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J. Differential Equations 240 (2007) 92–124
Journal of Differential Equations
www.elsevier.com/locate/jde

Polynomial decay rate for the dissipative wave equation ☆

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Received 24 January 2006; revised 11 March 2007

Available online 2 June 2007

Abstract

Using Fourier integral operators with special amplitude functions, we analyze the stabilization of the wave equation in a three-dimensional bounded domain on which exists a trapped ray bouncing up and down infinitely between two parallel parts of the boundary.

MSC: 35L05; 35B40; 93B07

Keywords: Wave equation; Stabilization; Trapped ray

1. Introduction and main result

This paper is devoted to the stabilization of the wave equation with a localized linear dissipation in a three-dimensional bounded domain on which exists a trapped ray. More precisely, we consider the following dissipative wave equation

$$\begin{cases} \partial_t^2 w - \Delta w + \alpha(x)\partial_t w = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ w = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \big(w(\cdot,0), \partial_t w(\cdot,0)\big) = (w_0, w_1) & \text{in } \Omega, \end{cases} \tag{1.1}$$

☆ This work was supported by the NSF of China under grant 10525105, the NCET of China under grant NCET-04-0882 and China Postdoctoral Science Foundation.
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0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.05.016

where Ω is a bounded domain in $\mathbb{R}^3$ with a boundary $\partial\Omega$ at least Lipschitz. Here, α is a non-negative function in $L^\infty(\Omega)$ and depends on a non-empty proper subset ω of Ω on which $1/\alpha \in L^\infty(\omega)$ (in particular, $\{x \in \Omega;\ \alpha(x) > 0\}$ is a non-empty open set). For non-identically zero initial data $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$, we denote the energy of the solution w of (1.1) at time t by

$$\mathcal{E}(w,t) = \int_\Omega \left(\left|\partial_t w(x,t)\right|^2 + \left|\nabla w(x,t)\right|^2\right) dx.$$

It is well known that $\mathcal{E}(w,t)$ is a continuous decreasing function of time and we have the following formula for any $0 \leqslant t_0 < t_1$,

$$\mathcal{E}(w,t_1) = \mathcal{E}(w,t_0) - 2\int_{t_0}^{t_1}\int_\Omega \alpha(x)\left|\partial_t w(x,t)\right|^2 dx\,dt. \tag{1.2}$$

Our main purpose is to investigate the energy decay rates for the damped wave equation (1.1). Among decay rates results for this equation, there are only two classes of such. First, C. Bardos, G. Lebeau and J. Rauch [3] introduced a geometric control condition saying that "any generalized bicharacteristic ray of $\partial_t^2 - \Delta$ parametrized by $t \in (0, T_c)$ meets $\omega \times (0, T_c)$ for some $T_c > 0$," which implies the uniform exponential decay rate of the energy (it is also required $\partial\Omega$ sufficiently smooth in order to deal with grazing contact on the boundary). Such a geometric control condition is almost necessary and establishes a link between Ω and ω. The second kind of results is due to G. Lebeau [9] and N. Burq [5, Remarque 4.1]) who established from resolvent estimates, a logarithmic decay rate for the dissipative wave equation without any assumption on the localization of the subdomain ω, but when more regularity on the initial data is assumed (and when α and $\partial\Omega$ are sufficiently smooth).

Our work is between these two categories. We will construct a geometry (Ω, ω) with a trapped ray (the geometric control condition is then not fulfilled) and establish a polynomial decay rate (therefore better that the logarithmic one) when $(w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

Now we begin by presenting precisely the geometry of our problem. Next, we present our main result. From now, $(w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

1.1. The geometry and the trapped ray

Let us introduce the geometry on which we work in this paper and explain why there is a trapped ray.

First, we set $D(r_1, r_2) = \{(x_1, x_2) \in \mathbb{R}^2;\ |x_1| < r_1,\ |x_2| < r_2\}$ where $r_1, r_2 > 0$. Next, let $m_1, m_2, \rho > 0$. We choose Ω a connected open set in $\mathbb{R}^3$ bounded by Γ_1, Γ_2, Υ where

$\Gamma_1 = \overline{D(m_1, m_2)} \times \{\rho\}$, with boundary $\partial\Gamma_1$,
$\Gamma_2 = \overline{D(m_1, m_2)} \times \{-\rho\}$, with boundary $\partial\Gamma_2$,
Υ is a surface with boundary $\partial\Upsilon = \partial\Gamma_1 \cup \partial\Gamma_2$.

Therefore, the boundary of Ω is $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Upsilon$. Further, we suppose that either $\partial\Omega$ is C^2 with $\Upsilon \subset (\mathbb{R}^2 \setminus D(m_1, m_2)) \times \mathbb{R}$ (in particular $\Upsilon \in C^2$) or Ω is convex (in particular Υ is Lipschitz).

Finally, we choose $\omega = \Omega \cap \Theta$ where Θ is a small neighborhood of Υ in $\mathbb{R}^3$ such that $\Theta \cap D(M_1, M_2) \times [-\rho, \rho] = \emptyset$ for some $M_1 \in (0, m_1)$ and $M_2 \in (0, m_2)$.

Now, recall that the bicharacteristics associated to $\partial_t^2 - \Delta$ in the whole space are curves in the space–time variables and their Fourier variables described by

$$\begin{cases} x(s) = x_o + 2\xi(s)s, \\ t(s) = t_o - 2\tau(s)s, \end{cases} \quad \text{and} \quad \begin{cases} \xi(s) = \xi_o, \\ \tau(s) = \tau_o, \end{cases}$$

with $|\xi(s)|^2 - \tau^2(s) = 0$ for $s \in [0, +\infty)$, when $(x_o, t_o, \xi_o, \tau_o) \in \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{0\}$. The rays associated to $\partial_t^2 - \Delta$ in the whole space are the projection of the bicharacteristics on the space–time domain. In particular, for $s \in [0, +\infty)$,

$$\begin{cases} x(s) - x_o - 2\xi_o s = 0, \\ t(s) + 2\tau_o s = 0, \\ |\xi_o|^2 - \tau_o^2 = 0, \end{cases} \tag{1.3}$$

here, $t_o = 0$ and $\tau_o \neq 0$. The definition of generalized bicharacteristic is given in [14] and under the assumption saying that the boundary is C^∞ and has no infinite order of contact with its tangents, a generalized bicharacteristic is uniquely determined by any one of its points (see also [4]).

The fact that our boundary $\partial\Omega$ (and more precisely Υ) is Lipschitz or of class C^2 will not create difficulties. Indeed, in our geometry, we define the ray starting at $x_o \in \Omega$ with direction $\xi_o \in \mathbb{S}^2$ (i.e., $|\xi_o| = 1$) by a continuous curve $x(s)$ parametrized by s satisfying the following rules: it is the solution of (1.3) with initial data $x(0) = x_o$ for $s \in [0, s_o]$ until it hits the boundary $\partial\Omega$ at $x(s_o)$; if for some $s_1 > 0$, $x(s_1) \in \overline{\Upsilon}$, the parametrization of the curve $x(s)$ stops; if for some $s_1 > 0$, $x(s_1) \in (\Gamma_1 \cup \Gamma_2) \setminus (\partial\Gamma_1 \cup \partial\Gamma_2)$, the curve $x(s)$ is reflected like a billiard ball following the rule of geometric optics "angle of incidence = angle of reflection" until it hits the boundary $\partial\Omega$ at $x(s_2)$ for some $s_2 > s_1$.

The key geometric observation in our setting is that any $(x_o, \xi_o) \in D(M_1, M_2) \times (-\rho/2, \rho/2) \times \{(0, 0, \pm 1)\}$ generates a trapped ray: a ray starting at x_o with direction ξ_o such that it will never go outside $\overline{D(M_1, M_2)} \times [-\rho, \rho]$. Consequently, thanks to an adequate Gaussian beam construction (see [15]), we cannot have a uniform exponential decay rate for the dissipative wave equation for any damping α only acting in ω (see [3]). In our geometry and with our assumption on α, a type of logarithmic decay rate still holds due to a logarithmic dependence inequality for the wave equation (reproducing the ideas and some techniques in [10,16] for the internal case). Now, we may hope a better decay rate because our trapped ray behaves quite simply by bouncing between Γ_1 and Γ_2 always in the same direction $\xi_o = (0, 0, \pm 1)$.

1.2. Main result

Now we are ready to formulate our main result. From now, we will only consider the geometry (Ω, ω) described in the previous subsection. Recall that the real function $\alpha \in L^\infty(\Omega)$ satisfies $\alpha \geqslant 0$ and $1/\alpha \in L^\infty(\omega)$.

Theorem. *There exist $C > 0$ and $\delta > 0$ such that for any $t > 0$ and any initial data $(w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the solution w of* (1.1) *satisfies*

$$\int_\Omega \left(\left|\partial_t w(x,t)\right|^2 + \left|\nabla w(x,t)\right|^2\right) dx \leqslant \frac{C}{t^\delta} \left\|(w_0, w_1)\right\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)}.$$

Remark 1. The polynomial decay rate for the damped wave equation holds in particular for the following two choices of a non-negative real function α: $\alpha > 0$ a.e. on X where X is a neighborhood of Υ in $\mathbb{R}^3$; $\alpha \in C(\overline{\Omega})$ such that $\alpha > 0$ on $\overline{\Upsilon}$. Indeed, with such a choice of α, we can choose ω as in the previous subsection.

Remark 2. In a two-dimensional square domain, the polynomial decay rate for the damped wave equation was obtained by Z. Liu and B. Rao [13]. It was generalized recently by N. Burq and M. Hitrik [6] for partially rectangular planar domain by using resolvent estimates. In one dimension, a sharp polynomial decay rate was established by X. Zhang and E. Zuazua [18] for a wave–heat coupled system where the dissipation acts thought the heat equation on a proper subdomain.

Our strategy to get such a polynomial decay rate will consist to establish a kind of observability estimate for the wave equation. More precisely, we have the following result.

Proposition. *The following two statements are equivalent.*

(i) *There exist $C > 0$ and $\delta > 0$ such that for any non-identically zero initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the solution u of the wave equation*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ \big(u(\cdot,0), \partial_t u(\cdot,0)\big) = (u_0, u_1) & \text{in } \Omega \end{cases} \tag{1.4}$$

satisfies

$$\left\|(u_0, u_1)\right\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leqslant C \int_0^{C\Lambda^{1/\delta}} \int_\Omega \alpha(x)\left(\left|\partial_t u(x,t)\right|^2 + \left|u(x,t)\right|^2\right) dx\, dt, \tag{1.5}$$

where $\Lambda = \dfrac{\|(u_0,u_1)\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)}}{\|(u_0,u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)}}$.

(ii) *There exist $C > 0$ and $\delta > 0$ such that for any $t > 0$ and any non-identically zero initial data $(w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the solution w of* (1.1) *satisfies*

$$\mathcal{E}(w,t) \leqslant \frac{C}{t^\delta} \left\|(w_0, w_1)\right\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)}.$$

Clearly, the proof of Theorem is now reduced to the one of Proposition and of the inequality (1.5). It is worthwhile to see that (1.5) looks like an observability estimate where the time of

observability depends on the quantity Λ which can be seen as a measure of the frequency of the initial data (u_0, u_1). Notice also that by easy minimization techniques, (1.5) is equivalent to the existence of $C > 0$ such that for any $h > 0$ sufficiently small, we have

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant C \int_0^{C(1/h)^{1/\delta}} \int_\Omega \alpha(x)\big(\big|\partial_t u(x,t)\big|^2 + \big|u(x,t)\big|^2\big)\,dx\,dt + h\|(u_0, u_1)\|^2_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)},$$

for any u, solution of (1.4) with initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Let us briefly say some words on the proof of (1.5). We will divide the proof of (1.5) into two steps: first, we will choose ω_o an adequate subset of $D(m_1, m_2) \times (-\frac{\rho}{4}, \frac{\rho}{4})$ such that any ray starting at any $x_o \in \Omega$ with any direction $\xi_o \in \mathbb{S}^2$ will meet a suitable compact set in $\omega_o \cup \omega$. It will imply an observability estimate with $\omega_o \cup \omega \times (0, T)$ being the domain of observation for some $T > 0$. Next, since $1/\alpha \in L^\infty(\omega)$, we only need to establish a kind of Hölder interpolation estimate traducing the fact that if $u = 0$ on $\omega \times (0, C(1/h)^{1/\delta})$ for any $h > 0$ sufficiently small, then $u = 0$ on $\omega_o \cup \omega \times (0, T)$. By a classical trace inequality valid for any solution of the wave equation with homogeneous Dirichlet boundary condition, in the kind of Hölder interpolation estimate searched the term $u|_{\omega\times(0,C(1/h)^{1/\delta})}$ will be replaced by $\partial_\nu u|_{\Upsilon\times(-C(1/h)^{1/\delta},C(1/h)^{1/\delta})}$ (see Theorem 1 in Section 4) where ν is the unit-outer normal to $\partial\Omega$. Let us now give some ideas on the proof of Theorem 1. We have to pay more attention on a ray bouncing up and down infinitely between the two parallel planes Γ_1 and Γ_2, because in some sense we would like to see how a solution u localized on a trapped ray could be bounded by $\partial_\nu u|_{\Upsilon\times(-C(1/h)^{1/\delta},C(1/h)^{1/\delta})}$. In same time, we need to estimate in a good way the dissipation phenomena in order to improve the logarithmic decay rate. To this end, we will apply some simple tools usually used for the study of the propagation of singularities for hyperbolic equations (see e.g. [1,17]). We will also have to link the s variable of the bicharacteristic flow and the number of reflections of the ray between Γ_1 and Γ_2 . On the other hand, we will work with the operator $i\partial_s + h(\Delta - \partial_t^2)$. Observe that the product of four, mono-dimensional, solutions of the Schrödinger equations $i\partial_s \pm h\partial^2$ can create a solution of $i\partial_s + h(\Delta - \partial_t^2)a(x,t,s) = 0$. The dispersive property of the linear Schrödinger equations $i\partial_s \pm h\partial^2$ on the s variable will be exploited.

