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ON THE EXISTENCE OF TIME OPTIMAL CONTROLS FOR LINEAR EVOLUTION EQUATIONS

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Abstract. This work concerns with the existence of the time optimal controls for some linear evolution equations without the *a priori* assumption on the existence of admissible controls. Both global and local existence results are presented. Some necessary conditions, sufficient conditions, and necessary and sufficient conditions for the existence of time optimal controls are derived by establishing the relationship between controllability and time optimal control problems.

1. **Introduction.** Let H be a reflexive Banach space with norm $\|\cdot\|$, U another Banach space with norm $\|\cdot\|_U$, $-A$ the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ in H, with domain $D(A)$. Let $B(\cdot)\in L^\infty(\mathbf{R}^+;L(H))$ and $D\in L(U,H)$ (Here, $\mathbf{R}^+=(0,\infty)$, $L(H)$ and $L(U,H)$ denote respectively the space of all linear bounded operators on H and that of all linear bounded operators from U into H). In the sequel, we shall fix a $\rho>0$ and an $\varepsilon\geq 0$.

One of the most important infinite dimensional control problems is the time optimal driving of the solution $y\equiv y(t;y_0,u)$ of the following controlled system

$$y_t+Ay+B(t)y=Du(t),\ t>0 \tag{1}$$

from an initial point $y_0\in H$ into a target set $Q\subset H$, i.e.,

$$y(0;y_0,u)=y_0 \text{ and } y(T;y_0,u)\in Q, \tag{2}$$

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by means of (strongly measurable) controls $u(t) : \mathbf{R}^+ \to U$ with a maximum-norm bound

$$\|u(t)\|_U \le \rho \quad \text{a.e. in } \mathbf{R}^+. \tag{3}$$

We call it a time optimal control problem governed by equation (1) with initial data y_0, target set Q and control set $\mathcal{U}_\rho$ where

$$\mathcal{U}_\rho = \{u : \mathbf{R}^+ \to U \text{ measurable}; \ \ \|u(t)\|_U \le \rho, \text{ a.e. } t \in \mathbf{R}^+\}.$$

In this paper, we shall take the target set Q as the following:

$$Q = B_\varepsilon(y_1) \equiv \{\bar{y} \in H; \ \|\bar{y} - y_1\| \le \varepsilon\}, \tag{4}$$

where $y_1 \in H$ is arbitrary but fixed, and consider the time optimal control problem

$(\mathbf{P}_1)$ $\quad \mathrm{Min}\{T; \ y(T; y_0, u) \in Q, \ u \in \mathcal{U}_\rho\}.$

Note that problem $(\mathbf{P}_1)$ depends on the controlled system (1), the initial data y_0, the target set Q and the control set $\mathcal{U}_\rho$. When system (1), y_0, Q and $\mathcal{U}_\rho$ are given, a function $u \in \mathcal{U}_\rho$ is called an *admissible control* for $(\mathbf{P}_1)$ if there exists a number $T \ge 0$ such that $y(T; y_0, u) \in Q$; while a function $u^* \in \mathcal{U}_\rho$ is called a *time optimal control* for $(\mathbf{P}_1)$ if there exists a number $T^* \ge 0$ such that $y(T^*; y_0, u^*) \in Q$ and

$$T^* = \mathrm{Min}\{T; \ y(T; y_0, u) \in Q, \ u \in \mathcal{U}_\rho\}.$$

We call this T^* the *optimal time* for $(\mathbf{P}_1)$.

When system (1), Q and $\mathcal{U}_\rho$ are given, we say that problem $(\mathbf{P}_1)$ has *admissible controls (or time optimal controls) globally* if for each $y_0 \in H$, it has admissible controls (or time optimal controls, respectively). We say that problem $(\mathbf{P}_1)$ has *admissible controls (or time optimal controls) locally* if there exists $\delta > 0$ such that for each $y_0 \in H$ with $\mathrm{dist}(y_0, Q) \le \delta$, it has admissible controls (or time optimal controls, respectively). By *the global or local existence of admissible controls or time optimal controls* for $(\mathbf{P}_1)$, we mean that $(\mathbf{P}_1)$ has admissible controls or time optimal controls globally or locally.

There exist numerous results on the characterization of time optimal controls governed by evolution systems (cf. [1], [4], [5], [9], [17]) where the existence of admissible controls are *a priori* assumed, from which the existence of time optimal control follows in many cases. We remark that the study of the existence of admissible controls has independent interest. Indeed, the answer to this question would provide the foundation for the various optimal control problems which study after that what would be an optimal choice, in one or another sense, of such trajectory. However, to the best of our knowledge, the problems whether $(\mathbf{P}_1)$ has admissible controls globally or not and in which conditions $(\mathbf{P}_1)$ has admissible controls globally have not been investigated clearly so far, even for the "simple" target set Q given by (4). Indeed, in this paper we observe that problem $(\mathbf{P}_1)$ may have no admissible control globally, as shown by our analysis in Section 2.

In this article, we shall also consider a special but important case of $(\mathbf{P}_1)$ in which $Q = \{0\}$. More precisely we shall study the following time optimal control problem

$(\mathbf{P}_2)$ $\quad \mathrm{Min}\{T; \ y(T; y_0, u) = 0, u \in \mathcal{U}_\rho\}.$

We shall see that even in the case where system (1) is a controlled heat equation with some potential, problem $(\mathbf{P}_2)$ may have no admissible control globally. Thus, it is quite interesting to establish some sufficient conditions, necessary condition, and/or sufficient and necessary conditions for the global existence (or even local existence) of admissible controls for $(\mathbf{P}_2)$. This turns out to be the main topic in this paper. In this work, we obtain some sufficient conditions for the global existence of time optimal controls for problem $(\mathbf{P}_2)$ and show that such sufficient conditions are also necessary in some sense. Also, for some linear heat and wave equations with potentials we establish sufficient and necessary conditions for the global existence of time optimal controls.

It is notable that the above problems, $(\mathbf{P}_1)$ with any given $\varepsilon > 0$, $(\mathbf{P}_1)$ with $\varepsilon = 0$ and $(\mathbf{P}_2)$, are respectively closely related to the constrained approximate, the constrained exact and the constrained null controllability of evolution systems, i.e., some constraints are added to the control function for the corresponding controllability. In the present situation, system (1) is said to be constrained approximate (or exact) controllable under the constraint control (3) if for any $y_1 \in H$ problem $(\mathbf{P}_1)$ with any given $\varepsilon > 0$ (or $\varepsilon = 0$ respectively) has admissible controls globally; while system (1) is said to be constrained null controllable under the constraint control (3) if problem $(\mathbf{P}_2)$ has admissible controls globally. Although great progresses have been made in the field of controllability theory without constraint ([2, 6, 7, 11, 8, 19] and the references cited therein), as far as we know, the study of constrained controllability theory, even for the linear evolution systems, is not well-understood (We refer to [14] for an interesting constrained controllability result for the wave equations). Indeed, it is well known that, in the case without constraint, for most linear systems, the approximate controllability and null controllability may hold simultaneously. However, this is not true for the constrained case. We show in the present work that the constrained approximate controllability is impossible even for the evolution systems with very weak dissipation mechanism, while the constrained null controllability is still possible for the evolution systems with linearly growing energies with respect to time variable t.

