Waves, Damped Wave and Observation*

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Abstract

This article describes some applications of two kinds of observation estimates for the wave equation and for the damped wave equation in a bounded domain where the geometric control condition of C. Bardos, G. Lebeau and J. Rauch may fail.

1 The wave equation and observation

We consider the wave equation in the solution $u = u(x,t)$

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R} , \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) & , \end{cases} \tag{1.1}$$

consisting in a bounded open set Ω in $\mathbb{R}^n$, $n \geq 1$, either convex or C^2, to be connected with boundary $\partial\Omega$. It is well known that for any initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the above problem is well posed and has a unique strong solution.

Linked to exact controllability and strong stabilization for the wave equation (see [Li]), it appears the following observability problem which consists in proving the following estimate

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \int_0^T \int_\omega |\partial_t u(x,t)|^2 \, dxdt$$

for some constant $C > 0$ independent of the initial data. Here, $T > 0$ and ω is a non-empty open subset in Ω. Due to finite speed of propagation,

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the time T has to be chosen to be large enough. Dealing with high frequency waves, i.e., waves which propagate according to the law of geometrical optics, the choice of ω can not be arbitrary. In other words, the existence of trapped rays (e.g, constructed with gaussian beams (see [Ra]) implies the requirement of some kinds of geometric conditions on (ω, T) (see [BLR]) in order that the above observability estimate may hold.

Now, we want to know what kind of estimate we may expect in a geometry with trapped rays. Let us introduce the quantity

$$\Lambda = \frac{\|(u_0, u_1)\|_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)}}{\|(u_0, u_1)\|_{H_0^1(\Omega) \times L^2(\Omega)}} ,$$

which can be seen as a measure of the frequency of the wave. In this paper, we present the two following inequalities

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq e^{C\Lambda^{1/\beta}} \int_0^T \int_\omega |\partial_t u\,(x,t)|^2\, dx dt \qquad (1.2)$$

and

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \int_0^{C\Lambda^{1/\gamma}} \int_\omega |\partial_t u\,(x,t)|^2\, dx dt \qquad (1.3)$$

where $\beta \in (0, 1)$, $\gamma > 0$. We will also give their applications to control theory.

The strategy to get estimate (1.2) is now well known (see [Ro2],[LR]) and a sketch of the proof will be given in Appendix for completeness. More precisely, we have the following results.

Theorem 1.1. *For any ω non-empty open subset in Ω, for any $\beta \in (0, 1)$, there exist $C > 0$ and $T > 0$ such that for any solution u of (1.1) with non-identically zero initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality (1.2) holds.*

Now, we can ask whether it is possible to get another weight function of Λ other than the exponential one, a polynomial weight function with a geometry (Ω, ω) with trapped rays in particular. Here we present the following results.

Theorem 1.2. *There exists a geometry (Ω, ω) with trapped rays such that for any solution u of (1.1) with non-identically zero initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality (1.3) holds for some $C > 0$ and $\gamma > 0$.*

The proof of Theorem 1.2 is given in [Ph1]. With the help of Theorem 2.1 below, it can also be deduced from [LiR], [BuH].

2 The damped wave equation and our motivation

We consider the following damped wave equation in the solution $w = w(x,t)$

$$\begin{cases} \partial_t^2 w - \Delta w + 1_\omega \partial_t w = 0 & \text{in } \Omega \times (0,+\infty) \ , \\ w = 0 & \text{on } \partial\Omega \times (0,+\infty) \ , \end{cases} \tag{2.1}$$

consisting in a bounded open set Ω in $\mathbb{R}^n$, $n \geq 1$, either convex or C^2, to be connected with boundary $\partial\Omega$. Here ω is a non-empty open subset in Ω with trapped rays and 1_ω denotes the characteristic function on ω. Further, for any $(w, \partial_t w)(\cdot, 0) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the above problem is well posed for any $t \geq 0$ and has a unique strong solution.

Denote for any $g \in C\left([0,+\infty); H_0^1(\Omega)\right) \cap C^1\left([0,+\infty); L^2(\Omega)\right)$,

$$E(g,t) = \frac{1}{2} \int_\Omega \left(|\nabla g(x,t)|^2 + |\partial_t g(x,t)|^2 \right) dx \ .$$

Then for any $0 \leq t_0 < t_1$, the strong solution w satisfies the following formula

$$E(w,t_1) - E(w,t_0) + \int_{t_0}^{t_1} \int_\omega |\partial_t w(x,t)|^2 \, dxdt = 0 \ . \tag{2.2}$$

2.1 The polynomial decay rate

Our motivation for establishing estimate (1.3) comes from the following result.

Theorem 2.1. *The following two assertions are equivalent. Let* $\delta > 0$.

(i) There exists $C > 0$ such that for any solution w of (2.1) with the non-null initial data $(w, \partial_t w)(\cdot, 0) = (w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, we have

$$\|(w_0, w_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \int_0^{C\left(\frac{E(\partial_t w, 0)}{E(w,0)}\right)^{1/\delta}} \int_\omega |\partial_t w(x,t)|^2 \, dxdt \ .$$

(ii) There exists $C > 0$ such that the solution w of (2.1) with the initial data $(w, \partial_t w)(\cdot, 0) = (w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ satisfies

$$E(w,t) \leq \frac{C}{t^\delta} \ \|(w_0, w_1)\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)} \quad \forall t > 0 \ .$$

Remark. It is not difficult to see (e.g., [Ph2]) by a classical decomposition method, a translation in time and (2.2), that the inequality (1.3) with the exponent γ for the wave equation implies the inequality of (i) in Theorem 2.1 with the exponent $\delta = 2\gamma/3$ for the damped wave equation. And conversely, the inequality of (i) in Theorem 2.1 with the exponent δ for the damped wave equation implies the inequality (1.3) with the exponent $\gamma = \delta/2$ for the wave equation.

Proof of Theorem 2.1.

