sgn}(\bar u-\tilde v(\tilde t,\tilde x))(f(\tilde v(\tilde t,\tilde x))-f(v(t,1^-)))\psi(t,\tilde t,1,\tilde x)\\
&\qquad\qquad - \mathrm{sgn}(\bar u-\tilde v(\tilde t,\tilde x))(f(\tilde v(\tilde t,\tilde x))-f(v(t,0^+))\psi(t,\tilde t,0,\tilde x)dtd\tilde x d\tilde t \geq 0. \quad (3.38)
\end{aligned}$$

Now consider ϕ a nonnegative function in $\mathcal{C}^1_c(\mathbb{R}^2)$ and ρ a nonnegative, even, $\mathcal{C}^\infty$ function with support in $[-1;1]$ satisfying also:

$$\int_{-1}^1 \rho(x)dx = 1.$$

We define the family (ψ_n) of nonnegative functions in $\mathcal{C}^1_c(\mathbb{R}^4)$ by:

$$\psi_n(t,\tilde t,x,\tilde x) = n^2\phi(\frac{t+\tilde t}{2},\frac{x+\tilde x}{2})\rho(n(t-\tilde t))\rho(n(x-\tilde x)). \quad (3.39)$$

It is clear that for all n in $\mathbb{N}$ and all $(t,\tilde t,x,\tilde x)$ in $\mathbb{R}^4$:

$$\begin{aligned}
\partial_t\psi_n(t,\tilde t,x,\tilde x)+\partial_{\tilde t}\psi_n(t,\tilde t,x,\tilde x) &= n^2\phi(\frac{t+\tilde t}{2},\frac{x+\tilde x}{2})\rho(n(t-\tilde t))\rho(n(x-\tilde x)),\\
\partial_x\psi_n(t,\tilde t,x,\tilde x)+\partial_{\tilde x}\psi_n(t,\tilde t,x,\tilde x) &= n^2\partial_2\phi(\frac{t+\tilde t}{2},\frac{x+\tilde x}{2})\rho(n(t-\tilde t))\rho(n(x-\tilde x)).
\end{aligned}$$

We substitute ψ_n in (3.38) and let n tend to infinity. We will do so term by term.

$$\begin{aligned}
&\int_0^{+\infty}\int_0^1\int_0^1 |\tilde u_0(\tilde x)-v(t,x)|\psi_n(t,0,x,\tilde x)d\tilde x dx dt\\
&\qquad = \int_0^{+\infty}\int_0^1\int_0^1 |\tilde u_0(\tilde x)-v(t,x)|n^2\phi(\frac{t}{2},\frac{x+\tilde x}{2})\rho(nt)\rho(n(x-\tilde x))d\tilde x dx dt\\
&\qquad\quad = \int_0^{+\infty}\int_{-n}^n\int_0^1 |\tilde u_0(X-\frac{\delta_x}{2n})-v(\frac{\delta_t}{n},X-\frac{\delta_x}{2n})|\phi(\frac{\delta_t}{2n},X)\rho(\delta_t)\rho(\delta_x)dXd\delta_x d\delta_t,
\end{aligned}$$

after the change of variable $(t,x,\tilde x)\to\left(\delta_t=nt,\delta_x=n(x-\tilde x),X=\frac{x+\tilde x}{2}\right)$. And since

$$\int_0^1|\tilde u_0(X-\frac{\delta_x}{2n})-v(\frac{\delta_t}{n},X-\frac{\delta_x}{2n})|\phi(\frac{\delta_t}{2n},X)dX \underset{n\to+\infty}{\to} \int_0^1|\tilde u_0(X)-u_0(X)|\phi(0,X)dX \;\; d\delta_t d\delta_x \;\; a.e.,$$

we obtain:

$$\int_0^{+\infty}\int_0^1\int_0^1 |\tilde u_0(\tilde x)-v(t,x)|\psi_n(t,0,x,\tilde x)d\tilde x dx dt \underset{n\to+\infty}{\to} \frac{1}{2}\int_0^1 |\tilde u_0(X)-u_0(X)|\phi(0,X)dX. \quad (3.40)$$

Note that the $\frac{1}{2}$ factor comes from integrating δ_t from 0 to $+\infty$ and because ρ is even.
The same type of reasoning imply:

$$\int_0^{+\infty}\int_0^1\int_0^1 |u_0(x) - \tilde{v}(\tilde{t},\tilde{x})|\psi_n(0,\tilde{t},x,\tilde{x})dxd\tilde{x}d\tilde{t} \underset{n\to+\infty}{\to} \frac{1}{2}\int_0^1 |\tilde{u}_0(X) - u_0(X)|\phi(0,X)dX, \quad (3.41)$$

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 |v(t,x) - \tilde{v}(\tilde{t},\tilde{x})|(\partial_t\psi_n(t,\tilde{t},x,\tilde{x}) + \partial_{\tilde{t}}\psi_n(t,\tilde{t},x,\tilde{x}))dxd\tilde{x}dtd\tilde{t} \underset{n\to+\infty}{\to} \int_0^{+\infty}\int_0^1 |v(T,X) - \tilde{v}(T,X)|\partial_T\phi(T,X)dXdT, \quad (3.42)$$

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \mathrm{sgn}(v(t,x)-\tilde{v}(\tilde{t},\tilde{x}))(f(v(t,x))-f(\tilde{v}(\tilde{t},\tilde{x})))(\partial_t\psi_n(t,\tilde{t},x,\tilde{x})+\partial_{\tilde{x}}\psi_n(t,\tilde{t},x,\tilde{x}))dxd\tilde{x}dtd\tilde{t} \underset{n\to+\infty}{\to} \int_0^{+\infty}\int_0^1 \mathrm{sgn}(v(T,X) - \tilde{v}(T,X))(f(v(T,X)) - f(\tilde{v}(T,X)))\partial_X\phi(T,X)dXdT. \quad (3.43)$$

In the derivation of (3.43) the fact that $(w,z) \to \mathrm{sgn}(z-w)(f(z)-f(w))$ is Lipschitz near $z = w$ is critical so the lack of regularity of $(w,z) \to \mathrm{sgn}(z-w)$ prevents the same argument to work for the remaining terms. However it is clear that

$$\mathrm{sgn}(v(t,x) - \tilde{v}(\tilde{t},\tilde{x}))(h(t) - \tilde{h}(\tilde{t}))\psi(t,\tilde{t},x,\tilde{x}) \leq |h(t) - \tilde{h}(\tilde{t})|\psi(t,\tilde{t},x,\tilde{x}).$$

So we get:

$$\limsup_{n\to+\infty} \int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \mathrm{sgn}(v(t,x) - \tilde{v}(\tilde{t},\tilde{x}))(h(t) - \tilde{h}(\tilde{t}))\psi_n(t,\tilde{t},x,\tilde{x})dxd\tilde{x}dtd\tilde{t} \leq \int_0^{+\infty}\int_0^1 |h(T) - \tilde{h}(T)|\phi(T,X)dXdT. \quad (3.44)$$

It only remains to control the boundary terms:

$$\int_0^{+\infty}\int_0^1\int_0^{+\infty} \mathrm{sgn}(\bar{u} - v(t,x))(f(v(t,x)) - f(\tilde{v}(\tilde{t},1^-)))\psi_n(t,\tilde{t},x,1) - \mathrm{sgn}(\bar{u} - \tilde{v}(t,x))(f(v(t,x)) - f(\tilde{v}(\tilde{t},0^+))\psi_n(t,\tilde{t},x,0)dtdxd\tilde{t},$$

$$\int_0^{+\infty}\int_0^1\int_0^{+\infty} \mathrm{sgn}(\bar{u} - \tilde{v}(\tilde{t},\tilde{x}))(f(\tilde{v}(\tilde{t},\tilde{x})) - f(v(t,1^-)))\psi_n(t,\tilde{t},1,\tilde{x}) - \mathrm{sgn}(\bar{u} - \tilde{v}(\tilde{t},\tilde{x}))(f(\tilde{v}(\tilde{t},\tilde{x})) - f(v(t,0^+))\psi_n(t,\tilde{t},0,\tilde{x})dtd\tilde{x}d\tilde{t}.$$

But if ϕ has a support in $\mathbb{R} \times (0,1)$ then for n large enough we have:

$$\psi_n(.,.,.,0) = \psi_n(.,.,.,1) = \psi_n(.,.,1,.) = \psi(.,.,0,.) = 0,$$

so both boundary terms tend to 0. Combining this with (3.40), (3.41), (3.42), (3.43) and (3.44), we see that for every nonnegative function ϕ in $\mathcal{C}_c^1(\mathbb{R} \times (0,1))$:

$$\int_0^{+\infty}\int_0^1 |v(t,x) - \tilde{v}(t,x)|\phi_t(t,x) + \mathrm{sgn}(v(t,x) - \tilde{v}(t,x))(f(v(t,x)) - f(\tilde{v}(t,x)))\phi_x(t,x) + |h(t) - \tilde{h}(t)|\phi(t,x)dxdt + \int_0^1 |u_0(x) - \tilde{u}_0(x)|\phi(0,x)dx \geq 0. \quad (3.45)$$

A density argument shows that the previous estimate holds for any Lipschitz function ϕ with compact support in $\mathbb{R} \times (0,1)$. Now for $T > 0$ and $n \in \mathbb{N}^*$, we define α_n and β_n as follows:

$$\alpha_n(t) = \begin{cases} 1 & \text{for } t \leq T, \\ 0 & \text{for } t \geq T + \frac{1}{n}, \\ 1 - n(t-T) & \text{for } T \leq t \leq T + \frac{1}{n}, \end{cases}$$

$$\beta_n(x) = \begin{cases} 1 & \text{for } x \in [\frac{1}{n}, 1 - \frac{1}{n}], \\ 2nx - 1 & \text{for } x \in [\frac{1}{2n}, \frac{1}{n}], \\ 2n(1-x) - 1 & \text{for } x \in [1 - \frac{1}{n}, 1 - \frac{1}{2n}] \\ 0 & \text{otherwise .} \end{cases}$$

Taking $\phi = \alpha_n(t)\beta_n(x)$ in (3.45) and letting n tend to infinity we end up with:

$$\int_0^T \operatorname{sgn}(v(t,0^+) - \tilde{v}(t,0^+))(f(v(t,0^+)) - f(\tilde{v}(t,0^+))) \\ - \operatorname{sgn}(v(t,1^-) - \tilde{v}(t,1^-))(f(v(t,1^-)) - f(\tilde{v}(t,1^-)))dt + \int_0^T |h(t) - \tilde{h}(t)|dt \\ + ||u_0 - \tilde{u}_0||_{L^1(0,1)} - ||v(T,.) - \tilde{v}(T,.)||_{L^1(0,1)} \geq 0. \quad (3.46)$$

Now for three numbers a, b, c we have:

$$\forall k \in I(a,b) \cap I(a,c) \cap I(b,c), \\ \operatorname{sgn}(c-b)(f(c) - f(b)) = \operatorname{sgn}(c-a)(f(c) - f(k)) + \operatorname{sgn}(b-a)(f(b) - f(k)).$$

Applying this identity with $(a = \bar{u}, b = v(t,1^-), c = \tilde{v}(t,1^-))$ or $(a = \bar{u}, b = v(t,0^+), c = \tilde{v}(t,0^+))$ and using the boundary conditions of (3.34) we obtain:

$$\operatorname{sgn}(v(t,0^+) - \tilde{v}(t,0^+))(f(v(t,0^+)) - f(\tilde{v}(t,0^+))) \leq 0 \quad dt \text{ a.e.}, \\ \operatorname{sgn}(v(t,1^-) - \tilde{v}(t,1^-))(f(v(t,1^-)) - f(\tilde{v}(t,1^-))) \geq 0 \quad dt \text{ a.e..}$$

Finally substituting those inequalities in (3.46) provides (3.35). □

We define $\mathcal{X}$ as the following function space:

$$\mathcal{X} = \{\alpha \in \mathcal{C}^0(\mathbb{R}^+) \mid ||\alpha||_{\mathcal{X}} := \sup_{t \geq 0}(|\alpha(t)|e^{-2L_G t}) < +\infty\},$$

where L_G is the Lipschitz constant of g (see 18). Consider the operator $\mathcal{F}$ which to $\alpha \in \mathcal{X}$ associates the function $||v(t,.) - \bar{u}||_{L^1(0,1)}$, with v the entropy solution of

$$\begin{cases} \partial_t v + \partial_x(f(v)) = g(\alpha(t)) & \text{on } (0,+\infty) \times (0,1), \\ v(0,.) = u_0 & \text{on } (0,1), \\ \operatorname{sgn}(v(t,1^-) - \bar{u})(f(v(t,1^-)) - f(k)) \geq 0 & \forall k \in I(\bar{u}, v(t,1^-)), \ dt \text{ a.e.}, \\ \operatorname{sgn}(v(t,0^+) - \bar{u})(f(v(t,0^+)) - f(k)) \leq 0 & \forall k \in I(\bar{u}, v(t,0^+)), \ dt \text{ a.e..} \end{cases} \quad (3.47)$$

Then we have the following result.