This paper is organized as follows. In Section 2, we give some necessary conditions for the global existence of admissible controls for problem $(\mathbf{P}_1)$. In Section 3, we establish sufficient conditions for the global existence and local existence of time optimal controls for problem $(\mathbf{P}_2)$. In Section 4, we obtain necessary and sufficient conditions for the global existence of time optimal control problems for the linear heat and wave equations with potentials.

2. Necessary conditions for the existence of admissible controls for $(\mathbf{P}_1)$.

In this section we shall give some necessary conditions for the global existence of admissible controls for problem $(\mathbf{P}_1)$ where $Q = B_\varepsilon(y_1)$ with $y_1 \in H$ and $\varepsilon \geq 0$. We shall see that for some evolution system under framework of (1), problem $(\mathbf{P}_1)$ may have no admissible control globally.

Let H^* be the dual space of H with norm $\|\cdot\|_*$. We denote by $F : H \to H^*$ and $F^* : H^* \to H^{**} \equiv H$ the duality mappings respectively. Then for any $\psi \in H$, $\varphi^* \in H^*$ and $\psi^* \in F(\psi)$, $\varphi \in F^*(\varphi^*)$, we have

$$\|\psi^*\|_*^2 = \|\psi\|^2 = \langle \psi, \psi^* \rangle \text{ and } \|\varphi\|^2 = \|\varphi^*\|_*^2 = \langle \varphi, \varphi^* \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the paring between H and H^*.

Let $T > 0$ be arbitrary but fixed and consider the adjoint equation of equation (1) on $(0, T)$

$$\begin{cases} p_t - A^*p - B^*(t)p = 0 \quad \text{in } (0,T), \\ p(T) = p_1, \end{cases} \tag{5}$$

where A^* and $B^*(t)$ are the dual operators of A and $B(t)$ respectively and $p_1 \in H^*$. Let $\{G(t,s); 0 \le s \le t < \infty\}$ be the evolution system generated by $-A - B(\cdot)$ (cf. [15]) and $G^*(t,s)$ be the dual of $G(t,s)$. Then the solution p of equation (5) can be represented by

$$p(t) = G^*(T,t)p_1. \tag{6}$$

We assume that

$(\mathbf{H}_1)$ There exist a constant $\Lambda > 0$ and a $p_1 \in F(y_1)$ such that

$$\|G^*(T,0)p_1\|_* + \int_0^T \|G^*(T,s)p_1\|_* \, ds \le \Lambda \|p_1\|_*, \qquad \forall\, T > 0. \tag{7}$$

The main result in this section is as follows:

Theorem 2.1. *Let $(\mathbf{H}_1)$ hold for $y_1 \in H$. Suppose that there exist $y_0 \in H$ and a control $u \in \mathcal{U}_\rho$ such that $y(T; y_0, u) \in B_\varepsilon(y_1)$ for some $T > 0$. Then*

$$\|y_1\| \le \Lambda(\|y_0\| + \rho\|D\|_{L(U,H)}) + \varepsilon. \tag{8}$$

Proof. From

$$y(T) = G(T,0)y_0 + \int_0^T G(T,s)Du(s)\, ds,$$

where $y(T) \equiv y(T; y_0, u)$ and noting (7), we get

$$\begin{aligned}
\langle y(T), p_1\rangle &= \langle y_0, G^*(T,0)p_1\rangle + \int_0^T \langle Du(s), G^*(T,s)p_1\rangle\, ds \\
&\le \|y_0\| \|G^*(T,0)p_1\|_* + \rho\|D\|_{L(U,H)} \int_0^T \|G^*(T,s)p_1\|_*\, ds \\
&\le (\|y_0\| + \rho\|D\|_{L(U,H)}) \Big(\|G^*(T,0)p_1\|_* + \int_0^T \|G^*(T,s)p_1\|_*\, ds \Big) \\
&\le \Lambda(\|y_0\| + \rho\|D\|_{L(U,H)})\|p_1\|_* \\
&= \Lambda(\|y_0\| + \rho\|D\|_{L(U,H)})\|y_1\|.
\end{aligned} \tag{9}$$

On the other hand, since $y(T) \in B_\varepsilon(y_1)$ and $p_1 \in F(y_1)$ we have

$$\begin{aligned}
\langle y(T), p_1\rangle &= \langle y_1, p_1\rangle + \langle y(T) - y_1, p_1\rangle \\
&\ge \|y_1\|^2 - \varepsilon\|y_1\|,
\end{aligned}$$

which together with (9) implies (8). This completes the proof of Theorem 2.1. $\square$

Remark 1. Theorem 2.1 gives a necessary condition for the global existence of admissible controls for problem $(\mathbf{P}_1)$ with $Q = B_\varepsilon(y_1)$. Note that the constants Λ and $\|D\|_{L(U,H)}$ appeared in (8) of Theorem 2.1, are independent of T, y_1, y_0, ρ and ε. By this, we see that for fixed y_0, ρ and ε, in order to have $u \in \mathcal{U}_\rho$ and $T > 0$ such that $y(T; y_0, u) \in B_\varepsilon(y_1)$, y_1 can not be far from original point in H, i.e., $\|y_1\|$ can not be large enough. Thus we could at most expect the local existence for the admissible controls of problem $(\mathbf{P}_1)$.

The following is an example such that $(\mathbf{H}_1)$ holds.

Let $\Omega \subset \mathbf{R}^n$ ($n \geq 1$) be a bounded domain with C^2 boundary $\partial\Omega$, b be a real number, $\omega \subset \Omega$ be an open subset and χ_ω be the characteristic function of ω. Consider the following heat equation

$$\begin{cases} y_t - \Delta y - by = \chi_\omega u & \text{in } \Omega \times \mathbf{R}^+, \\ y = 0 & \text{on } \partial\Omega \times \mathbf{R}^+. \end{cases} \tag{10}$$

Let $\{\lambda_i\}_{i\geq 1}, \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, be the eigenvalues of $-\Delta$ with the Dirichlet boundary condition and $\{e_i\}_{i\geq 1}$ be the corresponding eigenfunctions. The adjoint equation of (10) on $(0,T)$ is as follows

$$\begin{cases} p_t + \Delta p + bp = 0 & \text{in } \Omega \times (0,T), \\ p = 0 & \text{on } \partial\Omega \times (0,T), \\ p(T) = p_1 & \text{in } \Omega. \end{cases} \tag{11}$$

In the case that $b < \lambda_1$, one can easily check that for all $p_1 \in L^2(\Omega)$, the solution p of equation (11) satisfies

$$\|p(0)\|_{L^2(\Omega)} + (\lambda_1 - b)\int_0^T \|p(t)\|_{L^2(\Omega)}\, dt \leq \|p_1\|_{L^2(\Omega)}.$$

Hence assumption $(\mathbf{H}_1)$ holds for all $y_1 \in L^2(\Omega)$ (Note that the duality mapping $F : L^2(\Omega) \to L^2(\Omega)$ is the identity mapping). In the case that $b \geq \lambda_1$, we observe first that there is a λ_N such that $b < \lambda_N$ since $\lambda_i \to \infty$ as $i \to \infty$. Hence $(\mathbf{H}_1)$ holds for all $y_1 \in \operatorname{span}\{e_j, j \geq N\}$. Indeed, for $p_1 = \sum_{j\geq N} \eta_j e_j$ with $\sum_{j\geq N} \eta_j^2 < \infty$, the solution p of equation (11) is given by

$$p(t) = \sum_{j\geq N} e^{-(\lambda_j - b)(T-t)}\eta_j e_j,$$

and satisfies (7).