$(ii) \Rightarrow (i)$. Suppose that

$$E(w,T) \leq \frac{C}{T^\delta} \|(w_0,w_1)\|^2_{H^2\cap H^1_0(\Omega)\times H^1_0(\Omega)} \quad \forall T > 0 \ .$$

Therefore from (2.2)

$$E(w,0) \leq \frac{C}{T^\delta} \|(w_0,w_1)\|^2_{H^2\cap H^1_0(\Omega)\times H^1_0(\Omega)} + \int_0^T \int_\omega |\partial_t w(x,t)|^2 \, dxdt \ .$$

By choosing

$$T = \left(2C \frac{\|(w_0,w_1)\|^2_{H^2\cap H^1_0(\Omega)\times H^1_0(\Omega)}}{E(w,0)} \right)^{1/\delta} ,$$

we get the desired estimate

$$E(w,0) \leq 2 \int_0^{\left[2C \frac{\|(w_0,w_1)\|^2_{H^2\cap H^1_0(\Omega)\times H^1_0(\Omega)}}{E(w,0)} \right]^{1/\delta}} \int_\omega |\partial_t w(x,t)|^2 \, dxdt \ .$$

$(i) \Rightarrow (ii)$. Conversely, suppose the existence of a constant $c > 1$ such that the solution w of (2.1) with the non-null initial data $(w, \partial_t w)(\cdot, 0) = (w_0, w_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ satisfies

$$E(w,0) \leq c \int_0^{c\left(\frac{E(w,0)+E(\partial_t w,0)}{E(w,0)}\right)^{1/\delta}} \int_\omega |\partial_t w(x,t)|^2 \, dxdt \ .$$

We obtain the following inequalities by a translation on the time variable and by using (2.2). $\forall s \geq 0$

$$\begin{aligned} \frac{E(w,s)}{E(w,0)+E(\partial_t w,0))} &\leq c \int_s^{s+c\left(\frac{(E(w,0)+E(\partial_t w,0))}{E(w,s)}\right)^{1/\delta}} \int_\omega \frac{|\partial_t w(x,t)|^2}{E(w,0)+E(\partial_t w,0)} dxdt \\ &\leq c \left(\frac{E(w,s)}{E(w,0)+E(\partial_t w,0)} - \frac{E\left(w, s+c\left(\frac{E(w,0)+E(\partial_t w,0)}{E(w,s)}\right)^{1/\delta}\right)}{E(w,0)+E(\partial_t w,0)} \right) . \end{aligned}$$

Denoting $G(s) = \frac{E(w,s)}{E(w,0)+E(\partial_t w,0)}$, we deduce using the decreasing of G that

$$G\left(s + c\left(\frac{1}{G(s)}\right)^{1/\delta}\right) \leq G(s) \leq c\left[G(s) - G\left(s + c\left(\frac{1}{G(s)}\right)^{1/\delta}\right)\right]$$

which gives

$$G\left(s + c\left(\frac{1}{G(s)}\right)^{1/\delta}\right) \leq \frac{c}{1+c}G(s) \ .$$

Let $c_1 = \left(\frac{1+c}{c}\right)^{1/\delta} - 1 > 0$ and denote $d(s) = \left(\frac{c}{c_1}\frac{1}{s}\right)^{\delta}$. We distinguish two cases.

If $c_1 s \leq c\left(\frac{1}{G(s)}\right)^{1/\delta}$, then $G(s) \leq \left(\frac{c}{c_1}\frac{1}{s}\right)^{\delta}$ and

$$G((1+c_1)s) \leq d(s) \ .$$

If $c_1 s > c\left(\frac{1}{G(s)}\right)^{1/\delta}$, then $s + c\left(\frac{1}{G(s)}\right)^{1/\delta} < (1+c_1)s$ and the decreasing of G gives $G((1+c_1)s) \leq G\left(s + c\left(\frac{1}{G(s)}\right)^{1/\delta}\right)$ and then

$$G((1+c_1)s) \leq \frac{c}{1+c}G(s) \ .$$

Consequently, we have that $\forall s > 0$, $\forall n \in \mathbb{N}$, $n \geq 1$,

$$\begin{aligned} G((1+c_1)s) \leq \max\Big[& d(s), \tfrac{c}{1+c}d\left(\tfrac{s}{(1+c_1)}\right), \cdots, \\ & \left(\tfrac{c}{1+c}\right)^n d\left(\tfrac{s}{(1+c_1)^n}\right), \left(\tfrac{c}{1+c}\right)^{n+1} G\left(\tfrac{s}{(1+c_1)^n}\right)\Big] \ . \end{aligned}$$

Now, remark that with our choice of c_1, we get

$$\frac{c}{1+c}d\left(\frac{s}{(1+c_1)}\right) = d(s) \quad \forall s > 0 \ .$$

Thus, we deduce that $\forall n \geq 1$

$$\begin{aligned} G((1+c_1)s) &\leq \max\left(d(s), \left(\tfrac{c}{1+c}\right)^{n+1} G\left(\tfrac{s}{(1+c_1)^n}\right)\right) \\ &\leq \max\left(d(s), \left(\tfrac{c}{1+c}\right)^{n+1}\right) \qquad \text{because } G \leq 1 \ , \end{aligned}$$

and conclude that $\forall s > 0$

$$\frac{E(w,s)}{E(w,0) + E(\partial_t w, 0)} = G(s) \leq d\left(\frac{s}{1+c_1}\right) = \left(\frac{c(1+c_1)}{c_1}\right)^{\delta}\frac{1}{s^{\delta}} \ .$$

This completes the proof.

2.2 The approximate controllability

The goal of this section consists in giving an application of estimate (1.2).

For any ω non-empty open subset in Ω, for any $\beta \in (0,1)$, let $T > 0$ be given in Theorem 1.1.

Let $(v_0, v_1, v_{0d}, v_{1d}) \in \left(H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)\right)^2$ and u be the solution of (1.1) with initial data $(u, \partial_t u)(\cdot, 0) = (v_0, v_1)$.

For any integer $N > 0$, let us introduce

$$f_N(x,t) = -1_\omega \sum_{\ell=0}^{N} \left[\partial_t w^{(2\ell+1)}(x,t) + \partial_t w^{(2\ell)}(x, T-t)\right] , \qquad (2.3)$$

where $w^{(0)} \in C\left([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\right)$ is the solution of the damped wave equation (2.1) with initial data

$$\left(w^{(0)}, \partial_t w^{(0)}\right)(\cdot, 0) = (v_{0d}, -v_{1d}) - (u, -\partial_t u)(\cdot, T) \text{ in } \Omega ,$$

and for $j \geq 0$, $w^{(j+1)} \in C\left([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\right)$ is the solution of the damped wave equation (2.1) with initial data

$$\left(w^{(j+1)}, \partial_t w^{(j+1)}\right)(\cdot, 0) = \left(-w^{(j)}, \partial_t w^{(j)}\right)(\cdot, T) \text{ in } \Omega .$$

Introduce

$$M = \sup_{j \geq 0} \left\| w^{(j)}(\cdot, 0), \partial_t w^{(j)}(\cdot, 0)\right\|^2_{H^2(\Omega) \times H_0^1(\Omega)} .$$

Our main result is as follows.

