

Quantitative uniqueness for time-periodic heat equation with potential and its applications

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Abstract .- In this paper, we establish a quantitative unique continuation property for some time-periodic linear parabolic equations in a bounded domain Ω . We prove that for time-periodic heat equation with particular time-periodic potential, its solution $u(x, t)$ satisfies $\|u(\cdot, 0)\|_{L^2(\Omega)} \leq C \|u(\cdot, 0)\|_{L^2(\omega)}$ where $\omega \subset \Omega$. Also we deduce the asymptotic controllability for the heat equation with an even, time-periodic potential. Moreover, the controller belongs to a finite dimensional subspace and is explicitly computed.

Key Words .- time-periodic parabolic equation, Poincaré map, observability, controllability.

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1 Introduction and main results

Throughout this paper, Ω is a connected bounded domain in \mathbb{R}^d , $d \geq 1$, with a boundary $\partial\Omega$ of class C^2 , ω is a non-empty open subset of Ω . Let $\ell_1 > 0$ be the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition (i.e. the smallest strictly positive eigenvalue of $-\Delta$ in $H_0^1(\Omega)$). Let $T > 0$ be a real number, we denote $\|\cdot\|_\infty$ the usual norm in $L^\infty(\Omega \times (0, T))$.

In this paper, we study an infinite dimensional system generated by the following parabolic equation

$$\begin{cases} \partial_t y - \Delta y + ay = f & \text{in } \Omega \times (0, +\infty) , \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty) , \end{cases} \quad (1.1)$$

with an even, T -periodic, bounded potential $a = a(x, t) \in L^\infty(\Omega \times \mathbb{R})$ satisfying

$$\ell_1 \leq \|a\|_\infty \quad \text{and} \quad a(x, t + T) = a(x, t) = a(x, -t) \quad \text{in } \Omega \times \mathbb{R} . \quad (1.2)$$

We assume that $y(\cdot, 0) = y^o \in L^2(\Omega)$ and $f = f(x, t) \in L^1_{loc}(0, +\infty; L^2(\Omega))$, so that (1.1) admits a unique solution $y = y(x, t) \in C([0, +\infty); L^2(\Omega))$. Let $\{G(t, s)\}_{0 \leq s \leq t < +\infty}$ be the T -periodic evolutionary process on $L^2(\Omega)$, such that

$$y(\cdot, t) = G(t, s)y(\cdot, s) + \int_s^t G(t, r)f(\cdot, r)dr, \quad (1.3)$$

for all $0 \leq s \leq t$, where the T -periodicity of $G(t, s)$ means $G(t+T, s+T) = G(t, s)$.

Notice that any bounded function in $L^\infty(\Omega \times (0, T/2))$ can be extended to be an even, T -periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$. Here, we are only interested in the case where $(\|a\|_\infty - \ell_1) \geq 0$ and a not positive, because if $(\|a\|_\infty - \ell_1) < 0$ or $a \geq 0$, then the system (1.1) is clearly stable when $f = 0$.

The Poincaré map is usually defined by $G(T+t, t)$. We restrict your attention on the operator $G(T, 0)$

$$G(T, 0): \begin{array}{l} L^2(\Omega) \longrightarrow L^2(\Omega) \\ y(\cdot, 0) \longmapsto y(\cdot, T) \end{array} \quad (1.4)$$

It is well-known that $G(T, 0)$ is compact by the smoothing action of the diffusion. Since a is even and T -periodic, we see that $G(T, 0)$ is self-adjoint. Consequently, there is a complete orthonormal set in $L^2(\Omega)$ formed of eigenfunctions $(z_j^o)_{j \geq 1}$ of $G(T, 0)$ corresponding to eigenvalues $(\lambda_j)_{j \geq 1}$, where $\lambda_j = \lambda_j(T)$ depends on T , $\lambda_j \in \mathbb{R}$, $\lambda_j \rightarrow 0$ as $j \rightarrow +\infty$. Thus, we may arrange the eigenfunctions so that the sequence $(|\lambda_j|)_{j \geq 1}$ is non-increasing, $\dots \leq |\lambda_{m+1}| \leq |\lambda_m| \leq \dots \leq |\lambda_1|$. Clearly, $\lambda_j \neq 0$ from the backward uniqueness property for the linear parabolic equation. Furthermore, we will see that for all $T > 0$ and for all even, T -periodic, bounded potential a , any eigenvalue $\lambda_j(T)$ of $G(T, 0)$ satisfies

$$\ell_1 + \frac{\ln |\lambda_j(T)|}{T} \leq \|a\|_\infty. \quad (1.5)$$

The first result of the paper is as follows.

Theorem 1 .- *There exist two constants $c_o \in (0, 1)$ and $C > 0$, both only depending on ω and Ω , such that if we choose the time-periodicity $T \in (0, c_o]$ and an even, T -periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$ such that*

$$\ell_1 \leq \|a\|_\infty \leq \ell_1 \left(\frac{c_o}{T} \right)^{1/4},$$

then any eigenfunction z_j^o of $G(T, 0)$ in $L^2(\Omega)$, corresponding to the eigenvalue $\lambda_j(T)$ with $|\lambda_j(T)| \geq 1$, satisfies

$$\int_\Omega |G(t, 0)z_j^o(x)|^2 dx \leq e^{C\|a\|_\infty^{4/3}} \int_\omega |G(t, 0)z_j^o(x)|^2 dx$$

for all $t \geq 0$.

The knowledge of the eigencouple of the Poincaré map plays a key role in the study of periodic parabolic systems (see [14], [15]). Theorem 1 implies clearly that the eigenfunctions z_j^o of the Poincaré map corresponding to the eigenvalues λ_j with $|\lambda_j| \geq 1$ has unique continuation property.

It is classical in control theory for linear PDE that the unique continuation property is linked with the approximate controllability and more precisely, a quantitative uniqueness result yields an estimate of the cost of the approximate control (see e.g. [12]). Here, our second theorem establishes an asymptotic controllability (or open loop stabilizability) for the heat equation with time-periodic potential.

Theorem 2 .- *There exist two constants $c_o \in (0, 1)$ and $C > 0$, both only depending on Ω and ω , such that if we choose the time-periodicity $T \in (0, c_o]$ and an even, T -periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$ such that*

$$\ell_1 \leq \|a\|_\infty \leq \ell_1 \left(\frac{c_o}{T} \right)^{1/4},$$

and if there exists $m > 0$ such that

$$\cdots \leq |\lambda_{m+1}(T)| < 1 \leq |\lambda_m(T)| = \cdots = |\lambda_2(T)| = |\lambda_1(T)| ,$$

then for each initial data $y^o \in L^2(\Omega)$, the control function $f_c \in C([0, T]; L^2(\Omega))$ satisfying

$$\left\{ \begin{array}{l} (i) \quad f_c(x, T-t) = \sum_{j=1, \dots, m} \sigma_j(t) G(t, 0) z_j^o(x) , \\ (ii) \quad \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{pmatrix} = -\frac{1}{T} \left[\left(\int_{\omega} G(t, 0) z_{\kappa}^o(x) G(t, 0) z_j^o(x) dx \right)_{1 \leq \kappa, j \leq m} \right]^{-1} \begin{pmatrix} \int_{\Omega} y^o(x) G(T, 0) z_1^o(x) dx \\ \vdots \\ \int_{\Omega} y^o(x) G(T, 0) z_m^o(x) dx \end{pmatrix} , \\ (iii) \quad \int_0^T \|f_c(\cdot, s)\|_{L^2(\omega)} ds \leq e^{C\|a\|_{\infty}^{4/3}} e^{(\|a\|_{\infty} - \ell_1)T} \|y^o\|_{L^2(\Omega)} , \end{array} \right.$$

implies that the solution $y = y(x, t) \in C([0, +\infty); L^2(\Omega))$ of the following heat equation with potential a and control function f_c ,

$$\left\{ \begin{array}{l} \partial_t y - \Delta y + ay = f_c \cdot 1_{|\omega \times (0, T)} \quad \text{in } \Omega \times (0, +\infty) , \\ y = 0 \quad \text{on } \partial\Omega \times (0, +\infty) , \\ y(\cdot, 0) = y^o \quad \text{in } \Omega , \end{array} \right.$$

satisfies for all $t \geq T$,

$$\|y(\cdot, t)\|_{L^2(\Omega)} \leq C_m e^{-\gamma t} \|y^o\|_{L^2(\Omega)} ,$$

where

$$\gamma = \frac{-\ln |\lambda_{m+1}(T)|}{T} > 0 \quad \text{and} \quad C_m = \frac{2e^{C(\|a\|_{\infty}^{4/3} + \|a\|_{\infty} T)}}{|\lambda_{m+1}(T)|^2} .$$

The notion of asymptotic controllability is usual in the nonlinear control theory of finite dimensional systems (see e.g. [7]). Here, (i) says that our control has a finite dimensional structure (see also [4], [5]). (ii) implies that the operator associating the initial data with the control function is linear which gives a kind of robustness property of our control. (iii) gives us an explicit expression of the cost of the control, from which we see easily that we can act a control on the equation in a very short time T , but as a payment, evenness and T -periodicity for the potential a are required.

From the following property of the Poincaré map,

$$G(nT, 0) z_j^o = (\lambda_j)^n z_j^o \tag{1.6}$$

for all $n \in \mathbb{N}$, it is clear that if all eigenvalues λ_j satisfy $|\lambda_j| < 1$, then the system (1.1) is stable without control. On the other hand, if the modulus of some eigenvalues of $G(T, 0)$ are bigger than one, then the equation (1.1) with $f = 0$ is unstable. We call such eigenvalues as unstable eigenvalues. Theorem 2 amounts to saying that if all unstable eigenvalues have same absolute value then the equation (1.1) with $f = f_c \cdot 1_{|\omega \times (0, T)}$ can be stabilized by a control f_c in a finite dimension subspace of $L^2(\Omega)$ spanned by $(z_j^o)_{j=1, \dots, m}$ which are the eigenfunctions of $G(T, 0)$ corresponding to the unstable eigenvalues. So it is important to study the unique continuation of the eigenfunctions of $G(T, 0)$ corresponding to the unstable eigenvalues. This is why we restrict our attention in Theorem 1 to the case where $|\lambda_j| \geq 1$.

The unique continuation property or more precisely the number of zeros of a one-dimensional parabolic equation is well described in [1] which leads to the study of the asymptotic behaviour of periodic 1d quasilinear parabolic equation [3], or to the consideration of the structural stability of periodic 1d semilinear heat equation [6].

In order to get Theorem 1 and Theorem 2, we need a more general unique continuation result in bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, described as follows.

Theorem 3 .- *There exist two constants $c_o \in (0, 1)$ and $C > 0$, both only depending on Ω and ω , such that for all $T \in (0, c_o]$, $\beta \geq 0$ and $q \in L^\infty(\Omega \times (0, T))$ satisfying*

$$\ell_1 + \beta \leq \|q\|_\infty \leq \ell_1 \left(\frac{c_o}{T} \right)^{1/4} ,$$

we have

$$\int_\Omega |u(x, 0)|^2 dx \leq e^{C\|q\|_\infty^{4/3}} \int_\omega |u(x, 0)|^2 dx ,$$

where $u \in C([0, T]; L^2(\Omega))$ solves the following T -periodic linear heat equation

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 & \text{in } \Omega \times (0, T) , \\ u = 0 & \text{on } \partial\Omega \times (0, T) , \\ u(\cdot, 0) = u(\cdot, T) & \text{in } \Omega . \end{cases}$$

Strong unique continuation for heat equation with potential in bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 1$, has been widely studied, (see for instance [10], [11] and references therein). In [17], one obtains that any solution v of the linear heat equation with time-independent Lipschitz continuous potential such that $v(\cdot, t_o) = 0$ in ω for some $t_o \in (0, T)$, satisfies $v = 0$ in $\Omega \times (0, T)$. The idea of the proof in [17] is to transform the solution v to a solution of an elliptic equation where unique continuation property for elliptic operator can be used (see also [8]). Here we are interested in the unique continuation and its quantification of the linear time-periodic heat equation with time-dependent potential. The basic technique used here is to combine some properties of time-periodic solutions of heat equation with a kind of global Carleman estimate for an elliptic operator.

This paper is organized as follows. In section 2, we give an introduction of the Poincaré map and some related preliminary results, then we prove Theorem 1. In section 3 and section 4, we give the proofs of Theorem 2 and Theorem 3 respectively. In section 5, we point out some related results for a particular case where the potential depends only on time. Finally, in the Appendix, we give the proof of the Global Carleman inequality we need.

2 The Poincaré map

Let a be an even, T -periodic, bounded function in $L^\infty(\Omega \times \mathbb{R})$; more precisely $a(x, t + T) = a(x, t) = a(x, -t)$ in $\Omega \times \mathbb{R}$. We consider the following heat equation with the potential $a = a(x, t)$,

$$\begin{cases} \partial_t y - \Delta y + ay = f & \text{in } \Omega \times (s, +\infty) , \\ y = 0 & \text{on } \partial\Omega \times (s, +\infty) , \end{cases} \quad (2.1)$$

where $s \geq 0$ and $f \in L^1_{loc}(0, +\infty; L^2(\Omega))$. Let $y \in C([s, +\infty); L^2(\Omega))$ be the mild solution of (2.1) and $G(t, s)$ be the T -periodic evolutionary process generated by $-\Delta + aI$ in $L^2(\Omega)$. More precisely, $\{G(t, s)\}_{0 \leq s \leq t < +\infty}$ is a family of bounded linear operators from $L^2(\Omega)$ to itself, such that

$$\begin{cases} (i) & G(t, t) = I \quad \text{for all } t \geq 0 , \\ (ii) & G(t, r)G(r, s) = G(t, s) \quad \text{for all } 0 \leq s \leq r \leq t , \\ (iii) & \text{The map } (t, s) \mapsto G(t, s)g \text{ is continuous for every fixed } g \in L^2(\Omega) , \\ (iv) & G(t + T, s + T) = G(t, s) \quad \text{for all } 0 \leq s \leq t , \\ (v) & \|G(t, s)\| \leq e^{(\|a\|_\infty - \ell_1)(t-s)} \quad \text{for all } 0 \leq s \leq t . \end{cases} \quad (2.2)$$

Thus the mild solution y of equation (2.1) can be written in the following form

$$y(\cdot, t) = G(t, s)y(\cdot, s) + \int_s^t G(t, r)f(\cdot, r) dr \quad (2.3)$$

for all $0 \leq s \leq t$.

Now, we are going to focus our attention on the Poincaré map $G(T, 0)$ and obtain some properties in our case. Since a is even and T -periodic, we get that $G(T, 0)$ is self-adjoint. Indeed, let $p^T \in L^2(\Omega)$ and $p \in C([0, T]; L^2(\Omega))$ be the solution of

$$\begin{cases} -\partial_t p - \Delta p + ap = 0 & \text{in } \Omega \times (0, T) , \\ p = 0 & \text{on } \partial\Omega \times (0, T) , \\ p(\cdot, T) = p^T & \text{in } \Omega , \end{cases} \quad (2.4)$$

then for all $y^\circ \in L^2(\Omega)$,

$$\int_{\Omega} y^\circ(x) p(x, 0) dx = \int_{\Omega} [G(T, 0) y^\circ(x)] p^T(x) dx , \quad (2.5)$$

that is

$$G^*(T, 0) p^T(x) = p(x, 0) . \quad (2.6)$$

On the other hand, because $a(x, T-t) = a(x, t)$, the solution $w(x, t) = p(x, T-t)$ solves

$$\begin{cases} \partial_t w - \Delta w + aw = 0 & \text{in } \Omega \times (0, T) , \\ w = 0 & \text{on } \partial\Omega \times (0, T) , \\ w(\cdot, 0) = p^T & \text{in } \Omega , \\ w(\cdot, T) = p(\cdot, 0) & \text{in } \Omega . \end{cases} \quad (2.7)$$

Finally, for all $p^T \in L^2(\Omega)$,

$$G^*(T, 0) p^T(x) = G(T, 0) p^T(x) . \quad (2.8)$$

As $G(T, 0)$ is a compact selfadjoint operator on $L^2(\Omega)$, we deduce that there is a complete orthonormal set in $L^2(\Omega)$ formed by the eigenfunctions $(z_j^\circ)_{j \geq 1}$ of $G(T, 0)$ with corresponding eigenvalues $\lambda_j = \lambda_j(T) \in \mathbb{R}$, which converge to 0 as $j \rightarrow +\infty$. Thus, we can arrange the eigenfunctions so that the sequence $(|\lambda_j|)_{j \geq 1}$ is non-increasing.