Remark 2. Generally, Assumption $(\mathbf{H}_1)$ holds for any $y_1 \in H$ whenever there are two constants $C > 0$ and $r > 0$ so that

$$||G(t,0)||_{L(H)} \leq Ce^{-rt}, \qquad \forall\, t \geq 0,$$

i.e., $-A - B(\cdot)$ generates an exponentially decay evolution system.

The following is a weakened version of Assumption $(\mathbf{H}_1)$:

$(\mathbf{H}_1)'$ There exist a Banach subspace V of H^* with continuous embedding, a constant $\Lambda > 0$ and a $p_1 \in F(y_1) \cap V$, such that

$$\|G^*(T,0)p_1\|_* + \int_0^T \|G^*(T,s)p_1\|_*\, ds \leq \Lambda\|p_1\|_V, \qquad \forall\, T > 0.$$

Similar to the proof of Theorem 2.1, one can show that

Theorem 2.2. *Let $(\mathbf{H}_1)'$ hold for $y_1 \in H$. Suppose that there exist a control $u \in \mathcal{U}_\rho$ and an element $y_0 \in H$ such that $y(T; y_0, u) \in B_\varepsilon(y_1)$ for some $T > 0$. Then*

$$\|y_1\|^2 \leq \Lambda(\|y_0\| + \rho\|D\|_{L(U,H)})\|p_1\|_V + \varepsilon\|y_1\|. \tag{12}$$

By Theorem 2.2, one has the following consequences.

Corollary 1. *Let $(\mathbf{H}_1)'$ hold for some $y_1 \in H \setminus \{0\}$. Then problem $(\mathbf{P}_1)$ has no solution provided that the target Q is sufficiently away from 0, or in other words, system (1) is not constrained approximately controllable under the control constraint (3).*

Proof. It is easy to see that $my_1 \in H$, $m = 1, 2, \cdots$, satisfy $(\mathbf{H}_1)'$ (with the same constant Λ). Therefore, if problem $(\mathbf{P}_1)$ admits a solution for any given initial data y_0 and any target Q, then applying (12) in Theorem 2.2 to my_1, one finds

$$\|y_1\|^2 \le \frac{1}{m}\Big[\Lambda(\|y_0\| + \rho\|D\|_{L(U,H)})\|p_1\|_V + \varepsilon\|y_1\|\Big]. \tag{13}$$

Taking $m \to \infty$ in (13), we get $y_1 = 0$, which is a contradiction. This completes the proof of Corollary 1. □

Corollary 2. *Suppose that $B = 0$ and $-A^*$ generates a C_0-semigroup $S^*(t)$ in H^* with a polynomial decay rate, i.e., for some positive real number r and c,*

$$\|S^*(t)z_0^*\|_* \le \frac{c}{(1+t)^r}\|z_0^*\|_{D(A^*)}, \qquad \forall\, t > 0 \text{ and } z_0^* \in D(A^*). \tag{14}$$

Then problem $(\mathbf{P}_1)$ has no solution provided that the target Q is sufficiently away from 0.

Proof. In view of the semigroup property of $S^*(t)$, for $m = 1, 2, \cdots$, it follows from (14) that

$$\|S^*(mt)z_0^*\|_* \le \frac{c^m}{(1+t)^{mr}}\|z_0^*\|_{D((A^*)^m)}, \quad \forall\, t > 0,\ z_0^* \in D((A^*)^m). \tag{15}$$

We now fix a m so that $mr > 1$. Then, from (15) and using again the semigroup property of $S^*(t)$, it is easy to check that $(\mathbf{H}_1)'$ hold for any $z_0 \in F^*(z_0^*)$ with $z_0^* \in D((A^*)^m)$. Hence, the desired conclusion follows from Corollary 1. This completes the proof of Corollary 2. □

Remark 3. There are extensive studies on the polynomial decay rate for solutions to linear evolution equations. We refer to [12], [16] and the references cited therein for related results in this respect. Note however that, in order to have a polynomial decay rate for the underlying semigroup, generally, one only needs to introduce a weak dissipative mechanism into the system. Corollary 2 says that a weak dissipative mechanism of the system may lead that problem $(\mathbf{P}_1)$ has no solutions for "large" target Q. The following corollary shows that one has the same non-existence result for even weaker dissipative mechanism of the system.

Corollary 3. *Suppose that $B = 0$ and A^* has a pair of eigenvalue λ^* and eigenvector φ^* in H^* such that* $\mathrm{Re}\lambda^* < 0$. *Then problem $(\mathbf{P}_1)$ has no solution provided that the target Q is sufficiently away from* 0.

Proof. Fix a $\varphi \in F^*(\varphi^*)$. By $S^*(t)\varphi^* = e^{\lambda^* t}\varphi^*$, and noting $\mathrm{Re}\lambda^* < 0$, it is easy to see that φ satisfies $(\mathbf{H}_1)$. Therefore, Corollary 3 follows easily from Theorem 2.1 and the same argument as that in the proof of Corollary 1. □

Remark 4. It is well-known that A^* has at least one eigenvector whenever A^* has compact resolvent. Therefore, the assumptions on A^* in Corollary 3 hold if, further, $-A^*$ generates a strongly stable C_0-semigroup in H^*, i.e., for any $z \in H^*$, $S^*(t)z \to 0$ in H^* as $t \to \infty$. On the other hand, for most of systems with dissipative mechanism, it is not difficult to check the strong stability of the underlying semigroup. Note that Corollary 3 says that for this sort of systems, problem $(\mathbf{P}_1)$ has no solutions for "large" target Q, and constrained approximate controllability is impossible under the control constraint (3).

Remark 5. The solvability of problem ($\mathbf{P}_1$) is not clear when $-A-B(\cdot)$ generates an evolution system $G(t,s)$ in H such that $G^*(t,0)$ is unbounded at infinity, i.e., for some $z_0 \in H^*$,

$$\overline{\lim_{t\to\infty}}||G^*(t,0)z_0||_* = \infty.$$

3. Sufficient conditions for the existence of time optimal control for ($\mathbf{P}_2$).

3.1. **Global existence result.** In this subsection we shall give some sufficient conditions for the global existence of time optimal controls for problem ($\mathbf{P}_2$).