Lemma 15. *The operator $\mathcal{F}$ has the following properties.*

- *For any function α in $\mathcal{X}$, the function $\mathcal{F}(\alpha)$ belongs to $\mathcal{X}$.*
- *The operator $\mathcal{F}$ is $\frac{1}{2}$-Lipschitz on $\mathcal{X}$.*

Proof. We take α in $\mathcal{X}$ and v the entropy solution of (3.47). The constant function $\bar{u}$ is a solution of (3.34) with source term equal to 0 and initial data equal to $\bar{u}$. Using (3.35) we have:

$$\begin{aligned}\mathcal{F}(\alpha)(T) &= ||v(T,.) - \bar{u}||_{L^1(0,1)} \\ &\leq ||u_0 - \bar{u}||_{L^1(0,1)} + \int_0^T g(\alpha(t))dt.\end{aligned}$$

Therefore for any $T \geq 0$, we have:

$$\begin{aligned}e^{-2L_GT}\mathcal{F}(\alpha)(T) &\leq ||u_0 - \bar{u}||_{L^1(0,1)} + \int_0^T e^{-2L_GT} g(\alpha(t))dt \\ &\leq ||u_0 - \bar{u}||_{L^1(0,1)} + \int_0^T L_G e^{-2L_G(T-t)}|\alpha(t)|e^{-2L_Gt}dt \\ &\leq ||u_0 - \bar{u}||_{L^1(0,1)} + ||\alpha||_{\mathcal{X}} \int_0^T L_G e^{-2L_G(T-t)}dt \leq ||u_0 - \bar{u}||_{L^1(0,1)} + \frac{||\alpha||_{\mathcal{X}}}{2}.\end{aligned}$$

It follows that $\mathcal{F}(\alpha)$ is in $\mathcal{X}$.

In order to prove the second assertion let us consider α, β in $\mathcal{X}$ and v_α, v_β the corresponding entropy solutions of (3.47). Using (3.35) we see that for any nonnegative T:

$$\begin{aligned}|\mathcal{F}(\alpha)(T) - \mathcal{F}(\beta)(T)| &= |||v_\alpha(T,.) - \bar{u}||_{L^1(0,1)} - ||v_\beta(T,.) - \bar{u}||_{L^1(0,1)}| \\ &\leq ||v_\alpha(T,.) - v_\beta(T,.)||_{L^1(0,1)} \\ &\leq \int_0^T |g(\alpha(t)) - g(\beta(t))|dt.\end{aligned}$$

But for any $T \geq 0$:

$$\begin{aligned}e^{-2L_GT}|\mathcal{F}(\alpha) - \mathcal{F}(\beta)|(T) &\leq \int_0^T L_G e^{-2L_G(T-t)}|\alpha(t) - \beta(t)|e^{-2L_Gt}dt \\ &\leq ||\alpha - \beta||_{\mathcal{X}} \int_0^T L_G e^{-2L_G(T-t)}dt \\ &\leq \frac{||\alpha - \beta||_{\mathcal{X}}}{2}.\end{aligned}$$

□

Let us now go back to the proof of Proposition 18. Applying the Banach fixed point theorem to $\mathcal{F}$, we see that (3.33) has a unique entropy solution u such that $||u(T,.) - \bar{u}||_{L^1(0,1)}$ is in $\mathcal{X}$. But if v is an entropy solution of (3.33) and if we use (3.35) we have:

$$||v(T,.) - \bar{u}||_{L^1(0,1)} \leq ||u_0 - \bar{u}||_{L^1(0,1)} + \int_0^T L_G||v(t,.) - \bar{u}||_{L^1(0,1)}dt.$$

Using Gronwall's lemma we obtain:

$$||v(T,.) - \bar{u}||_{L^1(0,1)} \leq ||u_0 - \bar{u}||_{L^1(0,1)}e^{L_GT}.$$

Thus the application:

$$T \mapsto ||v(T,.) - \bar{u}||_{L^1(0,1)},$$

is in $\mathcal{X}$ and therefore $v = u$. Using Lemma 14 and Gronwall's lemma we have that for u and v the entropy solutions to (3.33) for initial data u_0 and v_0:

$$\forall t \geq 0, \qquad ||u(t,.) - v(t,.)||_{L^1(0,1)} \leq ||u_0 - v_0||_{L^1(0,1)} e^{L_G t}.$$

This concludes concludes the proof of Proposition 18.

3.3 Generalized characteristics and boundary conditions.

We begin by recalling a few definitions and results from [44]. We will refer in this section to the system:

$$\begin{cases} \partial_t u + \partial_x(f(u)) = h(t) & \text{on } (0,+\infty)\times(0,1), \\ u(0,.) = u_0 & \text{on } (0,1), \\ \operatorname{sgn}(u(t,1^-) - \bar{u})(f(u(t,1^-)) - f(k)) \geq 0 & \forall k \in I(\bar{u}, u(t,1^-)),\ dt\ a.e., \\ \operatorname{sgn}(u(t,0^+) - \bar{u})(f(u(t,0^+)) - f(k)) \leq 0 & \forall k \in I(\bar{u}, u(t,0^+)),\ dt\ a.e., \end{cases} \tag{3.48}$$

where $u_0 \in \mathrm{BV}(0,1)$, $h \in \mathcal{C}^0(\mathbb{R}^+)$, $\bar{u} \in \mathbb{R}$ and u is the unique entropy solution. Following [44] we introduce the following.

Definition 11. • *If γ is an absolutely continuous function defined on an interval $(a,b) \subset \mathbb{R}^+$ and with values in $(0,1)$, we say that γ is a generalized characteristic of (3.48) if:*

$$\dot{\gamma}(t) \in I(f'(u(t,\gamma(t)^-)), f'(u(t,\gamma(t)^+))) \qquad dt\ a.e..$$

This is the classical characteristic ODE taken in the weak sense of Filippov [55].

- *A generalized characteristic γ is said to be genuine on (a,b) if:*

$$u(t,\gamma(t)^+) = u(t,\gamma(t)^-) \quad dt\ a.e..$$

We recall the following results from [44].

Theorem 12. • *For any (t,x) in $(0,+\infty)\times(0,1)$ there exists at least one generalized characteristic γ such that $\gamma(t) = x$.*

- *If γ is a generalized characteristics defined on (a,b) then for almost all t in (a,b):*

$$\dot{\gamma}(t) = \begin{cases} f'(u(t,\gamma(t)) & \text{if } u(t,\gamma(t)^+) = u(t,\gamma(t)^-), \\ \frac{f(u(t,\gamma(t)^+)) - f(u(t,\gamma(t)^-))}{u(t,\gamma(t)^+) - u(t,\gamma(t)^-)} & \text{if } u(t,\gamma(t)^+) \neq u(t,\gamma(t)^-). \end{cases}$$

- *If γ is a genuine generalized characteristics on (a,b), then there exists a C^1 function v defined on (a,b) such that:*

$$\begin{gathered} u(b,\gamma(b)^+) \leq v(b) \leq u(b,\gamma(b)^-), \\ u(t,\gamma(t)^+) = v(t) = u(t,\gamma(t)^-) \quad \forall t \in (a,b), \\ u(a,\gamma(a)^-) \leq v(a) \leq u(a,\gamma(a)^+). \end{gathered} \tag{3.49}$$

Furthermore (γ, v) *satisfy the classical ODE equation:*

$$\forall t \in (a,b) \begin{cases} \dot{\gamma}(t) = f'(v(t)), \\ \dot{v}(t) = h(t). \end{cases} \tag{3.50}$$

- *Two genuine characteristics may intersect only at their endpoints.*
- *If* γ_1 *and* γ_2 *are two generalized characteristics defined on* (a,b)*, then we have:*

$$\forall t \in (a,b), \quad (\gamma_1(t) = \gamma_2(t) \Rightarrow \forall s \geq t,\ \gamma_1(s) = \gamma_2(s))\,.$$

- *For any* (t,x) *in* $\mathbb{R}^+ \times (0,1)$ *there exist two generalized characteristics* χ^+ *and* χ^- *called maximal and minimal and associated to* v^+ *and* v^- *by* (3.50)*, such that if* γ *is a generalized characteristic going through* (t,x) *then*

$$\begin{gathered} \forall s \leq t, \qquad \chi^-(s) \leq \gamma(s) \leq \chi^+(s), \\ \chi^+ \text{ and } \chi^- \text{ are genuine on } \{s < t\}\ , \\ v^+(t) = u(t,x^+) \qquad \text{and} \qquad v^-(t) = u(t,x^-). \end{gathered}$$

Note that in the previous theorem every property dealt only with the interior of the domain $\mathbb{R}^+ \times [0,1]$. We will now be interested in the influence of the boundary conditions on the generalized characteristics. Following the method of [44], we begin with a few technical identities.

Lemma 16. • *If* χ *is a Lipschitz function defined on* $[a,b]$ *and satisfying:*

$$\forall t \in (a,b), \quad 0 \leq \chi(t) < 1, \tag{3.51}$$

we have:

$$\int_0^{\chi(b)} u(b,x)dx - \int_0^{\chi(a)} u(a,x)dx = \int_a^b \chi(t)h(t)dt \\ + \int_a^b f(u(t,0^+)) - f(u(t,\chi(t)^+)) + \dot{\chi}(t)u(t,\chi(t)^+)dt. \tag{3.52}$$

- *If* χ *is a Lipschitz function defined on* $[a,b]$ *and such that:*

$$\forall t \in (a,b), \quad 1 \geq \chi(t) > 0,$$

we have:

$$\int_{\chi(b)}^1 u(b,x)dx - \int_{\chi(a)}^1 u(a,x)dx = \int_a^b (1-\chi(t))h(t)dt \\ + \int_a^b f(u(t,\chi(t)^-)) - f(u(t,1^-)) - \dot{\chi}(t)u(t,\chi(t)^-)dt. \tag{3.53}$$

- *Finally if* χ_1 *and* χ_2 *are two Lipschitz functions defined on* $[a,b]$ *and satisfying:*

$$\forall t \in (a,b), \quad 0 < \chi_1(t) < \chi_2(t) < 1,$$

the following holds:

$$\int_{\chi_1(b)}^{\chi_2(b)} u(b,x)dx - \int_{\chi_1(a)}^{\chi_2(a)} u(a,x)dx = \int_a^b h(t)(\chi_2(t) - \chi_1(t))dt$$
$$+ \int_a^b f(u(t,\chi_1(t)^-)) - f(u(t,\chi_2(t)^+)) + \dot{\chi_2}(t)u(t,\chi_2(t)^+) - \dot{\chi_1}(t)u(t,\chi_1(t)^-)dt. \quad (3.54)$$