Now we are able to give some properties of the eigencouple $(z_j^\circ, \lambda_j(T))$ associated to the compact selfadjoint operator $G(T, 0)$.

Lemma 2-1 .- For all $s \in [0, T]$ and $\phi \in L^2(\Omega)$,

$$\int_{\Omega} \phi(x) G(T-s, 0) z_j^\circ(x) dx = \int_{\Omega} G(T, s) \phi(x) z_j^\circ(x) dx .$$

It follows immediately from Lemma 2-1 that for all $s \in [0, T]$,

$$\int_{\Omega} G(s, 0) z_i^\circ(x) \frac{G(T-s, 0) z_j^\circ(x)}{\lambda_j} dx = \delta_{ij} . \quad (2.8)$$

Lemma 2-2 .- As soon as the time-periodicity $T > 0$ and an even, T -periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$ are chosen, then any eigenvalue $\lambda_j(T)$ of $G(T, 0)$ satisfies

$$\ell_1 + \frac{\ln |\lambda_j(T)|}{T} \leq \|a\|_\infty .$$

Note that in particular, under the hypothesis of Theorem 1, the eigenvalue $\lambda_j(T)$ have to satisfy

$$\ell_1 + \frac{\ln |\lambda_j(T)|}{T} \leq \|a\|_\infty \leq \ell_1 \left(\frac{c_o}{T}\right)^{1/4} . \quad (2.9)$$

2.1 Proof of Lemma 2-1

Let $g = g(x, t) \in C([s, T]; L^2(\Omega))$ be the solution of

$$\begin{cases} \partial_t g - \Delta g + ag = 0 & \text{in } \Omega \times (s, T) , \\ g = 0 & \text{on } \partial\Omega \times (s, T) , \\ g(\cdot, s) = \phi & \text{in } \Omega , \end{cases} \quad (2.10)$$

and $z_j = z_j(x, t) \in C([0, T]; L^2(\Omega))$ be the solution of

$$\begin{cases} \partial_t z_j - \Delta z_j + az_j = 0 & \text{in } \Omega \times (0, T) , \\ z_j = 0 & \text{on } \partial\Omega \times (0, T) , \\ z_j(\cdot, 0) = z_j^o & \text{in } \Omega . \end{cases} \quad (2.11)$$

Now, from the relation $a(\cdot, T-t) = a(\cdot, t)$, we obtain that the function $t \mapsto \int_{\Omega} g(x, t) z_j(x, T-t) dx$ is constant. Indeed,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} g(x, t) z_j(x, T-t) dx &= \int_{\Omega} \partial_t g(x, t) z_j(x, T-t) dx + \int_{\Omega} g(x, t) (-\partial_t z_j)(x, T-t) dx \\ &= \int_{\Omega} -(-\Delta + a(x, t)) g(x, t) z_j(x, T-t) dx \\ &\quad + \int_{\Omega} g(x, t) (-\Delta + a(x, t)) z_j(x, T-t) dx = 0 . \end{aligned} \quad (2.12)$$

Consequently, for all $t \in [s, T]$,

$$\int_{\Omega} g(x, s) z_j(x, T-s) dx = \int_{\Omega} g(x, t) z_j(x, T-t) dx = \int_{\Omega} g(x, T) z_j^o(x) dx . \quad (2.13)$$

That completes the proof of Lemma 2-1.

2.2 Proof of Lemma 2-2

Let

$$\tilde{\beta} = \frac{\ln |\lambda_j(T)|}{T} \in \mathbb{R} \quad (2.14)$$

and $u = u(x, t) \in C([0, 2T]; L^2(\Omega))$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + \tilde{\beta}u + au = 0 & \text{in } \Omega \times (0, 2T) , \\ u = 0 & \text{on } \partial\Omega \times (0, 2T) , \\ u(\cdot, 0) = z_j^o & \text{in } \Omega . \end{cases} \quad (2.15)$$

By an energy method, we have the following equality

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u(x, t)|^2 dx + \int_{\Omega} |\nabla u(x, t)|^2 dx + \tilde{\beta} \int_{\Omega} |u(x, t)|^2 dx + \int_{\Omega} a(x, t) |u(x, t)|^2 dx = 0 . \quad (2.16)$$

Also, notice that $u(x, t) = e^{-\tilde{\beta}t} z_j(x, t) = e^{-\tilde{\beta}t} G(t, 0) z_j^o(x) \neq 0$ and

$$u(\cdot, 2T) = u(\cdot, 0) \quad \text{in } \Omega . \quad (2.17)$$

From (2.16) and (2.17), we get

$$\left(\ell_1 + \tilde{\beta} - \|a\|_{\infty} \right) \int_0^{2T} \int_{\Omega} |u(x, t)|^2 dx \leq 0 . \quad (2.18)$$

Consequently

$$\ell_1 + \tilde{\beta} = \ell_1 + \frac{\ln |\lambda_j(T)|}{T} \leq \|a\|_{\infty} . \quad (2.19)$$

This is the desired inequality.

2.3 Proof of Theorem 1

For $|\lambda_j(T)| \geq 1$, we have that

$$\beta = \frac{\ln |\lambda_j(T)|}{T} \geq 0 \quad (2.20)$$

and $u(x, t) = e^{-\beta t} z_j(x, t) \in C([0, 2T]; L^2(\Omega))$ satisfies

$$\begin{cases} \partial_t u - \Delta u + \beta u + au = 0 & \text{in } \Omega \times (0, 2T) , \\ u = 0 & \text{on } \partial\Omega \times (0, 2T) , \\ u(\cdot, 2T) = u(\cdot, 0) & \text{in } \Omega . \end{cases} \quad (2.21)$$

Now, we apply Theorem 3 for the $2T$ -periodic solution u of (2.21) with $q = a \in L^\infty(\Omega \times (0, 2T))$ satisfying

$$\ell_1 + \frac{\ln |\lambda_j(T)|}{T} \leq \|a\|_\infty \leq \ell_1 \left(\frac{c_o}{2T} \right)^{1/4} \quad (2.22)$$

to obtain

$$\int_{\Omega} |u(x, 0)|^2 dx \leq e^{C\|a\|_\infty^{4/3}} \int_{\omega} |u(x, 0)|^2 dx . \quad (2.23)$$

It follows by translation in time and by the $2T$ -periodicity of u that for all $t \in [0, 2T]$,

$$\int_{\Omega} |u(x, t)|^2 dx \leq e^{C\|a\|_\infty^{4/3}} \int_{\omega} |u(x, t)|^2 dx . \quad (2.24)$$

Consequently, we obtain that for all $t \in [0, 2T]$,

$$\int_{\Omega} |G(t, 0) z_j^o(x)|^2 dx \leq e^{C\|a\|_\infty^{4/3}} \int_{\omega} |G(t, 0) z_j^o(x)|^2 dx . \quad (2.25)$$

Hence, if we choose c_o in Theorem 1 to be $\frac{c_o}{2}$ in Theorem 3 and same C , we complete the proof of Theorem 1.

3 The asymptotic controllability problem

In this Section, we prove Theorem 2. Recall that if $(\|a\|_\infty - \ell_1) < 0$, then the system (1.1) is already stable with $f = 0$. We consider the solution of the following heat equation with a potential $a = a(x, t) \in L^\infty(\Omega \times \mathbb{R})$ satisfying $a(x, t+T) = a(x, t) = a(x, -t)$ in $\Omega \times \mathbb{R}$ and $(\|a\|_\infty - \ell_1) \geq 0$,

$$\begin{cases} \partial_t y - \Delta y + ay = f_c \cdot 1_{\omega \times (0, T)} & \text{in } \Omega \times (0, +\infty) , \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty) , \\ y(\cdot, 0) = y^o & \text{in } \Omega , \end{cases} \quad (3.1)$$

where $y^o \in L^2(\Omega)$ and $f_c \in C([0, T]; L^2(\Omega))$. It is clear that for all integers $n \geq 1$, we have

$$y(\cdot, nT) = G(nT, 0) y^o + \int_0^{nT} G(nT, s) (f_c(\cdot, s) \cdot 1_{\omega \times (0, T)}) ds . \quad (3.2)$$

Let $(z_j^o, \lambda_j(T))$ be the eigencouple of the compact selfadjoint operator $G(T, 0)$, then

$$y^o = \sum_{j \geq 1} \left(\int_{\Omega} y^o(x) z_j^o(x) dx \right) z_j^o . \quad (3.3)$$

It follows from Lemma 2-1 that for any $s \in [0, T]$,

$$\begin{aligned} G(T, s) (f_c(x, s) \cdot 1_{|\omega \times (0, T)}) &= \sum_{j \geq 1} \left(\int_{\Omega} G(T, s) (f_c(x, s) \cdot 1_{|\omega \times (0, T)}) z_j^{\circ}(x) dx \right) z_j^{\circ}(x) \\ &= \sum_{j \geq 1} \left(\int_{\Omega} (f_c(x, s) \cdot 1_{|\omega \times (0, T)}) G(T - s, 0) z_j^{\circ}(x) dx \right) z_j^{\circ}(x) . \end{aligned} \quad (3.4)$$

By (3.3), (3.4) and by the periodicity of $G(t, s)$, we infer

$$\begin{aligned} G(nT, 0) y^{\circ} &= \sum_{j \geq 1} \left(\int_{\Omega} y^{\circ}(x) z_j^{\circ}(x) dx \right) G(nT, 0) z_j^{\circ} \\ &= \sum_{j \geq 1} \lambda_j^n \left(\int_{\Omega} y^{\circ}(x) z_j^{\circ}(x) dx \right) z_j^{\circ} \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} G(nT, s) (f_c(x, s) \cdot 1_{|\omega \times (0, T)}) &= G(nT, T) G(T, s) (f_c(x, s) \cdot 1_{|\omega \times (0, T)}) \\ &= \sum_{j \geq 1} \left(\int_{\Omega} (f_c(x, s) \cdot 1_{|\omega \times (0, T)}) G(T - s, 0) z_j^{\circ}(x) dx \right) G(nT, T) z_j^{\circ}(x) \\ &= \sum_{j \geq 1} (1_{|(0, T)} \cdot \int_{\omega} f_c(x, s) G(T - s, 0) z_j^{\circ}(x) dx) G((n-1)T, 0) z_j^{\circ}(x) \\ &= \sum_{j \geq 1} \lambda_j^{n-1} (1_{|(0, T)} \cdot \int_{\omega} f_c(x, s) G(T - s, 0) z_j^{\circ}(x) dx) z_j^{\circ}(x) . \end{aligned} \quad (3.6)$$

We conclude from (3.5) and (3.6) that $y(x, nT) = \sum_{j \geq 1} \lambda_j^n Y_j(T) z_j^{\circ}(x)$ for all integers $n \geq 1$, where

$$Y_j(T) = \int_{\Omega} y^{\circ}(x) z_j^{\circ}(x) dx + \frac{1}{\lambda_j} \int_0^T \int_{\omega} f_c(x, s) G(T - s, 0) z_j^{\circ}(x) dx ds . \quad (3.7)$$

Recall that we have arranged the eigenfunctions so that the sequence $(|\lambda_j|)_{j \geq 1}$ is non-decreasing. Let $m > 0$ be such that $|\lambda_j| \leq |\lambda_{m+1}| < 1$ for all $j \geq m+1$ and $1 \leq |\lambda_m| \leq |\lambda_j|$ for all $j = 1, \dots, m$. Now we decompose $y(x, nT)$ as follows

$$y(x, nT) = y_{1,m}(x, nT) + y_{2,m}(x, nT) , \quad (3.8)$$

where

$$y_{1,m}(x, nT) = \sum_{j=1, \dots, m} \lambda_j^n Y_j(T) z_j^{\circ}(x) \quad (3.9)$$

and

$$y_{2,m}(x, nT) = \sum_{j \geq m+1} \lambda_j^n Y_j(T) z_j^{\circ}(x) . \quad (3.10)$$

Note that $y_{1,m}$ is the projection of y in the subspace of $L^2(\Omega)$ spanned by eigenfunctions $(z_j^{\circ})_{j=1, \dots, m}$ of $G(T, 0)$ corresponding to the unstable eigenvalues $(\lambda_j)_{j=1, \dots, m}$ while $y_{2,m}$ is the projection of y in the subspace of $L^2(\Omega)$ spanned by eigenfunctions $(z_j^{\circ})_{j \geq m+1}$ of $G(T, 0)$ corresponding to the stable eigenvalues $(\lambda_j)_{j \geq m+1}$. Next, we estimate respectively $\int_{\Omega} |y_{2,m}(x, nT)|^2 dx$ and $\int_{\Omega} |y_{1,m}(x, nT)|^2 dx$.

It follows first from (3.10) that

$$\begin{aligned} \int_{\Omega} |y_{2,m}(x, nT)|^2 dx &= \sum_{j \geq m+1} |\lambda_j|^{2n} |Y_j(T)|^2 \\ &= \sum_{j \geq m+1} |\lambda_j|^{2n} \left| \int_{\Omega} y(x, T) \frac{z_j^{\circ}(x)}{\lambda_j} dx \right|^2 . \end{aligned} \quad (3.11)$$

Indeed, because of the evenness and T -periodicity of a , we have

$$\begin{aligned} &\int_0^T \int_{\omega} f_c(x, s) \frac{G(T-s, 0) z_j^{\circ}(x)}{\lambda_j} dx ds \\ &= \int_0^T \int_{\Omega} (\partial_t y(x, s) - \Delta y(x, s) + a(x, T-s) y(x, s)) \frac{G(T-s, 0) z_j^{\circ}(x)}{\lambda_j} dx ds \\ &= \int_{\Omega} y(x, T) \frac{z_j^{\circ}(x)}{\lambda_j} dx - \int_{\Omega} y^{\circ}(x) z_j^{\circ}(x) dx . \end{aligned} \quad (3.12)$$

As $|\lambda_j| \leq |\lambda_{m+1}| < 1$ for all $j \geq m+1$, we deduce from (3.11) that

$$\begin{aligned} \int_{\Omega} |y_{2,m}(x, nT)|^2 dx &\leq |\lambda_{m+1}|^{2(n-1)} \sum_{j \geq m+1} \left| \int_{\Omega} y(x, T) z_j^{\circ}(x) dx \right|^2 \\ &\leq |\lambda_{m+1}|^{2(n-1)} \int_{\Omega} |y(x, T)|^2 dx . \end{aligned} \quad (3.13)$$

On the other hand, we have

$$\int_{\Omega} |y_{1,m}(x, nT)|^2 dx = \sum_{j=1, \dots, m} |\lambda_j|^{2n} |Y_j(T)|^2$$

by (3.9) where

$$Y_j(T) = \int_0^T \left[\frac{1}{T} \int_{\Omega} y^o(x) z_j^o(x) dx + \frac{1}{\lambda_j} \int_{\omega} f_c(x, s) G(T-s, 0) z_j^o(x) dx \right] ds. \quad (3.14)$$

Now we are on the position to construct a control function $f_c = f_c(x, s) \in C([0, T]; L^2(\Omega))$ such that for all $x \in \Omega$, for all $s \in [0, T]$,

$$f_c(x, T-s) = \sum_{i=1, \dots, m} \sigma_i(s) G(s, 0) z_i^o(x), \quad (3.15)$$

where $\sigma_i = \sigma_i(s) \in C([0, T])$ and

$$\frac{1}{T} \int_{\Omega} y^o(x) z_j^o(x) dx + \frac{1}{\lambda_j} \int_{\omega} f_c(x, s) G(T-s, 0) z_j^o(x) dx = 0 \quad (3.16)$$

for all $j = 1, \dots, m$, which implies $\int_{\Omega} |y_{1,m}(x, nT)|^2 dx = 0$.