Theorem 2.2 . *Suppose that $M < +\infty$. Then there exists $C > 0$ such that for all $N > 0$, the control function f_N given by (2.3) drives the system*

$$\begin{cases} \partial_t^2 v - \Delta v = 1_{\omega \times (0,T)} f_N & \text{in } \Omega \times (0,T) , \\ v = 0 & \text{on } \partial\Omega \times (0,T) , \\ (v, \partial_t v)(\cdot, 0) = (v_0, v_1) & \text{in } \Omega , \end{cases}$$

to the desired data (v_{0d}, v_{1d}) approximately at time T, i.e.,

$$\|v(\cdot, T) - v_{0d}, \partial_t v(\cdot, T) - v_{1d}\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq \frac{C}{[\ln(1+2N)]^{2\beta}} M ,$$

and satisfies

$$\|f_N\|_{L^\infty(0,T; L^2(\Omega))} \leq C(N+1) \|(v_0, v_1, v_{0d}, v_{1d})\|_{(H_0^1(\Omega) \times L^2(\Omega))^2} .$$

Remark. For any $\varepsilon > 0$, we can choose N such that

$$\frac{C}{[\ln(1+2N)]^{2\beta}} M \simeq \varepsilon^2 \text{ and } (2N+1) \simeq e^{\left(\frac{\sqrt{CM}}{\varepsilon}\right)^{1/\beta}} ,$$

in order that

$$\|v(\cdot,T) - v_{0d}, \partial_t v(\cdot,T) - v_{1d}\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon ,$$

and

$$\|f\|_{L^\infty(0,T;L^2(\Omega))} \leq e^{\left[\left(\frac{C}{\varepsilon}\sqrt{M}\right)^{1/\beta}\right]} \|(v_0, v_1, v_{0d}, v_{1d})\|_{\left(H_0^1(\Omega)\times L^2(\Omega)\right)^2} .$$

In [Zu], a method was proposed to construct an approximate control. It consists of minimizing a functional depending on the parameter ε. However, no estimate of the cost is given. On the other hand, estimate of the form (1.2) was originally established by [Ro2] to give the cost (see [Le]). Here, we present a new way to construct an approximate control by superposing different waves. Given a cost to be not overcome, we construct a solution which will be closed in the above sense to the desired state. It takes ideas from [Ru] and [BF] like an iterative time reversal construction.

2.2.1 Proof

Consider the solution

$$V(\cdot,t) = \sum_{\ell=0}^{N} \left[w^{(2\ell+1)}(\cdot,t) + w^{(2\ell)}(\cdot,T-t) \right] .$$

We deduce that for $t \in (0,T)$

$$\begin{cases} \partial_t^2 V(\cdot,t) - \Delta V(\cdot,t) = -1_\omega \sum_{\ell=0}^{N} \left[\partial_t w^{(2\ell+1)}(\cdot,t) + \partial_t w^{(2\ell)}(\cdot,T-t)\right] , \\ V = 0 \quad \text{on } \partial\Omega \times (0,T) , \\ (V, \partial_t V)(\cdot,0) = 0 \quad \text{in } \Omega . \end{cases}$$

Now, from the definition of $w^{(0)}$, the property of $\left(w^{(j+1)}, \partial_t w^{(j+1)}\right)(\cdot,0)$ and a change of variable, we obtain that

$$\begin{aligned} (V, \partial_t V)(\cdot,T) &= \left(w^{(0)}, -\partial_t w^{(0)}\right)(\cdot,0) + \left(w^{(2N+1)}, \partial_t w^{(2N+1)}\right)(\cdot,T) \\ &= (v_{0d}, v_{1d}) - (u, \partial_t u)(\cdot,T) + \left(w^{(2N+1)}, \partial_t w^{(2N+1)}\right)(\cdot,T) \end{aligned}$$

Finally, the solution $v = V + u$ satisfies

$$\begin{cases} \partial_t^2 v - \Delta v = 1_{\omega\times(0,T)} f_N \quad \text{in } \Omega \times (0,T) , \\ v = 0 \quad \text{on } \partial\Omega \times (0,T) , \\ (v, \partial_t v)(\cdot,0) = (v_0, v_1) \quad \text{in } \Omega , \\ (v, \partial_t v)(\cdot,T) = (v_{0d}, v_{1d}) + \left(w^{(2N+1)}, \partial_t w^{(2N+1)}\right)(\cdot,T) \quad \text{in } \Omega . \end{cases}$$

Clearly,

$$\|v(\cdot,T)-v_{0d},\partial_t v(\cdot,T)-v_{1d}\|^2_{H_0^1(\Omega)\times L^2(\Omega)}=2E\left(w^{(2N+1)},T\right) .$$

It remains to estimate $E\left(w^{(2N+1)},T\right)$. We claim that

$$\exists C>0 \qquad \forall N\geq 1 \qquad E\left(w^{(2N+1)},T\right)\leq \frac{C}{\left[\ln(1+2N)\right]^{2\beta}}M .$$

Indeed, from Theorem 1.1, we can easily see by a classical decomposition method that there exist $C>0$ and $T>0$ such that for any $j\geq 0$,

$$\begin{aligned}&\left\|w^{(j+1)}(\cdot,0),\partial_t w^{(j+1)}(\cdot,0)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\ &\leq C\exp\left(C\frac{\left\|w^{(j+1)}(\cdot,0),\partial_t w^{(j+1)}(\cdot,0)\right\|_{H^2(\Omega)\times H^1(\Omega)}}{\left\|w^{(j+1)}(\cdot,0),\partial_t w^{(j+1)}(\cdot,0)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}}\right)^{1/\beta}\\ &\int_0^T\int_\omega\left|\partial_t w^{(j+1)}(x,t)\right|^2dxdt .\end{aligned}$$

Since

$$E\left(w^{(j+1)},0\right)=E\left(w^{(j)},T\right) \qquad \forall j\geq 0 ,$$

we deduce from (2.2) that for any $j\geq 0$

$$\begin{aligned}E\left(w^{(j+1)},0\right)&\leq C\exp\left(C\frac{M}{\left\|w^{(j+1)}(\cdot,0),\partial_t w^{(j+1)}(\cdot,0)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}\right)^{1/(2\beta)}\\ &\left[E\left(w^{(j)},T\right)-E\left(w^{(j+1)},T\right)\right] .\end{aligned}$$

Let

$$d_j=E\left(w^{(j+1)},T\right) .$$

By using the decreasing property of the sequence d_j, that is $d_j\leq d_{j-1}$, we obtain that for any integer $0\leq j\leq 2N$

$$d_j\leq Ce^{\left(C\frac{M}{d_{2N}}\right)^{1/(2\beta)}}\left[d_{j-1}-d_j\right] .$$

By summing over $[0,2N]$, we deduce that

$$(2N+1)\,d_{2N}\leq Ce^{\left(C\frac{M}{d_{2N}}\right)^{1/(2\beta)}}\left[d_{-1}-d_{2N}\right] .$$

Finally, using the fact that $d_{-1}\leq M$, it follows that

$$d_{2N}\leq\frac{C}{\left[\ln(1+2N)\right]^{2\beta}}M .$$

This completes the proof of our claim.