Proof. We begin with the proof of (3.52). We will prove the equality when (3.51) holds on $[a,b]$ and then extend since both sides of (3.52) are continuous in a and b. For $\epsilon > 0$ we define the following two functions:

$$\psi_\epsilon(t,x) = \begin{cases} 1 & \text{when } t \in [a,b] \text{ and } 0 \le x \le \chi(t), \\ 1 - \frac{x-\chi(t)}{\epsilon} & \text{when } t \in [a,b] \text{ and } \chi(t) \le x \le \chi(t)+\epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

$$\forall t \in \mathbb{R}^+, \quad \rho_\epsilon(t) = \begin{cases} 1 & \text{when } a+\epsilon \le t \le b, \\ \frac{b-t}{\epsilon} & \text{when } b-\epsilon \le t \le b, \\ \frac{t-a}{\epsilon} & \text{when } a \le t \le a+\epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The product $\rho_\epsilon \psi_\epsilon$ is Lipschitz and has compact support in $\mathbb{R}^+ \times [0,1]$. Since u is a weak solution of (3.48) we have:

$$\int_0^{+\infty}\int_0^1 u(t,x)(\partial_t \rho_\epsilon(t)\psi_\epsilon(t,x) + \rho_\epsilon(t)\partial_t\psi_\epsilon(t,x)) + f(u(t,x))\rho_\epsilon(t)\partial_x\psi_\epsilon(t,x) + h(t)dxdt$$
$$+ \int_0^T u_0(x)\rho_\epsilon(0)\psi_\epsilon(0,x)dx + \int_0^{+\infty} f(u(t,0^+))\rho_\epsilon(t)\psi_\epsilon(t,0) - f(u(t,1^-))\rho_\epsilon(t)\psi_\epsilon(t,1)dt = 0. \quad (3.55)$$

It is easy to see that:

- for $\epsilon > 0$, $\rho_\epsilon(0) = 0$,
- for ϵ small enough: $\rho(t)\psi_\epsilon(t,1) = 0$ for all t,
- when $\epsilon \to 0$, $\forall t \ge 0$, $\rho_\epsilon(t)\psi_\epsilon(t,0) \to \mathbb{1}_{[a,b]}(t)$,
- when $\epsilon \to 0$, we have:

$$\int_0^{+\infty}\int_0^1 u(t,x)\partial_t\rho_\epsilon(t)\psi_\epsilon(t,x)dxdt \to \int_0^{\chi(a)} u(a,x)dx - \int_0^{\chi(b)} u(b,x)dx,$$
$$\int_0^{+\infty}\int_0^1 u(t,x)\rho_\epsilon(t)\partial_t\psi_\epsilon(t,x)dxdt \to \int_a^b \dot{\chi}(t)u(t,\chi(t)^+)dt,$$
$$\int_0^{+\infty}\int_0^1 f(u(t,x))\rho_\epsilon(t)\partial_x\psi_\epsilon(t,x)dxdt \to -\int_a^b f(u(t,\chi(t)^+))dt.$$

Therefore taking the limit in (3.55) we get:

$$\int_0^{\chi(a)} u(a,x)dx - \int_0^{\chi(b)} u(b,x)dx + \int_a^T h(t)dt \\ + \int_a^T f(u(t,0^+)) - f(u(t,\chi(t)^+)) + \dot{\chi}(t)u(t,\chi(t)^+)dt = 0,$$

which is exactly (3.52).

The proof of (3.53) is symmetrical and is omitted. And for (3.54) we use the same ideas but with the following test functions:

$$\psi_\epsilon(t,x) = \begin{cases} 1 & \text{when } t \in [a,b] \text{ and } \chi_1(t) \leq x \leq \chi_2(t), \\ 1 - \frac{x-\chi_2(t)}{\epsilon} & \text{when } t \in [a,b] \text{ and } \chi_2(t) \leq x \leq \chi_2(t)+\epsilon, \\ \frac{x+\epsilon-\chi_1(t)}{\epsilon} & \text{when } t \in [a,b] \text{ and } \chi_1(t)-\epsilon \leq x \leq \chi_1(t), \\ 0 & \text{otherwise,} \end{cases}$$

$$\forall t \in \mathbb{R}^+, \quad \rho_\epsilon(t) = \begin{cases} 1 & \text{when } a+\epsilon \leq t \leq b-\epsilon, \\ \frac{b-t}{\epsilon} & \text{when } b-\epsilon \leq t \leq b, \\ \frac{t-a}{\epsilon} & \text{when } a \leq t \leq a+\epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

□

Let us also show the additional lemma.

Lemma 17. *Consider $t > 0$ and x in $\{0,1\}$ and suppose that one of the following conditions is satisfied:*

$$x = 1 \text{ and } f'(u(t,x^-)) > 0, \\ x = 0 \text{ and } f'(u(t,x^+)) < 0.$$

Then there is a genuine backward characteristic γ going through (t,x) and such that:

$$\dot{\gamma}(t) = \begin{cases} f'(u(t,1^-)) & \text{if} \quad x = 1, \\ f'(u(t,0^+)) & \text{if} \quad x = 0. \end{cases}$$

Proof. We will prove only the first case, the second one being identical. Let (x_n) be an increasing sequence in $(0,1)$ such that $x_n \underset{n\to+\infty}{\to} 1$. We immediately see that:

$$f'(u(t,x_n)) \underset{n\to+\infty}{\to} f'(u(t,1^-)), \tag{3.56}$$

and so we can suppose that:

$$\forall n \geq 0, \quad f'(u(t,x_n)) \geq \frac{f'(u(t,1^-))}{2}. \tag{3.57}$$

Now consider χ_n the maximal generalized backward characteristic going through (t,x_n) and v_n the functions associated to them by (3.50). Using (3.57) and the continuity of h, we deduce that there exists $\epsilon > 0$ independent of n such that if the functions χ_n, v_n are maximally defined on an interval I then $[t-\epsilon,t] \subset I$. Now a classical ODE result asserts that because x_n and $f'(u(t,x_n))$ converge then the functions χ_n and v_n converge uniformly toward two functions γ and v satisfying:

$$\forall s \in [t-\epsilon,t], \quad \begin{cases} \dot{v}(s) = h(s) & v(t) = u(t,1^-) \\ \dot{\gamma}(s) = f'(v(s)) & \gamma(t) = 1 \end{cases}. \tag{3.58}$$

It is know that the uniform limit of generalized characteristics is a generalized characteristic (see [68][Chapter 1] or [44][Chapter10]) therefore γ is a generalized characteristic. It is genuine since it satisfies (3.58). □

In the remaining part of this section, we will suppose:

$$f'(\bar{u}) \geq 0 \quad \text{and} \quad \forall t \in \mathbb{R}^+, \; h(t) \geq 0. \tag{3.59}$$

Remark 17. *Note that h being nonnegative and f being convex, any genuine generalized characteristic is also convex since it satisfies ODE* (3.50).

We will now describe the behavior of generalized characteristics at the boundary points.

Proposition 19. *There is no genuine generalized characteristic γ defined on (a,b) with $a > 0$ and such that:*

$$\gamma(t) \underset{t \to a^+}{\to} 1. \tag{3.60}$$

Proof. Let us suppose that (3.60) is false. We have a genuine characteristic γ defined on (a,b) such that $\gamma(a) = 1$ and $a > 0$. Thanks to Remark 17, we have

$$f'(v(a)) < 0.$$

Now consider $\epsilon > 0$ and apply (3.54) with $\chi_1(t) = \gamma(t) - \epsilon$ and $\chi_2 = \gamma$. Then:

$$\int_{\gamma(T)-\epsilon}^{\gamma(T)} u(T,x)dx - \int_{1-\epsilon}^{1} u(a,x)dx = \epsilon \int_a^T h(s)ds$$
$$+ \int_a^T f(u(s,\gamma(s)-\epsilon^-)) - f(u(s,\gamma(s)^+)) - \dot{\gamma}(s)(u(s,\gamma(s)-\epsilon^-) - u(s,\gamma(s)^+))ds.$$

Since $u(s,\gamma(s)^+) = v(s)$, $\dot{\gamma}(s) = f'(v(s))$ and f is convex we obtain:

$$\int_{\gamma(T)-\epsilon}^{\gamma(T)} u(T,x)dx - \int_{1-\epsilon}^{1} u(a,x)dx \geq \epsilon(v(T) - v(a)).$$

Therefore after dividing by ϵ and taking the limit $\epsilon \to 0$ we arrive at:

$$v(T) - u(a,1^-) \geq v(T) - v(a). \tag{3.61}$$

And so:

$$f'(u(a,1^-)) \leq f'(v(a)) < 0.$$

Since we supposed $f'(\bar{u}) \geq 0$ this forces f to be of type I. Thus we have a unique u^* such that $f'(u^*) = 0$ and the boundary condition at $x = 1$ becomes:

$$u(t,1^-) \geq u^*, \quad dt \ a.e. \tag{3.62}$$

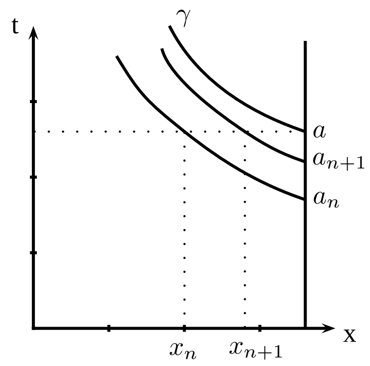


Figure 3.1:

Consider $(x_n)_{n\geq 0}$ satisfying:

$$\forall n \geq 0, \quad 0 < x_n < x_{n+1} < 1,$$
$$u(a,.) \text{ is continuous at every } x_n,$$
$$x_n \underset{n\to+\infty}{\to} 1,$$
$$\forall n \geq 0, \quad f'(u(a,x_n)) \leq \frac{f'(u(a,1^-))}{2}.$$

This sequence exists thanks to (3.61) and also because $u(a,.)$ is in $\mathrm{BV}(0,1)$. Using Theorem 12 we know that for any n, there exist a unique number $a_n < a$ and two regular functions γ_n and v_n solutions of:

$$\begin{cases} \dot{v_n}(t) = h(t), \\ v_n(a) = u(a,x_n), \end{cases} \tag{3.63}$$

$$\begin{cases} \dot{\gamma_n}(t) = f'(v_n(t)), \\ \gamma_n(a) = x_n, \end{cases} \tag{3.64}$$

$$\gamma_n(a_n) = 1, \tag{3.65}$$

maximally defined on (a_n, a). Furthermore γ_n is a genuine generalized characteristic on (a_n, a). Using the fact that x_n is increasing and Theorem 12, we can see that a_n is nondecreasing. Furthermore using $f'(u_n) \leq \frac{f'(u(a,1^-))}{2} < 0$, $h \geq 0$ and f convex, we obtain:

$$a_n \underset{n\to+\infty}{\to} a.$$

Suppose now that given n, we have a certain T such that:

$$a_n < T < a \quad \text{and} \quad f'(u(T,1^-)) > 0.$$

Using Lemma 17, we get a time $R < T$ and a backward characteristic δ issue from $(T,1)$, defined on $[R,T]$ and genuine on (R,T). We also have $R \geq a_n$ and $\delta(R) = 1$ because γ_n and δ do not cross. Additionally if w is the regular function associated to δ by (3.50), we have $w(T) = u(T,1^-)$.