We need the following assumptions:

($\mathbf{H}_2$) There exists a time duration $T_0 > 0$ such that for each natural number j, there is a positive constant $C_j = C_j(T_0, ||B_j^*(\cdot)||_{L^\infty(0,T_0;L(H^*))})$ such that

$$\|p^j(0)\|_* \le C_j\|D^*p^j\|_{L^1(0,T_0;U^*)}, \quad \forall\, p_1 \in H^*, \tag{16}$$

where $B_j^*(\cdot) = B^*\big((j-1)T_0 + \cdot\big)$, p^j is the solution to the following adjoint equation

$$\begin{cases} -p_t^j + A^*p^j + B_j^*(t)p^j = 0 \quad \text{in } (0,T_0), \\ p^j(T_0) = p_1. \end{cases} \tag{17}$$

($\mathbf{H}_3$) For $M_j = \|G^*(jT_0,0)\|_{L(H^*)}$, $j = 1,2,\cdots$, and T_0 and C_j appeared in Assumption ($\mathbf{H}_2$), it holds that

$$\sum_{j=1}^{\infty}\frac{1}{M_jC_j} = \infty. \tag{18}$$

Remark 6. Assumption ($\mathbf{H}_2$) is a kind of observability inequality associated with system (1). On the one hand, we observe that the right hand side of (16) is L^1-norm in time while the usual observability inequality in the literature is for L^2-norm in time. We refer [6, 7, 8, 13, 19] and the rich references cited therein for recent progress on the observability estimates (of L^2-norm in time) for PDEs. However, for time-reversible system, inequality (16) can be implied by the exact observability estimate

$$\|p_1\|_* \le C_j{}'\|D^*p^j\|_{L^2(0,T_0;U^*)}, \quad \forall\, p_1 \in H^*.$$

Indeed, by the later inequality and the time reversibility of system (17) or more precisely

$$\frac{1}{C_j{}''}\|p_1\|_* \le \|p^j(0)\|_* \le C_j{}''\|p_1\|_*, \quad \forall\, p_1 \in H^*,$$

we get

$$\begin{aligned}\|p^j(0)\|_* &\le C_j{}'''\Big(\sup_{t\in[0,T_0]}\|D^*p^j(t)\|_{U^*}\int_0^{T_0}\|D^*p^j(t)\|_{U^*}\,dt\Big)^{\frac12} \\ &\le C_j{}''''\Big(\|p^j(0)\|_*\int_0^{T_0}\|D^*p^j(t)\|_{U^*}\,dt\Big)^{\frac12}\end{aligned}$$

which yields (16) (with a observability constant $(C_j{}'''')^2$) where $\|\cdot\|_{U^*}$ denotes the norm in U^*. We are not clear whether for all time-irreversible systems, inequality (16) can be implied by the observability estimate

$$\|p^j(0)\|_* \le C_j{}'\|D^*p^j\|_{L^2(0,T_0;U^*)}, \quad \forall\, p_1 \in H^*,$$

(we refer to [6] for a positive result for the heat equation with potential).

On the other hand, in some typical cases, the constants C_j, $j = 1, 2, \cdots$, in (16) are bounded by a uniform constant $C \equiv C\left(T_0, \|B^*\|_{L^\infty(\mathbf{R}^+;L(H^*))}\right)$, i.e., $C_j \le C$, $\forall\, j$.

Remark 7. Suppose that $C_j \le C$ $(\forall j)$, where C_j are the positive constants in Assumption $(\mathbf{H}_2)$. Then Assumption $(\mathbf{H}_3)$ holds whenever $\{G^*(t,s);\ 0 \le s \le t < \infty\}$ is uniformly bounded in $L(H^*)$, i.e., there exists a constant $M > 0$ such that

$$\|G^*(t,s)\|_{L(H^*)} \le M, \quad \text{for all } 0 \le s \le t < \infty.$$

This is the case if $-A$ generates a contractive C_0-semigroup in H and $B = 0$. More generally, Assumption $(\mathbf{H}_3)$ holds if $C_j \le C$ $(\forall j)$ and $\{G^*(t,s);\ 0 \le s \le t < \infty\}$ grows at most linearly in $L(H^*)$, i.e., there exists a constant $M > 0$ such that

$$\|G^*(t,s)\|_{L(H^*)} \le M(1 + t - s), \quad \text{for all } 0 \le s \le t < \infty.$$

We have the following global existence of admissible controls for problem $(\mathbf{P}_2)$.

Theorem 3.1. *Suppose that U is reflexive, $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$ hold. Then problem $(\mathbf{P}_2)$ has admissible controls globally, or in other words, system (1) is constrained null controllable under the control constraint (3).*

Proof. The proof is divided into three steps.

Step 1. We claim that $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$ imply a refined observability estimate for solutions p to

$$\begin{cases} -p_t + A^*p + B^*(t)p = 0 & \text{in } (0, NT_0), \\ p(NT_0) = p_1, \end{cases} \tag{19}$$

with a observability constant of order $\frac{1}{\sum_{j=1}^N \frac{1}{M_jC_j}}$, where M_j and C_j are given by $(\mathbf{H}_3)$ and $(\mathbf{H}_2)$ respectively, N is a positive integer to be determined later.

To show this, we rewrite the solution of (19) as $p\,(t) = G^*\,(NT_0, t)\,p_1$. First, for all $j = 1, \cdots, N$ and all $s \in [0, T_0]$, we put

$$q^j(s) = p((j-1)T_0 + s). \tag{20}$$

Then,

$$\begin{cases} -q^j_s + A^*q^j + B^*((j-1)T_0 + s)q^j = 0 & \text{in } (0, T_0), \\ q^j(T_0) = p(jT_0). \end{cases} \tag{21}$$

Applying Assumption $(\mathbf{H}_2)$ to (21), we deduce that

$$\|q^j(0)\|_* \le C_j \int_0^{T_0} \left\|D^*q^j(s)\right\|_{U^*} ds, \tag{22}$$

where $C_j \equiv C_j\left(T_0, \|B_j^*(\cdot)\|_{L^\infty(0,T_0;L(H^*))}\right)$. Now, by (20), changing variable in (22) and noting $p((j-1)T_0 + s) \equiv G^*\,(NT_0, (j-1)T_0 + s)\,p_1$, we arrive at

$$\|G^*(NT_0, (j-1)T_0)p_1\|_* \le C_j \int_{(j-1)T_0}^{jT_0} \|D^*G^*\,(NT_0, s)\,p_1\|_{U^*}\, ds. \tag{23}$$

On the other hand, Assumption $(\mathbf{H}_3)$ implies that

$$\|G^*(NT_0, 0)p_1\|_* \le M_j\, \|G^*(NT_0, (j-1)T_0)p_1\|_*\,. \tag{24}$$

Combining (23) and (24), we conclude that for all $p_1 \in H^*$ and integer $N > 0$, it holds

$$\|G^*\,(NT_0, 0)\,p_1\|_* \le \frac{1}{\sum_{j=1}^N \frac{1}{M_jC_j}} \int_0^{NT_0} \|D^*G^*\,(NT_0, s)\,p_1\|_{U^*}\, ds. \tag{25}$$

Clearly, (25) can be viewed as a refined observability estimate where the observability constant $\frac{1}{\sum_{j=1}^{N}\frac{1}{M_jC_j}} \to 0$ as $N \to +\infty$.