On the other hand, the computation of the bound of f_N is immediate. Therefore, we check that for some $C > 0$ and $T > 0$,

$$\|f_N\|_{L^\infty(0,T;L^2(\Omega))} \leq C\,(N+1)\,\|(v_0, v_1, v_{0d}, v_{1d})\|_{\left(H_0^1(\Omega)\times L^2(\Omega)\right)^2} ,$$

$$\|v(\cdot,T) - v_{0d}, \partial_t v(\cdot,T) - v_{1d}\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \frac{C}{[\ln(1+2N)]^{2\beta}} M ,$$

for any $\beta \in (0,1)$ and any integer $N > 0$. This completes the proof of our Theorem.

2.2.2 Numerical experiments

Here, we perform numerical experiments to investigate the practical applicability of the approach proposed to construct an approximate control. For simplicity, we consider a square domain $\Omega = (0,1)\times(0,1)$, $\omega = (0,1/5)\times(0,1)$. The time of controllability is given by $T = 4$.

For convenience we recall some well-known formulas. Denote by $\{e_j\}_{j\geq 1}$ the Hilbert basis in $L^2(\Omega)$ formed by the eigenfunctions of the operator $-\Delta$ with eigenvalues $\{\lambda_j\}_{j\geq 1}$, such that $\|e_j\|_{L^2(\Omega)} = 1$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, i.e.,

$$\begin{cases} \lambda_j = \pi^2\left(k_j^2 + \ell_j^2\right), & k_j, \ell_j \in \mathbb{N}^*, \\ e_j(x_1,x_2) = 2\sin(\pi k_j x_1)\sin(\pi \ell_j x_2) . \end{cases}$$

The solution of

$$\begin{cases} \partial_t^2 v - \Delta v = f & \text{in } \Omega\times(0,T) , \\ v = 0 & \text{on } \partial\Omega\times(0,T) , \\ (v,\partial_t v)(\cdot,0) = (v_0,v_1) & \text{in } \Omega , \end{cases}$$

where f is in the form

$$f(x_1,x_2) = -1_\omega \sum_{j\geq 1} f_j(t)\, e_j(x_1,x_2) ,$$

is given by the formula

$$v(x_1,x_2,t) = \lim_{G\to+\infty} \sum_{j=1}^{G} \left\{ a_j^0 \cos\left(t\sqrt{\lambda_j}\right) + a_j^1 \tfrac{1}{\sqrt{\lambda_j}} \sin\left(t\sqrt{\lambda_j}\right) \right.$$
$$\left. + \tfrac{1}{\sqrt{\lambda_j}} \int_0^t \sin\left((t-s)\sqrt{\lambda_j}\right) R_j(s)\, ds \right\} e_j(x_1,x_2) ,$$

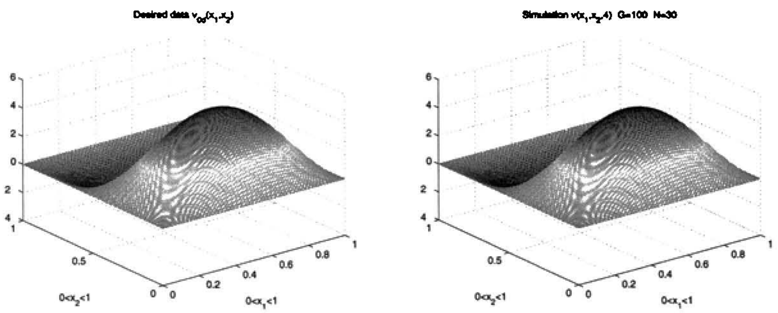
where

$$\begin{cases} v_0(x_1,x_2) = \lim\limits_{G\to+\infty} \sum\limits_{j=1}^{G} a_j^0 e_j(x_1,x_2), & \sum\limits_{j\geq 1} \lambda_j \left|a_j^0\right|^2 < +\infty , \\ v_1(x_1,x_2) = \lim\limits_{G\to+\infty} \sum\limits_{j=1}^{G} a_j^1 e_j(x_1,x_2), & \sum\limits_{j\geq 1} \left|a_j^1\right|^2 < +\infty , \\ R_j(t) = -\lim\limits_{G\to+\infty} \sum\limits_{i=1}^{G} \left(\int_\omega e_i e_j dx_1 dx_2\right) f_i(t) . \end{cases}$$

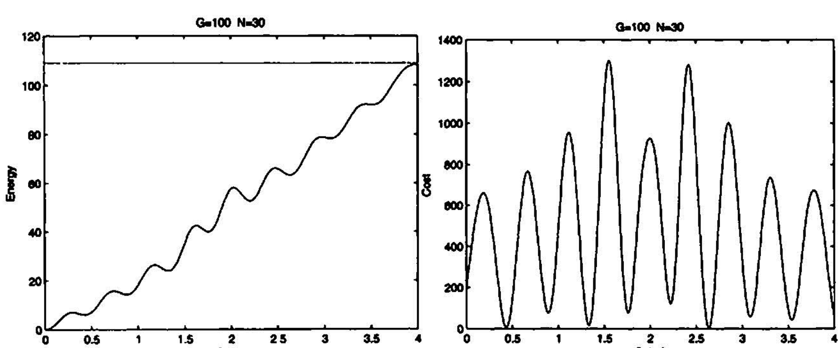
Here, G will be the number of Galerkin mode. The numerical results are shown below. The approximate solution of the damped wave equation is established via a system of ODE solved by MATLAB.