It follows that:

$$\begin{aligned} f'(u(T,1^-)) = f'(w(T)) &= \frac{1}{T-R}\int_R^T (f'(w(T)) - f'(w(t)))dt \\ &\leq \frac{||f''||_{L^\infty}}{T-R}\int_R^T\int_t^T h(s)dsdt \\ &\leq \frac{T-R}{2}||f''||_{L^\infty}||h||_{L^\infty(R,T)} \\ &\leq \frac{a-a_n}{2}||f''||_{L^\infty}||h||_{L^\infty(R,T)}. \end{aligned}$$

Combined with (3.62) this implies:

$$\underset{t\in[a-\nu,a]}{\text{essup}}\ f'(u(t,1^-)) \underset{\nu\to 0^+}{\to} 0. \tag{3.66}$$

Since f is convex and there is only one number u^* such that $f'(u^*) = 0$, we can deduce:

$$\underset{t\in[a-\nu,a]}{\text{essup}}\ f(u(t,1^-)) \underset{\nu\to 0^+}{\to} f(u^*). \tag{3.67}$$

Let us now consider any number u_i such that $f'(u_i) < 0$ and define $\chi_\epsilon(t) = 1-\epsilon+f'(u_i)(t-a)$. Applying (3.53) between $a(\epsilon) = a + \frac{\epsilon}{f'(u_i)}$ and a we obtain:

$$\begin{aligned} \int_{1-\epsilon}^1 u(a,x)dx &= \int_{a(\epsilon)}^a (1-\chi_\epsilon(t))h(t)dt + \int_{a(\epsilon)}^a (f(u(t,\chi_\epsilon(t)^-)) - f(u(t,1^-)) - \dot{\chi}_\epsilon(t)u(t,\chi_\epsilon(t)^-))dt \\ &= \int_{a(\epsilon)}^a (1-\chi_\epsilon(t))h(t)dt + \int_{a(\epsilon)}^a (f(u(t,\chi_\epsilon(t)^-)) - f(u_i) - \dot{\chi}_\epsilon(t)(u(t,\chi_\epsilon(t)^-) - u_i))dt \\ &\qquad + \int_{a(\epsilon)}^a (f(u_i) - f(u(t,1^-)))dt + \int_{a(\epsilon)}^a \dot{\chi}_\epsilon(t)u_i dt. \end{aligned}$$

Using the convexity of f and since $\dot{\chi}_\epsilon = f'(u_i)$ we have:

$$\forall t \in (a(\epsilon), a), \quad f(u(t,\chi_\epsilon(t)^-)) - f(u_i) - \dot{\chi}_\epsilon(t)(u(t,\chi_\epsilon(t)^-) - u_i) \geq 0. \tag{3.68}$$

We have supposed $f'(u_i) < 0$ therefore $f(u_i) > f(u^*)$ and then for ϵ small enough, $a(\epsilon)$ is close enough to a to guarantee, thanks to (3.67):

$$f(u_i) \geq f(u(t,1^-)) \quad dt\ a.e. \text{ on } (a(\epsilon), a).$$

Thus we have:

$$\int_{1-\epsilon}^1 u(a,x)dx \geq \int_{a(\epsilon)}^a (1-\chi_\epsilon(t))h(t)dt + \epsilon u_i.$$

And now dividing by ϵ and letting ϵ tend to 0 we obtain:

$$\forall u_i \ s.t.\ f'(u_i) < 0, \qquad u(a,1^-) \geq u_i.$$

In turn this implies $f'(u(a,1^-)) \geq 0$ and we have a contradiction. □

Proposition 20. *Consider a genuine generalized characteristic γ defined on (a,b) with $a > 0$ and v the regular function associated to γ by (3.50). If $\gamma(t) \underset{t\to a}{\to} 0$ then $v(t) \underset{t\to a}{\to} \bar{u}$.*

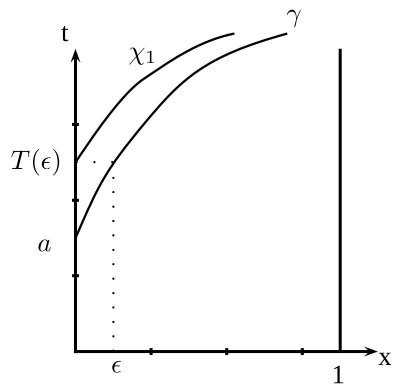


Figure 3.2:

Proof. We will proceed in two steps.

- First let us show that $v(a) \geq \bar{u}$. Once again we consider $\epsilon > 0$ small enough and define the time
$$T(\epsilon) = \inf\{t \in (a,b) \mid \gamma(t) \geq \epsilon\}.$$
Then if we apply (3.54) to $\chi_1(t) = \gamma(t) - \epsilon$ and $\chi_2(t) = \gamma(t)$ on $[T(\epsilon), b]$ we get:
$$\int_{\gamma(b)-\epsilon}^{\gamma(b)} u(b,x)dx - \int_0^{\epsilon} u(T(\epsilon),x)dx = \epsilon \int_{T(\epsilon)}^{b} h(t)dt$$
$$+ \int_{T(\epsilon)}^{b} (f(u(t,\gamma(t)-\epsilon^-)) - f(u(t,\gamma(t)^+)) + \dot{\gamma}(t)(u(t,\gamma(t)^+) - u(t,\gamma(t)-\epsilon^-)))dt.$$
We apply (3.52) to $\chi(t) = \gamma(t)$ on $[a, T(\epsilon)]$ to have:
$$\int_0^{\epsilon} u(T(\epsilon),x)dx - \int_0^{\gamma(a)} u(a,x)dx = \int_a^{T(\epsilon)} \gamma(t)h(t)dt$$
$$+ \int_a^{T(\epsilon)} (f(u(t,0^+)) - f(u(t,\gamma(t)^+)) + \dot{\gamma}(t)u(t,\gamma(t)^+))dt.$$
Adding the two previous equalities and remembering that:
$$\gamma(a) = 0,\ u(t,\gamma(t)) = v(t) \text{ and } \dot{\gamma}(t) = f'(v(t)),$$
we obtain:
$$\int_{\gamma(b)-\epsilon}^{\gamma(b)} u(b,x)dx = \epsilon \int_{T(\epsilon)}^{b} h(t)dt + \int_a^{T(\epsilon)} \gamma(t)h(t)dt$$
$$+ \int_{T(\epsilon)}^{b} (f(u(t,\gamma(t)-\epsilon^-)) - f(v(t)) + f'(v(t))(v(t) - u(t,\gamma(t)-\epsilon^-)))dt$$
$$+ \int_a^{T(\epsilon)} (f(u(t,0^+)) - f(v(t)) + f'(v(t))v(t))dt.$$

Now using the fact that $f(u(t,0^+)) \geq f(\bar{u})$ for almost all t and remembering that f is convex we have:

$$\begin{aligned}\int_{T(\epsilon)}^{b} (f(u(t,0^+)) - f(v(t)) + f'(v(t))v(t))dt &\geq \int_{T(\epsilon)}^{b} (f(\bar{u}) - f(v(t)) + f'(v(t))v(t))dt \\ &\geq \int_{T(\epsilon)}^{b} (f(\bar{u}) - f(v(t)) - f'(v(t))(\bar{u} - v(t)))dt \\ &\qquad + \bar{u}\int_{T(\epsilon)}^{b} \dot{\gamma}(t)dt \\ &\geq \epsilon\bar{u}.\end{aligned}$$

The convexity of f also implies:

$$\int_{T(\epsilon)}^{b} f(u(t,\gamma(t)-\epsilon^-)) - f(v(t)) + f'(v(t))(v(t) - u(t,\gamma(t)-\epsilon^-))dt \geq 0.$$

But thanks to (3.50) we know that:

$$\int_{T(\epsilon)}^{b} h(t)dt = v(b) - v(T(\epsilon)),$$

so in the end we have for any ϵ positive and small enough:

$$\int_{\gamma(b)-\epsilon}^{\gamma(b)} u(b,x)dx \geq \int_{a}^{T(\epsilon)} \gamma(t)h(t)dt + \epsilon(v(b) - v(T(\epsilon))) + \epsilon\bar{u}.$$

Dividing by ϵ and letting it tend to 0 provides:

$$v(b) \geq v(b) - v(a) + \bar{u},$$

which is indeed $v(a) \geq \bar{u}$.

- Now to prove $v(a) \leq \bar{u}$, let us suppose $v(a) > \bar{u}$.

  For ϵ positive if we apply (3.54) to $\chi_1 = \gamma$ and $\chi_2 = \gamma + \epsilon$ between a and $t > a$, we have:

$$\begin{aligned}\int_{\gamma(t)}^{\gamma(t)+\epsilon} u(t,x)dx - \int_{0}^{\epsilon} u(a,x)dx &= \epsilon\int_{a}^{t} h(s)ds + \int_{a}^{t} f(v(t)) - f(u(t,(\gamma(t)+\epsilon)^+)) \\ &\qquad + f'(v(t))(u(t,(\gamma(t)+\epsilon)^+) - v(t))dt \\ &\leq \epsilon\int_{a}^{t} h(s)ds \\ &\leq v(t) - v(a).\end{aligned}$$

Dividing by ϵ and letting it tend to 0 provides:

$$u(t,\gamma(t)^+) - u(a,0^+) \leq v(t) - v(a).$$

Since γ is genuine this implies:

$$\bar{u} < v(a) \leq u(a,0^+). \tag{3.69}$$

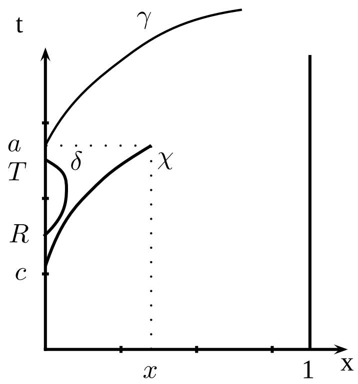


Figure 3.3:

Now consider x in $(0,1)$, χ the minimal generalized characteristic through (a,x) and w the function associated to it by (3.50). We can see that if x is close enough to 0 then thanks to $f'(u(a,0^+)) > 0$ there exists $c < a$ such that

$$\chi(c) = 0.$$

Consider T such that $c < T < a$ and suppose that $f'(u(T,0^+)) < 0$. Using Lemma 17 we get a generalized characteristic δ on $[R,T]$, genuine on (R,T) and such that $\delta(T) = 0$. But we would also have $R \geq c$ and $\delta(R) = 0$ because δ and γ do not cross (Theorem 12). This is impossible because thanks to Remark 17, δ is convex, therefore no such T exists. (see figure 3.3)

Using the boundary condition at $x = 0$ this implies:

$$\text{for almost all } t \text{ in } (c,a), \qquad u(t,0^+) = \bar{u}.$$

Now we consider u_i larger than $\bar{u}$. This implies that $f'(u_i)$ is positive. We define γ_ϵ and a_ϵ by:

$$\gamma_\epsilon(t) = \epsilon + f'(u_i)(t-a),$$
$$a_\epsilon = \frac{\epsilon}{f'(u_i)}.$$

If we apply (3.52) with $\chi = \gamma_\epsilon$ for ϵ small enough (so that $a_\epsilon \geq c$), we obtain:

$$\begin{aligned}\int_0^\epsilon u(a,x)dx &= \int_{a_\epsilon}^a \gamma_\epsilon(t)h(t)dt + \int_{a_\epsilon}^a (f(u(t,0^+)) - f(u(t,\gamma_\epsilon(t)^+)) + f'(u_i)u(t,\gamma_\epsilon(t)^+))dt \\ &\leq \int_{a_\epsilon}^a \gamma_\epsilon(t)h(t)dt + \int_{a_\epsilon}^a (f(u_i) - f(u(t,\gamma_\epsilon(t)^+)) + f'(u_i)(u(t,\gamma_\epsilon(t)^+) - u_i))dt + \epsilon u_i \\ &\leq \int_{a_\epsilon}^a \gamma_\epsilon(t)h(t)dt + \epsilon u_i.\end{aligned}$$

Dividing by ϵ and letting it tend to 0 provides:

$$u(a,0^+) \leq u_i.$$

Since u_i can be arbitrarily close to $\bar{u}$, we end up with:

$$v(a) \leq u(a,0^+) \leq \bar{u}.$$

□

Proposition 21. *If γ is a genuine generalized characteristic defined on (a,b) with $a>0$ and v is the regular function associated to γ by* (3.50)*. Suppose:*

$$\gamma(t) \underset{t\to b}{\to} 0 \text{ and } v(t) \underset{t\to b}{\to} \bar{v}, \tag{3.70}$$

then

$$f'(\bar{v}) \leq 0 \text{ and } f(\bar{v}) \geq f(\bar{u}). \tag{3.71}$$

Proof. Since γ is convex and since for t in (a,b) $\gamma(t)>0=\gamma(b)$, we have:

$$\forall t \in (a,b), \qquad \dot{\gamma}(t) \leq 0.$$

Letting t tend to b we have

$$f'(\bar{v}) = \lim_{t\to b} f'(v(t)) = \lim_{t\to b} \dot{\gamma}(t) \leq 0.$$

Now consider ϵ positive and define:

$$T(\epsilon) = \sup\{t \in (a,b) \mid \gamma(t) \geq \epsilon\}.$$

We apply (3.54) to $\chi_1(t)=\gamma(t)-\epsilon$ and $\chi_2(t)=\gamma(t)$ on $[a,T(\epsilon)]$ and get:

$$\begin{aligned}\int_0^{\epsilon} u(T(\epsilon),x)dx - \int_{\gamma(a)-\epsilon}^{\gamma(a)} u(a,x)dx = \epsilon \int_a^{T(\epsilon)} h(t)dt \\ + \int_a^{T(\epsilon)} (f(u(t,(\gamma(t)-\epsilon)^-)) - f(u(t,\gamma(t)^+)) + \dot{\gamma}(t)(u(t,\gamma(t)^+) - u(t,(\gamma(t)-\epsilon)^-)))dt.\end{aligned}$$