Step 2. We now deduce a controllability result for system (1) with an explicit estimate on the cost. The proof presented here was inspired by [1].

We claim that estimate (25) and the reflexivity of U imply the desired controllability result with a cost of order $\frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}}$.

Indeed, by (25), for all $p_1 \in H^*$ and integer $N > 0$, we have

$$\begin{aligned} -\left\langle G\left(NT_0,0\right)y_0,p_1\right\rangle &= -\left\langle y_0,G^*\left(NT_0,0\right)p_1\right\rangle \\ &\le \frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}}\int_0^{NT_0}\left\|D^*G^*\left(NT_0,s\right)p_1\right\|_{U^*}ds. \end{aligned} \tag{26}$$

On the other hand, by the reflexivity of U, one deduces that (cf. [3])

$$\left(L^1\left(0,NT_0;U^*\right)\right)^* = L^\infty\left(0,NT_0;U\right).$$

Therefore

$$\begin{aligned} &\frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}}\left\|D^*G^*(NT_0,\cdot)p_1\right\|_{L^1(0,NT_0;U^*)} \\ &= \sup\left\{\int_0^{NT_0}\left\langle u\left(s\right),D^*G^*(NT_0,s)p_1\right\rangle ds;\ \|u\|_{L^\infty(0,NT_0;U)} \le \frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}}\right\}. \end{aligned} \tag{27}$$

By (26) and (27), it follows that for all $p_1 \in H^*$ and integer $N > 0$,

$$\begin{aligned} &-\left\langle G\left(NT_0,0\right)y_0,p_1\right\rangle \\ &\le \sup\left\{\int_0^{NT_0}\left\langle G\left(NT_0,s\right)Du\left(s\right),p_1\right\rangle ds;\ \|u\|_{L^\infty(0,NT_0;U)} \le \frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}}\right\}. \end{aligned} \tag{28}$$

Let us introduce a subset $\mathcal{K}_N$ of H as follows

$$\mathcal{K}_N = \left\{\int_0^{NT_0}G\left(NT_0,s\right)Du\left(s\right)ds;\ \|u\|_{L^\infty(0,NT_0;U)} \le \frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}}\right\}.$$

It is easy to check that $\mathcal{K}_N$ is convex and closed in H. We claim that

$$-G(NT_0,0)y_0 \in \mathcal{K}_N\ . \tag{29}$$

To show this, we use contradiction argument. Suppose that $-G(NT_0,0)y_0 \notin \mathcal{K}_N$, then by Hahn-Banach theorem, there exists $\varphi^* \in H^*$ such that

$$\left\langle -G\left(NT_0,0\right)y_0,\varphi^*\right\rangle = 1,\ \text{and}\ \left\langle \zeta,\varphi^*\right\rangle = 0,\quad \forall\,\zeta \in \mathcal{K}_N.$$

By applying (28) with $p_1 = \varphi^*$, we arrive at a contradiction.

Now, (29) implies that for each natural number $N > 0$, there exists a function $u_N \in L^\infty\left(0,NT_0;U\right)$ such that

$$\|u_N\|_{L^\infty(0,NT_0;U)} \le \frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}} \tag{30}$$

and

$$G(NT_0,0)y_0 + \int_0^{NT_0}G(NT_0,s)Du_N(s)ds = 0. \tag{31}$$

Step 3. Note that, by $(\mathbf{H}_3)$, $\frac{\|y_0\|}{\sum_{j=1}^{N}\frac{1}{M_jC_j}} \to 0$ as $N \to +\infty$. It follows from (30) and (31) that for any given $y_0 \in H$, there exist a $N > 0$ and a control $u_N \in L^\infty(0, NT_0; U)$ such that

$$\|u_N\|_{L^\infty(0,NT_0;U)} \leq \rho$$

and

$$y(NT_0; y_0, u_N) = 0 .$$

This completes the proof of Theorem 3.1. □

As a consequence of Theorem 3.1, we may obtain sufficient conditions for the global existence of admissible controls for $(\mathbf{P}_1)$ where $Q = \{y_1\}$ with $y_1 \in H$ arbitrary but fixed.

We say that system (1) is time-reversible and well-posed if for any $T > 0$ and $y_1 \in H$, the system

$$\begin{cases} y_t + Ay + B(t)y = 0, & \text{in } (0,T), \\ y(T) = y_1 \end{cases}$$

has a unique solution $y(\cdot, y_1, T) \in C([0,T]; H)$, moreover, for each $T > 0$, there exists a positive constant C_T such that

$$\|y(t; y_1, T)\| \leq C_T\|y_1\| \quad \text{for all } \ y_1 \in H \text{ and } t \in [0,T].$$

Corollary 4. *Suppose that U is reflexive, $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$ hold. If system (1) is time-reversible and well-posed, then for any $y_1 \in H$, problem $(\mathbf{P}_1)$ with $Q = \{y_1\}$ has admissible controls globally, or in other words, system (1) is constrained exact controllable under the control constraint (3).*

Proof. By Theorem 3.1, we deduce that for any $y_0 \in H$ and $y_2 \in H$ there are a $T > 0$ and a control $u \in \mathcal{U}_\rho$ such that $y(T; y_0 - y_2, u) = 0$. By time reversibility, for any $y_1 \in H$ there is $y_3 \in H$ such $y(T; y_3, 0) = y_1$. Adding the two above solutions with $y_2 = y_3$, the desired result yields. This completes the proof of Corollary 4. □

Next we shall concern with the existence of time optimal controls for problem $(\mathbf{P}_2)$. The following lemma amounts to saying that the global (or local) existence of admissible controls for problem $(\mathbf{P}_1)$ implies the global (or local) existence of time optimal controls for problem $(\mathbf{P}_1)$.

Lemma 3.2. *If problem $(\mathbf{P}_1)$ has admissible controls globally (or locally), then it has time optimal controls globally (or locally, respectively).*

Proof. We shall only prove the global existence of time optimal controls. The local existence follows from the exact same way. Let $y_0 \in H$ be arbitrary but fixed. It follows from the global (or local) existence of admissible controls for problem $(\mathbf{P}_1)$ that there exist a non-increasing sequence $\{T_m\}_{m\geq 1} \subset \mathbf{R}^+$ and a sequence of controls $\{u_m\}_{m\geq 1} \subset \mathcal{U}_\rho$ such that $T_m \to T^*$, $u_m \to u^*$ weakly-star in $L^\infty(\mathbf{R}^+, U)$ and $y_m(T_m) \equiv y(T_m; y_0, u_m) \in Q$, where $T^* = \inf\{T;\ y(T; y_0, u) \in Q, u \in \mathcal{U}_\rho\}$.