Example 1 : low frequency The initial condition and desired target are specifically as follows: $(v_0, v_1) = (0,0)$ and $(v_{0d}, v_{1d}) = (e_1 + e_2, e_1)$. We take the number of Galerkin mode $G = 100$ and the number of iterations in the time reversal construction $N = 30$.

Below, we plot the graph of the desired initial data v_{0d} and the controlled solution $v(\cdot, t = T = 4)$.



Below, we plot the graph of the energy of the controlled solution and the cost of the control function.



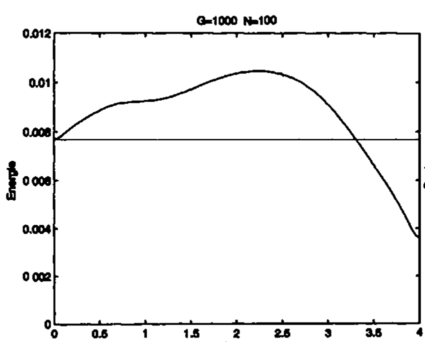
Example 2 : high frequency The initial condition and desired target are specifically as follows: $(v_{0d}, v_{1d}) = (0,0)$ and with $(k_o, a_o, b_o,) = (200, 1/2, 10000)$, for $(x_1, x_2) \in (0,1) \times (0,1)$,

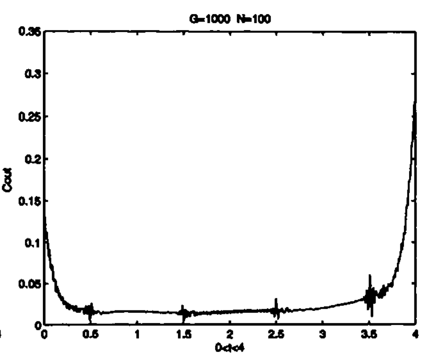
$$\begin{cases} v_0(x_1,x_2) = \sum_{j=1}^{G} \left(\int_0^1 \int_0^1 g_0(x_1,x_2) e_j(x_1,x_2) dx_1 dx_2 \right) e_j(x_1,x_2) \ , \\ v_1(x_1,x_2) = \sum_{j=1}^{G} \left(\int_0^1 \int_0^1 g_1(x_1,x_2) e_j(x_1,x_2) dx_1 dx_2 \right) e_j(x_1,x_2) \ , \\ g_0(x_1,x_2) = e^{-\frac{k_o a_o}{2}(x_1-x_{o1})^2} e^{-\frac{k_o b_o}{2}(x_2-x_{o2})^2} \cos(k_o(x_2-x_{o2})/2) \ , \\ g_1(x_1,x_2) = e^{-\frac{k_o a_o}{2}(x_1-x_{o1})^2} e^{-\frac{k_o b_o}{2}(x_2-x_{o2})^2} \\ \qquad [k_o b_o (x_2-x_{o2}) \cos(k_o(x_2-x_{o2})/2) \\ \qquad + (k_o/2 + a_o) \sin(k_o(x_2-x_{o2})/2) \\ \qquad -k_o a_o^2 (x_1-x_{o1})^2 \sin(k_o(x_2-x_{o2})/2)] \ . \end{cases}$$

Notice that we have chosen as initial data the G-first projections on the basis $\{e_j\}_{j\geq 1}$ of a gaussian beam $g(x_1,x_2,t)$ such that $g(\cdot, t=0) = g_0$, $\partial_t g(\cdot, t=0) = g_1$, which propagate in the direction $(0,1)$.

We take the number of Galerkin mode $G = 1000$ and the number of iterations in the time reversal construction $N = 100$.

Below, we plot the graph of the energy of the controlled solution and the cost of the control function.





3 Conclusion

In this paper, we have considered the wave equation in a bounded domain (eventually convex). Two kinds of inequalities are described when there occur trapped rays. Applications to control theory are given. First, we link such kind of estimate with the damped wave equation and its decay rate. Next, we describe the design of an approximate control function by an iterative time reversal method. We also provide a numerical simulation in a square domain. I'm grateful to Prof. Jean-Pierre Puel, the "French-Chinese Summer Institute on Applied Mathematics" and Fudan University for the kind invitation and the support to my visit.

4 Appendix

In this appendix, we recall most of the materials from the works by I. Kukavica [Ku2] and L. Escauriaza [E] for the elliptic equation and from the works by G. Lebeau and L. Robbiano [LR] for the wave equation.

In the original paper dealing with doubling property and frequency function, N. Garofalo and F.H. Lin [GaL] studied the monotonicity property of the following quantity

$$\frac{r\int_{B_{0,r}} |\nabla v(y)|^2 dy}{\int_{\partial B_{0,r}} |v(y)|^2 d\sigma(y)} .$$

However, it seems more natural in our context to consider the monotonicity properties of the frequency function (see [Ze]) defined by

$$\frac{\int_{B_{0,r}} |\nabla v(y)|^2 \left(r^2 - |y|^2\right) dy}{\int_{B_{0,r}} |v(y)|^2 dy} .$$

4.1 Monotonicity formula

Following the ideas of I. Kukavica ([Ku2], [Ku], [KN], see also [E], [AE]), one obtains the following three lemmas. Detailed proofs are given in [Ph3].

Lemma A. *Let $D \subset \mathbb{R}^{N+1}$, $N \geq 1$, be a connected bounded open set such that $\overline{B_{y_o,R_o}} \subset D$ with $y_o \in D$ and $R_o > 0$. If $v = v(y) \in H^2(D)$ is a solution of $\Delta_y v = 0$ in D, then*

$$\Phi(r) = \frac{\int_{B_{y_o,r}} |\nabla v(y)|^2 \left(r^2 - |y - y_o|^2\right) dy}{\int_{B_{y_o,r}} |v(y)|^2 dy}$$

is non-decreasing on $0 < r < R_o$, *and*

$$\frac{d}{dr}\ln \int_{B_{y_o,r}} |v(y)|^2 dy = \frac{1}{r}(N + 1 + \Phi(r)) .$$

Lemma B. *Let $D \subset \mathbb{R}^{N+1}$, $N \geq 1$, be a connected bounded open set such that $\overline{B_{y_o,R_o}} \subset D$ with $y_o \in D$ and $R_o > 0$. Let r_1, r_2, r_3 be three* *real numbers such that* $0 < r_1 < r_2 < r_3 < R_o$. *If* $v = v(y) \in H^2(D)$ *is a solution of* $\Delta_y v = 0$ *in* D, *then*

$$\int_{B_{y_o,r_2}} |v(y)|^2 \, dy \leq \left(\int_{B_{y_o,r_1}} |v(y)|^2 \, dy \right)^{\alpha} \left(\int_{B_{y_o,r_3}} |v(y)|^2 \, dy \right)^{1-\alpha} ,$$

where $\alpha = \frac{1}{\ln \frac{r_2}{r_1}} \left(\frac{1}{\ln \frac{r_2}{r_1}} + \frac{1}{\ln \frac{r_3}{r_2}} \right)^{-1} \in (0,1)$.