The next section describes the proof of Proposition. Section 3 focus on the inequality (1.5). In Section 4, we establish of a kind of Hölder interpolation estimate. We close this paper with two appendices devoted to prove some technical results.

Throughout this paper, c denotes a positive constant which only may depend on (m_1, m_2, ρ). And γ will denote an absolute constant larger than one. The value of $c > 0$ and $\gamma > 1$ may change from line to line.

2. Proof of Proposition

The proof of Proposition uses many classical techniques for hyperbolic systems (see e.g. [11]) as a decomposition argument in order to deal with the wave equation with a second member and as a useful transformation for deriving estimates with weaker norms from a stronger game of norms.

We begin to prove (ii) $\Rightarrow$ (i). First, we let $(u_0, u_1) = (w_0, w_1)$. Next, we combine the polynomial decay rate for $\mathcal{E}(w,t)$ and the formula (1.2) applied with $t_0 = 0$ and $t_1 = t$, in order to get by choosing

$$t = \left(\frac{2C\|(u_0,u_1)\|^2_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)}}{\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}} \right)^{1/\delta},$$

the following inequality

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant 4 \int_0^{(2C\Lambda)^{1/\delta}} \int_\Omega \alpha(x)\left|\partial_t w(x,t)\right|^2 dx\,dt.$$

Since $(u - w)$ solves a damped wave equation with a second member $\alpha\partial_t u$ and with identically zero initial data, we conclude that

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant 16 \int_0^{(2C\Lambda)^{1/\delta}} \int_\Omega \alpha(x)\left|\partial_t u(x,t)\right|^2 dx\,dt \tag{2.1}$$

holds with the same δ as the one of the statement (ii).

Now, let us give the proof of (i) $\Rightarrow$ (ii). We divide the proof into three steps. In the first step, we begin to let $(w_0, w_1) = (u_0, u_1)$. Using a similar decomposition argument than above, we first get from (i) the following estimate for the solution w of (1.1)

$$\mathcal{E}(w,0) \leqslant 2C\big(1 + c\|\alpha\|_{L^\infty(\Omega)}C^2\Lambda^{2/\delta}\big) \int_0^{C\Lambda^{1/\delta}} \int_\Omega \alpha(x)\big(\left|\partial_t w(x,t)\right|^2 + \left|w(x,t)\right|^2\big)\,dx\,dt,$$

which holds for any non-identically zero initial data $(w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. In the second step, we apply the previous inequality to $\partial_t\widetilde{w}$ where $\widetilde{w}$ is a solution of (1.1) with non-identically zero initial data $(\widetilde{w}(\cdot,0), \partial_t\widetilde{w}(\cdot,0)) = (\widetilde{w}_0, \widetilde{w}_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ satisfying the compatibility condition $\Delta\widetilde{w}_0 - \alpha\widetilde{w}_1 \in H_0^1(\Omega)$. Noticing that $\mathcal{E}(\widetilde{w},0) \leqslant c(1 + \|\alpha\|_{L^\infty(\Omega)})\mathcal{E}(\partial_t\widetilde{w},0)$, we obtain for some $c_1 > 0$ depending on $(\Omega,\omega,\alpha,\delta)$ the following inequality

$$\mathcal{E}(\widetilde{w},0) + \mathcal{E}(\partial_t\widetilde{w},0) \leqslant c_1\widetilde{\Lambda}^{2/\delta} \int_0^{c_1\widetilde{\Lambda}^{1/\delta}} \int_\Omega \alpha(x)\big(\left|\partial_t^2\widetilde{w}(x,t)\right|^2 + \left|\partial_t\widetilde{w}(x,t)\right|^2\big)\,dx\,dt,$$

with

$$\widetilde{\Lambda} = \frac{\mathcal{E}(\partial_t\widetilde{w},0) + \|(\widetilde{w}_1, \Delta\widetilde{w}_0 - \alpha\widetilde{w}_1)\|^2_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)}}{\mathcal{E}(\widetilde{w},0) + \mathcal{E}(\partial_t\widetilde{w},0)}.$$

By a translation on the time variable, it implies after using the formula (1.2) that for some $c_1 > 0$ depending on $(\Omega, \omega, \alpha, \delta)$ the following inequality holds

$$\mathcal{H}(s) \leqslant c_1 \left(\frac{1}{\mathcal{H}(s)}\right)^{2/\delta} \left[\mathcal{H}(s) - \mathcal{H}\left(c_1 \left(\frac{1}{\mathcal{H}(s)}\right)^{1/\delta} + s\right)\right] \quad \forall s \geqslant 0,$$

where

$$\mathcal{H}(s) = \sigma \frac{\mathcal{E}(\widetilde{w}, s) + \mathcal{E}(\partial_t \widetilde{w}, s)}{\mathcal{E}(\partial_t \widetilde{w}, 0) + \|(\widetilde{w}_1, \Delta \widetilde{w}_0 - \alpha \widetilde{w}_1)\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)}}$$

and $\sigma > 0$ is a constant only depending on (Ω, ω, α) and taken in order that $\mathcal{H}$ is bounded by one. Applying Lemma B in Appendix B, we get that there are $C > 0$ and $\delta > 0$, such that

$$\mathcal{E}(\partial_t \widetilde{w}, t) \leqslant \frac{C}{t^\delta} \left(\mathcal{E}(\partial_t \widetilde{w}, 0) + \left\|(\widetilde{w}_1, \Delta \widetilde{w}_0 - \alpha \widetilde{w}_1)\right\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)}\right) \quad \forall t > 0.$$

In the last step, we can use a well-known transformation in order to deduce from the previous step the desired statement (ii). Indeed, we apply the previous inequality to

$$\widetilde{w}(\cdot, t) = \int_0^t w(\cdot, \ell)\, d\ell - (-\Delta)^{-1}(w_1 + \alpha w_0) \quad \text{in } \Omega.$$

This completes the proof of Proposition.

Remark 3. We believe that the kind of Hölder interpolation estimate (2.1) may help to give an approach to solve an open problem mentioned by J.-L. Lions [12] concerning control problems for small perturbations of the domain Ω. Here, for example, for $\varepsilon > 0$ sufficiently small, one may consider Ω_ε a connected open set in $\mathbb{R}^3$ bounded by $\Gamma_{1,\varepsilon}$, Γ_2, Υ_ε where

$$\Gamma_{1,\varepsilon} = \left\{(x_1, x_2, x_3) \in \overline{D(m_1, m_2)} \times \mathbb{R};\ x_3 = \rho - \varepsilon(x_1 + m_1) - \varepsilon(x_2 + m_2)\right\},$$

and Υ_ε is a surface with boundary $\partial \Upsilon_\varepsilon = \partial \Gamma_{1,\varepsilon} \cup \partial \Gamma_2$. For such a domain Ω_ε, any ray with length of order $1/\varepsilon$ escapes $\overline{\Omega_\varepsilon}$ through ω. Such a geometrical observation implies an observability estimate with a time of observability of order $1/\varepsilon$ for the wave equation in Ω_ε with homogeneous Dirichlet boundary.

From now, we will only work with the wave equation u solution of (1.4) with initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

3. The wave equation and proof of (1.5)

In this section, we first introduce some notations linked to our geometry and used for the rest of this paper. Next, we recall some well-known results on the wave equation on Ω (Ω being of class C^2 or convex). Finally, we give the proof of (1.5).

3.1. The parameter r_o and the set ω_o

Let us introduce some useful notations, sets and a parameter r_o linked to our geometry.

Since Θ is a small neighborhood of Υ in $\mathbb{R}^3$ such that $\Theta \cap D(M_1, M_2) \times [-\rho, \rho] = \emptyset$ for some $M_1 \in (0, m_1)$ and $M_2 \in (0, m_2)$, there exists a positive real number $r_o < \min(m_1 - M_1, m_2 - M_2, \rho)/2$ such that $D(m_1, m_2) \setminus D(m_1 - 2r_o, m_2 - 2r_o) \times (\rho - 2r_o, \rho + 2r_o) \cup (-\rho - 2r_o, -\rho + 2r_o) \subset \Theta$. Now, let us introduce

$$\omega_o = D(m_1 - r_o, m_2 - r_o) \times \left(-\frac{\rho}{4}, \frac{\rho}{4}\right). \tag{3.1}$$

Notice that the hypothesis saying that $\Upsilon \subset (\mathbb{R}^2 \setminus D(m_1, m_2)) \times \mathbb{R}$ or Ω is convex, implies that $C_0^\infty(B(x_o, r_o/2)) \subset C_0^\infty(\Omega)$ for any $x_o \in \overline{\omega_o}$, where $B(x_o, r)$ denotes the ball of center x_o and radius r.

Finally, we will denote $D_\Theta = D(m_1, m_2) \setminus D(m_1 - 2r_o, m_2 - 2r_o)$.

3.2. Some useful inequalities for u

Let u be a solution of (1.4). First, for initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, then $u \in C(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}; H_0^1(\Omega)) \cap C^2(\mathbb{R}; L^2(\Omega))$. Next, denote by $\{\mu_j\}_{j \geqslant 1}$, $0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$, the eigenvalues of $-\Delta$ on Ω with Dirichlet boundary conditions and by $\{\ell_j\}_{j \geqslant 1}$ the corresponding normalized eigenfunctions, i.e., $\|\ell_j\|_{L^2(\Omega)} = 1$. Then,

$$u(x,t) = \sum_{j \geqslant 1} \left(\frac{b_j^0}{2} + \frac{b_j^1}{2i\sqrt{\mu_j}} \right) e^{it\sqrt{\mu_j}} \ell_j(x) + \sum_{j \geqslant 1} \left(\frac{b_j^0}{2} - \frac{b_j^1}{2i\sqrt{\mu_j}} \right) e^{-it\sqrt{\mu_j}} \ell_j(x), \tag{3.2}$$

where the sequences $(b_j^0)_{j \geqslant 1}$ and $(b_j^1)_{j \geqslant 1}$ are such that $\sum_{j \geqslant 1} \mu_j^2 |b_j^0|^2 < +\infty$ and $\sum_{j \geqslant 1} \mu_j |b_j^1|^2 < +\infty$.

Now, let us denote the energy of u at time t by

$$\mathcal{G}(u,t) = \int_\Omega \left(\left|\partial_t u(x,t)\right|^2 + \left|\nabla u(x,t)\right|^2 \right) dx.$$

It is well known that for any $t \in \mathbb{R}$,

$$\mathcal{G}(u,t) = \mathcal{G}(u,0), \tag{3.3}$$

$$\left\|(u, \partial_t u)(\cdot, t)\right\|_{L^2(\Omega) \times H^{-1}(\Omega)} = \left\|(u_0, u_1)\right\|_{L^2(\Omega) \times H^{-1}(\Omega)}. \tag{3.4}$$

On the other hand, utilizing localized energy estimates, we get for any domains Θ_1, Θ_2 in $\mathbb{R}^3$ such that $\Theta_1 \Subset \Theta_2$, the existence of $c > 0$ such that

$$\|\nabla u\|^2_{L^2((\Omega \cap \Theta_1) \times (-T_o, T_o))} \leqslant c\left(\|\partial_t u\|^2_{L^2((\Omega \cap \Theta_2) \times (-2T_o, 2T_o))} + \|u\|^2_{L^2((\Omega \cap \Theta_2) \times (-2T_o, 2T_o))} \right), \tag{3.5}$$

for any $T_o > 0$.

Finally, we have the following estimate for the normal derivative of u on the boundary Υ (see [11, p. 43]). There exists $c > 0$ such that for any $T_o > 0$,

$$\|\partial_\nu u\|_{L^2(\Upsilon\times(-T_o,T_o))} \leqslant c\big(\|\partial_t u\|_{L^2(\omega\times(-2T_o,2T_o))} + \|u\|_{L^2(\omega\times(-2T_o,2T_o))}\big). \tag{3.6}$$

Remark also that the multiplier techniques (which is also valid for convex domain [11, pp. 31, 43]) applied with the Laplacian allows to get the existence of $c > 0$ such that the following classical trace theorem holds. For any $t \in \mathbb{R}$,

$$\big\|\partial_\nu u(\cdot,t)\big\|_{L^2(\partial\Omega)} \leqslant c\big\|\Delta u(\cdot,t)\big\|_{L^2(\Omega)}.$$

Consequently, we obtain from (3.3) applied to $\partial_t u$, the existence of $c > 0$ such that for any $t \in \mathbb{R}$,

$$\frac{1}{c}\big\|\partial_\nu u(\cdot,t)\big\|^2_{L^2(\partial\Omega)} \leqslant \mathcal{G}(\partial_t u, 0) := \big\|(u_0,u_1)\big\|^2_{H^2\cap H^1_0(\Omega)\times H^1_0(\Omega)}. \tag{3.7}$$

3.3. Proof of (1.5)

We decompose the proof of (1.5) into two steps. In the first step, we will establish an observability estimate. In the second step, we will use a kind of Hölder interpolation estimate which will be proved in the next section.