We claim that $y_m(T_m) \to y^*(T^*)$ weakly in H, where $y^*(t) \equiv y(t; y_0, u^*)$. To this end, we prove first that

$$y_m(T^*) \to y^*(T^*) \quad \text{weakly in } H. \tag{32}$$

Indeed, for each $\varphi^* \in H^*$, we have

$$\langle y_m(T^*) - y^*(T^*), \varphi^* \rangle = \int_0^{T^*} \langle D(u_m(s) - u^*(s)), G^*(T, s)\varphi^* \rangle \, ds.$$

Since $D(u_m - u^*) \to 0$ weakly star in $L^\infty(0, T^*; H)$ it follows that

$$\langle y_m(T^*) - y^*(T^*), \varphi^* \rangle \to 0 \quad \text{as} \quad m \to \infty.$$

Next we shall prove that

$$y_m(T_m) - y_m(T^*) \to 0 \quad \text{weakly in } H. \tag{33}$$

Indeed, we have

$$\begin{aligned} y_m(T_m) - y_m(T^*) = \Big(G(T_m, 0) - G(T^*, 0)\Big) y_0 + \int_{T^*}^{T_m} G(T_m, s) D u_m(s)\, ds \\ + \int_0^{T^*} \Big(G(T_m, s) - G(T^*, s)\Big) D u_m(s)\, ds. \end{aligned} \tag{34}$$

Since $B(\cdot) \in L^\infty(\mathbf{R}^+, H)$, it follows by Gronwall's inequality that

$$\|G(t,s)\|_{L(H)} \le C(T) \quad \text{for all } 0 \le s \le t \le T, \tag{35}$$

where $T > T^*$ is an arbitrary but fixed number and $C(T)$ is a positive constant depending on T. From this, we obtain

$$\int_{T^*}^{T_m} G(T_m, s) D u_m(s)\, ds \to 0, \tag{36}$$

because $\|u_m\|_{L^\infty(\mathbf{R}^+; U)} \le \rho$.

By the continuity of $G(t,s)$, we get

$$\Big(G(T_m, 0) - G(T^*, 0)\Big) y_0 \to 0,$$

from which and by (34) and (36) we see that in order to show (33) it suffices to prove that

$$\int_0^{T^*} \Big(G(T_m, s) - G(T^*, s)\Big) D u_m(s)\, ds \to 0 \quad \text{weakly in } H.$$

To this end, we let $\varphi^* \in H^*$ and observe that for almost all $s \in (0, T^*)$,

$$\Big| \langle \big(G^*(T_m, s) - G^*(T^*, s)\big)\varphi^*, D u_m(s) \rangle \Big| \to 0,$$

because $\|u_m\|_{L^\infty(\mathbf{R}^+; U)} \le \rho$ and $G^*(T_m, s)\varphi^* \to G^*(T^*, s)\varphi^*$ as $m \to \infty$ for each $s \in (0, T^*)$. By (35) we see that

$$\Big| \langle \big(G^*(T_m, s) - G^*(T^*, s)\big)\varphi^*, D u_m(s) \rangle \Big| \le C \|\varphi\|_*,$$

where $C > 0$ is independent of m and $s \in (0, T^*)$. Then by Lebesgue dominated theorem, we obtain

$$\int_0^{T^*} \Big| \langle \big(G^*(T_m, s) - G^*(T^*, s)\big)\varphi^*, D u_m(s) \rangle \Big|\, ds \to 0$$

as desired.

Now it follows from (32) and (33) that $y_m(T_m) \to y^*(T^*)$ weakly in H. Since $y_m(T_m) \in Q$ and Q is a closed and convex subset in H, by Hahn-Banach theorem we get that $y^*(T^*) \in Q$. This completes the proof of Lemma 3.2. □

By Theorem 3.1 and Lemma 3.2 we have the following global existence of time optimal controls for problem $(\mathbf{P}_2)$.

Theorem 3.3. *Suppose that U is reflexive, $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$ hold. Then problem $(\mathbf{P}_2)$ has time optimal controls globally.*

As a consequence of Corollary 4 and Lemma 3.2 we have the following global existence of time optimal control for problem $(\mathbf{P}_1)$.

Theorem 3.4. *Suppose that U is reflexive, $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$ hold. If system (1) is time-reversible and well-posed, then for any $y_1 \in H$, problem $(\mathbf{P}_1)$ with $Q = \{y_1\}$ has time optimal controls globally.*

3.2. **Local existence result.** The following result concerns with the sufficient conditions for the local existence of time optimal controls for problem $(\mathbf{P}_2)$.

Theorem 3.5. *Suppose that U is reflexive and $(\mathbf{H}_2)$ holds. Then problem $(\mathbf{P}_2)$ has time optimal controls locally.*

Proof. By Lemma 3.2, it suffices to show that problem $(\mathbf{P}_2)$ has admissible controls locally. Fix $y_0 \in H$ arbitrarily, it suffices to show that the cost to transfer the state of system (1) from y_0 to 0 is of order $C||y_0||$. For this, following the same argument in the Step 2 in the proof of Theorem 3.1 (but using (16) directly rather than (25)), we deduce that there is a control $u \in L^\infty(0, T_0; U)$ such that $y(T_0; y_0, u) = 0$ and

$$||u||_{L^\infty(0,T_0;U)} \le C||y_0||.$$

Consequently, for all $y_0 \in H$ with $||y_0|| \le \frac{\rho}{C}$, the above u is an admissible control of problem $(\mathbf{P}_2)$. This completes the proof of Theorem 3.5. □

Remark 8. The local existence of time optimal controls governed by phase-field systems has been studied in [17] using another strategy to treat the $L^\infty(0, \infty; L^2(\Omega))$-norm for the control constraint.

3.3. **A counterexample.** In this subsection we shall show by an example that assumption $(\mathbf{H}_2)$ is not enough to guarantee the global existence of admissible controls for problem $(\mathbf{P}_2)$. Therefore assumption $(\mathbf{H}_3)$ is also necessary in some sense.

We give first the following assumption.

$(\mathbf{H}_4)$ $\quad B(\cdot) \equiv 0$ and there is an eigenvalue λ^* of A^* such that $\mathrm{Re}\lambda^* < 0$.

Remark 9. As we shall see in the next section, $(\mathbf{H}_2)$ and $(\mathbf{H}_4)$ may hold simultaneously in many situations. Note that, in these cases, $(\mathbf{H}_4)$ does break $(\mathbf{H}_3)$ (but not $(\mathbf{H}_2)$). Indeed, if $(\mathbf{H}_3)$ was also true, then, by Theorem 3.1, problem $(\mathbf{P}_2)$ would admit admissible controls globally, which contradicts the conclusion in Theorem 3.6 below.

Let $\varphi^* \in H^*$, with $\|\varphi^*\|_* = 1$, be an eigenvector of A^* corresponding to λ^*, i.e., $A^*\varphi^* = \lambda^*\varphi^*$. Let $\varphi \in F^*(\varphi^*)$, then $\langle \varphi, \varphi^* \rangle = \|\varphi\|^2 = \|\varphi^*\|_*^2 = 1$. We have the following negative result.

Theorem 3.6. *Suppose that $(\mathbf{H}_4)$ holds, and $y_0 = \mu\varphi$ with*

$$\mu \ge \frac{\rho\|D\|_{L(U,H)}}{-\mathrm{Re}\lambda^*}.$$

Then for any $T > 0$, there is no $u \in \mathcal{U}_\rho$ such that the corresponding solution $y(\cdot; y_0, u)$ of system (1) with initial data y_0 satisfies $y(T; y_0, u) = 0$.