The above two results are still available when we are closed to a part Γ of the boundary $\partial\Omega$ under the homogeneous Dirichlet boundary condition on Γ, as follows.

Lemma C. *Let* $D \subset \mathbb{R}^{N+1}$, $N \geq 1$, *be a connected bounded open set with boundary* ∂D. *Let* Γ *be a non-empty Lipschitz open subset of* ∂D. *Let* r_o, r_1, r_2, r_3, R_o *be five real numbers such that* $0 < r_1 < r_o < r_2 < r_3 < R_o$. *Suppose that* $y_o \in D$ *satisfies the following three conditions:*

i). $B_{y_o,r} \cap D$ *is star-shaped with respect to* y_o $\forall r \in (0, R_o)$,

ii). $B_{y_o,r} \subset D$ $\forall r \in (0, r_o)$,

iii). $B_{y_o,r} \cap \partial D \subset \Gamma$ $\forall r \in [r_o, R_o)$.

If $v = v(y) \in H^2(D)$ *is a solution of* $\Delta_y v = 0$ *in* D *and* $v = 0$ *on* Γ, *then*

$$\int_{B_{y_o,r_2} \cap D} |v(y)|^2 \, dy \leq \left(\int_{B_{y_o,r_1}} |v(y)|^2 \, dy \right)^{\alpha} \left(\int_{B_{y_o,r_3} \cap D} |v(y)|^2 \, dy \right)^{1-\alpha}$$

where $\alpha = \frac{1}{\ln \frac{r_2}{r_1}} \left(\frac{1}{\ln \frac{r_2}{r_1}} + \frac{1}{\ln \frac{r_3}{r_2}} \right)^{-1} \in (0,1)$.

4.1.1 Proof of Lemma B

Let

$$H(r) = \int_{B_{y_o,r}} |v(y)|^2 \, dy \ .$$

By applying Lemma A, we know that

$$\frac{d}{dr} \ln H(r) = \frac{1}{r} \left(N + 1 + \Phi(r) \right) \ .$$

Next, from the monotonicity property of Φ, one deduces the following two inequalities

$$\begin{aligned} \ln \left(\frac{H(r_2)}{H(r_1)} \right) &= \int_{r_1}^{r_2} \frac{N+1+\Phi(r)}{r} dr \\ &\leq \left(N + 1 + \Phi(r_2) \right) \ln \frac{r_2}{r_1} \ , \end{aligned}$$

$$\begin{array}{rl}\ln\left(\frac{H(r_3)}{H(r_2)}\right) & = \int_{r_2}^{r_3} \frac{N+1+\Phi(r)}{r} dr \\ & \geq (N+1+\Phi(r_2)) \ln\frac{r_3}{r_2} \ .\end{array}$$

Consequently,

$$\frac{\ln\left(\frac{H(r_2)}{H(r_1)}\right)}{\ln\frac{r_2}{r_1}} \leq (N+1) + \Phi(r_2) \leq \frac{\ln\left(\frac{H(r_3)}{H(r_2)}\right)}{\ln\frac{r_3}{r_2}} \ ,$$

and therefore the desired estimate holds

$$H(r_2) \leq (H(r_1))^\alpha (H(r_3))^{1-\alpha} \ ,$$

where $\alpha = \frac{1}{\ln\frac{r_2}{r_1}} \left(\frac{1}{\ln\frac{r_2}{r_1}} + \frac{1}{\ln\frac{r_3}{r_2}} \right)^{-1}$.

4.1.2 Proof of Lemma A

We introduce the following two functions H and D for $0 < r < R_o$:

$$\begin{array}{l} H(r) = \int_{B_{y_o,r}} |v(y)|^2 dy \ , \\ D(r) = \int_{B_{y_o,r}} |\nabla v(y)|^2 \left(r^2 - |y-y_o|^2\right) dy \ . \end{array}$$

First, the derivative of $H(r) = \int_0^r \int_{S^N} |v(\rho s + y_o)|^2 \rho^N d\rho d\sigma(s)$ is given by $H'(r) = \int_{\partial B_{y_o,r}} |v(y)|^2 d\sigma(y)$. Next, recall the Green formula

$$\begin{array}{l} \int_{\partial B_{y_o,r}} |v|^2 \partial_\nu G d\sigma(y) - \int_{\partial B_{y_o,r}} \partial_\nu \left(|v|^2\right) G d\sigma(y) \\ \quad = \int_{B_{y_o,r}} |v|^2 \Delta G dy - \int_{B_{y_o,r}} \Delta\left(|v|^2\right) G dy \ . \end{array}$$

We apply it with $G(y) = r^2 - |y-y_o|^2$ where $G_{|\partial B_{y_o,r}} = 0$, $\partial_\nu G_{|\partial B_{y_o,r}} = -2r$, and $\Delta G = -2(N+1)$. It gives

$$\begin{array}{rl} H'(r) & = \frac{1}{r} \int_{B_{y_o,r}} (N+1) |v|^2 dy + \frac{1}{2r} \int_{B_{y_o,r}} \Delta\left(|v|^2\right) \left(r^2 - |y-y_o|^2\right) dy \\ & = \frac{N+1}{r} H(r) + \frac{1}{r} \int_{B_{y_o,r}} div(v\nabla v) \left(r^2 - |y-y_o|^2\right) dy \\ & = \frac{N+1}{r} H(r) + \frac{1}{r} \int_{B_{y_o,r}} \left(|\nabla v|^2 + v\Delta v\right) \left(r^2 - |y-y_o|^2\right) dy \ . \end{array}$$

Consequently, when $\Delta_y v = 0$,

$$H'(r) = \frac{N+1}{r} H(r) + \frac{1}{r} D(r) \ , \tag{A.1}$$

that is $\frac{H'(r)}{H(r)} = \frac{N+1}{r} + \frac{1}{r}\frac{D(r)}{H(r)}$ the second equality in Lemma A.