Step 1. By applying the classical multiplier method [11, pp. 40–43] to the solution ψu where $\psi \in C_0^\infty(D(m_1,m_2)\times(0,2\rho))$ such that $0 \leqslant \psi \leqslant 1$, $\psi = 1$ on $D(m_1-2r_o, m_2-2r_o)\times(\rho/4,\rho)$, we obtain that for any $T_o > 0$,

$$\begin{aligned}&\int_0^{T_o}\int_{D(m_1-2r_o,m_2-2r_o)}\int_{\rho/4}^{\rho}\big(\big|\partial_t u(x,t)\big|^2 + \big|\nabla u(x,t)\big|^2\big)\,dx\,dt\\ &\quad\leqslant c\big(\mathcal{G}(u,0) + \|u\|^2_{H^1(D_\Theta\times(0,\rho)\times(0,T_o))} + \|u\|^2_{H^1(D(m_1,m_2)\times(0,\rho/4)\times(0,T_o))}\big).\end{aligned} \tag{3.8}$$

Here we used the fact that ψu solves a wave equation in the cubic domain $D(m_1,m_2)\times(0,\rho)$ with a second member supported in $D_\Theta\times(0,\rho)\cup D(m_1,m_2)\times(0,\rho/4)$ and with adequate boundary conditions. Reproducing similar arguments, we also have for any $T_o > 0$,

$$\begin{aligned}&\int_0^{T_o}\int_{D(m_1-2r_o,m_2-2r_o)}\int_{-\rho}^{-\rho/4}\big(\big|\partial_t u(x,t)\big|^2 + \big|\nabla u(x,t)\big|^2\big)\,dx\,dt\\ &\quad\leqslant c\big(\mathcal{G}(u,0) + \|u\|^2_{H^1(D_\Theta\times(-\rho,0)\times(0,T_o))} + \|u\|^2_{H^1(D(m_1,m_2)\times(-\rho/4,0)\times(0,T_o))}\big).\end{aligned} \tag{3.9}$$

Recall that in both estimates above, c does not depend on $T_o > 0$.

Combining (3.8), (3.9), (3.3) and using the fact that $D_\Theta\times(\rho-2r_o,\rho)\cup(-\rho,-\rho+2r_o)\subset\omega$ with (3.5), we deduce that there exist $C, T > 0$ and $\widetilde{\omega}\Subset\Omega$ such that

$$\mathcal{G}(u,0) \leqslant C\big(\|u\|^2_{H^1(\widetilde{\omega}\times(0,T))} + \|\partial_t u\|^2_{L^2(\omega\times(-2T,2T))} + \|u\|^2_{L^2(\omega\times(-2T,2T))}\big). \tag{3.10}$$

On the other hand, notice that there exist $\widetilde{\Theta} \Subset \Theta$ and $\widetilde{\omega}_o \Subset \omega_o$ (see (3.1) for its definition) such that any ray (introduced in Section 1) starting at any $x_o \in \Omega$ with any direction $\xi_o \in \mathbb{S}^2$ will meet $\widetilde{\omega}_o \cup (\Omega \cap \widetilde{\Theta})$. Consequently, by the propagation of the H^1 regularity along the ray (where in our geometry only hyperbolic contact on $\partial\Omega \setminus \overline{\Upsilon}$ occurs (see [14])) and the strategy developed by [3] (see e.g. [2, p. 35] and [8, p. XVI-4]), we obtain that for any $\widetilde{\omega} \Subset \Omega$ and any $T > 0$, there exist $C, C_T > 0$ (depending on T) such that

$$\|u\|^2_{H^1(\widetilde{\omega}\times(0,T))} \leqslant C\big(\|u\|^2_{H^1(\widetilde{\omega}_o\cup(\Omega\cap\widetilde{\Theta})\times(-C_T,C_T))} + \|u\|^2_{L^2(\Omega\times(-C_T,C_T))}\big). \tag{3.11}$$

Now, combining (3.10), (3.11), (3.4) and (3.5), we get after a translation on the time variable, that there exist $C, T > 0$ such that

$$\mathcal{G}(u,0) \leqslant C\big(\|\partial_t u\|^2_{L^2(\omega_o\cup\omega\times(0,T))} + \big\|(u_0,u_1)\big\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\big).$$

By a standard uniqueness-compactness argument (e.g. [11]), it implies that the solution u of (1.4) satisfies

$$\mathcal{G}(u,0) \leqslant C\big(\|\partial_t u\|^2_{L^2(\omega_o\times(0,T))} + \|\partial_t u\|^2_{L^2(\omega\times(0,T))}\big) \tag{3.12}$$

for some $C, T > 0$.

Step 2. We claim the following result which will be proved in the next section. For any $T > 0$, there exist $h_o, C > 0$ and an absolute constant $\gamma > 1$ such that for any $h \in (0, h_o]$ and initial data $(u_0, u_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$, the solution u of (1.4) satisfies

$$\begin{aligned}\int_0^T\int_{\omega_o} \big|\partial_t u(x,t)\big|^2\,dx\,dt \leqslant C\left(\frac{1}{h}\right)^{\gamma}\Bigg(\int_{|t|\leqslant\gamma(\frac{1}{h})^{\gamma}}\int_{\Upsilon}\big|\partial_\nu u(x,t)\big|^2\,dx\,dt\Bigg)^{1/2}\sqrt{\mathcal{G}(u,0)}\\ + C\sqrt{h}\,\sqrt{\mathcal{G}(u,0)}\,\sqrt{\mathcal{G}(\partial_t u,0)}.\end{aligned} \tag{3.13}$$

Consequently, by (3.12), (3.13), (3.6) and a translation on the time variable, we obtain the existence of $C > 0$ and $\gamma > 1$ such that the solution u of (1.4) satisfies for any $h \in (0, h_o]$,

$$\mathcal{G}(u,0) \leqslant C\left(\frac{1}{h}\right)^{\gamma}\int_0^{C(\frac{1}{h})^{\gamma}}\int_{\omega}\big(\big|\partial_t u(x,t)\big|^2 + \big|u(x,t)\big|^2\big)\,dx\,dt + Ch\mathcal{G}(\partial_t u,0).$$

By formula (3.3) and a translation on the time variable, this last inequality becomes, for any $N > 1$,

$$N\mathcal{G}(u,0) \leqslant C\left(\frac{1}{h}\right)^{\gamma}\sum_{n=0,\dots,N-1}\int_n^{(n+1)C(\frac{1}{h})^{\gamma}}\int_{\omega}\big(\big|\partial_t u(x,t)\big|^2 + \big|u(x,t)\big|^2\big)\,dx\,dt + ChN\mathcal{G}(\partial_t u,0).$$

We choose $N \in (C(\frac{1}{h})^\gamma, C(\frac{1}{h})^\gamma + 1]$. Therefore, there exists $C > 0$ such that for any $h \in (0, h_o]$,

$$\mathcal{G}(u,0) \leqslant \int_0^{C(\frac{1}{h})^\gamma} \int_\omega \left(\left|\partial_t u(x,t)\right|^2 + \left|u(x,t)\right|^2\right) dx\, dt + Ch\mathcal{G}(\partial_t u, 0).$$

Since $\mathcal{G}(u,0) \leqslant (c/h_o)h\mathcal{G}(\partial_t u, 0)$ for any $h \geqslant h_o$, the estimate

$$\mathcal{G}(u,0) \leqslant \int_0^{C(\frac{1}{h})^\gamma} \int_\omega \left(\left|\partial_t u(x,t)\right|^2 + \left|u(x,t)\right|^2\right) dx\, dt + \big(C + (c/h_o)\big)h\mathcal{G}(\partial_t u, 0)$$

holds for any $h > 0$. Finally, we choose

$$h = \frac{1}{2}\frac{\mathcal{G}(u,0)}{(C + (c/h_o))\mathcal{G}(\partial_t u, 0)},$$

and use the fact that $1/\alpha \in L^\infty(\omega)$ in order to get

$$\mathcal{G}(u,0) \leqslant 2\|1/\alpha\|_{L^\infty(\omega)} \int_0^{c(\frac{\mathcal{G}(\partial_t u,0)}{\mathcal{G}(u,0)})^\gamma} \int_\Omega \alpha(x)\left(\left|\partial_t u(x,t)\right|^2 + \left|u(x,t)\right|^2\right) dx\, dt.$$

This completes the proof of (1.5)

4. Interpolation inequality

The purpose of this section is to establish the following inequality. Recall (3.1) for the definition of ω_o and the existence of r_o in Section 3.1.

Theorem 1. *Let $T > 0$. There exist $h_o, C > 0$ and an absolute constant $\gamma > 1$ such that for any $h \in (0, h_o]$ and initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the solution u of* (1.4) *satisfies*

$$\int_0^T \int_{\omega_o} \left|\partial_t u(x,t)\right|^2 dx\, dt \leqslant C\left(\frac{1}{h}\right)^\gamma \left(\int_{|t| \leqslant \gamma(\frac{1}{h})^\gamma} \int_\Upsilon \left|\partial_\nu u(x,t)\right|^2 dx\, dt\right)^{1/2} \sqrt{\mathcal{G}(u,0)}$$
$$+ C\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u, 0)}.$$

The rest of this section is devoted to the proof of Theorem 1. The proof of Theorem 1 is divided into eight steps. Let us now describe our strategy to prove Theorem 1 step by step.

Step 1. Let $h > 0$ be sufficiently small. From any point $x_o \in \overline{\omega_o}$, we localize around x_o and introduce a particular weight function $a_o = a_o(x,t) \in C^\infty(\mathbb{R}^4)$ depending on $x_o \in \overline{\omega_o}$ and on h.

Next, we will see that in order to bound $\partial_t u$ on $\omega_o \times (0, T)$, it is sufficient to study two quantities but with the same following form

$$\int_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t) f(x,t) u(x,t)\, dx\, dt,$$

with $f = \partial_t u$ or $f = u$ and $\varphi(\cdot, t) \in C_0^\infty(\Omega)$ such that $\|\varphi(\cdot,t)\|_{L^\infty(\Omega)} \leqslant e^{-t^2/\gamma}$.

Step 2. Denote $(\xi, \tau) = (\xi_1, \xi_2, \xi_3, \tau)$ the Fourier variables, dual of $(x, t) = (x_1, x_2, x_3, t)$. We use the Fourier inversion formula applied to φf and we cut the integral over the (ξ_3, τ) Fourier variables as follows

$$\int_{\mathbb{R}} d\xi_3 \int_{\mathbb{R}} d\tau = \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \int_{\xi_{o3}-1}^{\xi_{o3}+1} d\xi_3 \int_{|\tau|<\lambda} d\tau + \int_{\mathbb{R}} d\xi_3 \int_{|\tau|\geqslant\lambda} d\tau,$$

for some $\lambda \geqslant 1$, where $(2\mathbb{Z}+1) = \{2n+1;\ n \in \mathbb{Z}\}$. Then, we focus on the quantity

$$a_o(x,t)\frac{1}{(2\pi)^4} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi f}(\xi,\tau)\, d\xi\, d\tau.$$

Step 3. We introduce a complex weight function $a = a(x,t,s) \in C^\infty(\mathbb{R}^5;\mathbb{C})$ with the properties of localization, propagation and dispersion and such that $(i\partial_s + h(\Delta - \partial_t^2))a = 0$ on the whole space and $a(x - x_o, t, 0) = a_o(x,t)$. Such a weight function was inspired by [19]. Denote $x_o = (x_{o1}, x_{o2}, x_{o3})$. Then, we may define a Fourier integral operator inspired by [7] in the following form

$$\begin{aligned}(A_0 f)(x,t,s) = {} & \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{ix\xi+t\tau}\, e^{-i(|\xi|^2-\tau^2)hs}\, \widehat{\varphi f}(\xi,\tau) \\ & \times a(x_1 - x_{o1} - 2\xi_1 hs, x_2 - x_{o2} - 2\xi_2 hs, x_3 - x_{o3} - 2\xi_3 hs, t + 2\tau hs, s)\, d\xi\, d\tau,\end{aligned}$$

which solves $(i\partial_s + h(\Delta - \partial_t^2))A_0 f = 0$ and such that $(A_0 f)(x,t,0)$ is equal to the quantity focused in step 2.