Proof. By contradiction, we assume that there were $T > 0$ and $u \in \mathcal{U}_\rho$ such that $y(T; y_0, u) = 0$. Recalling that we have assumed $B(\cdot) \equiv 0$ in this case, one has

$$0 = y(T) = S(T)(\mu\varphi) + \int_0^T S(T-s)Du(s)\,ds. \tag{37}$$

Let $p(t) = e^{\lambda^* t}\varphi^*$, then $p(t)$ solves

$$\begin{cases} p_t - A^* p = 0 \quad \text{in } (0,T), \\ p(0) = \varphi^*, \quad p(T) = e^{\lambda^* T}\varphi^*. \end{cases} \tag{38}$$

By (37) and (38) we get

$$\begin{aligned} 0 = \langle y(T), p(T)\rangle &= \langle \mu\varphi, S^*(T)p(T)\rangle + \int_0^T \langle Du(s), S^*(T-s)p(T)\rangle\, ds \\ &= \mu\langle \varphi, \varphi^*\rangle + \int_0^T \langle Du(s), p(s)\rangle\, ds. \end{aligned}$$

Since $\|\varphi^*\|_*^2 = \langle \varphi, \varphi^*\rangle = 1$, $u \in \mathcal{U}_\rho$ and $\mathrm{Re}\lambda^* < 0$, the latter implies

$$\mu \le \rho \|D\|_{L(U,H)} \int_0^T e^{s\mathrm{Re}\lambda^*}\, ds < \frac{\rho\|D\|_{L(U,H)}}{-\mathrm{Re}\lambda^*},$$

which contradicts our assumption. This completes the proof of Theorem 3.6. □

4. **Necessary and sufficient conditions for the existence of time optimal controls of some typical systems.** In this section we shall study problems $(\mathbf{P}_1)$ and $(\mathbf{P}_2)$ for some typical controlled systems like the controlled heat equations and controlled wave equations with a real number potential. We shall give necessary and sufficient conditions for the global existence of time optimal controls. Let $\Omega \subset \mathbf{R}^n$ $(n \ge 1)$ be a bounded domain with C^2 boundary $\partial\Omega$, $\omega \subset \Omega$ be an open subset and χ_ω be the characteristic function of ω. Let b be a real number. Throughout this section, we denote by $\{\lambda_j\}_{j\ge1}$, $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and by $\{e_j\}_{j\ge1}$ the corresponding normalized eigenfunctions, i.e., $\|e_j\|_{L^2(\Omega)} = 1$.

4.1. **The heat equation with potential.** We begin with the following time optimal control problem associated with the controlled heat equation

$$\begin{cases} y_t - \Delta y - by = \chi_\omega u & \text{in } \Omega \times (0,+\infty), \\ y = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{39}$$

where $u = u(x,t)$ is the control and $y_0 \in L^2(\Omega)$ is the initial data.

$(\mathbf{P}_3)$ Min $\left\{T;\ y(T; y_0, u) = 0, \|u\|_{L^\infty(\mathbf{R}^+; L^2(\Omega))} \le \rho\right\}$,
where $y(\cdot; y_0, u)$ is the solution of (39).

We recall first some result about null controllability for the parabolic equation (39). It follows from [6, Proposition 3.2] that for each $b \in \mathbf{R}$ there exists a constant $C \equiv C(\Omega, \omega, b) > 0$ such that

$$\|p(\cdot,0)\|_{L^2(\Omega)}^2 \le e^{C\left(T+\frac{1}{T}\right)} \left(\int_0^T \int_\omega |p(x,t)|\, dxdt\right)^2, \tag{40}$$

for any $p_1 \in L^2(\Omega)$ and $T > 0$, where $p = p(x,t)$ is solution to the adjoint equation of (39) in $\Omega \times (0,T)$

$$\begin{cases} p_t + \Delta p + bp = 0 & \text{in } \Omega \times (0,T), \\ p = 0 & \text{on } \partial\Omega \times (0,T), \\ p(\cdot,T) = p_1 & \text{in } \Omega. \end{cases}$$

Now we give a necessary and sufficient condition for the global existence of time optimal controls for $(\mathbf{P}_3)$.

Theorem 4.1. *Problem* $(\mathbf{P}_3)$ *has time optimal controls globally if and only if* $b \leq \lambda_1$.

Proof. Let $H = U = L^2(\Omega)$, $B = 0$, $A = -\Delta - bI$, where I is the identity operator on $L^2(\Omega)$, with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $D : L^2(\Omega) \to L^2(\Omega)$ be defined by $Dv = \chi_\omega v$ for each $v \in L^2(\Omega)$. Then problem $(\mathbf{P}_3)$ can be studied in the framework of $(\mathbf{P}_2)$.

We observe first that due to estimate (40), assumption $(\mathbf{H}_2)$ holds for all $b \in \mathbf{R}$.

If $b > \lambda_1$, it follows from Theorem 3.6 that problem $(\mathbf{P}_3)$ has no time optimal control.

If $b \leq \lambda_1$, then one can easily check that $(\mathbf{H}_3)$ holds. Thus the global existence of time optimal controls for $(\mathbf{P}_3)$ is a consequence of Theorem 3.3. This completes the proof of Theorem 4.1. □

Remark 10. The global existence of time optimal controls governed by some semilinear parabolic equations where the nonlinear term is monotone was obtained in [18] via the combination of feedback stabilization and local null controllability.

4.2. **The wave equation with potential.** We now consider the controlled wave equation

$$\begin{cases} w_{tt} - \Delta w - bw = \chi_\omega f & \text{in } \Omega \times (0, +\infty), \\ w = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ (w(\cdot, 0), w_t(\cdot, 0)) = (w_0, w_1) & \text{in } \Omega, \end{cases} \tag{41}$$

where $f = f(x,t)$ is the control and $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$ is the initial data.

The time optimal control problem for system (41) is formulated as follows. Let $(w_2, w_3) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$(\mathbf{P}_4) \qquad \min \left\{ T; (w(\cdot, T), w_t(\cdot, T)) = (w_2, w_3),\ \|f\|_{L^\infty(\mathbf{R}^+; L^2(\Omega))} \leq \rho \right\},$$

where $w = w(x,t)$ is the solution of (41).

First, recall that the approximate and exact controllability properties for the hyperbolic equation (41) are only true for time $T > 0$ large enough because of the finite speed of propagation of the waves. Further, we need to introduce some kind of geometric condition on ω because of the trapped ray. To this end, we introduce some definitions and notations [11]. Fix $x_o \in \mathbf{R}^n$, we define

$$\Gamma_o = \{x \in \partial\Omega; (x - x_o) \cdot \nu(x) > 0\},$$

where $\nu(x)$ is the unit outward normal vector of Ω at $x \in \partial\Omega \in C^2$. For any $\eta > 0$, the η-neighborhood of Γ_o in $\mathbf{R}^n$ is denoted by $\mathcal{O}_\eta(\Gamma_o)$. On another hand, following [2], we say that ω satisfies the geometric control condition when $\partial\Omega$ is C^∞, $\partial\Omega$ has no contacts of infinite order with its tangents and further any generalized bicharacteristic ray of $\partial_{tt} - \Delta$ meets $\omega \times (0, T)$ for some $T > 0$.