By (3.15) and (3.16), it suffices to obtain $\{\sigma_i(s)\}_{i=1, \dots, m}$ by solving the following linear system

$$\sum_{i=1, \dots, m} \sigma_i(s) \int_{\omega} z_i(x, s) z_j(x, s) dx = -\frac{\lambda_j}{T} \int_{\Omega} y^o(x) z_j^o(x) dx \quad (3.17)$$

that is

$$\begin{bmatrix} \int_{\omega} z_1(x, s) z_1(x, s) dx & \cdots & \int_{\omega} z_m(x, s) z_1(x, s) dx \\ \vdots & \ddots & \vdots \\ \int_{\omega} z_1(x, s) z_m(x, s) dx & \cdots & \int_{\omega} z_m(x, s) z_m(x, s) dx \end{bmatrix} \begin{pmatrix} \sigma_1(s) \\ \vdots \\ \sigma_m(s) \end{pmatrix} = -\frac{1}{T} \begin{pmatrix} \int_{\Omega} y^o(x) z_1(x, T) dx \\ \vdots \\ \int_{\Omega} y^o(x) z_m(x, T) dx \end{pmatrix}, \quad (3.18)$$

where $z_j = z_j(x, t) \in C([0, +\infty); L^2(\Omega))$ is the solution of

$$\begin{cases} \partial_t z_j - \Delta z_j + a z_j = 0 & \text{in } \Omega \times (0, +\infty), \\ z_j = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ z_j(\cdot, 0) = z_j^o & \text{in } \Omega. \end{cases} \quad (3.19)$$

The existence of such $\sigma_i(s) \in C([0, T])$ can be proved if and only if for all $s \in [0, T]$, the matrix $(\int_{\omega} G(s, 0) z_i^o(x) G(s, 0) z_j^o(x) dx)_{1 \leq i, j \leq m}$ is invertible.

3.1 Invertibility of $(\int_{\omega} G(s, 0) z_i^o(x) G(s, 0) z_j^o(x) dx)_{1 \leq i, j \leq m}$

By contradiction, we assume that the matrix $M(s) = [\int_{\omega} z_i(x, s) z_j(x, s) dx]_{1 \leq i, j \leq m}$ is not invertible for some $s \in [0, T]$ fixed. Then the lines of $M(s)$ are linearly dependent which implies that $\{z_i(x, s)\}_{i=1, \dots, m}$ are linearly dependent in $L^2(\omega)$. Hence there exists $(\mu_1(s), \dots, \mu_{m-1}(s))$ a non-zero vector in \mathbb{R}^{m-1} such that

$$\mu_1(s) z_1(\cdot, s) + \cdots + \mu_{m-1}(s) z_{m-1}(\cdot, s) - z_m(\cdot, s) = 0 \quad \text{in } \omega. \quad (3.20)$$

However, by the assumption that $|\lambda_{m+1}| < 1 \leq |\lambda_m| = \cdots = |\lambda_2| = |\lambda_1|$, it follows that the function

$$\varkappa_s(x, t) = \mu_1(s) z_1(x, t) + \cdots + \mu_{m-1}(s) z_{m-1}(x, t) - z_m(x, t) \quad (3.21)$$

satisfies

$$\begin{cases} \partial_t \varkappa_s - \Delta \varkappa_s + a \varkappa_s = 0 & \text{in } \Omega \times (0, 2T) , \\ \varkappa_s = 0 & \text{on } \partial\Omega \times (0, 2T) , \\ \varkappa_s(\cdot, 2T) = |\lambda_1|^2 \varkappa_s(\cdot, 0) & \text{in } \Omega . \end{cases} \quad (3.22)$$

Let $u(x, t) = e^{-\frac{\ln|\lambda_1|}{T}t} \varkappa_s(x, t)$, then $u \in C([0, 2T]; L^2(\Omega))$ solves

$$\begin{cases} \partial_t u - \Delta u + \frac{\ln|\lambda_1|}{T}u + au = 0 & \text{in } \Omega \times (0, 2T) , \\ u = 0 & \text{on } \partial\Omega \times (0, 2T) , \\ u(\cdot, 2T) = u(\cdot, 0) & \text{in } \Omega . \end{cases} \quad (3.23)$$

By Theorem 3, there exists a constant $c_1 > 0$ such that if we choose $a \in L^\infty(\Omega \times \mathbb{R})$ such that

$$\ell_1 \leq \|a\|_\infty \leq \ell_1 \left(\frac{c_1}{2T} \right)^{1/4} \quad (3.24)$$

and noting that

$$\ell_1 + \frac{\ln|\lambda_1|}{T} \leq \|a\|_\infty \quad \text{and} \quad \frac{\ln|\lambda_1|}{T} \geq 0 , \quad (3.25)$$

which follows from Lemma 2-2 and the fact that $|\lambda_1| \geq 1$, then the solution $u = u(x, t)$ of (3.23) satisfies

$$u(\cdot, 0) = 0 \text{ in } \omega \implies u(\cdot, 0) = 0 \text{ in } \Omega . \quad (3.26)$$

It follows by translation in time and by the $2T$ -periodicity of u that

$$u(\cdot, s) = 0 \text{ in } \omega \implies u(\cdot, s) = 0 \text{ in } \Omega . \quad (3.27)$$

So $\varkappa_s(\cdot, s) = 0$ in Ω and

$$\mu_1(s) z_1^o + \cdots + \mu_{m-1}(s) z_{m-1}^o - z_m^o = 0 \quad \text{in } \Omega . \quad (3.28)$$

This contradiction shows that the matrix $[\int_\omega z_i(x, s) z_j(x, s) dx]_{1 \leq i, j \leq m}$ is invertible for all $s \in [0, T]$ and

$$\begin{pmatrix} \sigma_1(s) \\ \vdots \\ \sigma_m(s) \end{pmatrix} = -\frac{1}{T} \begin{bmatrix} \int_\omega z_1(x, s) z_1(x, s) dx & \cdots & \int_\omega z_m(x, s) z_1(x, s) dx \\ \vdots & \ddots & \vdots \\ \int_\omega z_1(x, s) z_N(x, s) dx & \cdots & \int_\omega z_m(x, s) z_m(x, s) dx \end{bmatrix}^{-1} \cdot \begin{pmatrix} \int_\Omega y^o(x) z_1(x, T) dx \\ \vdots \\ \int_\Omega y^o(x) z_m(x, T) dx \end{pmatrix} . \quad (3.29)$$

Consequently, if we take the control function $f_c \in C([0, T]; L^2(\Omega))$ in equation (3.1) as

$$f_c(x, t) = \sum_{i=1, \dots, m} \sigma_i(T-t) z_i(x, T-t) , \quad (3.30)$$

then the corresponding solution $y \in C([0, +\infty); L^2(\Omega))$ satisfies

$$\begin{aligned} \|y(x, nT)\|_{L^2(\Omega)} &= \|y_{2,m}(x, nT)\|_{L^2(\Omega)} \leq |\lambda_{m+1}|^{(n-1)} \|y(\cdot, T)\|_{L^2(\Omega)} \\ &\leq |\lambda_{m+1}|^{(n-1)} \left(\|y^o\|_{L^2(\Omega)} e^{(\|a\|_\infty - \ell_1)T} + \int_0^T \|f_c(\cdot, s)\|_{L^2(\omega)} e^{(\|a\|_\infty - \ell_1)(T-s)} ds \right) , \end{aligned} \quad (3.31)$$

for all integers $n \geq 1$, where λ_{m+1} is the first eigenvalue such that $|\lambda_{m+1}| < 1$.

3.2 Estimate of $\|f_c(\cdot, s)\|_{L^2(\omega)}$

It follows from (3.31) that we would like to compute $\int_0^T \|f_c(\cdot, s)\|_{L^2(\omega)} e^{(\|a\|_\infty - \ell_1)(T-s)} ds$. We have

$$\begin{aligned} & (\sigma_1(s) \ \cdots \ \sigma_m(s)) \cdot \left[\left(\int_\omega z_i(x, s) z_j(x, s) dx \right)_{1 \leq i, j \leq m} \right] \begin{pmatrix} \sigma_1(s) \\ \vdots \\ \sigma_m(s) \end{pmatrix} \\ &= -\frac{1}{T} (\sigma_1(s) \ \cdots \ \sigma_m(s)) \cdot \begin{pmatrix} \int_\Omega y^\circ(x) z_1(x, T) dx \\ \vdots \\ \int_\Omega y^\circ(x) z_m(x, T) dx \end{pmatrix} \end{aligned} \quad (3.32)$$

or equivalently

$$\int_\omega \left| \sum_{i=1, \dots, m} \sigma_i(s) z_i(x, s) \right|^2 dx = -\frac{1}{T} \int_\Omega y^\circ(x) \sum_{i=1, \dots, m} \sigma_i(s) z_i(x, T) dx, \quad (3.33)$$

which implies

$$\begin{aligned} \left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\omega)}^2 &\leq \frac{1}{T} \|y^\circ\|_{L^2(\Omega)} \left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, T) \right\|_{L^2(\Omega)} \\ &\leq \frac{1}{T} \|y^\circ\|_{L^2(\Omega)} \left\| \sum_{i=1, \dots, m} \sigma_i(s) G(T, s) z_i(\cdot, s) \right\|_{L^2(\Omega)} \\ &\leq \frac{1}{T} \|y^\circ\|_{L^2(\Omega)} \left\| G(T, s) \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\Omega)} \\ &\leq \frac{1}{T} \|y^\circ\|_{L^2(\Omega)} e^{(\|a\|_\infty - \ell_1)(T-s)} \left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\Omega)}. \end{aligned} \quad (3.34)$$

Now we claim that there exists a constant $c > 0$ such that

$$\left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\Omega)} \leq c \left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\omega)}. \quad (3.35)$$

Indeed, since $|\lambda_{m+1}| < 1 \leq |\lambda_m| = \cdots = |\lambda_2| = |\lambda_1|$, then the function

$$\varkappa_s(x, t) = \sum_{i=1, \dots, m} \sigma_i(s) z_i(x, t) \quad (3.36)$$

satisfies

$$\begin{cases} \partial_t \varkappa_s - \Delta \varkappa_s + a \varkappa_s = 0 & \text{in } \Omega \times (0, 2T), \\ \varkappa_s = 0 & \text{on } \partial\Omega \times (0, 2T), \\ \varkappa_s(\cdot, 2T) = |\lambda_1|^2 \varkappa_s(\cdot, 0) & \text{in } \Omega. \end{cases} \quad (3.37)$$

Let $u(x, t) = e^{-\frac{\ln|\lambda_1|}{T}t} \varkappa_s(x, t)$, then $u \in C([0, 2T]; L^2(\Omega))$ satisfies

$$\begin{cases} \partial_t u - \Delta u + \frac{\ln|\lambda_1|}{T}u + au = 0 & \text{in } \Omega \times (0, 2T), \\ u = 0 & \text{on } \partial\Omega \times (0, 2T), \\ u(\cdot, 2T) = u(\cdot, 0) & \text{in } \Omega. \end{cases} \quad (3.38)$$

By Theorem 3, there exists a constant $c_1 > 0$, which is the same as that in (3.24) such that if we choose $a \in L^\infty(\Omega \times \mathbb{R})$ satisfying (3.24) and note (3.25) then the following estimate holds

$$\int_{\Omega} |u(x, 0)|^2 dx \leq e^{C\|a\|_\infty^{4/3}} \int_{\omega} |u(x, 0)|^2 dx , \quad (3.39)$$

where $C > 0$ is a constant which depends only on ω and Ω . It follows by translation in time and by the $2T$ -periodicity of u solution of (3.38) that for all $s \in [0, T]$,

$$\int_{\Omega} |u(x, s)|^2 dx \leq e^{C\|a\|_\infty^{4/3}} \int_{\omega} |u(x, s)|^2 dx , \quad (3.40)$$

which implies

$$\int_{\Omega} |\varkappa_s(x, s)|^2 dx \leq e^{C\|a\|_\infty^{4/3}} \int_{\omega} |\varkappa_s(x, s)|^2 dx \quad (3.41)$$

or equivalently

$$\left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\Omega)} \leq e^{C\|a\|_\infty^{4/3}} \left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\omega)} , \quad (3.42)$$

as desired. By (3.42) and (3.34) we get

$$\left\| \sum_{i=1, \dots, m} \sigma_i(s) z_i(\cdot, s) \right\|_{L^2(\omega)} \leq \frac{1}{T} \|y^o\|_{L^2(\Omega)} e^{(\|a\|_\infty - \ell_1)(T-s)} e^{C\|a\|_\infty^{4/3}} . \quad (3.43)$$

Now we conclude that if we take $c_o = \frac{c_1}{2}$ where c_1 is given in (3.39) then for all $a \in L^\infty(\Omega \times \mathbb{R})$ satisfying

$$\ell_1 \leq \|a\|_\infty \leq \ell_1 \left(\frac{c_o}{T} \right)^{1/4} , \quad (3.44)$$

we may take the control function $f_c \in C([0, T]; L^2(\Omega))$ as form (3.15) with estimates

$$\begin{aligned} \int_0^T \|f_c(\cdot, s)\|_{L^2(\omega)} ds &\leq \frac{1}{T} e^{C\|a\|_\infty^{4/3}} \|y^o\|_{L^2(\Omega)} \int_0^T e^{(\|a\|_\infty - \ell_1)(T-s)} ds \\ &\leq e^{C\|a\|_\infty^{4/3}} e^{(\|a\|_\infty - \ell_1)T} \|y^o\|_{L^2(\Omega)} \end{aligned} \quad (3.45)$$

and

$$\int_0^T \|f_c(\cdot, s)\|_{L^2(\omega)} e^{(\|a\|_\infty - \ell_1)(T-s)} ds \leq e^{C\|a\|_\infty^{4/3}} e^{2(\|a\|_\infty - \ell_1)T} \|y^o\|_{L^2(\Omega)} , \quad (3.46)$$

so that the corresponding solution $y \in C([0, +\infty); L^2(\Omega))$ of (3.1) satisfies (3.31).

3.3 Exponential decay of the solution

Now we turn to obtaining the exponential decay of the solution. Let $C_{T,a} = 1 + e^{C\|a\|_\infty^{4/3}} e^{(\|a\|_\infty - \ell_1)T}$. By (3.31) and (3.46), it follows that

$$\|y(\cdot, nT)\|_{L^2(\Omega)} \leq |\lambda_{m+1}|^{(n-1)} C_{T,a} e^{(\|a\|_\infty - \ell_1)T} \|y^o\|_{L^2(\Omega)} \quad (3.47)$$

for all integers $n \geq 1$.

For any $t \geq T$, we may write $t = nT + \sigma$ where $n \geq 1$ and $0 \leq \sigma < T$. Since $y(\cdot, t) = G(t, nT)y(\cdot, nT)$, it follows from (3.47) that

$$\begin{aligned} \|y(\cdot, t)\|_{L^2(\Omega)} &= \|y(\cdot, nT)\|_{L^2(\Omega)} e^{(\|a\|_\infty - \ell_1)(t-nT)} \leq \|y(\cdot, nT)\|_{L^2(\Omega)} e^{(\|a\|_\infty - \ell_1)T} \\ &\leq \exp\left(- (n-1) \ln\left(\frac{1}{|\lambda_{m+1}(T)|}\right)\right) \frac{1}{|\lambda_{m+1}(T)|^2} C_{T,a} e^{2(\|a\|_\infty - \ell_1)T} \|y^o\|_{L^2(\Omega)} , \end{aligned} \quad (3.48)$$

which implies

$$\|y(\cdot, t)\|_{L^2(\Omega)} \leq \exp\left(-\frac{t}{T} \ln\left(\frac{1}{|\lambda_{m+1}(T)|}\right)\right) \frac{C_{T,a} e^{2(\|a\|_\infty - \ell_1)T}}{|\lambda_{m+1}(T)|^2} \|y^o\|_{L^2(\Omega)} \quad (3.49)$$

for all $t \geq T$, and completes the proof of Theorem 2.

4 The quantitative unique continuation for time-periodic heat equation with potential

In this section, we shall show Theorem 3 by dividing the proof into three steps. In the first step, we study the solutions u locally in $\omega \times \{t = 0\}$ then, in the second step, we apply a global Carleman inequality for elliptic operators where time $t \in (0, T)$ is seen as a parameter. In the last step, we conclude the proof of Theorem 3 by choosing an adequate time T .