Step 4. Since we will need to deal with the boundary $\{x_3 = \pm\rho\}$, we will construct an appropriate sequence $(A_n f)_{n\in\mathbb{Z}}$ of Fourier integral operators satisfying $(i\partial_s + h(\Delta - \partial_t^2))A_n f = 0$. Next, we define a partial sum of the series $\sum_{n\in\mathbb{Z}} A_n f$,

$$\mathcal{A}_{P,Q} f = \sum_{n=-2Q}^{2P+1} A_n f \quad \text{for some } (P,Q) \in \mathbb{N}^2.$$

Such a construction looks like a mirror reflection and was inspired by [15]. Good properties for $\mathcal{A}_{P,Q} f$ on the boundary $\{x_3 = \pm\rho\}$ are checked.

Step 5. We multiply the equation $(i\partial_s + h(\Delta - \partial_t^2))\mathcal{A}_{P,Q} f = 0$ by u solution of (1.4) and integrate by parts over $(x,t,s) \in \Omega \times \mathbb{R} \times [0,L]$. Then, we will see thanks to steps 2 and 4 that the quantity expressed in step 1 depends on the following two terms: an internal term namely given by

$$\int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} (\mathcal{A}_{P,Q} f)(x,t,L) u(x,t)\,dx\,dt,$$

and a boundary term given by

$$ih \int_0^L \int_{\partial\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} (\mathcal{A}_{P,Q} f)(x,t,s)\partial_\nu u(x,t)\,dx\,dt\,ds.$$

Step 6. We let $L \geqslant 1$ and estimate the internal term explicitly with respect to h and L and uniformly with respect to (P,Q). We will need to check that the internal term is small for any (P,Q) and for sufficiently small h and large L. Here, some property of dispersion on the s variable of $a(x,t,s)$ is required.

Step 7. We estimate the boundary term. We treat the boundary parts Υ and $\Gamma_1 \cup \Gamma_2$ separately. Recall that we want to make appear $\int_\Upsilon \int_{|t|\leqslant\gamma(\frac{1}{h})^\gamma} |\partial_\nu u(x,t)|^2\,dt\,dx$. Also, we will need to choose (P,Q) depending on (ξ_{o3}, L) such that the boundary term on the part $\Gamma_1 \cup \Gamma_2$ is sufficiently small for large λ, L and small h.

Step 8. We finally make the appropriate choice for $\lambda \geqslant 1$ and $L \geqslant 1$ with respect to h in order to conclude the proof of Theorem 1.

Now, we give the details of each step.

4.1. Step 1: Localization around any point of $\overline{\omega_o}$

Let $h \in (0, h_o]$ where $h_o = \min(1, (r_o/8)^2)$. We begin by covering $\overline{\omega_o}$ with a finite collection of balls $B(x_o^i, 2\sqrt{h})$ for $i \in I$ with $x_o^i \in \overline{\omega_o}$ and where I is a countable set such that the number of elements of I is $\frac{c_o}{h\sqrt{h}}$ for some constant $c_o > 0$ independent of h. Then, for each x_o^i, we introduce $\chi_{x_o^i} \in C_0^\infty(B(x_o^i, r_o/2)) \subset C_0^\infty(\Omega)$ be such that $0 \leqslant \chi_{x_o^i} \leqslant 1$ and $\chi_{x_o^i} = 1$ on $B(x_o^i, r_o/4) \supset B(x_o^i, 2\sqrt{h})$. Consequently,

$$\begin{aligned}
\int_0^T \int_{\omega_o} \left|\partial_t u(x,t)\right|^2 dx\,dt &\leqslant e^{\frac{1}{2}T^2} \int_0^T \int_{\omega_o} e^{-\frac{1}{2}t^2} \left|\partial_t u(x,t)\right|^2 dx\,dt \\
&\leqslant e^{\frac{1}{2}T^2+2} \sum_{i\in I} \int_0^T \int_{B(x_o^i, 2\sqrt{h})} \chi_{x_o^i}(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o^i|^2+t^2)} \left|\partial_t u(x,t)\right|^2 dx\,dt \\
&\leqslant e^{\frac{1}{2}T^2+2} \sum_{i\in I} \int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x) \left|e^{-\frac{1}{4}(\frac{1}{h}|x-x_o^i|^2+t^2)} \partial_t u(x,t)\right|^2 dx\,dt. \qquad (4.1)
\end{aligned}$$

We claim that the quantity $\int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x)|e^{-\frac{1}{4}(\frac{1}{h}|x-x_o^i|^2+t^2)}\partial_t u(x,t)|^2\,dx\,dt$ can be bounded uniformly with respect to i. For simplicity, we omit the i indexation but we have in mind that the sum $\sum_{i\in I}$ will make appear $\frac{c_o}{h\sqrt{h}}$.

Now, for any $x_o \in \overline{\omega_o}$, we set

$$\chi = \chi_{x_o}, \quad a_o(x,t) = e^{-\frac{1}{4}(\frac{1}{h}|x-x_o|^2+t^2)} \quad \text{and} \quad \varphi(x,t) = \chi(x)a_o(x,t).$$

Then, we use such weight functions a_o and φ with u the solution of the wave equation (1.4) as follows. By integrations by parts,

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}} \chi(x)\big|a_o\partial_t u(x,t)\big|^2\,dx\,dt \\
&\quad = -\int_{\Omega\times\mathbb{R}} \chi(x)\partial_t\big(\big|a_o(x,t)\big|^2\big)\frac{1}{2}\partial_t\big(\big|u(x,t)\big|^2\big)\,dx\,dt \\
&\qquad - \int_{\Omega\times\mathbb{R}} \chi(x)\big|a_o(x,t)\big|^2\partial_t^2 u(x,t)u(x,t)\,dx\,dt \\
&\quad = \int_{\Omega\times\mathbb{R}} \chi(x)\frac{1}{2}\partial_t^2\big(\big|a_o(x,t)\big|^2\big)\big|u(x,t)\big|^2\,dx\,dt - \int_{\Omega\times\mathbb{R}} \chi(x)\big|a_o(x,t)\big|^2\partial_t^2 u(x,t)u(x,t)\,dx\,dt.
\end{aligned}$$

As $\partial_t^2(|a_o(x,t)|^2) = -|a_o(x,t)|^2 + t^2|a_o(x,t)|^2$ and $t^2|a_o(x,t)|^2 \leqslant 4|a_o(x,t/\sqrt{2})|^2$, we have

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}} \chi(x)\big|a_o\partial_t u(x,t)\big|^2\,dx\,dt \leqslant{}& 2\int_{\Omega\times\mathbb{R}} a_o(x,t/\sqrt{2})\varphi(x,t/\sqrt{2})\big|u(x,t)\big|^2\,dx\,dt \\
&+ \left|\int_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t)\partial_t^2 u(x,t)u(x,t)\,dx\,dt\right|. \qquad (4.2)
\end{aligned}$$

In the next subsection, we will first study the second term of the second member of (4.2),

$$\int_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t)f(x,t)u(x,t)\,dx\,dt \quad \text{when } f = \partial_t^2 u.$$

4.2. Step 2: The Fourier variables

Denote

$$\mathcal{F}(\varphi f)(\xi,\tau) := \widehat{\varphi f}(\xi,\tau) = \int_{\Omega\times\mathbb{R}} e^{-i(x\xi+t\tau)}\varphi(x,t)f(x,t)\,dx\,dt,$$

then for any $(x,t) \in \Omega \times \mathbb{R}$,

$$a_o(x,t)\varphi(x,t)f(x,t) = a_o(x,t)\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi\,d\tau.$$

Let $\lambda \geqslant 1$. We cut the integral over $\tau \in \mathbb{R}$ into two parts, $\{|\tau| \geqslant \lambda\}$ and $\{|\tau| < \lambda\}$. Next, for $\{|\tau| < \lambda\}$ and $\xi = (\xi_1, \xi_2, \xi_3)$, the integral over $\xi_3 \in \mathbb{R}$ is divided as follows:

$$\begin{aligned} a_o(x,t)\varphi(x,t)f(x,t) = {} & a_o(x,t)\frac{1}{(2\pi)^4}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi\,d\tau \\ & + a_o(x,t)\frac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|\geqslant\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi\,d\tau. \end{aligned}$$

Consequently,

$$\begin{aligned} & \int_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t)f(x,t)u(x,t)\,dx\,dt - R_0 \\ & = \int_{\Omega\times\mathbb{R}} a_o(x,t)\frac{1}{(2\pi)^4}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi\,d\tau\,u(x,t)\,dx\,dt \qquad (4.3) \end{aligned}$$

where

$$R_0 = \int_{\Omega\times\mathbb{R}} a_o(x,t)\frac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|\geqslant\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi\,d\tau\,u(x,t)\,dx\,dt.$$

Moreover, it holds

$$|R_0| \leqslant c\sqrt{\frac{1}{\lambda}}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)}. \qquad (4.4)$$

The proof of (4.4) is given in Appendix A (see (A.1)).

4.3. Step 3: The complex weight function

We set for any $(x,t,s) \in \mathbb{R}^5$,

$$a(x,t,s) = \left(\frac{1}{(is+1)^{3/2}}e^{-\frac{1}{4h}\frac{|x|^2}{is+1}}\right)\left(\frac{1}{\sqrt{-ihs+1}}e^{-\frac{1}{4}\frac{t^2}{-ihs+1}}\right).$$

Therefore, the following identities hold

$$a_o(x,t) = a(x - x_o, t, 0),$$
$$\big(i\partial_s + h\big(\Delta - \partial_t^2\big)\big)a(x,t,s) = 0 \quad \forall (x,t,s) \in \mathbb{R}^5,$$
$$\big|a(x,t,s)\big| = \frac{1}{(\sqrt{s^2+1}\,)^{3/2}} \frac{1}{(\sqrt{(hs)^2+1}\,)^{1/2}}\, e^{-\frac{1}{4h}\frac{|x|^2}{s^2+1}}\, e^{-\frac{1}{4}\frac{t^2}{(hs)^2+1}}. \tag{4.5}$$

4.4. Step 4: The partial sum of the series of FIO

Let $(x_o, \xi_{o3}) = (x_{o1}, x_{o2}, x_{o3}, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z}+1)$, we introduce for all $n \in \mathbb{Z}$,

$$\begin{aligned}
&\big(A_n(x_o,\xi_{o3})f\big)(x,t,s) := (A_n f)(x,t,s)\\
&= \frac{(-1)^n}{(2\pi)^4} \int\limits_{\mathbb{R}^2} \int\limits_{\xi_{o3}-1}^{\xi_{o3}+1} \int\limits_{|\tau|<\lambda} e^{i(x_1\xi_1 + x_2\xi_2 + t\tau)} e^{i[(-1)^n x_3 + 2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho]\xi_3} e^{-i(|\xi|^2-\tau^2)hs}\widehat{\varphi f}(\xi,\tau)\\
&\quad \times a\bigg(x_1 - x_{o1} - 2\xi_1 hs, x_2 - x_{o2} - 2\xi_2 hs, x_3 + (-1)^n\bigg[2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho - x_{o3} - 2\xi_3 hs\bigg],\\
&\quad t + 2\tau hs, s\bigg)\, d\xi\, d\tau.
\end{aligned}$$

Let $(P, Q) \in \mathbb{N}^2$. Consider the partial sum of the series of Fourier integral operators $\mathcal{A}_{P,Q} f := \mathcal{A}_{P,Q}(x_o,\xi_{o3})f$ given by

$$\mathcal{A}_{P,Q}(x_o,\xi_{o3})f = \sum_{n=-2Q}^{2P+1} A_n(x_o,\xi_{o3})f.$$

One can check that for any $(x_o, \xi_{o3}, P, Q) \in \overline{\omega_o} \times (2\mathbb{Z}+1) \times \mathbb{N}^2$,

$$\big(i\partial_s + h\big(\Delta - \partial_t^2\big)\big)\big(\mathcal{A}_{P,Q}(x_o,\xi_{o3})f\big)(x,t,s) = 0 \quad \forall (x,t,s) \in \mathbb{R}^5. \tag{4.6}$$

Further, we have the following two results.

Lemma 1. *For any $h \in (0, h_o]$, $\lambda \geqslant 1$ and any $(x_o, P, Q) \in \overline{\omega_o} \times \mathbb{N}^2$,*

$$\begin{aligned}
&\int\limits_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t)f(x,t)u(x,t)\,dx\,dt - R_0 - R_1\\
&\quad = \int\limits_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \big(\mathcal{A}_{P,Q}(x_o,\xi_{o3})f\big)(x,t,0)u(x,t)\,dx\,dt,
\end{aligned}$$

where R_0 satisfies (4.4) *and*

$$|R_1| \leqslant c\left(\frac{\lambda}{h}\right)^{\gamma} e^{-\frac{1}{ch}}\mathcal{G}(u,0). \tag{4.7}$$

Lemma 2. *For any* $(x_o, P, Q) \in \overline{\omega_o} \times \mathbb{N}^2$ *and any* $(x_1, x_2, \xi_{o3}, t, s) \in \mathbb{R}^2 \times (2\mathbb{Z}+1) \times \mathbb{R}^2$,

$$\big(\mathcal{A}_{P,Q}(x_o, \xi_{o3}) f\big)\left(x_1, x_2, \frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) = 0,$$

and

$$\begin{aligned}&\big(\mathcal{A}_{P,Q}(x_o, \xi_{o3}) f\big)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right)\\ &\quad = \big(A_{-2Q}(x_o, \xi_{o3}) f\big)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) + \big(A_{2P+1}(x_o, \xi_{o3}) f\big)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right).\end{aligned}$$

Now, we begin to prove Lemma 1.