We apply (3.52) to $\chi(t)=\gamma(t)$ on $[T(\epsilon),b]$ to obtain:

$$\begin{aligned}\int_0^{\gamma(b)} u(b,x)dx - \int_0^{\epsilon} u(T(\epsilon),x)dx = \int_{T(\epsilon)}^b \gamma(t)h(t)dt \\ + \int_{T(\epsilon)}^b (f(u(t,0^+)) - f(u(t,\gamma(t)^+)) + \dot{\gamma}(t)u(t,\gamma(t)^+))dt.\end{aligned}$$

As in the proof of Proposition 20, we add the two previous equalities and remember that:

$$\gamma(b)=0, \ u(t,\gamma(t))=v(t) \text{ and } \dot{\gamma}(t)=f'(v(t)),$$

to get:

$$\begin{aligned}-\int_{\gamma(a)-\epsilon}^{\gamma(a)} u(a,x)dx = \epsilon \int_a^{T(\epsilon)} h(t)dt + \int_{T(\epsilon)}^b \gamma(t)h(t)dt \\ + \int_a^{T(\epsilon)} (f(u(t,(\gamma(t)-\epsilon)^-)) - f(v(t)) + f'(v(t))(v(t) - u(t,(\gamma(t)-\epsilon)^-)))dt \\ + \int_{T(\epsilon)}^b (f(u(t,0^+)) - f(v(t)) + f'(v(t))v(t))dt.\end{aligned}$$

Using the fact that $f(u(t,0^+)) \geq f(\bar{u})$ for almost all t thanks to the boundary condition and remembering that f is convex, we have for any u_i such that $f(\bar{u}) \geq f(u_i)$:

$$\begin{aligned}\int_{T(\epsilon)}^{b} (f(u(t,0^+)) - f(v(t)) + f'(v(t))v(t))dt &\geq \int_{T(\epsilon)}^{b} f(u_i) - f(v(t)) + f'(v(t))v(t)dt \\ &\geq \int_{T(\epsilon)}^{b} (f(u_i) - f(v(t)) - f'(v(t))(u_i - v(t)))dt \\ &\qquad + u_i \int_{T(\epsilon)}^{b} \dot{\gamma}(t)dt \\ &\geq -\epsilon u_i.\end{aligned}$$

The convexity of f also implies:

$$\int_{a}^{T(\epsilon)} (f(u(t,(\gamma(t)-\epsilon)^-)) - f(v(t)) + f'(v(t))(v(t) - u(t,(\gamma(t)-\epsilon)^-)))dt \geq 0.$$

Thanks to (3.50) we know:

$$\int_{a}^{T(\epsilon)} h(t)dt = v(T(\epsilon)) - v(a).$$

We deduce that for any $\epsilon > 0$ small enough:

$$-\int_{\gamma(a)-\epsilon}^{\gamma(a)} u(a,x)dx \geq \int_{T(\epsilon)}^{b} \gamma(t)h(t)dt + \epsilon(v(T(\epsilon)) - v(a)) - \epsilon u_i.$$

And finally dividing by ϵ and letting ϵ tend to 0 we have:

$$-v(a) \geq \bar{v} - v(a) - u_i,$$

In the end:

$$\forall u_i \ s.t.\ f(u_i) \leq f(\bar{u}), \qquad \bar{v} \leq u_i.$$

Thus we have proven (3.71). ☐

We will now use the three previous propositions to prove two crucial estimates on the infimum and supremum of $u(t,.)$.

Proposition 22. *If u is the unique entropy solution of the system* (3.48) *then:*

$$\forall t \geq 0, \qquad \inf_{x\in(0,1)} u(t,x) \geq \min(\bar{u}, \inf_{x\in(0,1)} u_0(x) + \int_0^t h(s)ds), \tag{3.72}$$

$$\forall t \geq 0, \qquad \sup_{x\in(0,1)} u(t,x) \leq \max(\bar{u}, \sup_{x\in(0,1)} u_0(x)) + \int_0^t h(s)ds. \tag{3.73}$$

Proof. Take (t,x) in $(0,+\infty)\times(0,1)$ and consider χ^+ the maximal backward generalized characteristic going through (t,x), and v the function associated to χ^+ by (3.50). We suppose that χ^+ is maximally defined on $[a,b]$ for a certain a. If $\chi^+(a) = 0$ we have, thanks to Proposition 20:

$$v(a) \geq \bar{u}.$$

Therefore using the last part of Theorem 12 we get:

$$u(t,x^+) = v(t) = v(a) + \int_a^t h(s)ds \geq \bar{u}.$$

Now if $\chi^+(a) > 0$ then thanks to Proposition 19 we get $a = 0$ and using (3.49) we have:

$$u(t,x^+) = v(t) = v(0) + \int_0^t h(s)ds \geq u(0,\chi^+(0)^-) \geq \inf_{(0,1)} u_0.$$

And since u is an entropy solution $u(t,x^-) \geq u(t,x^+)$ for $t > 0$ thus we have (3.72). The same kind of reasoning provides (3.73). ☐

Let us now prove a simple estimate on the characteristic speed.

Lemma 18. *Let us consider u the entropy solution of* (3.48) *then for any positive t and any x in $(0,1)$ we have:*

$$f'(u(t,x^+)) \geq \frac{x-1}{t}. \tag{3.74}$$

Proof. Let χ^+ be the maximal backward generalized characteristic going through (t,x). Then thanks to Theorem 12 we know that:

$$\dot{\chi}^+(t) = f'(u(t,x^+)).$$

But thanks to Remark 17, χ^+ is convex and so we get:

$$\forall s \leq t, \qquad \dot{\chi}^+(s) \leq f'(u(t,x^+)).$$

Finally thanks to Proposition 19, χ^+ cannot cross $x = 1$ at a positive time, which implies (3.74). ☐

3.4 Proof of Theorem 10

In this section we will prove the remaining parts of Theorem 10 in the case $f'(\bar{u}) > 0$. The first point was already proven in Section 3.2, (3.24) will be proven in Proposition 24, (3.25) and (3.26) will be proven in Proposition 25 and finally the regularization property will be proven in Proposition 23.

Since the feedback law (3.16) satisfies:

$$\forall z \in L^1(0,1), \qquad \mathcal{G}_1(z) \geq 0, \tag{3.75}$$

we can apply all the results of the preceding section to the entropy solution u of (3.14), (3.16) and (3.17). We begin the proof of Theorem 10 with the following geometric lemma.

Lemma 19. *Let us define the time T_2 in two ways depending on the type of the flux function f introduced in Definition 10:*

$$T_2 = \begin{cases} \frac{1}{2}\frac{2\bar{u}-\hat{u}-u^*}{f(\bar{u})-f(\frac{u^*+\hat{u}}{2})} - \frac{1}{f'(\frac{u^*+\hat{u}}{2})} & \text{if } f \text{ is of type I,} \\ \frac{1}{f'(\bar{u}-\|u_0-\bar{u}\|_{L^\infty(0,1)})} & \text{if } f \text{ is of type II.} \end{cases}$$

Then for any t larger than T_2 and any x in $(0,1)$ if χ^- is the minimal backward generalized characteristic going through (t,x) there exists a positive time a such that:

$$\chi^-(a) = 0.$$

Proof. We begin with the case where f is of type I. Let us define:

$$T_1 = -\frac{1}{f'(\frac{u^*+\hat{u}}{2})},$$

and thanks to the hypothesis on f, we have $0 < T_1 < +\infty$. Using Lemma 18 we see:

$$\forall x \in (0,1),\ \forall t \geq T_1, \qquad f'(u(t,x)) \geq -\frac{1}{T_1} = f'(\frac{\hat{u}+u^*}{2}).$$

Since f is strictly convex we deduce:

$$\forall x \in (0,1),\ \forall t \geq T_1, \qquad u(t,x) \geq \frac{\hat{u}+u^*}{2}. \tag{3.76}$$

Looking at the boundary condition (3.23) we see that this also implies:

$$\text{for almost all } t \text{ in } [T_1,+\infty), \quad u(t,0^+) = \bar{u}.$$

Consider $b > T_1$ and such that $u(b,0^+) = \bar{u}$. Then for x sufficiently close to 0 we have:

$$f'(u(b,x)) \geq \frac{f'(\bar{u})}{2} > 0. \tag{3.77}$$

Let χ be the minimal backward characteristic going through (b,x), a the time such that χ is maximally defined on $[a,b]$, and v the function associated to χ by (3.50). For x sufficiently close to 0 we have thanks to (3.77),(3.50) and Proposition 20:

$$a \geq T_1, \qquad v(a) = \bar{u}. \tag{3.78}$$

Now let γ be the forward characteristic going through (b,x) and maximally defined on $[b,c)$ for a certain c (eventually infinite). Thanks to (3.78), we see that for any t in (b,c), the minimal backward characteristic through $(t,\gamma(t))$ cross $x=0$ at a time a_1 such that $a_1 \geq a \geq T_1 > 0$. Using (3.50) and Proposition 20 we deduce:

$$\forall t \in (b,c), \qquad u(t,\gamma(t)^-) \geq \bar{u}.$$

Using (3.76) we obtain:

$$\text{for almost all } t \text{ in } (b,c), \qquad \dot{\gamma}(t) \geq 2\frac{f(\bar{u}) - f(\frac{\hat{u}+u^*}{2})}{2\bar{u}-\hat{u}-u^*} > 0.$$

This implies that c is finite and that $\gamma(c) = 1$. Consequently if $c \leq T_2$ we have finished, otherwise:

$$\gamma(T_2) \geq 2(T_2 - b)\frac{f(\bar{u}) - f(\frac{\hat{u}+u^*}{2})}{2\bar{u}-\hat{u}-u^*}.$$

The number b can be chosen as close to T_1 as we want and

$$2(T_2 - T_1)\frac{f(\bar{u}) - f(\frac{\hat{u}+u^*}{2})}{2\bar{u}-\hat{u}-u^*} = 1.$$

This concludes the proof in the case of a flux f of type I.

Now we suppose that f is of type II. Using Proposition 22 we see that:

$$\begin{aligned}\forall x \in (0,1),\ \forall t \geq 0, \qquad u(t,x) &\geq \min(\bar{u}, \inf_{y\in(0,1)}(u_0(y))) \\ &\geq \bar{u} + \min(0, \inf_{y\in(0,1)}(u_0(y)-\bar{u})) \\ &\geq \bar{u} - ||u_0 - \bar{u}||_{L^\infty(0,1)}.\end{aligned}$$

Obviously since f is of type II and since $f'(\bar{u}) > 0$ we have:

$$f'(\bar{u} - ||u_0 - \bar{u}||_{L^\infty(0,1)}) > 0 \text{ and thus } T_2 < +\infty.$$

Now for t larger than T_2 and for x in $(0,1)$ consider χ the minimal backward characteristic through (t,x) and a the nonnegative number such that χ is maximally defined on $[a,t]$. Using (3.50) we see

$$\forall s \in (a,t) \qquad \dot{\chi}(s) \geq f'(\bar{u} - ||u_0 - \bar{u}||_{L^\infty(0,1)}),$$

and since

$$T_2 f'(\bar{u} - ||u_0 - \bar{u}||_{L^\infty(0,1)}) = 1,$$

we deduce $a > 0$, which concludes the proof. □

Proposition 23. *The unique entropy solution u of* (3.14), (3.16) *and* (3.17) *is regular on* $(T_2, +\infty) \times (0,1)$ *and satisfies:*

$$\forall t \geq T_2, \quad ||u(t,.) - \bar{u}||_{L^1(0,1)} \leq ||u(T_2,.) - \bar{u}||_{L^1(0,1)} e^{-\frac{f'(\bar{u})}{2}(t-T_2)}. \tag{3.79}$$