4.1 Step 1: Local property of u near $t = 0$

The goal of this subsection is to prove the following result:

Lemma 4-1 .- *There exist a non-empty open subset $\tilde{\omega}$ of ω and a constant $C_\omega > 1$ depending only on ω and Ω such that for all $T > 0$ and integers $M \geq 2$, we have*

$$\begin{aligned} \int_{\tilde{\omega}} |u(x, t)|^2 dx &\leq e^{2\|q\|_\infty t} \left(C_\omega (M+2)^{2+\frac{d}{M}} t\right)^M \int_{\Omega} |u(x, 0)|^2 dx \\ &\quad + e^{2\|q\|_\infty t} (M+2)^{\frac{d}{M}} \left(\frac{1-(C_\omega(M+2)^2 t)^M}{1-C_\omega(M+2)^2 t}\right) \int_{\omega} |u(x, 0)|^2 dx \end{aligned}$$

for all $t \in (0, T)$, where the solution u solves the following linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 & \text{in } \Omega \times (0, T) , \\ u = 0 & \text{on } \partial\Omega \times (0, T) , \\ u(\cdot, 0) \in L^2(\Omega) . \end{cases}$$

Proof of Lemma 4-1 .- Let $x^o \in \mathbb{R}^d$ and $r_o > 0$; we denote

$$\begin{aligned} B(x^o, r) &= \{x \in \mathbb{R}^d \mid |x - x^o| \leq r_o\} , \\ D(x^o, r) &= \{x \in \mathbb{R}^d \mid |x_i - x_i^o| \leq r_o, \text{ for all } i = 1, \dots, d\} . \end{aligned} \quad (4.1)$$

Without loss of generality, we suppose that there exists $r > 0$ such that $D(0, 2r) \subset \omega$. Then $B(x^o, r) \subset \omega$ for all $x^o \in [-r, r]^d$. Let $N \geq 4$ be an arbitrary but fixed integer; we have

$$B(x^o, r/N) \subset B(x^o, 2r/N) \subset \dots \subset B(x^o, r) \subset \omega . \quad (4.2)$$

Let us define a sequence of smooth function $\Phi_n \in C_0^\infty(B(x^o, nr/N))$ for $n = 2, \dots, N$ such that

$$\begin{cases} 0 \leq \Phi_n \leq 1 , \\ \Phi_n = 1 \text{ in } B(x^o, (n-1)r/N) , \\ |\nabla \Phi_n| \leq 2N/r . \end{cases} \quad (4.3)$$

The construction of such sequence Φ_n is standard.

Let $\Phi \in C_0^\infty(B(x^\circ, r_o))$; multiplying the equation in Lemma 4-1 by $\Phi^2 u$, we get the usual energy equality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Phi u(x, t)|^2 dx + \int_{\Omega} |\Phi \nabla u(x, t)|^2 dx + \beta \int_{\Omega} |\Phi u(x, t)|^2 dx \\ & = - \int_{\Omega} q(x, t) |\Phi u(x, t)|^2 dx - 2 \int_{\Omega} \Phi(x) \nabla u(x, t) \cdot \nabla \Phi(x) u(x, t) dx , \end{aligned} \quad (4.4)$$

from which, after some calculation involving Cauchy-Schwarz inequality, we infer

$$\frac{1}{2} \frac{d}{dt} \int_{B(x^\circ, r_o)} |\Phi u(x, t)|^2 dx \leq \|q\|_\infty \int_{B(x^\circ, r_o)} |\Phi u(x, t)|^2 dx + \|\nabla \Phi\|_{L^\infty(B(x^\circ, r_o))}^2 \int_{B(x^\circ, r_o)} |u(x, t)|^2 dx , \quad (4.5)$$

Multiplying (4.5) by $e^{-2\|q\|_\infty t}$, we obtain

$$\frac{d}{dt} \left(\int_{B(x^\circ, r_o)} |\Phi u(x, t)|^2 dx e^{-2\|q\|_\infty t} \right) \leq 2 \|\nabla \Phi\|_{L^\infty((x^\circ, r_o))}^2 \left(\int_{B(x^\circ, r_o)} |u(x, t)|^2 dx e^{-2\|q\|_\infty t} \right) . \quad (4.6)$$

Taking $\Phi = \Phi_n$ and $r_o = nr/N$ in (4.6), using (4.3), it follows that

$$\frac{d}{dt} g_n(t) \leq 8(N/r)^2 g_{n+1}(t) \quad (4.7)$$

for all $n = 2, \dots, N-1$, where

$$g_n(t) = \int_{B(x^\circ, nr/N)} |\Phi_n u(x, t)|^2 dx e^{-2\|q\|_\infty t} . \quad (4.8)$$

Integrating (4.7) over $(0, t)$, we obtain that for all $n = 2, \dots, N-1$,

$$g_n(t) \leq 8(N/r)^2 \int_0^t g_{n+1}(\tau) d\tau + g_n(0) . \quad (4.9)$$

By induction, we deduce that for $n = 0, \dots, N-3$,

$$g_2(t) \leq \left(8(N/r)^2\right)^{n+1} t^n \int_0^t g_{n+3}(\tau) d\tau + \sum_{j=0, \dots, n} \left(8(N/r)^2 t\right)^j g_{j+2}(0) . \quad (4.10)$$

By (4.8), (4.3) and (4.10), after some computation, we have

$$\begin{aligned} & \int_{D(x^\circ, r/(N\sqrt{d}))} |u(x, t)|^2 dx e^{-2\|q\|_\infty t} \leq \int_{B(x^\circ, r/N)} |u(x, t)|^2 dx e^{-2\|q\|_\infty t} \\ & \leq \int_{B(x^\circ, 2r/N)} |\Phi_2 u(x, t)|^2 dx e^{-2\|q\|_\infty t} \leq g_2(t) \\ & \leq \left(8(N/r)^2\right)^{N-2} t^{N-3} \int_0^t g_N(\tau) d\tau + \sum_{j=0, \dots, N-3} \left(8(N/r)^2 t\right)^j g_{j+2}(0) \\ & \leq \left(8(N/r)^2\right)^{N-2} t^{N-3} \int_0^t \int_{B(x^\circ, r)} |\Phi_N u(x, \tau)|^2 dx e^{-2\|q\|_\infty \tau} d\tau \\ & \quad + \sum_{j=0, \dots, N-3} \left(8(N/r)^2 t\right)^j \int_{B(x^\circ, (j+2)r/N)} |\Phi_{j+2} u(x, 0)|^2 dx \\ & \leq \left(8(N/r)^2\right)^{N-2} t^{N-3} \int_0^t \int_{\Omega} |u(x, \tau)|^2 dx e^{-2\|q\|_\infty \tau} d\tau + \left(\frac{1 - (8(N/r)^2 t)^{N-2}}{1 - 8(N/r)^2 t}\right) \int_{\omega} |u(x, 0)|^2 dx . \end{aligned} \quad (4.11)$$

On the other hand, it is easy to check that for all $\tau \in (0, T)$,

$$\int_{\Omega} |u(x, \tau)|^2 dx e^{-2\|q\|_\infty \tau} \leq \int_{\Omega} |u(x, 0)|^2 dx , \quad (4.12)$$

which together with (4.11) implies

$$\begin{aligned} & \int_{D(x^\circ, r/(N\sqrt{d}))} |u(x, t)|^2 dx \\ & \leq e^{2\|q\|_\infty t} \left[\left(8(N/r)^2\right)^{N-2} t^{N-2} \int_{\Omega} |u(x, 0)|^2 dx + \left(\frac{1 - (8(N/r)^2 t)^{N-2}}{1 - 8(N/r)^2 t}\right) \int_{\omega} |u(x, 0)|^2 dx \right] . \end{aligned} \quad (4.13)$$

Now, we break the rectangle $\left[0, \frac{r}{\sqrt{d}}\right]^d \subset \mathbb{R}^d$ into N^d small rectangles D_j , $j = 1, \dots, N^d$, having same shape, by dividing each interval $\left[0, \frac{r}{\sqrt{d}}\right]$ into N small pieces (i.e. $\left[0, \frac{r}{\sqrt{d}}\right] = \bigcup_{k=0, \dots, N-1} \left[k \frac{r}{N\sqrt{d}}, (k+1) \frac{r}{N\sqrt{d}}\right]$) and $\left[0, \frac{r}{\sqrt{d}}\right]^2 = \left(\bigcup_{k=0, \dots, N-1} \left[k \frac{r}{N\sqrt{d}}, (k+1) \frac{r}{N\sqrt{d}}\right]\right) \times \left(\bigcup_{k=0, \dots, N-1} \left[k \frac{r}{N\sqrt{d}}, (k+1) \frac{r}{N\sqrt{d}}\right]\right)$, ...etc. It is clear that $meas(D_{j_1} \cap D_{j_2}) = 0$ for $j_1 \neq j_2$ and also

$$\int_{\left[0, \frac{r}{\sqrt{d}}\right]^d} |u(x, t)|^2 dx = \sum_{j=1, \dots, N^d} \int_{D_j} |u(x, t)|^2 dx . \quad (4.14)$$

One can check that for each D_j , $j = 1, \dots, N^d$,

$$D_j \subset D \left(x^o, k \frac{r}{N\sqrt{d}} \right) , \quad (4.15)$$

if $x^o = (x_1^o, \dots, x_d^o) \in [-r, r]^d$ with x_i^o , $j = 1, \dots, d$, equals $k \left(r/N\sqrt{d} \right)$ for some $k = 0, \dots, N-1$, which implies by (4.14) that

$$\begin{aligned} & \int_{\left[0, \frac{r}{\sqrt{d}}\right]^d} |u(x, t)|^2 dx \\ & \leq N^d e^{2\|q\|_\infty t} \left[\left(8(N/r)^2 t \right)^{N-2} \int_{\Omega} |u(x, 0)|^2 dx + \left(\frac{1 - (8(N/r)^2 t)^{N-2}}{1 - 8(N/r)^2 t} \right) \int_{\omega} |u(x, 0)|^2 dx \right] . \end{aligned} \quad (4.16)$$

Let $M = N - 2 \geq 2$; we have

$$\begin{aligned} \int_{\frac{r}{\sqrt{d}}[0,1]^d} |u(x, t)|^2 dx & \leq e^{2\|q\|_\infty t} \left[\left(\frac{8}{r^2} (M+2)^{2+\frac{d}{M}} t \right)^M \int_{\Omega} |u(x, 0)|^2 dx \right] \\ & \quad + e^{2\|q\|_\infty t} \left[(M+2)^{\frac{d}{M}} \left(\frac{1 - \left(\frac{8}{r^2} (M+2)^2 t \right)^M}{1 - \frac{8}{r^2} (M+2)^2 t} \right) \int_{\omega} |u(x, 0)|^2 dx \right] . \end{aligned} \quad (4.17)$$

Lemma 4-1 follows.

4.2 Step 2: Observability estimate for periodic-parabolic solutions

The goal of this section is to prove the following observability inequality:

Theorem 4-1 .- *Let $\tilde{\omega}$ be a non-empty open subset of ω . There exists a constant $c > 0$ such that for all $T > 0$, $\beta \geq 0$ and $q \in L^\infty(\Omega \times (0, T))$, we have*

$$\int_{\Omega} |u(x, 0)|^2 dx \leq \frac{e^{2\|q\|_\infty T}}{T} \exp \left(c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3} \right) \right) \int_0^T \int_{\tilde{\omega}} |u(x, t)|^2 dx dt ,$$

where $u \in C([0, T]; L^2(\Omega))$ solves the following T -periodic in time linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 & \text{in } \Omega \times (0, T) , \\ u = 0 & \text{on } \partial\Omega \times (0, T) , \\ u(\cdot, 0) = u(\cdot, T) & \text{in } \Omega . \end{cases}$$

As we know, for initial condition problems of parabolic equation, the constant of observability grows in an exponentially fast way of order $e^{1/T}$ (see [12]). However, Theorem 4-1 shows that the constant of observability explodes for small time only in a polynomially fast way of order $1/\sqrt{T}$ for parabolic equations with time-periodic condition.

In order to prove Theorem 4-1, we need the following propositions.

Proposition 4-1 .- *Let ω_o be a non-empty open subset of Ω . There exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $c > 0$, such that for all $T > 0$, $\beta \geq 0$, $q \in L^\infty(\Omega \times (0, T))$ and $\theta \geq c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right)$, we have*

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \frac{1}{2} \beta \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dx - \theta^2 \int_\Omega e^{2\theta\varphi(x)} |\nabla\varphi|^2 |u(x, t)|^2 dx \right) \\ & + \frac{1}{c} \left(\int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^2 \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \\ & \leq c \left(\theta \int_{\omega_o} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \end{aligned}$$

for almost every $t \in (0, T)$, where $u = u(x, t)$ solves the following linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 & \text{in } \Omega \times (0, T) , \\ u = 0 & \text{on } \partial\Omega \times (0, T) , \\ u(\cdot, 0) \in H^2(\Omega) \cap H_0^1(\Omega) . \end{cases}$$

Proposition 4-2 (Global Carleman inequality for the operator $\Delta - \beta$).- *Let ω_o be a non-empty open subset of Ω . There exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $c > 0$, such that for all $\beta \geq 0$, $\tilde{q} \in L^\infty(\Omega)$, $\theta > c \left(1 + \beta^{2/3} + \|\tilde{q}\|_{L^\infty(\Omega)}^{4/3}\right)$ and $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\begin{aligned} & \|2\theta \nabla\varphi \cdot \nabla(e^{\theta\varphi} w) - \tilde{q}w\|_{L^2(\Omega)}^2 + \frac{1}{c} \left(\theta \int_\Omega e^{2\theta\varphi(x)} |\nabla w(x)|^2 dx + \theta^3 \int_\Omega e^{2\theta\varphi(x)} |w(x)|^2 dx \right) \\ & \leq \|e^{\theta\varphi} (\Delta - \beta) w\|_{L^2(\Omega)}^2 + c \left(\theta \int_{\omega_o} e^{2\theta\varphi(x)} |\nabla w(x)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)} |w(x)|^2 dx \right) . \end{aligned}$$

The proof of Proposition 4-2 is given in the Appendix.

Proof of Proposition 4-1 .- We apply Proposition 4-2 to the solution u of the linear heat equation with potential, then we deduce that there exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $c > 0$, such that for all $\beta \geq 0$, $q \in L^\infty(\Omega \times (0, T))$, and $\theta > c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right)$, we have

$$\begin{aligned} & \|2\theta \nabla\varphi \cdot \nabla(e^{\theta\varphi} u(\cdot, t)) - q(\cdot, t) e^{\theta\varphi} u(\cdot, t)\|_{L^2(\Omega)}^2 \\ & + \frac{1}{c} \left(\theta \int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^3 \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \\ & \leq \|e^{\theta\varphi} (\Delta - \beta) u(\cdot, t)\|_{L^2(\Omega)}^2 + c \left(\theta \int_{\omega_o} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \end{aligned} \quad (4.18)$$

for almost every $t \in (0, T)$.

On the other hand, we multiply equation $-\partial_t u + (\Delta - \beta) u - qu = 0$ by $e^{2\theta\varphi} (\Delta - \beta) u$ and integrate it over Ω to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \frac{1}{2} \beta \frac{d}{dt} \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dx + \|e^{\theta\varphi} (\Delta - \beta) u(\cdot, t)\|_{L^2(\Omega)}^2 \\ & = \int_\Omega q(x, t) u(x, t) (\Delta - \beta) u(x, t) e^{2\theta\varphi(x)} dx - \int_\Omega \partial_t u(x, t) 2\theta \nabla\varphi(x) \cdot \nabla u(x, t) e^{2\theta\varphi(x)} dx \\ & = \int_\Omega q(x, t) u(x, t) (\Delta - \beta) u(x, t) e^{2\theta\varphi(x)} dx \\ & \quad - \int_\Omega \partial_t u(x, t) e^{\theta\varphi(x)} 2\theta \nabla\varphi(x) \cdot \nabla(e^{\theta\varphi(x)} u(x, t)) dx \\ & \quad + \int_\Omega \partial_t u(x, t) e^{\theta\varphi(x)} 2\theta \nabla\varphi(x) \cdot (\theta \nabla\varphi(x) e^{\theta\varphi(x)}) u(x, t) dx \end{aligned} \quad (4.19)$$

which implies

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \frac{1}{2} \beta \frac{d}{dt} \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dx - \theta^2 \frac{d}{dt} \int_\Omega |\nabla\varphi(x)|^2 e^{2\theta\varphi(x)} |u(x, t)|^2 dx \\ & + \|e^{\theta\varphi} (\Delta - \beta) u(\cdot, t)\|_{L^2(\Omega)}^2 - \int_\Omega q(x, t) u(x, t) (\Delta - \beta) u(x, t) e^{2\theta\varphi(x)} dx \\ & = - \int_\Omega \partial_t u(x, t) e^{\theta\varphi(x)} 2\theta \nabla\varphi(x) \cdot \nabla(e^{\theta\varphi(x)} u(x, t)) dx . \end{aligned} \quad (4.20)$$