4.4.1. Proof of Lemma 1. At $s = 0$

Since

$$\big(A_0(x_o, \xi_{o3}) f\big)(x, t, 0) = a_o(x, t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi + t\tau)}\, \widehat{\varphi f}(\xi, \tau)\, d\xi\, d\tau,$$

we then obtain from (4.3) that

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}} a_o(x, t)\varphi(x, t) f(x, t) u(x, t)\, dx\, dt - R_0\\ &\quad = \int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \big(A_0(x_o, \xi_{o3}) f\big)(x, t, 0) u(x, t)\, dx\, dt.\end{aligned}$$

Observe that

$$\mathcal{A}_{P,Q}(x_o, \xi_{o3}) f = A_0(x_o, \xi_{o3}) f + \sum_{n=1}^{2P+1} A_n(x_o, \xi_{o3}) f + \sum_{-2Q\leqslant n\leqslant -1} A_n(x_o, \xi_{o3}) f,$$

with the convention that for $Q = 0$, $\sum_{-2Q\leqslant n\leqslant -1} A_n(x_o, \xi_{o3}) f = 0$. So, we deduce that the equality of Lemma 1 holds with

$$R_1 = -\int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left[\sum_{n=1}^{2P+1} \big(A_n(x_o, \xi_{o3}) f\big) + \sum_{-2Q\leqslant n\leqslant -1} \big(A_n(x_o, \xi_{o3}) f\big)\right](x, t, 0) u(x, t)\, dx\, dt.$$

Now, we estimate R_1 uniformly with respect to (P, Q) as follows:

$$
\begin{aligned}
|R_1| &\leqslant \int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \sum_{n\in\mathbb{Z}\setminus\{0\}} \big|\big(A_n(x_o,\xi_{o3})f\big)(x,t,0)\big|\big|u(x,t)\big|\,dx\,dt \\
&\leqslant \int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \big|\widehat{\varphi f}(\xi,\tau)\big|\,d\xi\,d\tau \\
&\quad\times \sum_{n\in\mathbb{Z}\setminus\{0\}} a\left(x_1-x_{o1}, x_2-x_{o2}, (-1)^n x_3 + 2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho - x_{o3}, t, 0\right)\big|u(x,t)\big|\,dx\,dt \\
&\leqslant \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \big|\widehat{\varphi f}(\xi,\tau)\big|\,d\xi\,d\tau \\
&\quad\times \int_{\Omega\times\mathbb{R}} \sum_{n\in\mathbb{Z}\setminus\{0\}} a\big(x_1-x_{o1}, x_2-x_{o2}, \big|(-1)^n x_3 - x_{o3}\big| - 2|n|\rho, t, 0\big)\big|u(x,t)\big|\,dx\,dt. \qquad (4.8)
\end{aligned}
$$

On the other hand, the hypothesis saying that either $\Upsilon \subset (\mathbb{R}^2 \setminus D(m_1,m_2)) \times \mathbb{R}$ or Ω is convex, implies that for any $(x_{o1},x_{o2},x_{o3}) \in \overline{\omega_o}$ and $(x_1,x_2,x_3) \in \Omega$ such that $(x_1,x_2) \notin \overline{D(m_1-r_o/2, m_2-r_o/2)}$, we have $(x_1-x_{o1})^2 + (x_2-x_{o2})^2 \geqslant (r_o/2)^2$. But, if $(x_1,x_2,x_3) \in \Omega \cap \overline{D(m_1-r_o/2, m_2-r_o/2)} \times \mathbb{R}$, then we get $x_3 \in (-\rho,\rho)$ and therefore $||(-1)^n x_3 - x_{o3}| - 2|n|\rho| \geqslant \frac{3}{4}\rho$ for any $(x_{o1},x_{o2},x_{o3}) \in \overline{\omega_o}$ and any $n \in \mathbb{Z}\setminus\{0\}$. Consequently, for any $n \in \mathbb{Z}\setminus\{0\}$,

$$
\begin{aligned}
&a\big(x_1-x_{o1}, x_2-x_{o2}, \big|(-1)^n x_3 - x_{o3}\big| - 2|n|\rho, t, 0\big) \\
&\quad= e^{-\frac{1}{4h}[(x_1-x_{o1})^2+(x_2-x_{o2})^2+(|(-1)^n x_3-x_{o3}|-2|n|\rho)^2]}\, e^{-\frac{t^2}{4}} \\
&\quad\leqslant e^{-\frac{1}{8h}[\min(\frac{r_o}{2},\frac{3}{4}\rho)]^2} e^{-\frac{1}{8h}[|(-1)^n x_3-x_{o3}|-2|n|\rho]^2} e^{-\frac{t^2}{4}}.
\end{aligned}
$$

Hence, with the help of (3.3),

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}} \sum_{n\in\mathbb{Z}\setminus\{0\}} a\big(x_1-x_{o1}, x_2-x_{o2}, \big|(-1)^n x_3 - x_{o3}\big| - 2|n|\rho, t, 0\big)\big|u(x,t)\big|\,dx\,dt \\
&\quad\leqslant c e^{-\frac{1}{ch}} \sqrt{\mathcal{G}(u,0)}. \qquad (4.9)
\end{aligned}
$$

Finally, it remains to compute $\int_{\mathbb{R}^3}\int_{|\tau|<\lambda} |\widehat{\varphi f}(\xi,\tau)|\,d\xi\,d\tau$. We can check that there exists $\gamma > 1$ such that

$$
\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \big|\widehat{\varphi f}(\xi,\tau)\big|\,d\xi\,d\tau \leqslant c\left(\frac{\lambda}{h}\right)^{\gamma} \sqrt{\mathcal{G}(u,0)}. \qquad (4.10)
$$

The proof of (4.10) is given in Appendix A (see (A.2)).

Taking into account (4.8)–(4.10), we get the desired estimate for R_1 uniformly with respect to (P, Q) and this completes the proof of Lemma 1.

Now, the proof of Lemma 2 is as follows.

4.4.2. Proof of Lemma 2. On the boundary $\{x_3 = \pm\rho\}$

Since $a(x_1, x_2, x_3, t, s) = a(x_1, x_2, -x_3, t, s)$, the following identity holds. For any $n \in \mathbb{Z}$,

$$\big(A_n(x_o, \xi_{o3})f\big)\left(x_1, x_2, (-1)^n \frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) = -\big(A_{n+1}(x_o, \xi_{o3})f\big)\left(x_1, x_2, (-1)^n \frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right).$$

Thus, for any $n \in \mathbb{Z}$,

$$\big(A_{2n}(x_o, \xi_{o3})f\big)\left(x_1, x_2, \frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) = -\big(A_{2n+1}(x_o, \xi_{o3})f\big)\left(x_1, x_2, \frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right),$$

and

$$\big(A_{2n+1}(x_o, \xi_{o3})f\big)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) = -\big(A_{2n+2}(x_o, \xi_{o3})f\big)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right).$$

Therefore, the formula in Lemma 2 follows immediately.

4.5. Step 5: A key identity

By multiplying (4.6) by $u(x,t)$ and integrating by parts over $\Omega \times \mathbb{R} \times [0, L]$, we have that for all $h \in (0, h_o]$, $x_o \in \overline{\omega_o}$, $\lambda \geqslant 1$, $\xi_{o3} \in (2\mathbb{Z}+1)$, $(P, Q) \in \mathbb{N}^2$ and all $L > 0$,

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}} \big(\mathcal{A}_{P,Q}(x_o, \xi_{o3})f\big)(x, t, 0)u(x,t)\,dx\,dt \\
&\quad = \int_{\Omega\times\mathbb{R}} \big(\mathcal{A}_{P,Q}(x_o, \xi_{o3})f\big)(x, t, L)u(x,t)\,dx\,dt \\
&\qquad + ih\int_0^L \int_{\partial\Omega\times\mathbb{R}} \big(\mathcal{A}_{P,Q}(x_o, \xi_{o3})f\big)(x, t, s)\partial_\nu u(x,t)\,dx\,dt\,ds. \qquad (4.11)
\end{aligned}$$

Here, we have the possibility to take (P, Q) depending on (ξ_{o3}, L) in (4.11). Therefore, we write $P = P(\xi_{o3}, L)$ and $Q = Q(\xi_{o3}, L)$. Now, combining (4.11) and the equality in Lemma 1 of step 4, we have the following key identity. For any $h \in (0, h_o]$, $x_o \in \overline{\omega_o}$, $\lambda \geqslant 1$, and any $L > 0$,

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t)f(x,t)u(x,t)\,dx\,dt - R_0 - R_1 \\
&\quad = \int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \big(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_o, \xi_{o3})f\big)(x, t, L)u(x,t)\,dx\,dt \\
&\qquad + ih\int_0^L \int_{\partial\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \big(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_o, \xi_{o3})f\big)(x, t, s)\ \partial_\nu u(x,t)\,dx\,dt\,ds. \qquad (4.12)
\end{aligned}$$

Our strategy is as follows. First, L will be taken large enough, in order that the first term of the second member of (4.12) is polynomially small like $\frac{c}{\sqrt{L}}$ uniformly with respect to $(P(\xi_{o3}, L), Q(\xi_{o3}, L))$ by using the dispersion property on the s variable of the complex weight function a from (4.5). Next, $(P(\xi_{o3}, L), Q(\xi_{o3}, L))$ will be chosen in order that the second term of the second member of (4.12) is exponentially small with respect to h on the boundary $\Gamma_1 \cup \Gamma_2$ thanks to Lemma 2 of step 4.

From now, $L \geqslant 1$.

4.6. Step 6: The internal term

In this subsection, we study the internal term appearing in (4.12) and prove the following inequality.

Lemma 3. *For any $h \in (0, h_o]$, $x_o \in \overline{\omega_o}$, $\lambda \geqslant 1$, $(P, Q) \in \mathbb{N}^2$ and any $L \geqslant 1$,*

$$\left| \int_{\Omega \times \mathbb{R}} \sum_{\xi_{o3} \in (2\mathbb{Z}+1)} \big(\mathcal{A}_{P,Q}(x_o, \xi_{o3}) f\big)(x, t, L) u(x, t)\, dx\, dt \right| \leqslant c \frac{1}{\sqrt{L}} \left(\frac{\lambda}{h}\right)^{\gamma} \sqrt{\mathcal{G}(u, 0)} \sqrt{\mathcal{G}(\partial_t u, 0)}.$$

Proof. First, we have a uniform bound with respect to (P, Q). Indeed,

$$\begin{aligned}
&\left| \int_{\Omega \times \mathbb{R}} \sum_{\xi_{o3} \in (2\mathbb{Z}+1)} \big(\mathcal{A}_{P,Q}(x_o, \xi_{o3}) f\big)(x, t, L) u(x, t)\, dx\, dt \right| \\
&\quad = \left| \int_{\Omega \times \mathbb{R}} \sum_{\xi_{o3} \in (2\mathbb{Z}+1)} \left[\sum_{n=-2Q}^{2P+1} \big(A_n(x_o, \xi_{o3}) f\big)(x, t, L) \right] u(x, t)\, dx\, dt \right| \\
&\quad \leqslant \sum_{\xi_{o3} \in (2\mathbb{Z}+1)} \sum_{n \in \mathbb{Z}} \left| \int_{\Omega \times \mathbb{R}} \big(A_n(x_o, \xi_{o3}) f\big)(x, t, L) u(x, t)\, dx\, dt \right|. \qquad (4.13)
\end{aligned}$$

Next, we can check that the following identity holds from the definition of $A_n(x_o, \xi_{o3}) f$ and (3.2).