Proof. Let us take t larger than T_2 and x, y in $(0,1)$ with $x < y$. Consider γ_1 and γ_2 the minimal backward generalized characteristics going through (t,x) and (t,y) and v_1, v_2 the functions associated to them by (3.50). Thanks to Lemma 19, we get two times a_1 and a_2 larger than T_1 such that:

$$\gamma_1(a_1) = \gamma_2(a_2) = 0.$$

Furthermore since two genuine characteristics may cross only at their endpoints we have $a_2 \leq a_1$. But using (3.50) we have:

$$\begin{aligned}u(t,y) = v_2(t) &= \bar{u} + \int_{a_2}^t \mathcal{G}_1(u(s,.))ds \\ &\geq \bar{u} + \int_{a_1}^t \mathcal{G}_1(u(s,.))ds = v_1(t) = u(t,x).\end{aligned} \tag{3.80}$$

So for any time t larger than T_2, $u(t,.)$ is nondecreasing, using Oleinik inequality (3.4) this implies that it is continuous on $(0,1)$. The previous calculation also shows that:

$$\forall t \geq T_2,\ \forall x \in (0,1), \quad u(t,x) \geq \bar{u}. \tag{3.81}$$

Since $f'(\bar{u}) > 0$ we can see that as y tends to 0, a_2 tends to t and therefore using (3.50) we obtain:

$$\forall t \geq T_2, \quad u(t,0^+) = \bar{u}. \tag{3.82}$$

Let us now prove the regularity of u. For the sake of convenience let us put:

$$\forall t \geq 0, \quad g(t) = \mathcal{G}_1(u(t,.)).$$

Using the definition of $\mathcal{G}_1$ and the result of Section 3.2 we already know that g is continuous. We introduce the following auxiliary function B:

$$\forall t \geq 0, e \geq 0, x \in (0,1), u \in \mathbb{R}, \quad \mathrm{B}(t,x,e,u) = \left(u - \bar{u} - \int_e^t g(s)ds, x - \int_e^t f'\Big(\bar{u} + \int_e^s g(r)dr\Big)ds\right).$$

For t larger than T_2 and x in $(0,1)$, let $e(t,x)$ be the time for which the genuine backward characteristic γ going through (t,x) satisfies:

$$\gamma(e(t,x)) = 0.$$

Using (3.50) and Proposition 20 we can see that the following holds:

$$\mathrm{B}(t,x,e(t,x),u(t,x)) = (0,0).$$

It is clear that:

$$\partial_u \mathrm{B}(t,x,e,u) = (1,0),$$
$$\partial_e \mathrm{B}(t,x,e,u) = (g(e), f'(\bar{u})(1 + \int_e^t f''(\bar{u} + \int_e^s g(r)dr)ds)).$$

And since $f'(\bar{u}) > 0$ and $f'' \geq 0$, the regularity of u comes as a consequence of the implicit function theorem.

To show (3.79) let us consider s and t satisfying $T_2 < s < t$. Using (3.81) and Lemma 16 we get:

$$\begin{aligned}
||u(t,.) - \bar{u}||_{L^1(0,1)} - ||u(s,.) - \bar{u}||_{L^1(0,1)} &= \int_0^1 (|u(t,x) - \bar{u}| - |u(s,x) - \bar{u}|)dx \\
&= \int_0^1 (u(t,x) - u(s,x))dx \\
&= \int_s^t \mathcal{G}_1(u(r,.))dr + \int_s^t (f(u(r,0^+)) - f(u(r,1^-)))dr \\
&= \int_s^t \mathcal{G}_1(u(r,.))dr + \int_s^t (f(\bar{u}) - f(u(r,1^-)))dr \\
&\leq \int_s^t \mathcal{G}_1(u(r,.))dr + \int_s^t f'(\bar{u})(\bar{u} - u(r,1^-))dr.
\end{aligned}$$

Thanks to (3.80) and (3.82) we also have:

$$\forall x \in (0,1), \quad |u(r,x) - \bar{u}| = u(r,x) - \bar{u} \leq u(r,1^-) - \bar{u},$$

and therefore

$$\int_0^1 |u(r,x) - \bar{u}|dx \leq u(r,1^-) - \bar{u}.$$

Using the two previous inequalities and the definition of $\mathcal{G}_1$ we then deduce:

$$\begin{aligned}
||u(t,.) - \bar{u}||_{L^1(0,1)} - ||u(s,.) - \bar{u}||_{L^1(0,1)} &\leq \frac{f'(\bar{u})}{2} \int_s^t ||u(r,.) - \bar{u}||_{L^1(0,1)}dr \\
&\quad - f'(\bar{u}) \int_s^t ||u(r,.) - \bar{u}||_{L^1(0,1)}dr \\
&\leq -\frac{f'(\bar{u})}{2} \int_s^t ||u(r,.) - \bar{u}||_{L^1(0,1)}dr.
\end{aligned}$$

Applying Gronwall's lemma we obtain (3.79). □

We end this section with the last remaining estimate of Theorem 10.

Proposition 24. *If u is the entropy solution associated to the initial data u_0 we have:*

$$\forall t \geq 0, \qquad ||u(t,.) - \bar{u}||_{L^1(0,1)} \leq e^{f'(\bar{u})T_2} e^{-\frac{f'(\bar{u})}{2}t} ||u_0 - \bar{u}||_{L^1(0,1)}. \tag{3.83}$$

Proof. The constant function $\bar{u}$ is the unique entropy solution of (3.14), (3.16), (3.17) associated to the constant initial data $\bar{u}$. Therefore comparing u and $\bar{u}$ with Lemma 14 gives us:

$$\begin{aligned}\forall t \geq 0, \quad ||u(t,.) - \bar{u}||_{L^1(0,1)} &\leq ||u_0 - \bar{u}||_{L^1(0,1)} + \int_0^t \mathcal{G}_1(u(s,.)) - \mathcal{G}_1(\bar{u}) ds \\ &\leq ||u_0 - \bar{u}||_{L^1(0,1)} + \frac{f'(\bar{u})}{2} \int_0^t ||u(s,.) - \bar{u}||_{L^1(0,1)} ds.\end{aligned}$$

Using Gronwall's lemma we get:

$$\forall t \in [0, T_2], \quad ||u(t,.) - \bar{u}||_{L^1(0,1)} \leq ||u_0 - \bar{u}||_{L^1(0,1)} e^{\frac{f'(\bar{u})}{2}t} \leq ||u_0 - \bar{u}||_{L^1(0,1)} e^{\frac{f'(\bar{u})}{2}T_2}. \tag{3.84}$$

Combining the last estimate with (3.79) we obtain indeed (3.83). □

Proposition 25. *The state $\bar{u}$ is asymptotically stable in $L^\infty(0,1)$ for the system* (3.14), (3.16) *and* (3.17), *and if u is the entropy solution associated to the initial data u_0 we have:*

$$\forall t \geq 0, \quad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq e^{2f'(\bar{u})T_2 + 1} ||u_0 - \bar{u}||_{L^\infty(0,1)} e^{-\frac{f'(\bar{u})}{2}t}. \tag{3.85}$$

Proof. Using Proposition 22 we have:

$$\forall t \in [0, T_2], \quad \inf_{x \in (0,1)} (u(t,x) - \bar{u}) \geq \min(0, \inf_{x \in (0,1)} (u_0(x) - \bar{u})) \geq -||u_0 - \bar{u}||_{L^\infty(0,1)}.$$

Using Proposition 22 and (3.84) we obtain:

$$\begin{aligned}\forall t \in [0, T_2], \quad \sup_{x \in (0,1)} (u(t,x) - \bar{u}) &\leq \max(0, \sup_{x \in (0,1)} (u_0(x) - \bar{u}) + \int_0^t \mathcal{G}_1(u(s,.)) ds) \\ &\leq ||u_0 - \bar{u}||_{L^\infty(0,1)} + \int_0^t \frac{f'(\bar{u})}{2} e^{\frac{f'(\bar{u})}{2}s} ||u_0 - \bar{u}||_{L^1(0,1)} ds \\ &\leq e^{\frac{f'(\bar{u})}{2}T_2} ||u_0 - \bar{u}||_{L^\infty(0,1)}.\end{aligned}$$

Thus we get:

$$\forall t \in [0, T_2], \quad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq e^{\frac{f'(\bar{u})}{2}T_2} ||u_0 - \bar{u}||_{L^\infty(0,1)}. \tag{3.86}$$

Thanks to (3.80) and (3.81) we have:

$$\forall t \geq T_2, \quad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} = u(t, 1^-) - \bar{u}.$$

If we take χ_t the minimal backward generalized characteristic going through $(t,1)$ and v_t the function associated to it by (3.50) we know that, thanks to Proposition 20, there exists a_t positive such that:

$$\chi_t(a_t) = 0 \quad \text{and} \quad v_t(a_t) = \bar{u}.$$

Now using (3.81) and (3.50) we see:

$$\begin{aligned}\chi_t(t)-\chi_t(a_t) &= \int_{a_t}^{t} f'(v_t(s))ds \\ &\geq (t-a_t)f'(\bar{u}).\end{aligned}$$

In turn this shows:

$$a_t \geq t - \frac{1}{f'(\bar{u})} \underset{t\to+\infty}{\to} +\infty. \tag{3.87}$$

Thanks to estimate (3.83) and to the definition of $\mathcal{G}_1$ we also have:

$$\begin{aligned}\forall T\geq 0, \quad \int_T^{+\infty} \mathcal{G}_1(u(s,.))ds &\leq \int_T^{+\infty} \frac{f'(\bar{u})}{2} e^{-\frac{f'(\bar{u})}{2}s}||u_0-\bar{u}||_{L^1(0,1)}e^{f'(\bar{u})T_2}ds \\ &\leq e^{f'(\bar{u})T_2}||u_0-\bar{u}||_{L^\infty(0,1)}e^{-\frac{f'(\bar{u})}{2}T}.\end{aligned}$$

So using the last two estimates we obtain:

$$\begin{aligned}\forall t\geq T_2, \quad ||u(t,.)-\bar{u}||_{L^\infty(0,1)} = u(t,1^-)-\bar{u} = v_t(t)-\bar{u} &= \int_{a_t}^{t} \mathcal{G}_1(u(s,.))ds \\ &\leq \int_{a_t}^{+\infty} \mathcal{G}_1(u(s,.))ds \\ &\leq e^{f'(\bar{u})T_2}||u_0-\bar{u}||_{L^\infty(0,1)}e^{-\frac{f'(\bar{u})}{2}a_t} \\ &\leq e^{1+f'(\bar{u})T_2}||u_0-\bar{u}||_{L^\infty(0,1)}e^{-\frac{f'(\bar{u})}{2}t}.\end{aligned} \tag{3.88}$$

Combining (3.86) and (3.88) we obtain (3.85). □

3.5 Proof of Theorem 11

In this section we will prove the remaining parts of Theorem 11, therefore $f'(\bar{u})=0$. Therefore f is necessarily of type I and $\bar{u}=u^*=\hat{u}$. We will consider in the following the unique entropy solution u of (3.14), (3.16) and (3.17). We begin by proving the following Lemma which describes two alternative behaviors for u.

Lemma 20. *If the following condition holds:*

$$\int_0^{+\infty} \mathcal{G}_2(u(t,.))dt \leq \bar{u} - \inf_{x\in(0,1)} u_0(x), \tag{3.89}$$

then we have both:

$$||u(t,.)-\bar{u}||_{L^\infty(0,1)} \underset{t\to+\infty}{\to} 0, \tag{3.90}$$

and

$$\forall t\geq 0, \quad ||u(t,.)-\bar{u}||_{L^\infty(0,1)} \leq 2||u_0-\bar{u}||_{L^\infty(0,1)}. \tag{3.91}$$

Otherwise with T_1 the smallest time such that:

$$\int_0^{T_1} \mathcal{G}_2(u(t,.))dt = \bar{u} - \inf_{x\in(0,1)} u_0(x), \tag{3.92}$$

we have:

$$\forall t\geq T_1,\ \forall x\in(0,1), \quad u(t,x)\geq \bar{u}. \tag{3.93}$$

Proof. First let assume that condition (3.89) holds. Then thanks to Proposition 22 we have:

$$\begin{aligned}\forall t \geq 0, \quad \inf_{x\in(0,1)} u(t,x) - \bar{u} &\geq \min(0, \inf_{x\in(0,1)} (u_0(x) - \bar{u}) + \int_0^t \mathcal{G}_2(u(s,.))ds) \\ &\geq \min(0, \inf_{x\in(0,1)} (u_0(x) - \bar{u})) \geq -||u_0 - \bar{u}||_{L^\infty(0,1)}.\end{aligned}$$

Moreover:

$$\begin{aligned}\forall t \geq 0, \quad \sup_{x\in(0,1)} (u(t,x) - \bar{u}) &\leq \max(0, \sup_{x\in(0,1)} (u_0(x) - \bar{u})) + \int_0^t \mathcal{G}_2(u(s,.))ds \\ &\leq ||u_0 - \bar{u}||_{L^\infty(0,1)} + \int_0^{+\infty} \mathcal{G}_2(u(s,.))ds \leq 2||u_0 - \bar{u}||_{L^\infty(0,1)}.\end{aligned}$$

Thus we have (3.91). In order to prove (3.90) let us take T such that:

$$\int_T^{+\infty} \mathcal{G}_2(u(s,.))ds > 0,$$

and define u_i by:

$$u_i = \bar{u} + \int_T^{+\infty} \mathcal{G}_2(u(s,.))ds > \bar{u}. \tag{3.94}$$

Let us also define δ_T by:

$$\delta_T = \frac{1}{f'(u_i)}.$$

This is a finite number because $u_i > \bar{u}$, $f'(\bar{u}) = 0$ and f strictly convex.