Replacing $\partial_t u$ by $(\Delta - \beta)u - qu$ in the right hand side of (4.20), we deduce that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} e^{2\theta\varphi} |\nabla u(x, t)|^2 dx + \frac{1}{2} \beta \frac{d}{dt} \int_{\Omega} e^{2\theta\varphi} |u(x, t)|^2 dx - \theta^2 \frac{d}{dt} \int_{\Omega} |\nabla\varphi|^2 e^{2\theta\varphi} |u|^2 dx \\
& + \left\| e^{\theta\varphi} (\Delta - \beta) u(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
& + \int_{\Omega} e^{\theta\varphi(x)} (\Delta - \beta) u(x, t) \left[2\theta \nabla\varphi(x) \cdot \nabla (e^{\theta\varphi(x)} u(x, t)) - q(x, t) e^{\theta\varphi(x)} u(x, t) \right] dx \\
& = \int_{\Omega} q(x, t) u(x, t) e^{\theta\varphi(x)} 2\theta \nabla\varphi(x) \cdot \nabla (e^{\theta\varphi(x)} u(x, t)) dx .
\end{aligned} \tag{4.21}$$

By Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
& \int_{\Omega} q(x, t) u(x, t) e^{\theta\varphi(x)} 2\theta \nabla\varphi(x) \cdot \nabla (e^{\theta\varphi(x)} u(x, t)) dx \\
& = 2\theta \int_{\Omega} q(x, t) u(x, t) e^{\theta\varphi(x)} \nabla\varphi(x) \cdot (\theta \nabla\varphi(x) e^{\theta\varphi(x)} u(x, t) + e^{\theta\varphi(x)} \nabla u(x, t)) dx \\
& \leq 2\theta^2 \|q\|_{\infty} \|\nabla\varphi\|_{L^{\infty}(\Omega)}^2 \|e^{\theta\varphi} u\|_{L^2(\Omega)}^2 + 2\theta \|q\|_{\infty} \|\nabla\varphi\|_{L^{\infty}(\Omega)} \|e^{\theta\varphi} u\|_{L^2(\Omega)} \|e^{\theta\varphi} \nabla u\|_{L^2(\Omega)} .
\end{aligned} \tag{4.22}$$

Then, we conclude by (4.21) and (4.22) that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \beta \int_{\Omega} e^{2\theta\varphi(x)} |u(x, t)|^2 dx - 2\theta^2 \int_{\Omega} |\nabla\varphi(x)|^2 e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right] \\
& + \left\| e^{\theta\varphi} (\Delta - \beta) u(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
& + \int_{\Omega} e^{\theta\varphi(x)} (\Delta - \beta) u(x, t) \left[2\theta \nabla\varphi(x) \cdot \nabla (e^{\theta\varphi(x)} u(x, t)) - q(x, t) e^{\theta\varphi(x)} u(x, t) \right] dx \\
& \leq 2\theta^2 \|q\|_{\infty} \|\nabla\varphi\|_{L^{\infty}(\Omega)}^2 \|e^{\theta\varphi} u\|_{L^2(\Omega)}^2 \\
& + 2\theta \|q\|_{\infty} \|\nabla\varphi\|_{L^{\infty}(\Omega)} \|e^{\theta\varphi} u\|_{L^2(\Omega)} \|e^{\theta\varphi} \nabla u\|_{L^2(\Omega)} .
\end{aligned} \tag{4.23}$$

Consequently, there exists a constant $\tilde{c} > 0$ such for all $\beta \geq 0$ and $q \in L^{\infty}(\Omega \times (0, T))$, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \beta \int_{\Omega} e^{2\theta\varphi(x)} |u(x, t)|^2 dx - 2\theta^2 \int_{\Omega} |\nabla\varphi(x)|^2 e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right] \\
& + \frac{1}{2} \left\| e^{\theta\varphi} (\Delta - \beta) u(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
& + \frac{1}{2} \left\| e^{\theta\varphi} (\Delta - \beta) u(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
& + \int_{\Omega} e^{\theta\varphi(x)} (\Delta - \beta) u(x, t) \left[2\theta \nabla\varphi(x) \cdot \nabla (e^{\theta\varphi(x)} u(x, t)) - q(x, t) e^{\theta\varphi(x)} u(x, t) \right] dx \\
& \leq \tilde{c} \|q\|_{\infty} \left(\int_{\Omega} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^2 \int_{\Omega} e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right)
\end{aligned} \tag{4.24}$$

for almost every $t \in (0, T)$.

By (4.18) and (4.24), we get

$$\begin{aligned}
& \frac{1}{2} \left\| 2\theta \nabla\varphi \cdot \nabla (e^{\theta\varphi} u(\cdot, t)) - q(\cdot, t) e^{\theta\varphi} u(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
& + \frac{1}{2c} \left(\theta \int_{\Omega} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^3 \int_{\Omega} e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \\
& - \frac{c}{2} \left(\theta \int_{\omega_o} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \\
& \leq \frac{1}{2} \left\| e^{\theta\varphi} (\Delta - \beta) u(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
& \leq -\frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \beta \int_{\Omega} e^{2\theta\varphi(x)} |u(x, t)|^2 dx - 2\theta^2 \int_{\Omega} |\nabla\varphi(x)|^2 e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right] \\
& - \frac{1}{2} \left\| e^{\theta\varphi} (\Delta - \beta) u(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
& - \frac{1}{2} \int_{\Omega} e^{\theta\varphi(x)} (\Delta - \beta) u(x, t) \left[2\theta \nabla\varphi(x) \cdot \nabla (e^{\theta\varphi(x)} u(x, t)) - q(x, t) e^{\theta\varphi(x)} u(x, t) \right] dx \\
& + \tilde{c} \|q\|_{\infty} \left(\int_{\Omega} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^2 \int_{\Omega} e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right)
\end{aligned} \tag{4.25}$$

from which it follows that

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} e^{2\theta\varphi} |\nabla u(x, t)|^2 dx + \frac{1}{2} \beta \int_{\Omega} e^{2\theta\varphi} |u(x, t)|^2 dx - \theta^2 \int_{\Omega} |\nabla\varphi|^2 e^{2\theta\varphi} |u|^2 dx \right) \\
& + \frac{1}{2} \left\| \left[e^{\theta\varphi} (\Delta - \beta) u(\cdot, t) \right] + \left[2\theta \nabla\varphi \cdot \nabla (e^{\theta\varphi} u(\cdot, t)) - q(\cdot, t) e^{\theta\varphi} u(\cdot, t) \right] \right\|_{L^2(\Omega)}^2 \\
& + \frac{1}{2c} \theta \left(\int_{\Omega} e^{2\theta\varphi} |\nabla u(x, t)|^2 dx + \theta^2 \int_{\Omega} e^{2\theta\varphi} |u(x, t)|^2 dx \right) \\
& \leq \tilde{c} \|q\|_{\infty} \left(\int_{\Omega} e^{2\theta\varphi} |\nabla u(x, t)|^2 dx + \theta^2 \int_{\Omega} e^{2\theta\varphi} |u(x, t)|^2 dx \right) \\
& + \frac{c}{2} \left(\theta \int_{\omega_o} e^{2\theta\varphi} |\nabla u(x, t)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi} |u(x, t)|^2 dx \right) .
\end{aligned} \tag{4.26}$$

By taking $\theta \geq c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right) + 4c\tilde{c}\|q\|_\infty$ in (4.26), we deduce that the inequality

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \frac{1}{2}\beta \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dx - \theta^2 \int_\Omega |\nabla\varphi(x)|^2 e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \\ & + c \left(\int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^2 \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \\ & \leq \frac{c}{2} \left(\theta \int_{\omega_o} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)} |u(x, t)|^2 dx \right) \end{aligned} \quad (4.27)$$

holds for all $\theta \geq c_1 \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right)$ where $c_1 > 0$ is large enough. This completes the proof of Proposition 1.

Proof of Theorem 4-1 .- Through the proof of Theorem 4-1, c denotes several positive constants only depending on the geometry. By integrating the estimate in Proposition 4-1 over $(0, T)$, we get

$$\begin{aligned} & \int_0^T \int_\Omega e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dxdt + \theta^2 \int_0^T \int_\Omega e^{2\theta\varphi(x)} |u(x, t)|^2 dxdt \\ & \leq c \left(\theta \int_0^T \int_{\omega_o} e^{2\theta\varphi(x)} |\nabla u(x, t)|^2 dxdt + \theta^3 \int_0^T \int_{\omega_o} e^{2\theta\varphi(x)} |u(x, t)|^2 dxdt \right), \end{aligned} \quad (4.28)$$

thanks to the T -periodicity of u . With an adequate choice of $\theta > c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right)$ in (4.28), it follows that

$$\int_0^T \int_\Omega |u(x, t)|^2 dxdt \leq \exp \left(c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right) \right) \left(\int_0^T \int_{\omega_o} |u(x, t)|^2 dx + \int_0^T \int_{\omega_o} |\nabla u(x, t)|^2 dx \right). \quad (4.29)$$

By the usual energy method (i.e. equality like (4.4)), we also have

$$\int_0^T \int_{\omega_o} |\nabla u(x, t)|^2 dxdt \leq (c + \|q\|_\infty) \int_0^T \int_{\tilde{\omega}} |u(x, t)|^2 dx \quad (4.30)$$

for some $\omega_o \subset \tilde{\omega} \subset \omega$, using the T -periodicity of u .

Consequently, for all $T > 0$, $\beta \geq 0$ and $q \in L^\infty(\Omega \times (0, T))$,

$$\int_0^T \int_\Omega |u(x, t)|^2 dxdt \leq \exp \left(c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right) \right) \int_0^T \int_{\tilde{\omega}} |u(x, t)|^2 dx. \quad (4.31)$$

where u solves the following T -periodic in time linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u(\cdot, T) & \text{in } \Omega, \\ u(\cdot, 0) \in H^2(\Omega) \cap H_0^1(\Omega). \end{cases} \quad (4.32)$$

By density of $H^2(\Omega) \cap H_0^1(\Omega)$ in $L^2(\Omega)$, this last estimate is also true for solutions u with initial data in $L^2(\Omega)$.

On the other hand, we recall that

$$e^{2\|q\|_\infty t} \int_\Omega |u(x, T)|^2 dx \leq e^{2\|q\|_\infty T} \int_\Omega |u(x, t)|^2 dx \quad (4.33)$$

and

$$T \leq \frac{1}{2\|q\|_\infty} \left(e^{2\|q\|_\infty T} - 1 \right) = \int_0^T e^{2\|q\|_\infty t} dt. \quad (4.34)$$

Finally, it follows from (4.33), (4.34) and (4.31) that

$$\begin{aligned} \int_\Omega |u(x, 0)|^2 dx = \int_\Omega |u(x, T)|^2 dx & \leq \frac{e^{2\|q\|_\infty T}}{T} \int_0^T \int_\Omega |u(x, t)|^2 dxdt \\ & \leq \frac{e^{2\|q\|_\infty T}}{T} \exp \left(c \left(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\right) \right) \int_0^T \int_{\tilde{\omega}} |u(x, t)|^2 dxdt. \end{aligned} \quad (4.35)$$

This completes the proof of Theorem 4-1.

4.3 Step 3: Choice of T

Now we are able to conclude the proof of Theorem 3 as follows. By integrating over $(0, T)$ the estimate in Lemma 4-1, we have

$$\begin{aligned} \frac{1}{T} \int_0^T \int_{\bar{\omega}} |u(x, t)|^2 dx &\leq e^{2\|q\|_{\infty} T} \left(C_{\omega} (M+2)^{2+\frac{d}{M}} T \right)^M \int_{\Omega} |u(x, 0)|^2 dx \\ &+ e^{2\|q\|_{\infty} T} (M+2)^{\frac{d}{M}} \left(\frac{1-(C_{\omega}(M+2)^2 T)^M}{1-C_{\omega}(M+2)^2 T} \right) \int_{\omega} |u(x, 0)|^2 dx . \end{aligned} \quad (4.36)$$

Combining the later with Theorem 4-1, we deduce that there exists a constant $c > 1$ such that for all $T > 0$, $q \in L^{\infty}(\Omega \times (0, T))$ and integers $M \geq 2$, we have

$$\begin{aligned} &\int_{\Omega} |u(x, 0)|^2 dx \\ &\leq \exp \left(c \left(1 + \beta^{2/3} + \|q\|_{\infty}^{4/3} \right) \right) e^{4\|q\|_{\infty} T} \left(C_{\omega} (M+2)^{2+d/M} T \right)^M \int_{\Omega} |u(x, 0)|^2 dx \\ &+ \exp \left(c \left(1 + \beta^{2/3} + \|q\|_{\infty}^{4/3} \right) \right) e^{4\|q\|_{\infty} T} (M+2)^{d/M} \left(\frac{1-(C_{\omega}(M+2)^2 T)^M}{1-C_{\omega}(M+2)^2 T} \right) \int_{\omega} |u(x, 0)|^2 dx . \end{aligned} \quad (4.37)$$

From this we infer that there exists a constant $c > 1$ such that for all $T > 0$, $\beta \geq 0$ and $q \in L^{\infty}(\Omega \times (0, T))$ satisfying $\ell_1 + \beta \leq \|q\|_{\infty}$, the inequality

$$\begin{aligned} \int_{\Omega} |u(x, 0)|^2 dx &\leq e^{c(1+T)\|q\|_{\infty}^{4/3}} \left(C_{\omega} (M+2)^{2+d/M} T \right)^M \int_{\Omega} |u(x, 0)|^2 dx \\ &+ e^{c(1+T)\|q\|_{\infty}^{4/3}} (M+2)^{d/M} \left(\frac{1-(C_{\omega}(M+2)^2 T)^M}{1-C_{\omega}(M+2)^2 T} \right) \int_{\omega} |u(x, 0)|^2 dx \end{aligned} \quad (4.38)$$

holds for all integers $M \geq 2$.

First we choose an integer $M \geq 2$ so that

$$M < 2 \|q\|_{\infty}^{4/3} \left(\frac{1+d}{\ell_1^{4/3}} + 2c \right) \leq M+1 , \quad (4.39)$$

which can be done because $\|q\|_{\infty} \geq \ell_1$. By (4.39), it follows that

$$\begin{cases} 1+d \leq M , \\ 1+2c \|q\|_{\infty}^{4/3} \leq M . \end{cases} \quad (4.40)$$

By (4.40) and (4.38), we infer

$$\begin{aligned} \int_{\Omega} |u(x, 0)|^2 dx &\leq e^{M-1+c(T-1)\|q\|_{\infty}^{4/3}} \left(C_{\omega} (2M)^3 T \right)^M \int_{\Omega} |u(x, 0)|^2 dx \\ &+ e^{c(1+T)\|q\|_{\infty}^{4/3}} (2M) \left(\frac{1-(C_{\omega}(2M)^2 T)^M}{1-C_{\omega}(2M)^2 T} \right) \int_{\omega} |u(x, 0)|^2 dx . \end{aligned} \quad (4.41)$$

Next, we choose $T > 0$, so that $T \leq \frac{1}{e C_{\omega} \left(4 \|q\|_{\infty}^{4/3} \left(\frac{1+d}{\ell_1^{4/3}} + 2c \right) \right)^3} < \frac{1}{e C_{\omega} (2M)^3} \leq 1$, which can be done

because of (4.39). Then by (4.41), we get

$$\int_{\Omega} |u(x, 0)|^2 dx \leq e^{-1} \int_{\Omega} |u(x, 0)|^2 dx + e^{2c\|q\|_{\infty}^{4/3}} (2M)^2 \int_{\omega} |u(x, 0)|^2 dx , \quad (4.42)$$

which gives

$$(1 - e^{-1}) \int_{\Omega} |u(x, 0)|^2 dx \leq e^{2c\|q\|_{\infty}^{4/3}} \left[4 \|q\|_{\infty}^{4/3} \left(\frac{1+d}{\ell_1^{4/3}} + 2c \right) \right]^2 \int_{\omega} |u(x, 0)|^2 dx . \quad (4.43)$$

Now let $c_o = \frac{1}{eC_\omega \left(4 \left(\frac{1+d}{\ell_1^{4/3}} + 2c\right)\right)^3}$, then by (4.43), we conclude that if $\ell_1 + \beta \leq \|q\|_\infty$ and $\|q\|_\infty \leq$

$\ell_1 \left(\frac{c_o}{T}\right)^{1/4}$, then there exists a constant $C > 0$ such that

$$\int_{\Omega} |u(x, 0)|^2 dx \leq e^{C\|q\|_\infty^{4/3}} \int_{\omega} |u(x, 0)|^2 dx , \quad (4.44)$$

where the solution $u \in C([0, T]; L^2(\Omega))$ satisfies the following T -periodic in time linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 & \text{in } \Omega \times (0, T) , \\ u = 0 & \text{on } \partial\Omega \times (0, T) , \\ u(\cdot, 0) = u(\cdot, T) & \text{in } \Omega . \end{cases} \quad (4.45)$$

The proof of Theorem 3 is now completed.