Now, we compute the derivative of $D(r)$.

$$\begin{aligned} D'(r) &= \tfrac{d}{dr}\left(r^2 \int_0^r \int_{S^N} \left|(\nabla v)_{|\rho s+y_o}\right|^2 \rho^N d\rho d\sigma(s)\right) \\ &\quad - \int_{S^N} r^2 \left|(\nabla v)_{|rs+y_o}\right|^2 r^N d\sigma(s) \\ &= 2r \int_0^r \int_{S^N} \left|(\nabla v)_{|\rho s+y_o}\right|^2 \rho^N d\rho d\sigma(s) \\ &= 2r \int_{B_{y_o,r}} |\nabla v|^2 dy \ . \end{aligned} \tag{A.2}$$

On the other hand, we have by integrations by parts that

$$\begin{aligned} 2r \int_{B_{y_o,r}} |\nabla v|^2 dy = \tfrac{N+1}{r} D(r) + \tfrac{4}{r} \int_{B_{y_o,r}} |(y-y_o)\cdot\nabla v|^2 dy \\ -\tfrac{1}{r}\int_{B_{y_o,r}} \nabla v\cdot(y-y_o)\,\Delta v\left(r^2-|y-y_o|^2\right) dy \ . \end{aligned} \tag{A.3}$$

Therefore,

$$\begin{aligned} &(N+1)\int_{B_{y_o,r}} |\nabla v|^2 \left(r^2-|y-y_o|^2\right) dy \\ &= 2r^2 \int_{B_{y_o,r}} |\nabla v|^2 dy - 4\int_{B_{y_o,r}} |(y-y_o)\cdot\nabla v|^2 dy \\ &\quad +2\int_{B_{y_o,r}} (y-y_o)\cdot\nabla v \Delta v\left(r^2-|y-y_o|^2\right) dy \ , \end{aligned}$$

and this is the desired estimate (A.3).

Consequently, from (A.2) and (A.3), we obtain, when $\Delta_y v = 0$, the following formula

$$D'(r) = \frac{N+1}{r} D(r) + \frac{4}{r}\int_{B_{y_o,r}} |(y-y_o)\cdot\nabla v|^2 dy \ . \tag{A.4}$$

The computation of the derivative of $\Phi(r) = \frac{D(r)}{H(r)}$ gives

$$\Phi'(r) = \frac{1}{H^2(r)}\left[D'(r)H(r) - D(r)H'(r)\right] \ ,$$

which implies using (A.1) and (A.4) that

$$H^2(r)\,\Phi'(r) = \frac{1}{r}\left(4\int_{B_{y_o,r}} |(y-y_o)\cdot\nabla v|^2 dy H(r) - D^2(r)\right) \geq 0 \ ,$$

indeed, thanks to an integration by parts and using Cauchy-Schwarz inequality, we have

$$\begin{aligned} D^2(r) &= 4\left(\int_{B_{y_o,r}} v\nabla v\cdot(y-y_o)\,dy\right)^2 \\ &\leq 4\left(\int_{B_{y_o,r}} |(y-y_o)\cdot\nabla v|^2 dy\right)\left(\int_{B_{y_o,r}} |v|^2 dy\right) \\ &\leq 4\left(\int_{B_{y_o,r}} |(y-y_o)\cdot\nabla v|^2 dy\right) H(r) \ . \end{aligned}$$

Therefore, we have proved the desired monotonicity for Φ and this completes the proof of Lemma A.

4.1.3 Proof of Lemma C

Under the assumption $B_{y_o,r} \cap \partial D \subset \Gamma$ for any $r \in [r_o, R_o)$, we extend v by zero in $\overline{B_{y_o,R_o} \backslash D}$ and denote by $\overline{v}$ its extension. Since $v = 0$ on Γ, we have

$$\begin{cases} \overline{v} = v 1_D & \text{in } \overline{B_{y_o,R_o}} , \\ \overline{v} = 0 & \text{on } B_{y_o,R_o} \cap \partial D , \\ \nabla \overline{v} = \nabla v 1_D & \text{in } B_{y_o,R_o} . \end{cases}$$

Now, we denote $\Omega_r = B_{y_o,r} \cap D$, when $0 < r < R_o$. Particularly, $\Omega_r = B_{y_o,r}$, when $0 < r < r_o$. We introduce the following three functions:

$$H(r) = \int_{\Omega_r} |v(y)|^2 \, dy ,$$
$$D(r) = \int_{\Omega_r} |\nabla v(y)|^2 \left(r^2 - |y - y_o|^2\right) dy ,$$

and

$$\Phi(r) = \frac{D(r)}{H(r)} \geq 0 .$$

Our goal is to show that Φ is a non-decreasing function. Indeed, we will prove that the following equality holds

$$\frac{d}{dr} \ln H(r) = (N+1) \frac{d}{dr} \ln r + \frac{1}{r} \Phi(r) . \tag{C.1}$$

Therefore, from the monotonicity of Φ, we will deduce (in a similar way to that in the proof of Lemma A) that

$$\frac{\ln\left(\frac{H(r_2)}{H(r_1)}\right)}{\ln \frac{r_2}{r_1}} \leq (N+1) + \Phi(r_2) \leq \frac{\ln\left(\frac{H(r_3)}{H(r_2)}\right)}{\ln \frac{r_3}{r_2}} ,$$

and this will imply the desired estimate

$$\int_{\Omega_{r_2}} |v(y)|^2 \, dy \leq \left(\int_{B_{y_o,r_1}} |v(y)|^2 \, dy \right)^{\alpha} \left(\int_{\Omega_{r_3}} |v(y)|^2 \, dy \right)^{1-\alpha} ,$$

where $\alpha = \frac{1}{\ln\frac{r_2}{r_1}}\left(\frac{1}{\ln\frac{r_2}{r_1}} + \frac{1}{\ln\frac{r_3}{r_2}}\right)^{-1}$.