$$\begin{aligned}
&\int_{\Omega \times \mathbb{R}} \big(A_n(x_o, \xi_{o3}) f\big)(x, t, L) u(x, t)\, dx\, dt \\
&\quad = \frac{1}{(iL+1)^{3/2}} \frac{(-1)^n}{(2\pi)^4} \sum_{j \geqslant 1} \int_{\Omega} \ell_j(x) \\
&\qquad \times \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x_1\xi_1 + x_2\xi_2)}\, e^{i[(-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|}\rho]\xi_3}\, e^{-i(|\xi|^2 - \tau^2)hL} \widehat{\varphi f}(\xi, \tau)\, d\xi \\
&\qquad \times e^{-\frac{1}{4h} \frac{(x_1 - x_{o1} - 2\xi_1 hL)^2}{iL+1}}\, e^{-\frac{1}{4h} \frac{(x_2 - x_{o2} - 2\xi_2 hL)^2}{iL+1}}\, e^{-\frac{1}{4h} \frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|}\rho - x_{o3} - 2\xi_3 hL)^2}{iL+1}}\, dx
\end{aligned}$$

$$\times \int_{\mathbb{R}} e^{it\tau}\left(\frac{1}{\sqrt{-ihL+1}}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{-ihL+1}}\right)\left[\left(\frac{b_j^0}{2}+\frac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{it\sqrt{\mu_j}}\right.$$
$$\left.+\left(\frac{b_j^0}{2}-\frac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{-it\sqrt{\mu_j}}\right]dt\,d\tau. \tag{4.14}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}} e^{i(\tau\pm\sqrt{\mu_j})t}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{-ihL+1}}\,dt &= \int_{\mathbb{R}} e^{i(\tau\pm\sqrt{\mu_j})(t-2\tau hL)}e^{-\frac{1}{4}\frac{t^2}{-ihL+1}}\,dt\\ &= e^{i(\tau\pm\sqrt{\mu_j})(-2\tau hL)}\int_{\mathbb{R}} e^{i(\tau\pm\sqrt{\mu_j})t}e^{-\frac{1}{4}\frac{t^2}{-ihL+1}}\,dt\\ &= e^{i(\tau\pm\sqrt{\mu_j})(-2\tau hL)}\widetilde{\mathcal{F}}\big(e^{-\frac{1}{4}\frac{t^2}{-ihL+1}}\big)\big(-(\tau\pm\sqrt{\mu_j})\big)\\ &= e^{i(\tau\pm\sqrt{\mu_j})(-2\tau hL)}2\sqrt{\pi}\,\sqrt{-ihL+1}\,e^{-(-ihL+1)(\tau\pm\sqrt{\mu_j})^2},\end{aligned} \tag{4.15}$$

where we have used the following formula

$$\widetilde{\mathcal{F}}\big(e^{-\frac{z}{2}t^2}\big)(\tau) := \int_{\mathbb{R}} e^{-it\tau}e^{-\frac{z}{2}t^2}\,dt = \frac{\sqrt{2\pi}}{\sqrt{z}}\,e^{-\frac{1}{2z}\tau^2}, \quad \operatorname{Re} z \geqslant 0 \text{ and } z \neq 0.$$

Consequently, by (4.14) and (4.15), we obtain the following identity:

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}} \big(A_n(x_o,\xi_{o3})f\big)(x,t,L)u(x,t)\,dx\,dt\\ &= \frac{1}{(iL+1)^{3/2}}\frac{(-1)^n}{(2\pi)^4}\sum_{j\geqslant 1}\int_{\Omega}\ell_j(x)\\ &\quad\times\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} e^{i(x_1\xi_1+x_2\xi_2)}\,e^{i[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho]\xi_3}e^{-i(|\xi|^2-\tau^2)hL}\widehat{\varphi f}(\xi,\tau)\\ &\quad\times e^{-\frac{1}{4h}\frac{(x_1-x_{o1}-2\xi_1 hL)^2}{iL+1}}\,e^{-\frac{1}{4h}\frac{(x_2-x_{o2}-2\xi_2 hL)^2}{iL+1}}\,e^{-\frac{1}{4h}\frac{((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL)^2}{iL+1}}\,dx\,d\xi\\ &\quad\times\left[\left(\frac{b_j^0}{2}+\frac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{i(\tau+\sqrt{\mu_j})(-2\tau hL)}2\sqrt{\pi}e^{-(-ihL+1)(\tau+\sqrt{\mu_j})^2}\right.\\ &\quad\left.+\left(\frac{b_j^0}{2}-\frac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{i(\tau-\sqrt{\mu_j})(-2\tau hL)}2\sqrt{\pi}e^{-(-ihL+1)(\tau-\sqrt{\mu_j})^2}\right]d\tau.\end{aligned}$$

Thus, taking its complex modulus, one finds

$$\left|\int_{\Omega\times\mathbb{R}} \big(A_n(x_o,\xi_o)f\big)(x,t,L)u(x,t)\,dx\,dt\right|$$
$$\leqslant \frac{1}{L\sqrt{L}}\frac{2\sqrt{\pi}}{(2\pi)^4}\sum_{j\geqslant 1}\int_{\Omega}|\ell_j(x)|\left(\left|\frac{b_j^0}{2}+\frac{b_j^1}{2i\sqrt{\mu_j}}\right|+\left|\frac{b_j^0}{2}-\frac{b_j^1}{2i\sqrt{\mu_j}}\right|\right)$$
$$\times\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\tau\, e^{-\frac{1}{4h}((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL)^2\frac{1}{L^2+1}}\,d\xi\,dx.$$

Then, by summing over n, it comes

$$\sum_{n\in\mathbb{Z}}\left|\int_{\Omega\times\mathbb{R}} \big(A_n(x_o,\xi_{o3})f\big)(x,t,L)u(x,t)\,dx\,dt\right|$$
$$\leqslant c\frac{1}{\sqrt{L}}\sum_{j\geqslant 1}\left(\left|b_j^0\right|+\left|\frac{b_j^1}{\sqrt{\mu_j}}\right|\right)\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi\,d\tau, \tag{4.16}$$

where we have used

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{4h}((-1)^n x_3\frac{|\xi_{o3}|}{\xi_{o3}}+2n\rho-x_{o3}\frac{|\xi_{o3}|}{\xi_{o3}}-2\xi_3 hL\frac{|\xi_{o3}|}{\xi_{o3}})^2\frac{1}{L^2+1}}\leqslant cL,$$

and $\int_\Omega |\ell_j(x)|dx\leqslant c\|\ell_j\|_{L^2(\Omega)}=c$. On the other hand, using Schwarz inequality, we have

$$\sum_{j\geqslant 1}\left(\left|b_j^0\right|+\left|\frac{b_j^1}{\sqrt{\mu_j}}\right|\right)\leqslant\sqrt{\sum_{j\geqslant 1}\mu_j^2\left(\left|b_j^0\right|+\left|\frac{b_j^1}{\sqrt{\mu_j}}\right|\right)^2}\sqrt{\sum_{j\geqslant 1}\left|\frac{1}{\mu_j}\right|^2}$$
$$\leqslant c\sqrt{\mathcal{G}(\partial_t u,0)}, \tag{4.17}$$

because, from Weyl formula,

$$\sum_{j\geqslant 1}\left|\frac{1}{\mu_j}\right|^2\leqslant c\sum_{j\geqslant 1}\left|\frac{1}{j^{2/3}}\right|^2<+\infty.$$

We conclude the proof of Lemma 3 by using (4.13), (4.16), (4.17) and (4.10). □

4.7. Step 7: The boundary term

In this subsection, we study the boundary term appearing in (4.12). Recall that $\partial\Omega=\Gamma_1\cup\Gamma_2\cup\Upsilon$. We establish the following result.

Lemma 4. *The following two statements hold.*

(i) *For any $h \in (0, h_o]$, $x_o \in \overline{\omega_o}$, $\lambda \geqslant 1$ and any $L \geqslant 1$,*

$$\left| ih \int_0^L \int_{(\Gamma_1 \cup \Gamma_2)\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_o,\xi_{o3})f\right)(x,t,s)\partial_\nu u(x,t)\,dx\,dt\,ds \right|$$
$$\leqslant c\left(\frac{L\lambda}{h}\right)^\gamma e^{-\frac{1}{4h}} \sqrt{\mathcal{G}(\partial_t u,0)}\sqrt{\mathcal{G}(u,0)},$$

where $(P(\xi_{o3}, L), Q(\xi_{o3}, L))$ have been chosen adequately.

(ii) *For any $h \in (0, h_o]$, $x_o \in \overline{\omega_o}$, $\lambda \geqslant 1$, $(P, Q) \in \mathbb{N}^2$ and any $L \geqslant 1$,*

$$\left| ih \int_0^L \int_{\Upsilon\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left(\mathcal{A}_{P,Q}(x_o,\xi_{o3})f\right)(x,t,s)\ \partial_\nu u(x,t)\,dx\,dt\,ds \right|$$
$$\leqslant c\left(\frac{L\lambda}{h}\right)^\gamma \left[\left(\int_\Upsilon \int_{|t|\leqslant\frac{4\lambda L}{\sqrt{h}}} \left|\partial_\nu u(x,t)\right|^2 dx\,dt\right)^{1/2} + e^{-\frac{1}{8h}}\sqrt{\mathcal{G}(\partial_t u,0)}\right]\sqrt{\mathcal{G}(u,0)}.$$

Proof of (i) of Lemma 4. First, it holds

$$\left| \int_0^L \int_{(\Gamma_1 \cup \Gamma_2)\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_{o3},\xi_{o3})f\right)(x,t,s)\partial_\nu u(x,t)\,dx\,dt\,ds \right|$$
$$\leqslant \int_0^L \int_{-m_1}^{m_1} \int_{-m_2}^{m_2} \int_{\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left|\left(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_{o3},\xi_{o3})f\right)\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right|$$
$$\times \left|\partial_{x_3} u\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| dx_1\,dx_2\,dt\,ds$$
$$+ \int_0^L \int_{-m_1}^{m_1} \int_{-m_2}^{m_2} \int_{\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left|\left(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_{o3},\xi_{o3})f\right)\left(x_1,x_2,\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right|$$
$$\times \left|\partial_{x_3} u\left(x_1,x_2,\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| dx_1\,dx_2\,dt\,ds. \tag{4.18}$$

Next, from Lemma 2 of step 4, we only have to take care on $(A_{-2Q}(x_o,\xi_{o3})f)(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s)$ and on $(A_{2P+1}(x_o,\xi_{o3})f)(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s)$. Now, we can check that for any $(P, Q) \in \mathbb{N}^2$,

$$\left|\left(A_{-2Q}(x_o,\xi_{o3})f\right)\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right|$$
$$\leqslant \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}$$
$$\times\left|a\left(0,0,(4Q+1)\frac{\xi_{o3}}{|\xi_{o3}|}\rho+x_{o3}+2\xi_3hs,0,s\right)\right|d\xi\,d\tau,$$

and

$$\left|\left(A_{2P+1}(x_o,\xi_{o3})f\right)\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right|$$
$$\leqslant \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}$$
$$\times\left|a\left(0,0,(4P+3)\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3hs,0,s\right)\right|d\xi\,d\tau.$$

But, for $L\geqslant 1$, $\xi_{o3}\in(2\mathbb{Z}+1)$, we can choose (P,Q) large enough, for example

$$\begin{cases} Q=Q(\xi_{o3},L)\geqslant\dfrac{1}{4\rho}(L+1),\\ P=P(\xi_{o3},L)\geqslant\dfrac{1}{4\rho}\big[(L+1)+2\big(|\xi_{o3}|+1\big)h_oL\big],\end{cases}$$

in order that for any $s\in[0,L]$, $h\in(0,h_o]$ and $(x_{o3},\xi_3)\in[-\frac{\rho}{4},\frac{\rho}{4}]\times[\xi_{o3}-1,\xi_{o3}+1]$,

$$\begin{cases}\left|a\left(0,0,(4Q+1)\dfrac{\xi_{o3}}{|\xi_{o3}|}\rho+x_{o3}+2\xi_3hs,0,s\right)\right|\leqslant e^{-\frac{1}{4h}}\\ \left|a\left(0,0,-(4P+3)\dfrac{\xi_{o3}}{|\xi_{o3}|}\rho+x_{o3}+2\xi_3hs,0,s\right)\right|\leqslant e^{-\frac{1}{4h}},\end{cases}$$

where we have used (4.5). Consequently, one gets

$$\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\left|\left(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_{o3},\xi_{o3})f\right)\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right|\left|\partial_{x_3}u\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right|$$
$$\leqslant e^{-\frac{1}{4h}}\left(\left|\partial_{x_3}u(x_1,x_2,-\rho,t)\right|+\left|\partial_{x_3}u(x_1,x_2,\rho,t)\right|\right)\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\,d\xi\,d\tau. \tag{4.19}$$

We conclude, combining (4.18), (4.19), Lemma 2 of step 4, (3.7) and (4.10), that

$$\begin{aligned}
&\left| ih \int_0^L \int_{(\Gamma_1\cup\Gamma_2)\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \big(\mathcal{A}_{P(\xi_{o3},L),Q(\xi_{o3},L)}(x_{o3},\xi_{o3})f\big)(x,t,s)\partial_\nu u(x,t)\,dx\,dt\,ds \right| \\
&\quad\leqslant e^{-\frac{1}{4h}} c \int_0^L \int_{\mathbb{R}} \|\partial_\nu u(\cdot,t)\|_{L^2(\partial\Omega)} \int_{\mathbb{R}^3}\int_{|\tau|<\lambda} \big|\widehat{\varphi f}(\xi,\tau)\big| e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\,d\xi\,d\tau\,dt\,ds \\
&\quad\leqslant e^{-\frac{1}{4h}}\, c\sqrt{\mathcal{G}(\partial_t u,0)} \int_0^L \int_{\mathbb{R}} e^{-\frac{t^2}{4}\frac{1}{(hs)^2+1}}\,dt\,ds \int_{\mathbb{R}^3}\int_{|\tau|<\lambda} \big|\widehat{\varphi f}(\xi,\tau)\big|\,d\xi\,d\tau \\
&\quad\leqslant c\left(\frac{L\lambda}{h}\right)^{\gamma} e^{-\frac{1}{4h}} \sqrt{\mathcal{G}(\partial_t u,0)}\,\sqrt{\mathcal{G}(u,0)}.
\end{aligned}$$