5 The autonomous case

Let us denote by ℓ_i , with $0 < \ell_1 < \ell_2 \leq \ell_3 \leq \dots$, and by e_i for $i \geq 1$, the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$. Recall here that $(e_i)_{i \geq 1}$ forms an orthonormal basis of the Hilbert space $L^2(\Omega)$.

Let $a = a(t) \in L^\infty(\mathbb{R})$ be an even, T -periodic potential. From a Galerkin method, $G(t, s)$ can be explicitly expressed by $G(t, s) = e^{-\int_s^t a(\tau) dt} e^{\Delta(t-s)}$ and one can check easily that the eigencouple (z_j^o, λ_j) of the compact selfadjoint operator $G(T, 0)$ satisfies

$$\begin{cases} \lambda_j = e^{-\ell_j T - \int_0^T a(\tau) d\tau} , \\ z_j^o = e_j . \end{cases}$$

Furthermore, we obtain the following asymptotic controllability result.

Theorem 5-1 .- *Suppose that $a = a(t) \in L^\infty(\mathbb{R})$ is a space-independent potential satisfying*

$$\begin{cases} (i) & a(t+T) = a(t) = a(-t) \quad \text{in } \mathbb{R} , \\ (ii) & \ell_1 \leq -\frac{1}{T} \int_0^T a(t) dt < \ell_2 ; \end{cases}$$

then for all initial data $y^o \in L^2(\Omega)$, the control function $f \in L^1(0, T; L^2(\Omega))$ given by

$$f(x, t) = -\frac{1}{T} e^{-\ell_1 t - \int_{T-t}^T a(\tau) d\tau} \left(\frac{\int_{\Omega} y_o(x) e_1(x) dx}{\int_{\omega} |e_1(x)|^2 dx} \right) e_1(x)$$

implies that the solution $y = y(x, t) \in C([0, +\infty); L^2(\Omega))$ of the following heat equation with potential $a = a(t)$,

$$\begin{cases} \partial_t y - \Delta y + ay = f \cdot 1_{|\omega \times (0, T)} & \text{in } \Omega \times (0, +\infty) , \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty) , \\ y(\cdot, 0) = y^o & \text{in } \Omega , \end{cases}$$

satisfies for all $t \geq T$,

$$\|y(t)\|_{L^2(\Omega)} \leq e^{C(\|a\|_\infty^{3/2} + \|a\|_\infty T)} \exp\left(-\left(\ell_2 + \frac{1}{T} \int_0^T a(\tau) d\tau\right)t\right) \|y^o\|_{L^2(\Omega)}$$

and

$$\int_0^T \|f(s)\|_{L^2(\omega)} ds \leq e^{C\|a\|_\infty^{3/2}} e^{(\|a\|_\infty - \ell_1)T} \|y^o\|_{L^2(\Omega)} ,$$

where $C > 0$ is a constant which only depends on the geometry and $\alpha \geq 0$. More precisely, $\alpha = \frac{2}{3}$ for general non-empty subdomain $\omega \subset \Omega$ and $\alpha = 0$ under the Geometrical control condition of the work of Bardos-Lebeau-Rauch (i.e. $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is of class C^∞ , there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristics of $\partial_t^2 - \Delta$, and all generalized bicharacteristic ray of $\partial_t^2 - \Delta$ meets $\omega \times (0, T_c)$ for some $0 < T_c < +\infty$).

Remark that no hypotheses on $T > 0$ are required with a space-independent potential which satisfies $(\|a\|_\infty - \ell_1) \geq 0$ thanks to (ii). We omit details of the proof here. For general non-empty subdomain $\omega \subset \Omega$, we apply Theorem A (see Appendix) to get a quantification of the unique continuation property for the eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ and deduce $\alpha = \frac{2}{3}$. When the Geometrical control condition of the work of Bardos-Lebeau-Rauch [2] is satisfied, then we apply the observability estimate for the wave equation for a fixed frequency and deduce $\alpha = 0$.

If the even, T -periodic bounded potential $a = a(t)$ depends only on time t , then $G(T, 0)$ is a positive operator and the first eigenvalue is of multiplicity one (see Krein-Rutman theorem in e.g. [15]). We do not know if such result is still true for a potential $a = a(x, t)$ depending on space and time. Observe that here no positivity hypotheses on the potential or on the initial data are imposed as in [15] or as in [9].

6 Appendix: Global Carleman inequality for the operator $(\Delta - \beta)$

Carleman estimates are widely used in applied problems (see [16] for inverse problems or e.g. [13], [12], [18] for controllability problems). In what follows, we present the proof of the Global Carleman inequality for the operator $(\Delta - \beta)$ that we have used (see Proposition 4-2 or Theorem A below). In our inequality in Theorem A, the presence of the first term in the left hand side plays a key role.

Theorem A .- *Let Ω be a bounded connected domain in \mathbb{R}^d , $d \geq 1$, with a boundary $\partial\Omega$ of class C^2 , and ω_o be a non-empty open subset of Ω . Then there exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $C > 0$, such that for all $\tilde{\beta} \in \mathbb{R}$, for all $\tilde{q} \in L^\infty(\Omega)$, for all $\theta > C \left(1 + |\tilde{\beta}|^{2/3} + \|\tilde{q}\|_{L^\infty(\Omega)}^{4/3}\right)$, for all $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\begin{aligned} & \|2\theta \nabla \varphi \cdot \nabla (e^{\theta \varphi} w) - \tilde{q} w\|_{L^2(\Omega)}^2 + \frac{1}{C} \left(\theta \int_{\Omega} e^{2\theta \varphi(x)} |\nabla w(x)|^2 dx + \theta^3 \int_{\Omega} e^{2\theta \varphi(x)} |w(x)|^2 dx \right) \\ & \leq \left\| e^{\theta \varphi} (\Delta - \tilde{\beta}) w \right\|_{L^2(\Omega)}^2 + C \left(\theta \int_{\omega_o} e^{2\theta \varphi(x)} |\nabla w(x)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta \varphi(x)} |w(x)|^2 dx \right). \end{aligned}$$

6.1 An equality with weight

We denote $\|\cdot\| = \left(\int (\cdot)^2 dx \right)^{1/2}$ the usual norm in $L^2(\Omega)$ associated to the scalar product $(\cdot, \cdot) = \int (\cdot, \cdot) dx$. From now, we shall omit all x s in the functions of x .

Let $\theta > 0$, $\varphi = \varphi(x) \in C^2(\mathbb{R}^d)$ be arbitrary but given. We shall prove first that function $v(x) = e^{\theta \varphi(x)} w(x)$ where $w \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies the following

$$\left\{ \begin{aligned} & \left\| \Delta v - \theta \Delta \varphi v + \theta^2 (\nabla \varphi)^2 v - \tilde{\beta} v - \tilde{q} v \right\|^2 + \|2\theta \nabla \varphi \nabla v + \tilde{q} v\|^2 \\ & + 2(\text{boundary terms}) \\ & + 4\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 + 2\theta^3 \int \nabla \varphi \nabla \left((\nabla \varphi)^2 \right) (v)^2 \\ & + 4\theta \int \sum_i \sum_{j \neq i} \partial_{ij} \varphi \partial_i v \partial_j v \\ & = \left\| e^{\theta \varphi} (\Delta - \tilde{\beta}) w \right\|^2 + 2\theta^2 \int \text{div} (\Delta \varphi \nabla \varphi) (v)^2 \\ & - 4\theta \int \nabla \varphi \tilde{q} v \nabla v - 2 \int \left[\Delta v - \theta \Delta \varphi v + \theta^2 (\nabla \varphi)^2 v - \tilde{\beta} v - \tilde{q} v \right] \tilde{q} v \\ & + 2\theta \tilde{\beta} \int \Delta \varphi (v)^2 + 2\theta \int \Delta \varphi (\nabla v)^2 - 2\theta^3 \int \Delta \varphi (\nabla \varphi)^2 (v)^2, \end{aligned} \right. \quad (\text{A1})$$

$$\left\{ \begin{array}{l} 2\theta\tilde{\beta} \int \Delta\varphi(v)^2 + 2\theta \int \Delta\varphi(\nabla v)^2 - 2\theta^3 \int \Delta\varphi(\nabla\varphi)^2(v)^2 \\ = -2\theta \int \Delta\varphi v \left[\Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \tilde{\beta}v - \tilde{q}v \right] \\ -2\theta^2 \int (\Delta\varphi)^2(v)^2 - 2\theta \int \Delta\varphi\tilde{q}(v)^2 + \theta \int \Delta^2\varphi(v)^2 \end{array} \right. \quad (\text{A2})$$

and

$$\text{boundary terms} = -\theta \int_{\partial\Omega} (\partial_n v)^2 \partial_n \varphi . \quad (\text{A3})$$

Proof of (A1) .- We shall start from computation of all derivatives of $v = e^{\theta\varphi}w$ and then write term $e^{\theta\varphi}(\Delta - \tilde{\beta})w$ as the sum of P_1v and P_2v where P_1 is an differential operator of order 1 and P_2 is an differential operator of order 2 given below; after that we calculate $\int_{\Omega} (P_2v + P_1v)^2 dx$ where the term $\int_{\Omega} (P_2v, P_1v) dx$ plays a key role.

One can check easily that

$$\begin{aligned} \Delta v &= \Delta e^{\theta\varphi}w + 2\nabla e^{\theta\varphi}\nabla w + e^{\theta\varphi}\Delta w \\ &= \theta\Delta\varphi v - \theta^2(\nabla\varphi)^2 v + 2\theta\nabla\varphi\nabla v + e^{\theta\varphi}\Delta w \end{aligned} \quad (\text{A4})$$

and

$$P_2v + P_1v = e^{\theta\varphi}(\Delta - \tilde{\beta})w , \quad (\text{A5})$$

where

$$\begin{aligned} P_1v &= -2\theta\nabla\varphi\nabla v + \tilde{q}v \\ P_2v &= \Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \tilde{\beta}v - \tilde{q}v . \end{aligned} \quad (\text{A6})$$

Then we have

$$\left\| e^{\theta\varphi}(\Delta - \tilde{\beta})w \right\|^2 = \left\| \Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \tilde{\beta}v - \tilde{q}v \right\|^2 + \left\| -2\theta\nabla\varphi\nabla v + \tilde{q}v \right\|^2 + 2(P_2v, P_1v) . \quad (\text{A7})$$

However, a direct calculation leads us to

$$\begin{aligned} &(P_2v, P_1v) \\ &= \left(\Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \tilde{\beta}v - \tilde{q}v, -2\theta\nabla\varphi\nabla v + \tilde{q}v \right) \\ &= -2\theta \int \nabla\varphi\nabla v\Delta v + 2\theta^2 \int \Delta\varphi\nabla\varphi v\nabla v - 2\theta^3 \int \nabla\varphi(\nabla\varphi)^2 v\nabla v + 2\theta\tilde{\beta} \int \nabla\varphi v\nabla v \\ &\quad + 2\theta \int \nabla\varphi\tilde{q}v\nabla v + (P_2v, \tilde{q}v) \\ &= -2\theta \int \sum_i \partial_i\varphi\partial_i v\partial_i^2 v - 2\theta \int \sum_i \partial_i\varphi\partial_i v \sum_{j\neq i} \partial_j^2 v \\ &\quad + \theta^2 \int \Delta\varphi\nabla\varphi\nabla(v)^2 \\ &\quad - \theta^3 \int \nabla\varphi(\nabla\varphi)^2\nabla(v)^2 \\ &\quad + \theta\tilde{\beta} \int \nabla\varphi\nabla(v)^2 \\ &\quad + 2\theta \int \nabla\varphi\tilde{q}v\nabla v + (P_2v, \tilde{q}v) \\ &= -\theta \int \sum_i \partial_i\varphi\partial_i(\partial_i v)^2 \\ &\quad - 2\theta \int \sum_i \sum_{j\neq i} \partial_j(\partial_i\varphi\partial_i v\partial_j v) + 2\theta \int \sum_i \sum_{j\neq i} \partial_{ij}\varphi\partial_i v\partial_j v + \theta \int \sum_i \sum_{j\neq i} \partial_i\varphi\partial_i(\partial_j v)^2 \\ &\quad + \theta^2 \int \text{div} \left((\Delta\varphi\nabla\varphi)(v)^2 \right) - \theta^2 \int \text{div} (\Delta\varphi\nabla\varphi)(v)^2 \\ &\quad - \theta^3 \int \text{div} \left((\nabla\varphi(\nabla\varphi)^2)(v)^2 \right) + \theta^3 \int \text{div} (\nabla\varphi(\nabla\varphi)^2)(v)^2 \\ &\quad - \theta\tilde{\beta} \int \Delta\varphi(v)^2 \\ &\quad + 2\theta \int \nabla\varphi\tilde{q}v\nabla v + (P_2v, \tilde{q}v) . \end{aligned} \quad (\text{A8})$$

From this we get by integrations by parts

$$\begin{aligned}
& \left(\Delta v - \theta \Delta \varphi v + \theta^2 (\nabla \varphi)^2 v - \tilde{\beta} v - \tilde{q} v, -2\theta \nabla \varphi \nabla v + \tilde{q} v \right) \\
&= (\text{boundary terms}) \\
&+ \theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 \\
&+ 2\theta \int \sum_i \sum_{j \neq i} \partial_{ij} \varphi \partial_i v \partial_j v - \theta \int \sum_i \sum_{j \neq i} \partial_i^2 \varphi (\partial_j v)^2 \\
&- \theta^2 \int \operatorname{div} (\Delta \varphi \nabla \varphi) (v)^2 \\
&+ \theta^3 \int \operatorname{div} (\nabla \varphi (\nabla \varphi)^2) (v)^2 \\
&- \theta \tilde{\beta} \int \Delta \varphi (v)^2 \\
&+ 2\theta \int \nabla \varphi \tilde{q} v \nabla v + (P_2 v, \tilde{q} v) .
\end{aligned} \tag{A9}$$

Then by (A7) and (A9), we infer

$$\begin{aligned}
& \left\| e^{\theta \varphi} (\Delta - \tilde{\beta}) w \right\|^2 - \left(\left\| \Delta v - \theta \Delta \varphi v + \theta^2 (\nabla \varphi)^2 v - \tilde{\beta} v - \tilde{q} v \right\|^2 + \left\| -2\theta \nabla \varphi \nabla v + \tilde{q} v \right\|^2 \right) \\
&= 2 \left(\Delta v - \theta \Delta \varphi v + \theta^2 (\nabla \varphi)^2 v - \tilde{\beta} v - \tilde{q} v, -2\theta \nabla \varphi \nabla v + \tilde{q} v \right) \\
&= 2 (\text{boundary terms}) \\
&+ 2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 \\
&+ 4\theta \int \sum_i \sum_{j \neq i} \partial_{ij} \varphi \partial_i v \partial_j v - 2\theta \int \sum_i \sum_{j \neq i} \partial_i^2 \varphi (\partial_j v)^2 \\
&- 2\theta^2 \int \operatorname{div} (\Delta \varphi \nabla \varphi) (v)^2 \\
&+ 2\theta^3 \int \operatorname{div} (\nabla \varphi (\nabla \varphi)^2) (v)^2 \\
&- 2\theta \tilde{\beta} \int \Delta \varphi (v)^2 \\
&+ 4\theta \int \nabla \varphi \tilde{q} v \nabla v + 2 (P_2 v, \tilde{q} v)
\end{aligned} \tag{A10}$$

or equivalently

$$\begin{aligned}
& \left\| P_2 v \right\|^2 + \left\| 2\theta \nabla \varphi \nabla v - \tilde{q} v \right\|^2 + 2 (\text{boundary terms}) \\
&+ 2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 + 2\theta^3 \int \operatorname{div} (\nabla \varphi (\nabla \varphi)^2) (v)^2 \\
&+ 4\theta \int \sum_i \sum_{j \neq i} \partial_{ij} \varphi \partial_i v \partial_j v \\
&= \left\| e^{\theta \varphi} (\Delta - \tilde{\beta}) w \right\|^2 + 2\theta^2 \int \operatorname{div} (\Delta \varphi \nabla \varphi) (v)^2 - 4\theta \int \nabla \varphi \tilde{q} v \nabla v - 2 (P_2 v, \tilde{q} v) \\
&+ 2\theta \int \sum_i \sum_{j \neq i} \partial_i^2 \varphi (\partial_j v)^2 + 2\theta \tilde{\beta} \int \Delta \varphi (v)^2 .
\end{aligned} \tag{A11}$$

By adding $2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2$ on the both side of (A11) and noting that $\operatorname{div} (\nabla \varphi (\nabla \varphi)^2) = \nabla \varphi \nabla ((\nabla \varphi)^2) + \Delta \varphi (\nabla \varphi)^2$, we get

$$\begin{aligned}
& \left\| P_2 v \right\|^2 + \left\| 2\theta \nabla \varphi \nabla v - \tilde{q} v \right\|^2 + 2 (\text{boundary terms}) \\
&+ 4\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 + 2\theta^3 \int \nabla \varphi \nabla ((\nabla \varphi)^2) (v)^2 \\
&+ 4\theta \int \sum_i \sum_{j \neq i} \partial_{ij} \varphi \partial_i v \partial_j v \\
&= \left\| e^{\theta \varphi} (\Delta - \tilde{\beta}) w \right\|^2 \\
&+ 2\theta^2 \int \operatorname{div} (\Delta \varphi \nabla \varphi) (v)^2 - 4\theta \int \nabla \varphi \tilde{q} v \nabla v - 2 (P_2 v, \tilde{q} v) \\
&+ 2\theta \int \sum_i \sum_{j \neq i} \partial_i^2 \varphi (\partial_j v)^2 + 2\theta \tilde{\beta} \int \Delta \varphi (v)^2 \\
&+ 2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 - 2\theta^3 \int \Delta \varphi (\nabla \varphi)^2 (v)^2 \\
&= \left\| e^{\theta \varphi} (\Delta - \tilde{\beta}) w \right\|^2 \\
&+ 2\theta^2 \int \operatorname{div} (\Delta \varphi \nabla \varphi) (v)^2 - 4\theta \int \nabla \varphi \tilde{q} v \nabla v - 2 (P_2 v, \tilde{q} v) \\
&+ 2\theta \tilde{\beta} \int \Delta \varphi (v)^2 + 2\theta \int \Delta \varphi (\nabla v)^2 - 2\theta^3 \int \Delta \varphi (\nabla \varphi)^2 (v)^2 ,
\end{aligned} \tag{A12}$$

as desired.