For t larger than $T+\delta_T$ and x in $(0,1)$, consider γ the minimal backward characteristic going through (t,x) and the number a such that γ is maximally defined on $[a,t]$. Consider also the function v associated to γ by (3.50) we have:

$$\begin{aligned}\forall s \in [\max(a,T),t], \quad v(s) &= v(t) - \int_s^t \mathcal{G}_2(u(r,.))dr \\ &\geq u(t,x^-) - \int_T^{+\infty} \mathcal{G}_2(u(r,.))dr \\ &\geq u(t,x^-) + \bar{u} - u_i.\end{aligned}$$

We also have:

$$\begin{aligned}\forall s \in [\max(a,T),t], \quad \gamma(t) - \gamma(s) &= \int_s^t f'(v(r))dr \\ &\geq \int_s^t f'(u(t,x^-) + \bar{u} - \hat{u})dr \\ &\geq (t-s)f'(u(t,x^-) + \bar{u} - \hat{u}).\end{aligned} \tag{3.95}$$

Let us now suppose that the following holds:

$$u(t,x^-) \geq 2u_i - \bar{u}. \tag{3.96}$$

Then if we suppose $a \leq T$, we have thanks to (3.95) and since we took t larger than $T + \delta_T$:

$$\gamma(T) \leq x - \delta T f'(u_i) < 0,$$

which is not possible, therefore $a > T$. Thanks to Proposition 20 we deduce:

$$\gamma(a) = 0 \quad \text{and} \quad v(a) = \bar{u}.$$

Using (3.50) we can deduce:

$$u(t,x^-) = v(t) = v(a) + \int_a^t \mathcal{G}_2(u(r,.))dr \leq \bar{u} + \int_T^{+\infty} \mathcal{G}_2(u(r,.))dr \leq \bar{u} + u_i - \bar{u} \leq u_i.$$

However this contradicts (3.94) and (3.96) therefore:

$$\forall t \geq T + \delta T, \qquad \sup_{x\in(0,1)} u(t,x) \leq 2u_i - \bar{u} \underset{T\to+\infty}{\to} \bar{u}.$$

Now Lemma 18 provides:

$$\liminf_{t\to+\infty} \inf_{x\in(0,1)} f'(u(t,x)) \geq 0,$$

and we know thanks to the strict convexity of f that $\bar{u}$ is the only number such that $f'(\bar{u}) = 0$. This ends the proof of Lemma 20. □

Lemma 21. *If there exists a time $T_1 < +\infty$ as in* (3.92)*, then for any time $a \geq T_1$, there exists a generalized characteristic γ going through $(a, 0)$, defined on $[a, b]$ for a certain b, and which satisfies:*

$$\forall t \in [a,b], \quad \gamma(t) \geq \int_a^t f'\Big(\bar{u} + \int_a^s \mathcal{G}_2(u(r,.))dr\Big)ds. \tag{3.97}$$

Proof. Consider a time a larger or equal to T_1, the numbers b_n (larger than a and possibly infinite) and the functions γ_n such that γ_n is the unique forward generalized characteristic going through $(a, \frac{1}{n})$ and it is maximally defined on $[a, b_n]$. Thanks to Theorem 12 we have:

$$\text{for almost all } t \text{ in } (a,b_n), \quad \dot{\gamma}_n(t) = \begin{cases} f'(u(t,\gamma_n(t))) & \text{if } u(t,\gamma_n(t)^-) = u(t,\gamma_n(t)^+), \\ \frac{f(u(t,\gamma_n(t)^-)) - f(u(t,\gamma_n(t)^+))}{u(t,\gamma_n(t)^-) - u(t,\gamma_n(t)^+)} & \text{otherwise.} \end{cases} \tag{3.98}$$

Thanks to (3.4) u satisfies:

$$\forall (t,x) \in (0,+\infty) \times (0,1), \; u(t,x^-) \geq u(t,x^+).$$

Therefore we deduce:

$$\text{for almost all } t \text{ in } (a,b_n), \quad \dot{\gamma}_n(t) \geq f'(u(t,\gamma_n(t)^+)). \tag{3.99}$$

For any t in (a, b_n) the maximal backward generalized characteristic going through $(t, \gamma_n(t))$ is necessarily defined at least on (a, t). Indeed it cannot cross $x = 0$ at a time $s > a$ because it is maximal, and it cannot cross $x = 1$ because of Proposition 19. Since $a \geq T_1$ we have, using (3.50):

$$u(t,\gamma_n(t)^+) \geq \bar{u} + \int_a^t \mathcal{G}_2(u(s,.))ds.$$

After substituting in (3.99) we get:

$$\forall t \in (a,b_n), \quad \gamma_n(t) = \frac{1}{n} + \int_a^t \dot{\gamma}_n(s)ds \tag{3.100}$$

$$\geq \int_a^t f'\Big(\bar{u} + \int_a^s \mathcal{G}_2(u(r,.))dr\Big)ds. \tag{3.101}$$

Since $a \geq T_1$ the characteristics γ_n may leave $(0,1)$ only at $x = 1$. Thanks to Theorem 12 there is a only one forward generalized characteristic going through a point of $(0,+\infty)\times(0,1)$. So the sequence b_n is nondecreasing. We can choose b larger than a such that all characteristics γ_n are defined on $[a,b]$.

Furthermore using (3.98) we see that the family γ_n is uniformly Lipschitz on $[a,b]$ with values in $[0,1]$. Using the Arzela-Ascoli theorem we can suppose that there is an absolutely continuous curve γ defined on $[a,b]$ and such that:

$$\sup_{t\in[a,b]} |\gamma(t) - \gamma_n(t)| \underset{n\to+\infty}{\to} 0.$$

But it is known ([68, Chapter 1]) that the uniform limit of generalized characteristics is a generalized characteristic. Therefore γ is a generalized characteristic and as the limit of the curves γ_n it also satisfies:

$$\forall t \in [a,b], \quad \gamma(t) \geq \int_a^t f'(\bar{u} + \int_a^s \mathcal{G}_2(u(r,.))dr)ds,$$
$$\gamma(a) = 0.$$

□

We now prove a first asymptotic result in the case where we have a time $T_1 < +\infty$.

Proposition 26. *We have:*

$$||u(t,.) - \bar{u}||_{L^\infty(0,1)} \underset{t\to+\infty}{\to} 0.$$

Proof. Let us first remark that should $u(t,.)$ be equal to $\bar{u}$ for some t, it remains at $\bar{u}$ thanks to the uniqueness of the constant solution $\bar{u}$ of the system (3.14), (3.28), (3.17) and the proof is finished. Otherwise we have:

$$\forall t \geq 0, \qquad ||u(t,.) - \bar{u}||_{L^1(0,1)} > 0. \tag{3.102}$$

Thanks to the definition of $\mathcal{G}_2$ this implies that the function

$$t \mapsto \int_{T_1}^t \mathcal{G}_2(u(s,.))ds,$$

is positive and nondecreasing on $(0,+\infty)$. Since f is strictly convex and $f'(\bar{u}) = 0$, we know that f' is positive and increasing on $(\bar{u},+\infty)$. Thus we obtain:

$$\int_{T_1}^T f'\Big(\bar{u} + \int_{T_1}^t \mathcal{G}_2(u(s,.))ds\Big)dt \underset{T\to+\infty}{\to} +\infty. \tag{3.103}$$

Let us take T_2 the smallest time such that:

$$\int_{T_1}^{T_2} f'\Big(\bar{u} + \int_{T_1}^t \mathcal{G}_2(u(s,.))ds\Big)dt = 1.$$

Thanks to Lemma 21, we see that the generalized characteristic γ going through $(T_1, 0)$ has reached $x = 1$ by T_2 at the latest. Therefore for any (t,x) in $[T_2,+\infty)\times(0,1)$, if γ is the minimal backward characteristic through (t,x) there is a time a which is at least equal to T_1 and such that:

$$\gamma(a) = 0.$$

Consider $0 < x < y < 1$ and $t > T_2$. Let χ_1, χ_2 be the minimal generalized characteristics going through (t,x) and (t,y), and v_1, v_2 the functions associated to them by (3.50). Thanks to the choice of T_2 and since genuine characteristics may cross only at their endpoints (Theorem 12), we have a_1 and a_2 such that:

$$T_1 \leq a_2 \leq a_1, \quad \chi_1(a_1) = 0, \quad \chi_2(a_2) = 0.$$

Using Proposition 20 we also get:

$$v_1(a_1) = v_2(a_2) = \bar{u}.$$

But using (3.50), Proposition 20 and the positivity of $\mathcal{G}_2(u(s,.))$ we can see that:

$$u(t,x) = v_1(t) = \bar{u} + \int_{a_1}^{t} \mathcal{G}_2(u(s,.))ds \leq \bar{u} + \int_{a_2}^{t} \mathcal{G}_2(u(s,.))ds = v_2(a_2) = u(t,y).$$

Thus $u(t,.)$ is non decreasing on $(0,1)$, using additionally (3.81), we arrive at:

$$\forall t \geq T_2, \quad ||u(t,.) - \bar{u}||_{L^1(0,1)} = \int_0^1 |u(t,x) - \bar{u}|dx = \int_0^1 (u(t,x) - \bar{u})dx \tag{3.104}$$

$$\leq u(t,1^-) - \bar{u}. \tag{3.105}$$

And now for $t \geq T_2$ and $h > 0$ we get thanks to (3.104) and Lemma 16:

$$||u(t+h,.) - \bar{u}||_{L^1(0,1)} - ||u(t,.) - \bar{u}||_{L^1(0,1)} = \int_0^1 u(t+h,x) - u(t,x)dx \tag{3.106}$$

$$= \int_t^{t+h} \mathcal{G}_2(u(s,.))ds + \int_t^{t+h} f(u(s,0^+)) - f(u(s,1^-))ds. \tag{3.107}$$

So using (3.105) and the facts that $u(s,.)$ is non decreasing on $(0,1)$, f is convex and $f'(\bar{u}) = 0$, we end up with:

$$\begin{aligned} f(u(s,1^-)) - f(u(s,0^+)) &= f(\bar{u} + u(s,1^-) - \bar{u}) - f(\bar{u}) \\ &\geq f(\bar{u} + ||u(s,.) - \bar{u}||_{L^1(0,1)}) - f(\bar{u}) \geq 2\mathcal{G}_2(u(s,.)). \end{aligned}$$

Combining the two previous estimates we have:

$$||u(t+h,.) - \bar{u}||_{L^1(0,1)} - ||u(t,.) - \bar{u}||_{L^1(0,1)} \leq -\int_t^{t+h} \mathcal{G}_2(u(s,.))ds.$$