Now, it is well-known (see [11], [2]) that if $\omega = \mathcal{O}_\eta(\Gamma_o) \cap \Omega$ for some $\eta > 0$ or that ω satisfies the geometric control condition, then for all $b \in \mathbf{R}$ the system (41) is exactly controllable and we have the following exact observability estimate. There exist $T > 0$ and $C > 0$ such that for any $(\phi_0, \phi_1) \in L^2(\Omega \times H^{-1}(\Omega)$,

$$\|(\phi_0, \phi_1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega |\phi(x,t)|^2 \, dxdt , \tag{42}$$

where $\phi = \phi(x,t)$ is solution to

$$\begin{cases} \phi_{tt} - \Delta\phi - b\phi = 0 & \text{in } \Omega \times (0,T)\ , \\ \phi = 0 & \text{on } \partial\Omega \times (0,T)\ , \\ (\phi(\cdot,0), \phi_t(\cdot,0)) = (\phi_0, \phi_1) & \text{in } \Omega\ . \end{cases} \tag{43}$$

The main result in this subsection is as follows.

Theorem 4.2. *Suppose that $\omega = \mathcal{O}_\eta(\Gamma_o) \cap \Omega$ for some $\eta > 0$ or that ω satisfies the geometric control condition. Then problem $(\mathbf{P}_4)$ has time optimal controls globally if and only if $b \le \lambda_1$.*

Proof. First notice that the system (41) can be written in our abstract framework (1) when $y_0 = (w_0, w_1)$, $H = H_0^1(\Omega) \times L^2(\Omega, U = \big(L^2(\Omega)\big)^2$, $A = \begin{pmatrix} 0 & -I \\ -\Delta - b & 0 \end{pmatrix}$, $B = 0$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & \chi_\omega I \end{pmatrix}$. Then problem $(\mathbf{P}_4)$ can be studied in the framework of $(\mathbf{P}_1)$ with $Q = \{y_1 = (w_2, w_3)\}$.

Note that the adjoint system of (41) is similar to (43) by the time-reversibility of the wave equation. Clearly, $(\mathbf{H}_2)$ follows from Remark 6 and (42). Recall that $B = 0$. Therefore, in order to apply Theorem 3.5 to show that problem $(\mathbf{P}_4)$ has time optimal controls, it suffices to prove that $(\mathbf{H}_3)$ holds if $b \le \lambda_1$.

To end of this, consider the wave equation (43) where $\phi_0 = \sum_{j=1}^\infty a_j e_j$ and $\phi_1 = \sum_{j=1}^\infty d_j e_j$ such that

$$\|\phi_1\|^2_{H^{-1}(\Omega)} = \sum_{j=1}^\infty \frac{1}{\lambda_j} d_j^2 < \infty \quad \text{and} \quad \|\phi_0\|^2_{L^2(\Omega)} = \sum_{j=1}^\infty a_j^2 < \infty.$$

One can then check that

$$\phi(\cdot,t) = \sum_{j=1}^\infty \Big[a_j \cos(t\sqrt{\lambda_j - b}) + \frac{d_j}{\sqrt{\lambda_j - b}} \sin(t\sqrt{\lambda_j - b})\Big] e_j.$$

Here, in the critical case $b = \lambda_1$, we agree that $\frac{\sin(t0)}{0} = t$.

If $b \le \lambda_1$, then we conclude that there is a constant $C > 0$ such that for all $(\phi_0, \phi_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and all $t \ge 0$, it holds that

$$\begin{aligned} &\|\phi(\cdot,t)\|^2_{L^2(\Omega)} + \|\phi_t(\cdot,t)\|^2_{H^{-1}(\Omega)} \\ &= \textstyle\sum_{j=1}^\infty \Big(a_j \cos(t\sqrt{\lambda_j - b}) + \frac{d_j}{\sqrt{\lambda_j - b}} \sin(t\sqrt{\lambda_j - b})\Big)^2 \\ &\quad + \textstyle\sum_{j=1}^\infty \frac{1}{\lambda_j} \Big(-a_j\sqrt{\lambda_j - b}\sin(t\sqrt{\lambda_j - b}) + d_j \cos(t\sqrt{\lambda_j - b})\Big)^2 \\ &\le C(1+t)^2 \textstyle\sum_{j=1}^\infty \Big[a_j^2 + \frac{1}{\lambda_j} d_j^2\Big] = C(1+t)^2 \Big[\|\phi_0\|^2_{L^2(\Omega)} + \|\phi_1\|^2_{H^{-1}(\Omega)}\Big], \end{aligned}$$

from which, and noting Remark 7, we see that $(\mathbf{H}_3)$ holds when $b \le \lambda_1$.

Conversely, suppose that $(\mathbf{P}_4)$ has time optimal controls, then we want to show that $b \le \lambda_1$. By contradiction, we assume that $b > \lambda_1$. The existence of time optimal controls implies that for all $w_1 \in H^{-1}(\Omega)$, there exists a control $f \in L^\infty(\mathbf{R}^+; L^2(\Omega))$ with the constraint

$$\|f(\cdot,t)\|_{L^2(\Omega)} \le \rho, \qquad \text{a.e. in } \mathbf{R}^+,$$

and there is a time $T > 0$ such that the solution (w, w_t) of (41) satisfies $w(\cdot,0) = w(\cdot,T) = w_t(\cdot,T) = 0$.

Let $\phi(\cdot,t)=e^{-t\sqrt{b-\lambda_1}}e_1$, then ϕ solves

$$\begin{cases} \phi_{tt}-\Delta\phi-b\phi=0 & \text{in } \Omega\times(0,+\infty), \\ \phi=0 & \text{on } \partial\Omega\times(0,+\infty)\ , \\ \phi(\cdot,0)=e_1, \quad \phi_t(\cdot,0)=-\sqrt{b-\lambda_1}e_1 & \text{in } \Omega, \end{cases}$$

Multiplying ϕ by (41) and integrating it in $\Omega\times(0,T)$, we get

$$-\int_\Omega w_1e_1dx=\int_0^T\int_\omega e^{-t\sqrt{b-\lambda_1}}e_1(x)f(x,t)dxdt.$$

Hence, for all $w_1\in H^{-1}(\Omega)$, there exists a time $T>0$, such that

$$\begin{aligned} -\int_\Omega w_1e_1dx &\le \int_0^T e^{-t\sqrt{b-\lambda_1}}\left\|e_1\right\|_{L^2(\Omega)}\left\|f(\cdot,t)\right\|_{L^2(\Omega)}dt \\ &\le \frac{\rho}{-\sqrt{b-\lambda_1}}\left(e^{-T\sqrt{b-\lambda_1}}-1\right). \end{aligned}$$

We have a contradiction by choosing $w_1=-\frac{\rho}{\sqrt{b-\lambda_1}}e_1$. This completes the proof of Theorem 4.2. □

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