Proof of (A2) .- By integrations by parts and the fact that $v \in H^2(\Omega) \cap H_0^1(\Omega)$, it follows that

$$\begin{aligned}
& 2\theta \int \Delta\varphi \nabla v \nabla v \\
&= -2\theta \int \Delta\varphi v \Delta v - 2\theta \int v \nabla \Delta\varphi \nabla v \\
&= -2\theta \tilde{\beta} \int \Delta\varphi (v)^2 - 2\theta \int \Delta\varphi v \left(\Delta - \tilde{\beta} \right) v - 2\theta \int v \nabla \Delta\varphi \nabla v \\
&= -2\theta \tilde{\beta} \int \Delta\varphi (v)^2 - 2\theta \int \Delta\varphi v \left[P_2 v + \theta \Delta\varphi v - \theta^2 (\nabla\varphi)^2 v + \tilde{q}v \right] - 2\theta \int \nabla \Delta\varphi \frac{1}{2} \nabla (v^2)
\end{aligned} \tag{A13}$$

or equivalently

$$\begin{aligned}
& 2\theta \tilde{\beta} \int \Delta\varphi (v)^2 + 2\theta \int \Delta\varphi (\nabla v)^2 - 2\theta^3 \int \Delta\varphi (\nabla\varphi)^2 (v)^2 \\
&= -2\theta \int \Delta\varphi v [P_2 v] - 2\theta^2 \int (\Delta\varphi)^2 (v)^2 - 2\theta \int \Delta\varphi \tilde{q} (v)^2 + \theta \int \Delta^2\varphi (v)^2,
\end{aligned} \tag{A14}$$

as desired.

Proof of (A3) .- By (A8) and (A9) we can check that

$$\begin{aligned}
\text{boundary terms} &= -\theta \int \sum_i \partial_i \left(\partial_i \varphi (\partial_i v)^2 \right) \\
&\quad - 2\theta \int \sum_i \sum_{j \neq i} \partial_j \left(\partial_i \varphi \partial_i v \partial_j v \right) + \theta \int \sum_i \sum_{j \neq i} \partial_i \left(\partial_i \varphi (\partial_j v)^2 \right) \\
&\quad + \theta^2 \int \operatorname{div} \left((\Delta\varphi \nabla\varphi) (v)^2 \right) - \theta^3 \int \operatorname{div} \left((\nabla\varphi (\nabla\varphi)^2) (v)^2 \right) \\
&= -\theta \int_{\partial\Omega} \sum_i \partial_i \varphi (\partial_i v)^2 n_i \\
&\quad - 2\theta \int_{\partial\Omega} \sum_i \sum_{j \neq i} \partial_i \varphi \partial_i v \partial_j v n_j + \theta \int_{\partial\Omega} \sum_i \sum_{j \neq i} \partial_i \varphi (\partial_j v)^2 n_i \\
&\quad + \theta^2 \int_{\partial\Omega} (\Delta\varphi \nabla\varphi) (v)^2 \cdot n - \theta^3 \int_{\partial\Omega} \nabla\varphi (\nabla\varphi)^2 (v)^2 \cdot n.
\end{aligned} \tag{A15}$$

Since $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we deduce from (A15) that

$$\begin{aligned}
\text{boundary terms} &= -\theta \int_{\partial\Omega} \sum_i \partial_i \varphi (\partial_i v)^2 n_i \\
&\quad - 2\theta \int_{\partial\Omega} \sum_i \sum_{j \neq i} \partial_i \varphi \partial_i v \partial_j v n_j + \theta \int_{\partial\Omega} \sum_i \sum_{j \neq i} \partial_i \varphi (\partial_j v)^2 n_i \\
&= -2\theta \int_{\partial\Omega} \sum_i \partial_i \varphi (\partial_i v)^2 n_i - 2\theta \int_{\partial\Omega} \sum_i \sum_{j \neq i} \partial_i \varphi \partial_i v \partial_j v n_j \\
&\quad + \theta \int_{\partial\Omega} \sum_i \sum_{j \neq i} \partial_i \varphi (\partial_j v)^2 n_i + \theta \int_{\partial\Omega} \sum_i \partial_i \varphi (\partial_i v)^2 n_i \\
&= -2\theta \int_{\partial\Omega} \sum_i \sum_j \partial_i \varphi \partial_i v \partial_j v n_j + \theta \int_{\partial\Omega} \sum_i \sum_j \partial_i \varphi (\partial_j v)^2 n_i \\
&= -2\theta \int_{\partial\Omega} \nabla\varphi \nabla v \partial_n v + \theta \int_{\partial\Omega} (\nabla v)^2 \partial_n \varphi.
\end{aligned} \tag{A16}$$

Since $v|_{\partial\Omega} = 0$, $\nabla v = (\nabla v \cdot n) n$ and then we have

$$\begin{aligned}
\text{boundary terms} &= -2\theta \int_{\partial\Omega} (\partial_n v)^2 \partial_n \varphi + \theta \int_{\partial\Omega} (\nabla v)^2 \partial_n \varphi \\
&= -\theta \int_{\partial\Omega} (\partial_n v)^2 \partial_n \varphi.
\end{aligned} \tag{A17}$$

This completes the proof of (A3).

6.2 Choice of the weight function

We shall prove that with a particular choice of $\varphi = \varphi(x) \geq 1$ there exists a constant $C > 0$ such that for all $\tilde{\beta} \in \mathbb{R}$, for all $\tilde{q} \in L^\infty(\Omega)$, and $\theta > C \left(1 + |\tilde{\beta}|^{2/3} + \|\tilde{q}\|_{L^\infty(\Omega)}^{4/3} \right)$, the estimate

$$\begin{aligned}
& \left\| 2\theta \nabla\varphi \nabla (e^{\theta\varphi} w) - \tilde{q} e^{\theta\varphi} w \right\|^2 + \frac{1}{C} \left(\theta^3 \int_\Omega (e^{\theta\varphi} w)^2 + \theta \int_\Omega (\nabla (e^{\theta\varphi} w))^2 \right) \\
& \leq \left\| e^{\theta\varphi} (\Delta - \tilde{\beta}) w \right\|^2 + C \left(\theta^3 \int_{\omega_o} (e^{\theta\varphi} w)^2 + \theta \int_{\omega_o} (\nabla (e^{\theta\varphi} w))^2 \right)
\end{aligned} \tag{A18}$$

holds for all $w \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof of (A18) .- We study the particular case that $\varphi(x) = e^{\rho\psi(x)}$ where $\rho > 0$ and ψ is a smooth non-negative function. One can check that

$$\Delta\varphi = \rho^2 (\nabla\psi)^2 \varphi + \rho\Delta\psi\varphi \quad (\text{A19})$$

and

$$\begin{aligned} \partial_i\varphi\partial_{ij}^2\varphi &= \rho(\partial_i\varphi)^2\partial_j\psi + \partial_i\varphi\rho\partial_{ij}^2\psi\varphi \\ \partial_i\varphi\partial_j\varphi\partial_{ij}^2\varphi &= \rho^2(\partial_j\psi)^2(\partial_i\varphi)^2\varphi + \rho\partial_i\varphi\partial_j\varphi\partial_{ij}^2\psi\varphi. \end{aligned} \quad (\text{A20})$$

Next we compute $\nabla\varphi\nabla((\nabla\varphi)^2) = \sum_i\partial_i\varphi\partial_i(\nabla\varphi)^2$. Since

$$\begin{aligned} \partial_i\varphi\partial_i(\nabla\varphi)^2 &= 2\partial_i\varphi\nabla\varphi\nabla\partial_i\varphi = 2\partial_i^2\varphi(\partial_i\varphi)^2 + 2\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\varphi \\ &= 2\left[\rho^2(\partial_i\psi)^2\varphi + \rho\partial_i^2\psi\varphi\right](\partial_i\varphi)^2 \\ &\quad + 2\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\left[\rho^2\partial_i\psi\varphi\partial_j\psi + \rho\partial_{ij}^2\psi\varphi\right] \\ &= 2\left[\rho^2(\partial_i\psi)^2\varphi + \rho\partial_i^2\psi\varphi\right](\partial_i\varphi)^2 \\ &\quad + 2\partial_i\varphi\sum_{j\neq i}(\rho\partial_j\psi\varphi)(\rho\partial_i\psi\varphi)\rho\partial_j\psi + 2\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\rho\partial_{ij}^2\psi\varphi \\ &= 2\left[\rho^2(\partial_i\psi)^2\varphi + \rho\partial_i^2\psi\varphi\right](\partial_i\varphi)^2 \\ &\quad + 2\rho^2\partial_i\varphi\sum_{j\neq i}(\partial_j\psi)^2\varphi\partial_i\varphi + 2\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\rho\partial_{ij}^2\psi\varphi \\ &= 2\rho^2\sum_j(\partial_j\psi)^2\varphi(\partial_i\varphi)^2 \\ &\quad + 2\rho\left[\partial_i^2\psi\varphi(\partial_i\varphi)^2 + \partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\varphi\right], \end{aligned} \quad (\text{A21})$$

it follows that

$$\begin{aligned} \nabla\varphi\nabla((\nabla\varphi)^2) &= 2\rho^2(\nabla\psi)^2\varphi(\nabla\varphi)^2 \\ &\quad + 2\rho\sum_i\left[\partial_i^2\psi(\partial_i\varphi)^2 + \partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\right]\varphi. \end{aligned} \quad (\text{A22})$$

On the other hand, by (A1) and (A2), we infer

$$\begin{aligned} &\|P_2v\|^2 + \|2\theta\nabla\varphi\nabla v + \tilde{q}v\|^2 + 2(\text{boundary terms}) \\ &+ 4\theta\int\sum_i\partial_i^2\varphi(\partial_iv)^2 + 2\theta^3\int\nabla\varphi\nabla((\nabla\varphi)^2)(v)^2 \\ &+ 4\theta\int\sum_i\sum_{j\neq i}\partial_{ij}\varphi\partial_iv\partial_jv \\ &= \left\|e^{\theta\varphi}(\Delta - \tilde{\beta})w\right\|^2 + 2\theta^2\int\text{div}(\Delta\varphi\nabla\varphi)(v)^2 - 4\theta\int\nabla\varphi\tilde{q}v\nabla v - 2(P_2v, \tilde{q}v) \\ &- 2\theta\int\Delta\varphi v[P_2v] - 2\theta^2\int(\Delta\varphi)^2(v)^2 - 2\theta\int\Delta\varphi\tilde{q}(v)^2 + \theta\int\Delta^2\varphi(v)^2 \end{aligned} \quad (\text{A23})$$

which together with (A22) gives

$$\begin{aligned} &\|P_2v\|^2 + \|2\theta\nabla\varphi\nabla v - \tilde{q}v\|^2 + 2(\text{boundary terms}) \\ &+ 4\theta\rho^2\int\varphi\sum_i\left((\partial_i\psi)^2(\partial_iv)^2 + \sum_{j\neq i}\partial_i\psi\partial_j\psi\partial_iv\partial_jv\right) \\ &+ 4\theta\rho\int\varphi\sum_i\left(\partial_i^2\psi(\partial_iv)^2 + \sum_{j\neq i}\partial_{ij}^2\psi\partial_iv\partial_jv\right) \\ &+ 4\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi(v)^2 \\ &+ 4\theta^3\rho\int\sum_i\left[\partial_i^2\psi(\partial_i\varphi)^2 + \partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\right]\varphi(v)^2 \\ &= \left\|e^{\theta\varphi}(\Delta - \tilde{\beta})w\right\|^2 + 2\theta^2\int\text{div}(\Delta\varphi\nabla\varphi)(v)^2 - 4\theta\int\nabla\varphi\tilde{q}v\nabla v - 2(P_2v, \tilde{q}v) \\ &- 2\theta\int\Delta\varphi v[P_2v] - 2\theta^2\int(\Delta\varphi)^2(v)^2 - 2\theta\int\Delta\varphi\tilde{q}(v)^2 + \theta\int\Delta^2\varphi(v)^2 \end{aligned} \quad (\text{A24})$$

or equivalently

$$\begin{aligned} &\|P_2v\|^2 + \|2\theta\nabla\varphi\nabla v - \tilde{q}v\|^2 + 2(\text{boundary terms}) \\ &+ 4\theta\rho^2\int\varphi(\sum_i\partial_i\psi\partial_iv)^2 \\ &+ 4\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi(v)^2 \\ &= \left\|e^{\theta\varphi}(\Delta - \tilde{\beta})w\right\|^2 + 2\theta^2\int\text{div}(\Delta\varphi\nabla\varphi)(v)^2 - 4\theta\int\nabla\varphi\tilde{q}v\nabla v - 2(P_2v, \tilde{q}v) \\ &- 2\theta\int\Delta\varphi v[P_2v] - 2\theta^2\int(\Delta\varphi)^2(v)^2 - 2\theta\int\Delta\varphi\tilde{q}(v)^2 + \theta\int\Delta^2\varphi(v)^2 \\ &- 4\theta\rho\int\sum_i\varphi\left(\partial_i^2\psi(\partial_iv)^2 + \sum_{j\neq i}\partial_{ij}^2\psi\partial_iv\partial_jv\right) \\ &- 4\theta^3\rho\int\sum_i\left[\partial_i^2\psi(\partial_i\varphi)^2 + \partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\right]\varphi(v)^2. \end{aligned} \quad (\text{A25})$$

Next we compute the term $2\theta\rho^2 \int (\nabla\psi)^2 \varphi (\nabla v)^2$. To this end, we observe by (A2) that

$$\begin{aligned}
& 2\theta \int \left[\rho^2 (\nabla\psi)^2 \varphi + \rho \Delta\psi\varphi \right] (\nabla v)^2 = 2\theta \int \Delta\varphi (\nabla v)^2 \\
& = -2\theta\tilde{\beta} \int \Delta\varphi (v)^2 + 2\theta^3 \int \Delta\varphi (\nabla\varphi)^2 (v)^2 \\
& \quad - 2\theta \int \Delta\varphi v [P_2v] - 2\theta^2 \int (\Delta\varphi)^2 (v)^2 - 2\theta \int \Delta\varphi\tilde{q}(v)^2 + \theta \int \Delta^2\varphi (v)^2 \\
& = -2\theta\tilde{\beta} \int \Delta\varphi (v)^2 + 2\theta^3 \int \left[\rho^2 (\nabla\psi)^2 \varphi + \rho \Delta\psi\varphi \right] (\nabla\varphi)^2 (v)^2 \\
& \quad - 2\theta \int \Delta\varphi v [P_2v] - 2\theta^2 \int (\Delta\varphi)^2 (v)^2 - 2\theta \int \Delta\varphi\tilde{q}(v)^2 + \theta \int \Delta^2\varphi (v)^2
\end{aligned} \tag{A26}$$

which implies

$$\begin{aligned}
& 2\theta\rho^2 \int (\nabla\psi)^2 \varphi (\nabla v)^2 \\
& = -2\theta\rho \int \Delta\psi\varphi (\nabla v)^2 \\
& \quad + 2\theta^3\rho^2 \int (\nabla\psi)^2 (\nabla\varphi)^2 \varphi (v)^2 \\
& \quad + 2\theta^3\rho \int \Delta\psi (\nabla\varphi)^2 \varphi (v)^2 \\
& \quad - 2\theta \int \Delta\varphi v [P_2v] - 2\theta^2 \int (\Delta\varphi)^2 (v)^2 - 2\theta \int \Delta\varphi \left(\tilde{q} + \tilde{\beta} \right) (v)^2 + \theta \int \Delta^2\varphi (v)^2 .
\end{aligned} \tag{A27}$$