This completes the proof of (i) of Lemma 4. □

Proof of (ii) of Lemma 4. First, we have a uniform bound with respect to (P,Q). Indeed,

$$\big|\big(\mathcal{A}_{P,Q}(x_{o3},\xi_{o3})f\big)(x,t,s)\big| \leqslant \sum_{n\in\mathbb{Z}} \big|\big(A_n(x_o,\xi_{o3})f\big)(x,t,s)\big|.$$

Next, we can check that from (4.5), the following estimate holds

$$\begin{aligned}
\big|\big(A_n(x_o,\xi_{o3})f\big)(x,t,s)\big| \leqslant{}& \int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} \big|\widehat{\varphi f}(\xi,\tau)\big| e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}} \\
&\times e^{-\frac{1}{4h}((-1)^n x_3\frac{|\xi_{o3}|}{\xi_{o3}}+2n\rho-x_{o3}\frac{|\xi_{o3}|}{\xi_{o3}}-2\xi_3 hs\frac{|\xi_{o3}|}{\xi_{o3}})^2\frac{1}{s^2+1}}\,d\xi\,d\tau.
\end{aligned}$$

Now, noticing

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{4h}((-1)^n x_3\frac{|\xi_{o3}|}{\xi_{o3}}+2n\rho-x_{o3}\frac{|\xi_{o3}|}{\xi_{o3}}-2\xi_3 hs\frac{|\xi_{o3}|}{\xi_{o3}})^2\frac{1}{s^2+1}} \leqslant c(1+\sqrt{h}s),$$

we deduce that

$$\begin{aligned}
&\big|\big(\mathcal{A}_{P,Q}(x_{o3},\xi_{o3})f\big)(x,t,s)\big| \\
&\quad\leqslant c(1+\sqrt{h}s)\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} \big|\widehat{\varphi f}(\xi,\tau)\big| e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\,d\xi\,d\tau. \qquad (4.20)
\end{aligned}$$

On the other hand, we get

$$\int_{\Upsilon\times\mathbb{R}} e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\left|\partial_\nu u(x,t)\right| dx\,dt$$
$$\leqslant \int_{\Upsilon}\int_{|t+2\tau hs|\leqslant\sqrt{\frac{(hs)^2+1}{h}}}\left|\partial_\nu u(x,t)\right| dx\,dt + e^{-\frac{1}{8h}}\int_{\Upsilon\times\mathbb{R}} e^{-\frac{(t+2\tau hs)^2}{8}\frac{1}{(hs)^2+1}}\left|\partial_\nu u(x,t)\right| dx\,dt$$
$$\leqslant \int_{\Upsilon}\int_{|t|\leqslant|2\tau hs|+\sqrt{h}s+\frac{1}{\sqrt{h}}}\left|\partial_\nu u(x,t)\right| dx\,dt + c(1+hs)e^{-\frac{1}{8h}}\sqrt{\mathcal{G}(\partial_t u,0)}, \tag{4.21}$$

by cutting the integral over $t\in\mathbb{R}$ into two parts and using (3.7). Using (4.20), (4.21) and (4.10), we conclude that

$$\left|ih\int_0^L\int_{\Upsilon\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\left(\mathcal{A}_{P,Q}(x_{o3},\xi_{o3})f\right)(x,t,s)\partial_\nu u(x,t)\,dx\,dt\,ds\right|$$
$$\leqslant h\int_0^L\int_{\Upsilon\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\left|\left(\mathcal{A}_{P,Q}(x_{o3},\xi_{o3})f\right)(x,t,s)\right|\left|\partial_\nu u(x,t)\right| dx\,dt\,ds$$
$$\leqslant ch\int_0^L(1+\sqrt{h}s)\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right| d\xi$$
$$\times\left(\int_{\Upsilon\times\mathbb{R}} e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\left|\partial_\nu u(x,t)\right| dx\,dt\right) d\tau\,ds$$
$$\leqslant ch\left[\left(\frac{\lambda}{h}\right)^\gamma\sqrt{\mathcal{G}(u,0)}\right]\int_0^L(1+\sqrt{h}s)$$
$$\times\left(\int_{\Upsilon}\int_{|t|\leqslant 2\lambda hL+\sqrt{h}L+\frac{1}{\sqrt{h}}}\left|\partial_\nu u(x,t)\right| dx\,dt + c(1+hs)e^{-\frac{1}{8h}}\sqrt{\mathcal{G}(\partial_t u,0)}\right) ds$$
$$\leqslant c\left(\frac{L\lambda}{h}\right)^\gamma\left(\int_{\Upsilon}\int_{|t|\leqslant\frac{4\lambda L}{\sqrt{h}}}\left|\partial_\nu u(x,t)\right|^2 dx\,dt\right)^{\frac{1}{2}}\sqrt{\mathcal{G}(u,0)} + c\left(\frac{L\lambda}{h}\right)^\gamma e^{-\frac{1}{8h}}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)}.$$

This completes the proof of (ii) of Lemma 4. □

4.8. *Step 8: The choice of* λ *and* L

By the key identity (4.12), (4.4), (4.7), Lemma 3 of step 6 and Lemma 4 of step 7, we obtain, when $f=\partial_t^2 u$, that for any $h\in(0,h_o]$, $x_o\in\overline{\omega_o}$, $\lambda\geqslant 1$, and any $L\geqslant 1$,

$$\begin{aligned}
&\Big|\int_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t)\partial_t^2 u(x,t)u(x,t)\,dx\,dt\Big| \\
&\quad\leqslant c\frac{1}{\sqrt{\lambda}}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + c\left(\frac{\lambda}{h}\right)^{\gamma} e^{-\frac{1}{ch}}\mathcal{G}(u,0) + c\frac{1}{\sqrt{L}}\left(\frac{\lambda}{h}\right)^{\gamma}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} \\
&\qquad + c\left(\frac{L\lambda}{h}\right)^{\gamma}\Big(\int_{\Upsilon}\int_{|t|\leqslant\frac{4\lambda L}{\sqrt{h}}}\big|\partial_\nu u(x,t)\big|^2\,dx\,dt\Big)^{1/2}\sqrt{\mathcal{G}(u,0)} \\
&\qquad + c\left(\frac{L\lambda}{h}\right)^{\gamma} e^{-\frac{1}{\gamma h}}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)}.
\end{aligned}$$

We choose $\lambda \geqslant 1$ and $L \geqslant 1$ be such that $\frac{1}{h\sqrt{h}}\frac{1}{\sqrt{\lambda}} \leqslant c\sqrt{h}$ and $\frac{1}{h\sqrt{h}}\frac{1}{\sqrt{L}}(\frac{\lambda}{h})^{\gamma} \leqslant c\sqrt{h}$ in order that

$$\begin{aligned}
&\Big|\frac{1}{h\sqrt{h}}\int_{\Omega\times\mathbb{R}} a_o(x,t)\varphi(x,t)\partial_t^2 u(x,t)u(x,t)\,dx\,dt\Big| \\
&\quad\leqslant c\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + c\left(\frac{1}{h}\right)^{\gamma}\Big(\int_{\Upsilon}\int_{|t|\leqslant\gamma(\frac{1}{h})^{\gamma}}\big|\partial_\nu u(x,t)\big|^2\,dx\,dt\Big)^{1/2}\sqrt{\mathcal{G}(u,0)}.
\end{aligned} \tag{4.22}$$

By replacing from steps 2 and 3, f by u and a by $\widetilde{a}$, respectively, where $\widetilde{a}$ is given by

$$\widetilde{a}(x,t,s) = \left(\frac{1}{(is+1)^{3/2}}e^{-\frac{1}{4h}\frac{|x|^2}{is+1}}\right)\left(\frac{\sqrt{2}}{\sqrt{-ihs+2}}\,e^{-\frac{1}{4}\frac{t^2}{-ihs+2}}\right),$$

and satisfies $\widetilde{a}(x-x_o,t,0) = a_o(x,t/\sqrt{2})$ and $(i\partial_s + h(\Delta-\partial_t^2))\widetilde{a}(x,t,s) = 0$, we can argue in a similar fashion than in the precedent steps to show that

$$\begin{aligned}
&\frac{1}{h\sqrt{h}}\int_{\Omega\times\mathbb{R}} a_o(x,t/\sqrt{2})\big[\varphi(x,t/\sqrt{2})u(x,t)\big]u(x,t)\,dx\,dt \\
&\quad\leqslant c\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + c\left(\frac{1}{h}\right)^{\gamma}\Big(\int_{\Upsilon}\int_{|t|\leqslant\gamma(\frac{1}{h})^{\gamma}}\big|\partial_\nu u(x,t)\big|^2\,dx\,dt\Big)^{1/2}\sqrt{\mathcal{G}(u,0)}.
\end{aligned} \tag{4.23}$$

Consequently, from (4.2), (4.22) and (4.23), we obtain that for any $\{x_o^i\}_{i\in I} \in \overline{\omega_o}$,

$$\begin{aligned}
&\frac{1}{h\sqrt{h}}\int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x)\big|e^{-\frac{1}{4}(\frac{1}{h}|x-x_o^i|^2+t^2)}\partial_t u(x,t)\big|^2\,dx\,dt \\
&\quad\leqslant c\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + c\left(\frac{1}{h}\right)^{\gamma}\Big(\int_{\Upsilon}\int_{|t|\leqslant\gamma(\frac{1}{h})^{\gamma}}\big|\partial_\nu u(x,t)\big|^2\,dx\,dt\Big)^{1/2}\sqrt{\mathcal{G}(u,0)},
\end{aligned}$$

and finally, by (4.1), the desired kind of Hölder interpolation inequality is established.

Appendix A

The goal of this appendix is to prove, with the notations of the above sections, the two following inequalities.

Lemma A. *For any $h \in (0, h_o]$, $\lambda \geqslant 1$ and any u solution of* (1.4) *with initial data* $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$,

$$\begin{aligned}
&\left| \int_{\Omega\times\mathbb{R}} a_o(x,t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geqslant\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi u}(\xi,\tau)\, d\xi\, d\tau\, u(x,t)\, dx\, dt \right| \\
&\qquad + \left| \int_{\Omega\times\mathbb{R}} a_o(x,t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geqslant\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi \partial_t^2 u}(\xi,\tau)\, d\xi\, d\tau\, u(x,t)\, dx\, dt \right| \\
&\qquad \leqslant c\sqrt{\frac{1}{\lambda}} \sqrt{\mathcal{G}(u,0)}\, \sqrt{\mathcal{G}(\partial_t u, 0)}
\end{aligned} \tag{A.1}$$

and

$$\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left|\widehat{\varphi u}(\xi,\tau)\right| d\xi\, d\tau + \int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left|\widehat{\varphi \partial_t^2 u}(\xi,\tau)\right| d\xi\, d\tau \leqslant c\left(\frac{\lambda}{h}\right)^{\gamma} \sqrt{\mathcal{G}(u,0)}. \tag{A.2}$$

To this end, we essentially use the formula of conservation of energy (3.3) applied to u and to $\partial_t u$.

Proof of (A.1). Introduce

$$\mathcal{R}(g) = \int_{\Omega\times\mathbb{R}} a_o(x,t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geqslant\lambda} e^{i(x\xi+t\tau)}\, \widehat{\varphi g}(\xi,\tau)\, d\xi\, d\tau\, u(x,t)\, dx\, dt.$$

Thus,

$$\begin{aligned}
\left|\mathcal{R}(g)\right| &= \left| \int_{\Omega\times\mathbb{R}} a_o(x,t) \partial_t \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geqslant\lambda} \frac{1}{i\tau} e^{i(x\xi+t\tau)} \widehat{\varphi g}(\xi,\tau)\, d\xi\, d\tau \right) u(x,t)\, dx\, dt \right| \\
&= \left| \int_{\Omega\times\mathbb{R}} \partial_t\big(a_o u(x,t)\big) \left(\frac{1}{2\pi} \int_{|\tau|\geqslant\lambda} \frac{1}{i\tau} e^{it\tau} \left[\int_{\mathbb{R}} e^{-i\theta\tau} (\varphi g)(x,\theta)\, d\theta \right] d\tau \right) dx\, dt \right|.
\end{aligned}$$

It follows using Cauchy–Schwarz inequality and Parseval identity that

$$\left|\mathcal{R}(g)\right| \leqslant \int_{\Omega\times\mathbb{R}} \left|\partial_t\big(a_o u(x,t)\big)\right| \left(\frac{1}{2\pi} \left[\int_{|\tau|\geqslant\lambda} \frac{1}{\tau^2}\, d\tau \right]^{1/2} \left[\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-i\theta\tau} (\varphi g)(x,\theta)\, d\theta \right|^2 d\tau \right]^{1/2} \right) dx\, dt$$

$$\leqslant \int_{\Omega\times\mathbb{R}} \left|\partial_t\big(a_o u(x,t)\big)\right| \left(\frac{1}{2\pi}\left[\int_{|\tau|\geqslant\lambda} \frac{1}{\tau^2}\,d\tau\right]^{1/2}\left[2\pi\int_{\mathbb{R}} \left|(\varphi g)(x,\theta)\right|^2 d\theta\right]^{1/2}\right) dx\,dt$$

$$\leqslant \int_{\Omega\times\mathbb{R}} \left|\partial_t\big(a_o u(x,t)\big)\right| \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\lambda}}\left\|(\varphi g)(x,\cdot)\right\|_{L^2(\mathbb{R})}\right) dx\,dt$$

$$\leqslant \frac{1}{\sqrt{\pi}}\sqrt{\frac{1}{\lambda}}\int_{\mathbb{R}} \left\|\partial_t(a_o u)(\cdot,t)\right\|_{L^2(\Omega)} dt\,\|\varphi g\|_{L^2(\Omega\times\mathbb{R})}.$$