With A the function introduced in (3.27) and Q the function

$$Q : t \mapsto ||u(t,.) - \bar{u}||_{L^1(0,1)}, \tag{3.108}$$

this implies that for any t larger than T_2:

$$\dot{Q}(t) \leq -\mathrm{A}(Q(t)). \tag{3.109}$$

Therefore if we introduce the solution Q_1 of:

$$\begin{cases} \dot{Q}_1(t) = -\mathrm{A}(Q_1(t)), \\ Q_1(T_2) = Q(T_2), \end{cases} \tag{3.110}$$

the comparison principle provides:

$$\forall t \geq T_2, \quad 0 \leq Q(t) \leq Q_1(t).$$

Finally since f is strictly increasing on $(\bar{u}, +\infty)$, so is A on $(0, +\infty)$. Therefore Q_1 is strictly decreasing on $(T_2, +\infty)$. In turn this implies that $\dot{Q}_1$ is increasing on $(T_2, +\infty)$. So Q_1 is strictly convex, decreasing and positive therefore $\dot{Q}_1(t) \to 0$ when $t \to +\infty$. So $Q_1(t) \to 0$ and:

$$0 \leq ||u(t,.) - \bar{u}||_{L^1(0,1)} = Q(t) \leq Q_1(t) \underset{t\to+\infty}{\to} 0.$$

We recall that thanks to the choices of T_1 and T_2 we have (3.80) and (3.81) and then:

$$\forall t \geq T_2, \quad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} = u(t, 1^-) - \bar{u}.$$

For $t \geq T_2$ consider the number a_t and the function χ such that χ is the minimal backward characteristic through $(t, 1)$, maximally defined on $[a_t, t]$. Using (3.50) we have:

$$u(t, 1^-) - \bar{u} = \int_{a_t}^{t} \mathcal{G}_2(u(s,.))ds.$$

We have seen that if Q_2 is the solution to:

$$\begin{cases} \dot{Q}_2(s) = -\mathrm{A}(Q_2(s)), \\ Q_2(a_t) = Q(a_t), \end{cases}$$

we can deduce:

$$\forall s \geq T_2, \quad \mathcal{G}_2(u(s,.)) = \mathrm{A}(Q(s)) \leq \mathrm{A}(Q_2(s)) = -\dot{Q}_2(s).$$

So we see that:

$$0 \leq u(t, 1^-) - \bar{u} \leq \int_{a_t}^{t} -\dot{Q}_2(s)ds = Q_2(a_t) - Q_2(t) \leq Q_2(a_t) = Q(a_t).$$

Thanks to Lemma 21, we see that for any time $a_1 \geq T_1$, we have a time $c_1 > a_1$ a generalized characteristic γ_1 maximally defined on $[a_1, c_1]$ such that $\gamma_1(a_1) = 0$ and:

$$\forall s \in (a_1, c_1), \qquad \gamma_1(s) \geq \int_{a_1}^{s} f'\Big(\bar{u} + \int_{a_1}^{r} \mathcal{G}_2(u(\omega,.))d\omega\Big)dr.$$

Combining this estimate with (3.102) and using the same reasoning as the one leading to (3.103), we obtain $c_1 < +\infty$. Therefore we get:

$$a_t \underset{t\to+\infty}{\to} +\infty.$$

□

This concludes the proof of the first part of Theorem 11. The remaining part is proven in the next Proposition.

Proposition 27. *If we suppose additionally that there exists a positive number α such that:*

$$\forall z \in \mathbb{R} \quad f''(z) \geq \alpha, \tag{3.111}$$

then we have:

$$\begin{aligned}\forall t \geq 0, \quad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq & 2||u_0 - \bar{u}||_{L^\infty(0,1)} \\ & + \mathrm{A}^{-1}\left(\frac{e}{e-1}\left(\frac{(f'(\bar{u}+1))^2}{4\alpha} + \mathrm{A}(2||u_0-\bar{u}||_{L^\infty(0,1)})\right)\right) \\ & + \sqrt{\frac{2e}{\alpha(e-1)}\left(\frac{(f(1+\bar{u}))^2}{4\alpha} + \mathrm{A}(2||u_0-\bar{u}||_{L^\infty(0,1)})\right)}.\end{aligned} \tag{3.112}$$

Proof. Looking at (3.91) in the proof of Lemma 20, it is clear that we only need to prove the result in the cases where we have $T_1 < +\infty$ such that:

$$\int_0^{T_1} \mathcal{G}_2(u(s,.))ds = \bar{u} - \inf_{x\in(0,1)} u_0(x).$$

Using Proposition 22 as in the proof of Lemma 20 we have:

$$\forall t \in [0, T_1], \qquad ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq 2||u_0 - \bar{u}||_{L^\infty(0,1)}. \tag{3.113}$$

Looking at the proof of Proposition 26 we see that:

$$\forall t \geq T_2, \qquad ||u(t,.) - \bar{u}||_{L^1(0,1)} \leq ||u(T_2,.) - \bar{u}||_{L^1(0,1)}.$$

Therefore we only need to estimate $||u(t,.) - \bar{u}||_{L^1(0,1)}$ on $[T_1, T_2]$ to conclude.

Thanks to (3.111) we have:

$$\forall z \geq 0, \qquad f'(\bar{u}+z) \geq \alpha z. \tag{3.114}$$

As in the proof of Proposition 26 we take Q given by (3.108) and additionally:

$$\forall t \geq 0, \quad \mathcal{I}(t) = \mathrm{A}(Q(t)).$$

Using Lemma 14 and Gronwall's lemma we have:

$$\forall t \geq 0, \qquad \dot{\mathcal{I}}(t) = \dot{Q}(t)\mathrm{A}'(Q(t)) \leq \frac{f'(\bar{u}+1)}{2}\mathcal{I}(t). \tag{3.115}$$

In the following we will also use the notation:

$$L = \frac{f'(\bar{u}+1)}{2}.$$

Thanks to the definition of T_2 we have for any T in $(T_1, T_2]$:

$$\int_{T_1}^{T} f'(\bar{u} + \int_{T_1}^{t} \mathcal{I}(s)ds)dt \leq 1.$$

So using (3.114) we have:

$$\alpha \int_{T_1}^{T}\int_{T_1}^{t} \mathcal{I}(s)dsdt \leq 1. \tag{3.116}$$

Using (3.115) we see that

$$\forall s \in (T_1, T], \qquad \mathcal{I}(s) \geq \mathcal{I}(T) e^{-LT} e^{Ls}.$$

Therefore we have:

$$\forall T \in [T_1, T_2], \quad \alpha \mathcal{I}(T) \int_{T_1}^{T} \int_{T_1}^{t} e^{L(s-T)} ds dt \leq 1,$$

which becomes:

$$\frac{\alpha}{L^2} \left(\mathcal{I}(T)(1 - L(T - T_1) e^{-L(T-T_1)}) - \mathcal{I}(T) e^{-L(T-T_1)} \right) \leq 1.$$

We also have:

$$\forall T \geq T_1, \qquad L(T - T_1) e^{-L(T-T_1)} \leq \frac{1}{e}.$$

Thus we get:

$$\mathcal{I}(T)(1 - \frac{1}{e}) \leq \frac{L^2}{\alpha} + \mathcal{I}(T) e^{-L(T-T_1)}.$$

Finally using (3.115), the fact that A is increasing and (3.113) we obtain:

$$\begin{aligned} \forall T \in [T_1, T_2], \quad Q(T) &\leq \mathrm{A}^{-1} \left(\frac{e}{e-1} \left(\frac{L^2}{\alpha} + \mathcal{I}(T_1) \right) \right) \\ &\leq \mathrm{A}^{-1} \left(\frac{e}{e-1} \left(\frac{L^2}{\alpha} + \mathrm{A}(2 \|u_0 - \bar{u}\|_{L^\infty(0,1)}) \right) \right). \end{aligned}$$

Let us now introduce the constant K and the function $\mathcal{J}$ by:

$$K = \frac{e}{e-1} \left(\frac{L^2}{\alpha} + \mathrm{A}(2 \|u_0 - \bar{u}\|_{L^\infty(0,1)}) \right)$$

$$\forall t \in [T_1, T_2], \quad \mathcal{J}(t) = \int_{T_1}^{t} \mathcal{I}(s) ds.$$

We have thanks to the (3.111):

$$\int_{T_1}^{T_2} \mathcal{J}(s) ds \leq \frac{1}{\alpha}. \tag{3.117}$$

We also have the estimate:

$$\forall t \in [T_1, T_2], \quad \mathcal{J}(t) \geq \begin{cases} \mathcal{J}(T_2) - K(T_2 - t) & \text{if } t \geq T_2 - \frac{\mathcal{J}(T_2)}{K}, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting in (3.117) provides:

$$\mathcal{J}(T_2) \leq \sqrt{\frac{2K}{\alpha}}.$$

Combining with Proposition 22 we get:

$$\forall t \in [0, T_2], \quad \|u(t,.) - \bar{u}\|_{L^\infty(0,1)} \leq 2 \|u_0 - \bar{u}\|_{L^\infty(0,1)} + \sqrt{\frac{2K}{\alpha}}.$$

But using (3.109) we have:

$$\int_{T_2}^{+\infty} \mathcal{I}(t) dt \leq \int_{T_2}^{+\infty} -\dot{Q}(s) ds = Q(T_2).$$

Therefore we obtain:

$$\forall t \geq T_2, ||u(t,.) - \bar{u}||_{L^\infty(0,1)} \leq 2||u_0 - \bar{u}||_{L^\infty(0,1)} + \sqrt{\frac{2K}{\alpha}} + \mathrm{A}^{-1}(K), \tag{3.118}$$

which concludes the proof. □

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Résumé

Dans cette thèse nous étudierons plusieurs problèmes de la théorie du contrôle portant sur des modèles non-linéaires issus de la mécanique des fluides.

Dans le chapitre 1, nous étudions l'équation de Camassa-Holm sur un intervalle compact de $\mathbb{R}$. Après avoir introduit de bonnes conditions aux bords et une notion de solution faible, nous montrons un théorème d'existence et un théorème d'unicité fort-faible pour le problème mixte. Dans une seconde partie nous fournissons une loi de retour pour les données aux bords qui nous permet de stabiliser asymptotiquement l'état stationnaire naturel de l'équation.

Dans le chapitre 2, nous étudions le problème de la contrôlabilité exacte d'une loi de conservation scalaire à flux convexe, posée sur un intervalle compact et dans le cadre des solutions entropiques. On fournit des conditions suffisantes sur des fonctions de BV pour qu'elles soient atteignables en temps arbitraire depuis n'importe quelle donnée initiale. On contrôle l'équation via les données aux bords et aussi grâce à un terme source agissant uniformément en espace.

Enfin le chapitre 3 est consacré au problème de la stabilisation asymptotique des états stationnaires constants d'une loi de conservation scalaire à flux convexe, posée sur un intervalle compact et dans le cadre des solutions entropiques. On contrôle à nouveau l'équation via les données aux bords et un terme source agissant uniformément en espace. Nous fournissons deux lois de retour stationnaires (suivant que l'état à stabiliser est de vitesse critique ou non) qui nous permettent de montrer la stabilisation asymptotique globale.

Mots-clés: Contrôlabilité exacte, stabilisation asymptotique, équation de Camassa-Holm, lois de conservation, solutions entropiques, problème mixte.

Abstract

In this thesis, we study some problems from control theory on several models from fluid mechanics.

In chapter 1, we study the Camassa-Holm equation on a compact interval. After introducing our boundary conditions and a notion of weak solution, we prove an existence result and a weak-strong uniqueness result for the non-homogeneous initial boundary value problem. In a second part, we establish a result on the global asymptotic stabilization problem by means of a boundary feedback law.

In chapter 2, we study the exact controllability problem for a 1-D scalar conservation law with convex flux, on a compact interval and in the context of entropy solution. We provide several sufficient conditions for a BV function to be reachable in any time and from any initial data in BV. We control the equation by means of the boundary data and also through a source term acting uniformly in space.

Finally in chapter 3, we investigate the asymptotic stabilization problem of the constant stationary solutions of a scalar conservation laws with a convex flux, on a compact interval and in the context of entropy solutions. Once again we control the equation through the boundary data and a source term acting uniformly in space. We provide two stationary feedback laws (depending on whether the state to stabilize has critical speed or not) which allow us to prove the global asymptotic stabilization property.

Keywords: Exact controllability, asymptotic stabilization, Camassa-Holm, conservation laws, entropy solutions, initial boundary value problem.