Finally, it follows from (A25) and (A27) that

$$\begin{aligned}
& \|P_2v\|^2 + \|2\theta\nabla\varphi\nabla v - \tilde{q}v\|^2 + 2(\text{boundary terms}) + 4\theta\rho^2 \int (\sum_i \partial_i\psi\partial_iv)^2 \varphi \\
& + 4\theta^3\rho^2 \int (\nabla\psi)^2 (\nabla\varphi)^2 \varphi (v)^2 \\
& + 2\theta\rho^2 \int (\nabla\psi)^2 \varphi (\nabla v)^2 \\
& = \left\| e^{\theta\varphi} \left(\Delta - \tilde{\beta} \right) w \right\|^2 + 2\theta^2 \int \text{div}(\Delta\varphi\nabla\varphi)(v)^2 - 4\theta \int \nabla\varphi\tilde{q}v\nabla v - 2(P_2v, \tilde{q}v) \\
& + 2 \left(-2\theta \int \Delta\varphi v [P_2v] - 2\theta^2 \int (\Delta\varphi)^2 (v)^2 - 2\theta \int \Delta\varphi \left(\tilde{q} + \frac{\tilde{\beta}}{2} \right) (v)^2 + \theta \int \Delta^2\varphi (v)^2 \right) \\
& - 4\theta\rho \int \sum_i \left(\partial_i^2\psi\varphi (\partial_iv)^2 + \sum_{j\neq i} \partial_{ij}^2\psi\varphi\partial_iv\partial_jv \right) \\
& - 4\theta^3\rho \int \sum_i \left[\partial_i^2\psi (\partial_i\varphi)^2 + \partial_i\varphi \sum_{j\neq i} \partial_j\varphi\partial_{ij}^2\psi \right] \varphi (v)^2 \\
& - 2\theta\rho \int \Delta\psi\varphi (\nabla v)^2 + 2\theta^3\rho^2 \int (\nabla\psi)^2 (\nabla\varphi)^2 \varphi (v)^2 + 2\theta^3\rho \int \Delta\psi (\nabla\varphi)^2 \varphi (v)^2
\end{aligned} \tag{A28}$$

or equivalently

$$\begin{aligned}
& \|P_2v\|^2 + \|2\theta\nabla\varphi\nabla v - \tilde{q}v\|^2 + 2(\text{boundary terms}) + 4\theta\rho^2 \int (\sum_i \partial_i\psi\partial_iv)^2 \varphi \\
& + 2\theta^3\rho^2 \int (\nabla\psi)^2 (\nabla\varphi)^2 \varphi (v)^2 + 2\theta\rho^2 \int (\nabla\psi)^2 \varphi (\nabla v)^2 \\
& = \left\| e^{\theta\varphi} \left(\Delta - \tilde{\beta} \right) w \right\|^2 + 2\theta^2 \int \text{div}(\Delta\varphi\nabla\varphi)(v)^2 - 4\theta \int \nabla\varphi\tilde{q}v\nabla v - 2(P_2v, \tilde{q}v) \\
& + 2 \left(-2\theta \int \Delta\varphi v [P_2v] - 2\theta^2 \int (\Delta\varphi)^2 (v)^2 - 2\theta \int \Delta\varphi \left(\tilde{q} + \frac{\tilde{\beta}}{2} \right) (v)^2 + \theta \int \Delta^2\varphi (v)^2 \right) \\
& - 4\theta\rho \int \sum_i \left(\partial_i^2\psi\varphi (\partial_iv)^2 + \sum_{j\neq i} \partial_{ij}^2\psi\varphi\partial_iv\partial_jv \right) \\
& - 4\theta^3\rho \int \sum_i \left[\partial_i^2\psi (\partial_i\varphi)^2 + \partial_i\varphi \sum_{j\neq i} \partial_j\varphi\partial_{ij}^2\psi \right] \varphi (v)^2 \\
& - 2\theta\rho \int \Delta\psi\varphi (\nabla v)^2 + 2\theta^3\rho \int \Delta\psi\varphi (\nabla\varphi)^2 (v)^2 .
\end{aligned} \tag{A29}$$

Now, we choose ψ such that $\psi \in C^2(\overline{\Omega})$, $\psi(x) > 0$ in Ω , $\psi(x) = 0$ on $\partial\Omega$ and

$$|\nabla\psi(x)| > 0 \quad \forall x \in \overline{\Omega \setminus \omega_o} , \tag{A30}$$

where $\omega_o \subset \omega$ is an arbitrary fixed subdomain of Ω such that $\overline{\omega_o} \subset \omega$. The existence of such function ψ was proved in [13]. Then the boundary terms (A3) are non-negative. More precisely, for all $\rho > 0$, $\theta > 0$,

$$\text{boundary terms} = -\theta \int_{\partial\Omega} (\partial_n v)^2 (\rho \partial_n \psi) \varphi \geq 0 . \tag{A31}$$

By (A30), (A31) and (A29), there exists a constant $C > 0$, such that for all $\rho > 0$ there is a constant

$C_\rho > 0$ such that for all $\theta > 0$, we have

$$\begin{aligned}
& \|P_2 v\|^2 + \|2\theta \nabla \varphi \nabla v - \tilde{q} v\|^2 \\
& + \frac{1}{C} \left(\theta^3 \rho^2 \int_{\Omega \setminus \omega_o} (\nabla \varphi)^2 \varphi(v)^2 + \theta \rho^2 \int_{\Omega \setminus \omega_o} \varphi(\nabla v)^2 \right) \\
\leq & \|P_2 v\|^2 + \|2\theta \nabla \varphi \nabla v - \tilde{q} v\|^2 + \left[2(\text{boundary terms}) + 4\theta \rho^2 \int (\sum_i \partial_i \psi \partial_i v)^2 \varphi \right] \\
& + 2\theta^3 \rho^2 \int (\nabla \psi)^2 (\nabla \varphi)^2 \varphi(v)^2 + 2\theta \rho^2 \int (\nabla \psi)^2 \varphi(\nabla v)^2 \\
= & \left\| e^{\theta \varphi} \left(\Delta - \tilde{\beta} \right) w \right\|^2 + 2\theta^2 \int \operatorname{div} (\Delta \varphi \nabla \varphi)(v)^2 - 4\theta \int \nabla \varphi \tilde{q} v \nabla v - 2(P_2 v, \tilde{q} v) \\
& + 2 \left(-2\theta \int \Delta \varphi v [P_2 v] - 2\theta^2 \int (\Delta \varphi)^2 (v)^2 - 2\theta \int \Delta \varphi \left(\tilde{q} + \frac{\tilde{\beta}}{2} \right) (v)^2 + \theta \int \Delta^2 \varphi (v)^2 \right) \\
& - 4\theta \rho \int \sum_i \left(\partial_i^2 \psi \varphi (\partial_i v)^2 + \sum_{j \neq i} \partial_{ij}^2 \psi \varphi \partial_i v \partial_j v \right) \\
& - 4\theta^3 \rho \int \sum_i \left[\partial_i^2 \psi (\partial_i \varphi)^2 + \partial_i \varphi \sum_{j \neq i} \partial_j \varphi \partial_{ij}^2 \psi \right] \varphi(v)^2 \\
& - 2\theta \rho \int \Delta \psi \varphi (\nabla v)^2 + 2\theta^3 \rho \int \Delta \psi \varphi (\nabla \varphi)^2 (v)^2 .
\end{aligned} \tag{A32}$$

Using Cauchy-Schwarz inequality in (A32), we get

$$\begin{aligned}
& \|P_2 v\|^2 + \|2\theta \nabla \varphi \nabla v - \tilde{q} v\|^2 \\
& + \frac{1}{C} \left(\theta^3 \rho^2 \int_{\Omega \setminus \omega_o} (\nabla \varphi)^2 \varphi(v)^2 + \theta \rho^2 \int_{\Omega \setminus \omega_o} \varphi(\nabla v)^2 \right) \\
\leq & \left\| e^{\theta \varphi} \left(\Delta - \tilde{\beta} \right) w \right\|^2 \\
& + \theta^2 C_\rho \int (v)^2 + \theta^2 \|\tilde{q}\|_{L^\infty}^2 C_\rho \int (v)^2 + \int (\nabla v)^2 + 2\|\tilde{q}\|_{L^\infty}^2 \int (v)^2 \\
& + \frac{1}{2} \|P_2 v\|^2 + \theta \left(\|\tilde{q}\|_{L^\infty} + |\tilde{\beta}| \right) C_\rho \int (v)^2 + \theta C_\rho \int (v)^2 \\
& + C\theta \rho \int \varphi(\nabla v)^2 \\
& + C\theta^3 \rho \int (\nabla \varphi)^2 \varphi(v)^2 \\
& + C\theta \rho \int \varphi(\nabla v)^2 + C\theta^3 \rho \int \varphi(\nabla \varphi)^2 (v)^2 .
\end{aligned} \tag{A33}$$

Finally, we obtain that, (recall that $\varphi \geq 1$) for all $\rho > 1$, for all $\theta > 1$,

$$\begin{aligned}
& \frac{1}{2} \|P_2 v\|^2 + \|2\theta \nabla \varphi \nabla v - \tilde{q} v\|^2 \\
& + \frac{1}{C} \rho^2 \left(\theta^3 \int_{\Omega \setminus \omega_o} (\nabla \varphi)^2 \varphi(v)^2 + \theta \int_{\Omega \setminus \omega_o} \varphi(\nabla v)^2 \right) \\
\leq & \left\| e^{\theta \varphi} \left(\Delta - \tilde{\beta} \right) w \right\|^2 \\
& + \theta^2 \left(1 + |\tilde{\beta}| + \|\tilde{q}\|_{L^\infty} + \|\tilde{q}\|_{L^\infty}^2 \right) C_\rho \int (v)^2 \\
& + C\rho \left(\theta^3 \int (\nabla \varphi)^2 \varphi(v)^2 + \theta \int \varphi(\nabla v)^2 \right) \\
\leq & \left\| e^{\theta \varphi} \left(\Delta - \tilde{\beta} \right) w \right\|^2 \\
& + 2\theta^2 \left(1 + |\tilde{\beta}| + \|\tilde{q}\|_{L^\infty}^2 \right) C_\rho \int (v)^2 \\
& + C\rho \left(\theta^3 \int_{\omega_o} (\nabla \varphi)^2 \varphi(v)^2 + \theta \int_{\omega_o} \varphi(\nabla v)^2 \right) \\
& + C\rho \left(\theta^3 \int_{\Omega \setminus \omega_o} (\nabla \varphi)^2 \varphi(v)^2 + \theta \int_{\Omega \setminus \omega_o} \varphi(\nabla v)^2 \right) ,
\end{aligned} \tag{A34}$$

where $C > 0$ is a constant independent of ρ and θ . Now we may choose and fix $\rho > 1$ large enough in (A34) to get

$$\begin{aligned}
& \|2\theta \nabla \varphi \nabla v - \tilde{q} v\|^2 + \frac{1}{C} \left(\theta^3 \int_{\Omega \setminus \omega_o} (\nabla \varphi)^2 \varphi(v)^2 + \theta \int_{\Omega \setminus \omega_o} \varphi(\nabla v)^2 \right) \\
\leq & \left\| e^{\theta \varphi} \left(\Delta - \tilde{\beta} \right) w \right\|^2 \\
& + \theta^2 \left(1 + |\tilde{\beta}| + \|\tilde{q}\|_{L^\infty}^2 \right) C \left(\int_{\Omega \setminus \omega_o} (v)^2 + \int_{\omega_o} (v)^2 \right) \\
& + C \left(\theta^3 \int_{\omega_o} (\nabla \varphi)^2 \varphi(v)^2 + \theta \int_{\omega_o} \varphi(\nabla v)^2 \right) ,
\end{aligned} \tag{A35}$$

where $C > 0$ is a constant independent of θ . Then by (A35), using properties of ψ , we may choose θ

large enough so that $\theta^{3/2} > C_1 \left(1 + |\tilde{\beta}| + \|\tilde{q}\|_{L^\infty(\Omega)}^2\right)$ to get

$$\begin{aligned} & \|2\theta\nabla\varphi\nabla v - \tilde{q}v\|^2 + \frac{1}{C_2} \left(\theta^3 \int_{\Omega \setminus \omega_o} (v)^2 + \theta \int_{\Omega \setminus \omega_o} (\nabla v)^2\right) \\ & \leq \left\| e^{\theta\varphi} (\Delta - \tilde{\beta}) w \right\|^2 + C_2 \left(\theta^3 \int_{\omega_o} (\nabla\varphi)^2 \varphi(v)^2 + \theta \int_{\omega_o} \varphi(\nabla v)^2\right), \end{aligned} \quad (\text{A36})$$

where C_1 and C_2 are two constants independent of θ .

Finally, we have proved that there exists a constant $C > 0$ such that for all $\tilde{\beta} \in \mathbb{R}$, $\tilde{q} \in L^\infty(\Omega)$ and $\theta > C \left(1 + |\tilde{\beta}|^{2/3} + \|\tilde{q}\|_{L^\infty(\Omega)}^{4/3}\right)$, the following estimate holds

$$\begin{aligned} & \|2\theta\nabla\varphi\nabla v - \tilde{q}v\|^2 + \frac{1}{C} \left(\theta^3 \int_{\Omega} (v)^2 + \theta \int_{\Omega} (\nabla v)^2\right) \\ & \leq \left\| e^{\theta\varphi} (\Delta - \tilde{\beta}) w \right\|^2 + C \left(\theta^3 \int_{\omega_o} (v)^2 + \theta \int_{\omega_o} (\nabla v)^2\right). \end{aligned} \quad (\text{A37})$$

Thus, coming back to the function w , we obtain that for all $\theta > C \left(1 + |\tilde{\beta}|^{2/3} + \|\tilde{q}\|_{L^\infty(\Omega)}^{4/3}\right)$,

$$\begin{aligned} & \|2\theta\nabla\varphi\nabla (e^{\theta\varphi}w) - \tilde{q}e^{\theta\varphi}w\|^2 + \frac{1}{C} \left(\theta^3 \int_{\Omega} (e^{\theta\varphi}w)^2 + \theta \int_{\Omega} (\nabla (e^{\theta\varphi}w))^2\right) \\ & \leq \left\| e^{\theta\varphi} (\Delta - \tilde{\beta}) w \right\|^2 + C \left(\theta^3 \int_{\omega_o} (e^{\theta\varphi}w)^2 + \theta \int_{\omega_o} (\nabla (e^{\theta\varphi}w))^2\right), \end{aligned} \quad (\text{A38})$$

which completes the proof of (A18).

Finally, we deduce from (A38),

$$\begin{aligned} & \|2\theta\nabla\varphi\nabla (e^{\theta\varphi}w) - \tilde{q}e^{\theta\varphi}w\|^2 + \frac{1}{C} \left(\theta^3 \int_{\Omega} (e^{\theta\varphi}w)^2 + \theta \int_{\Omega} (e^{\theta\varphi}\nabla w)^2\right) \\ & \leq \left\| e^{\theta\varphi} (\Delta - \tilde{\beta}) w \right\|^2 + C \left(\theta^3 \int_{\omega_o} (e^{\theta\varphi}w)^2 + \theta \int_{\omega_o} (e^{\theta\varphi}\nabla w)^2\right). \end{aligned} \quad (\text{A39})$$

which implies Theorem A.

This closes the Appendix on the Global Carleman inequality for the operator $\Delta - \tilde{\beta}$ where $\tilde{\beta} \in \mathbb{R}$.

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