Since we have the following estimates

$$\|\varphi u\|_{L^2(\Omega\times\mathbb{R})} + \left\|\varphi\partial_t^2 u\right\|_{L^2(\Omega\times\mathbb{R})} \leqslant c\left[\int_{\mathbb{R}} e^{-\frac{1}{\gamma}t^2}\int_{\Omega}\left|\Delta u(x,t)\right|^2 dx\,dt\right]^{1/2} \quad \text{because } \partial_t^2 u = \Delta u$$

$$\leqslant c\sqrt{\mathcal{G}(\partial_t u,0)},$$

$$\int_{\mathbb{R}} \left\|\partial_t(a_o u)(\cdot,t)\right\|_{L^2(\Omega)} dt$$

$$\leqslant \int_{\mathbb{R}}\left[\int_{\Omega}\left|\partial_t a_o u(x,t)\right|^2 dx\right]^{1/2} dt + \int_{\mathbb{R}}\left[\int_{\Omega}\left|a_o\partial_t u(x,t)\right|^2 dx\right]^{1/2} dt$$

$$\leqslant \int_{\mathbb{R}} |t|e^{-\frac{1}{\gamma}t^2}\left[\int_{\Omega}\left|u(x,t)\right|^2 dx\right]^{1/2} dt + \int_{\mathbb{R}} e^{-\frac{1}{\gamma}t^2}\left[\int_{\Omega}\left|\partial_t u(x,t)\right|^2 dx\right]^{1/2} dt$$

$$\leqslant c\sqrt{\mathcal{G}(u,0)},$$

we conclude that

$$\left|\mathcal{R}(u)\right| + \left|\mathcal{R}\big(\partial_t^2 u\big)\right| \leqslant c\sqrt{\frac{1}{\lambda}}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)}.$$

This completes the proof of (A.1). □

Proof of (A.2). We estimate $\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}|\widehat{\varphi g}(\xi,\tau)|\,d\xi\,d\tau$ where g solves $\partial_t^2 g - \Delta g = 0$ in $\Omega\times\mathbb{R}$. By Cauchy–Schwarz inequality and Parseval identity,

$$\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi g}(\xi,\tau)\right| d\xi\,d\tau$$

$$= \int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\frac{1}{1+|\xi|^2}\left|\big(1+|\xi|^2\big)\widehat{\varphi g}(\xi,\tau)\right| d\xi\,d\tau$$

$$\leqslant \int_{|\tau|<\lambda} \left[\int_{\mathbb{R}^3} \frac{1}{(1+|\xi|^2)^2}\, d\xi \right]^{1/2} \left[\int_{\mathbb{R}^3} \left|\mathcal{F}\big((1-\Delta)(\varphi g)\big)(\xi,\tau)\right|^2 d\xi \right]^{1/2} d\tau$$

$$\leqslant c \int_{|\tau|<\lambda} \left[\int_{\mathbb{R}^3} \left|\mathcal{F}\big((1-\Delta)(\varphi g)\big)(\xi,\tau)\right|^2 d\xi \right]^{1/2} d\tau$$

$$\leqslant c\sqrt{\lambda} \left[\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left|\mathcal{F}\big((1-\Delta)(\varphi g)\big)(\xi,\tau)\right|^2 d\xi\, d\tau \right]^{1/2}.$$

Observe that $\varphi(x,t) = \varphi(x_o,t)\varphi(x,0)$ and

$$\begin{aligned}
\Delta(\varphi g) &= \varphi \Delta g + 2\nabla\varphi\nabla g + \Delta\varphi g \\
&= \varphi \partial_t^2 g + 2\varphi(x_o,t)\nabla\varphi(x,0)\nabla g + \Delta\varphi g \\
&= \partial_t^2(\varphi g) - 2\partial_t(\partial_t\varphi g) + \big(\partial_t^2\varphi + \Delta\varphi\big) g + 2\varphi(x_o,t)\nabla\varphi(x,0)\nabla g.
\end{aligned}$$

As a result,

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left|\widehat{\varphi g}(\xi,\tau)\right| d\xi\, d\tau \leqslant{}& c\sqrt{\lambda}\lambda^2 \left[\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \sum_{k=0,1,2} \left|\mathcal{F}\big(\partial_t^{2-k}\varphi g\big)(\xi,\tau)\right|^2 d\xi\, d\tau \right]^{1/2} \\
&+ c\sqrt{\lambda} \left[\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left|\mathcal{F}(\Delta\varphi g)(\xi,\tau)\right|^2 d\xi\, d\tau \right]^{1/2} \\
&+ c\sqrt{\lambda} \left[\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left|\mathcal{F}(\varphi(x_o,t)\nabla\varphi(x,0)\nabla g)(\xi,\tau)\right|^2 d\xi\, d\tau \right]^{1/2}.
\end{aligned}$$

In particular, when $g = u$ we obtain, using Parseval identity

$$\begin{aligned}
&\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left|\widehat{\varphi u}(\xi,\tau)\right| d\xi\, d\tau \\
&\quad \leqslant c\sqrt{\lambda}\lambda^2 \sum_{k=0,1,2} \left\|\partial_t^{2-k}\varphi u\right\|_{L^2(\Omega\times\mathbb{R})} + c\sqrt{\lambda}\|\Delta\varphi u\|_{L^2(\Omega\times\mathbb{R})} \\
&\qquad + c\sqrt{\lambda}\left\|\varphi(x_o,t)\nabla\varphi(x,0)\nabla u\right\|_{L^2(\Omega\times\mathbb{R})} \\
&\quad \leqslant c\left(\sqrt{\lambda}\lambda^2 + c\sqrt{\lambda}\frac{1}{h}\right)\left[\int_{\mathbb{R}} e^{-\frac{1}{\gamma}t^2} \left(\int_{\Omega} \left|u(x,t)\right|^2 dx \right) dt \right]^{1/2} + c\sqrt{\lambda}\frac{1}{\sqrt{h}}\sqrt{\mathcal{G}(u,0)}.
\end{aligned}$$

With similar arguments, when $g = \partial_t^2 u$ and using the fact that $|\tau| < \lambda$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_{|\tau|<\lambda} \left|\widehat{\varphi\partial_t^2 u}(\xi,\tau)\right| d\xi\, d\tau \\
&\quad\leqslant c\sqrt{\lambda}\lambda^3\Bigg[\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\sum_{k=0,1,2,3}\left|\mathcal{F}\big(\partial_t^{3-k}\varphi\partial_t u\big)(\xi,\tau)\right|^2 d\xi\, d\tau\Bigg]^{1/2} \\
&\qquad+ c\sqrt{\lambda}\lambda\Bigg[\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\sum_{k=0,1}\left|\mathcal{F}\big(\partial_t^{1-k}\Delta\varphi\partial_t u\big)(\xi,\tau)\right|^2 d\xi\, d\tau\Bigg]^{1/2} \\
&\qquad+ c\sqrt{\lambda}\lambda^2\Bigg[\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\sum_{k=0,1,2}\left|\mathcal{F}\big(\partial_t^{2-k}\varphi(x_o,t)\nabla\varphi(x,0)\nabla u\big)(\xi,\tau)\right|^2 d\xi\, d\tau\Bigg]^{1/2} \\
&\quad\leqslant c\left(\sqrt{\lambda}\lambda^3+\sqrt{\lambda}\lambda\frac{1}{h}+\sqrt{\lambda}\lambda^2\frac{1}{\sqrt{h}}\right)\sqrt{\mathcal{G}(u,0)}.
\end{aligned}
$$

We conclude that there exist $c>0$ and $\gamma>1$ such that for any $h\in(0,h_o]$ and $\lambda\geqslant 1$,

$$
\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi u}(\xi,\tau)\right| d\xi\, d\tau+\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi\partial_t^2 u}(\xi,\tau)\right| d\xi\, d\tau\leqslant c\left(\frac{\lambda}{h}\right)^{\gamma}\sqrt{\mathcal{G}(u,0)}.
$$

This completes the proof of (A.2). □

Appendix B

In this appendix, we give a criterion for polynomial decay rate.

Lemma B. *Let $\mathcal{H}$ be a continuous positive decreasing real function on $[0,+\infty)$ and bounded by one. Suppose that there are four constants $c_1>1$ and $c_2,\beta,\gamma>0$ such that*

$$
\mathcal{H}(s)\leqslant c_1\left(\frac{1}{\mathcal{H}(s)}\right)^{\beta}\left(\mathcal{H}(s)-\mathcal{H}\left(\left(\frac{c_2}{\mathcal{H}(s)}\right)^{\gamma}+s\right)\right)\quad\forall s>0.
$$

Then there exist $C>0$ and $\delta>0$ such that for any $t>0$,

$$
\mathcal{H}(t)\leqslant\frac{C}{t^{\delta}}.
$$

Proof. Let $t>0$. We distinguish two cases: if $(\frac{\mathcal{H}(s)}{c_2})^{\gamma}<\frac{1}{t}$ then $\mathcal{H}(s)\leqslant\frac{c_2}{t^{1/\gamma}}$; if $\frac{1}{t}\leqslant(\frac{\mathcal{H}(s)}{c_2})^{\gamma}$ then $(\frac{c_2}{\mathcal{H}(s)})^{\gamma}+s\leqslant t+s$, thus $\mathcal{H}(t+s)\leqslant\mathcal{H}((\frac{c_2}{\mathcal{H}(s)})^{\gamma}+s)$ and therefore

$$
\mathcal{H}(s)\leqslant(c_1)^{\frac{1}{\beta+1}}\big\{\mathcal{H}(s)-\mathcal{H}(t+s)\big\}^{\frac{1}{\beta+1}}.
$$

Consequently,

$$
\mathcal{H}(s)\leqslant(c_1)^{\frac{1}{\beta+1}}\big\{\mathcal{H}(s)-\mathcal{H}(t+s)\big\}^{\frac{1}{\beta+1}}+\frac{c_2}{t^{1/\gamma}}\quad\forall s,t>0.
$$

Let

$$\psi_t(s) = \frac{1}{(\frac{c_1 t}{s})^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}}.$$

Again, we distinguish two cases: if $\mathcal{H}(s) - \mathcal{H}(t+s) \leqslant \frac{t}{t+s}$, then $\mathcal{H}(s) \leqslant (\frac{c_1 t}{t+s})^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}$ and thus $\psi_t(t+s)\mathcal{H}(t+s) \leqslant 1$; if $\frac{t}{t+s} < \mathcal{H}(s) - \mathcal{H}(t+s)$, then $\frac{t}{t+s}\mathcal{H}(s) \leqslant \frac{t}{t+s} < \mathcal{H}(s) - \mathcal{H}(t+s)$ and therefore

$$\begin{aligned}\psi_t(t+s)\mathcal{H}(t+s) &< \frac{s}{t+s}\mathcal{H}(s)\psi_t(t+s) = \psi_t(s)\mathcal{H}(s)\left(\frac{\frac{\psi_t(t+s)}{t+s}}{\frac{\psi_t(s)}{s}}\right)\\ &< \psi_t(s)\mathcal{H}(s),\end{aligned}$$

by using the decreasing property of $\zeta \mapsto \frac{\psi_t(\zeta)}{\zeta}$. Consequently, we have proved that for any $s, t > 0$, we have either $\psi_t(t+s)\mathcal{H}(t+s) \leqslant 1$, or $\psi_t(t+s)\mathcal{H}(t+s) < \psi_t(s)\mathcal{H}(s)$. In particular, we deduce that for any $t > 0$ and $n \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned}\text{either}\quad & \psi_t\big((n+1)t\big)\mathcal{H}\big((n+1)t\big) \leqslant 1\\ \text{or}\quad & \psi_t\big((n+1)t\big)\mathcal{H}\big((n+1)t\big) < \psi_t(nt)\mathcal{H}(nt).\end{aligned}$$

Then inductively, it implies that

$$\psi_t\big((n+1)t\big)\mathcal{H}\big((n+1)t\big) \leqslant \max\big(1, \psi_t(t)\mathcal{H}(t)\big) = 1,$$

where we have used that $c_1 > 1$ and $\mathcal{H} \leqslant 1$. Hence for all $t > 0$ and $n \in \mathbb{N} \setminus \{0\}$,

$$\mathcal{H}\big((n+1)t\big) \leqslant \left(\frac{c_1}{n+1}\right)^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}.$$

We choose n such that $n+1 \leqslant t < n+2$ and we obtain that for all $t \geqslant 2$,

$$\mathcal{H}(t^2) \leqslant \left(\frac{2c_1}{t}\right)^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}.$$

The desired result now follows immediately. □

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