First, we compute the derivative of $H(r) = \int_{B_{y_o,r}} |\overline{v}(y)|^2 dy$.

$$\begin{aligned} H'(r) &= \int_{S^N} |\overline{v}(rs+y_o)|^2 r^N d\sigma(s) \\ &= \tfrac{1}{r}\int_{S^N} |\overline{v}(rs+y_o)|^2 rs\cdot sr^N d\sigma(s) \\ &= \tfrac{1}{r}\int_{B_{y_o,r}} div\left(|\overline{v}(y)|^2 (y-y_o)\right) dy \\ &= \tfrac{1}{r}\int_{B_{y_o,r}} \left((N+1)|\overline{v}(y)|^2 + \nabla|\overline{v}(y)|^2\cdot(y-y_o)\right) dy \\ &= \tfrac{N+1}{r}H(r) + \tfrac{2}{r}\int_{\Omega_r} v(y)\nabla v(y)\cdot(y-y_o)\,dy\ . \end{aligned} \tag{C.2}$$

Next, when $\Delta_y v = 0$ in D and $v_{|\Gamma} = 0$, we remark that

$$D(r) = 2\int_{\Omega_r} v(y)\nabla v(y)\cdot(y-y_o)\,dy\ , \tag{C.3}$$

indeed,

$$\begin{aligned} &\int_{\Omega_r} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right) dy \\ &= \int_{\Omega_r} div\left[v\nabla v\left(r^2-|y-y_o|^2\right)\right] dy - \int_{\Omega_r} v div\left[\nabla v\left(r^2-|y-y_o|^2\right)\right] dy \\ &= -\int_{\Omega_r} v\Delta v\left(r^2-|y-y_o|^2\right) dy - \int_{\Omega_r} v\nabla v\cdot\nabla\left(r^2-|y-y_o|^2\right) dy \\ &\quad \text{because on } \partial B_{y_o,r},\ r=|y-y_o| \text{ and } v_{|\Gamma}=0 \\ &= 2\int_{\Omega_r} v\nabla v\cdot(y-y_o)\,dy \quad \text{because } \Delta_y v = 0 \text{ in } D\ . \end{aligned}$$

Consequently, from (C.2) and (C.3), we obtain

$$H'(r) = \frac{N+1}{r}H(r) + \frac{1}{r}D(r)\ , \tag{C.4}$$

and this is (C.1).

On the other hand, the derivative of $D(r)$ is

$$\begin{aligned} D'(r) &= 2r\int_0^r \int_{S^N} \left|(\nabla\overline{v})_{|\rho s+y_o}\right|^2 \rho^N d\rho d\sigma(s) \\ &= 2r\int_{\Omega_r} |\nabla v(y)|^2 dy\ . \end{aligned} \tag{C.5}$$

Here, when $\Delta_y v = 0$ in D and $v_{|\Gamma} = 0$, we will remark that

$$\begin{aligned} 2r\int_{\Omega_r} |\nabla v(y)|^2 dy &= \tfrac{N+1}{r}D(r) + \tfrac{4}{r}\int_{B_{y_o,r}} |(y-y_o)\cdot\nabla v(y)|^2 dy \\ &\quad + \tfrac{1}{r}\int_{\Gamma\cap B_{y_o,r}} |\partial_\nu v|^2 \left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y) \end{aligned} \tag{C.6}$$

indeed,

$$
\begin{aligned}
&(N+1)\int_{\Omega_r}|\nabla v|^2\left(r^2-|y-y_o|^2\right)dy\\
&=\int_{\Omega_r}div\left(|\nabla v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\right)dy\\
&\quad-\int_{\Omega_r}\nabla\left(|\nabla v|^2\left(r^2-|y-y_o|^2\right)\right)\cdot(y-y_o)\,dy\\
&=\int_{\Gamma\cap B_{y_o,r}}|\nabla v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y)\\
&\quad-\int_{\Omega_r}\partial_{y_i}\left(|\nabla v|^2\left(r^2-|y-y_o|^2\right)\right)(y_i-y_{oi})\,dy\\
&=\int_{\Gamma\cap B_{y_o,r}}|\nabla v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y)\\
&\quad-\int_{\Omega_r}2\nabla v\partial_{y_i}\nabla v\left(r^2-|y-y_o|^2\right)(y_i-y_{oi})\,dy\\
&\quad+2\int_{\Omega_r}|\nabla v|^2|y-y_o|^2\,dy\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{\Omega_r}\partial_{y_j}v\partial^2_{y_iy_j}v\left(r^2-|y-y_o|^2\right)(y_i-y_{oi})\,dy\\
&=-\int_{\Omega_r}\partial_{y_j}\left((y_i-y_{oi})\,\partial_{y_j}v\partial_{y_i}v\left(r^2-|y-y_o|^2\right)\right)dy\\
&\quad+\int_{\Omega_r}\partial_{y_j}(y_i-y_{oi})\,\partial_{y_j}v\partial_{y_i}v\left(r^2-|y-y_o|^2\right)dy\\
&\quad+\int_{\Omega_r}(y_i-y_{oi})\,\partial^2_{y_j}v\partial_{y_i}v\left(r^2-|y-y_o|^2\right)dy\\
&\quad+\int_{\Omega_r}(y_i-y_{oi})\,\partial_{y_j}v\partial_{y_i}v\partial_{y_j}\left(r^2-|y-y_o|^2\right)dy\\
&=-\int_{\Gamma\cap B_{y_o,r}}\nu_j\left((y_i-y_{oi})\,\partial_{y_j}v\partial_{y_i}v\left(r^2-|y-y_o|^2\right)\right)d\sigma(y)\\
&\quad+\int_{\Omega_r}|\nabla v|^2\left(r^2-|y-y_o|^2\right)dy\\
&\quad+0\quad\text{because }\Delta_y v=0\text{ in }D\\
&\quad-\int_{\Omega_r}2\,|(y-y_o)\cdot\nabla v|^2\,dy\ .
\end{aligned}
$$

Therefore, when $\Delta_y v=0$ in D, we have

$$
\begin{aligned}
&(N+1)\int_{\Omega_r}|\nabla v|^2\left(r^2-|y-y_o|^2\right)dy\\
&=\int_{\Gamma\cap B_{y_o,r}}|\nabla v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y)\\
&\quad-2\int_{\Gamma\cap B_{y_o,r}}\partial_{y_j}v\nu_j\left((y_i-y_{oi})\,\partial_{y_i}v\right)\left(r^2-|y-y_o|^2\right)d\sigma(y)\\
&\quad+2r^2\int_{\Omega_r}|\nabla u|^2\,dy-4\int_{\Omega_r}|(y-y_o)\cdot\nabla v|^2\,dy\ .
\end{aligned}
$$

By the fact that $v_{|\Gamma}=0$, we get $\nabla v=(\nabla v\cdot\nu)\,\nu$ on Γ and deduce that

$$
\begin{aligned}
&(N+1)\int_{\Omega_r}|\nabla v|^2\left(r^2-|y-y_o|^2\right)dy\\
&=-\int_{\Gamma\cap B_{y_o,r}}|\partial_\nu v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y)\\
&\quad+2r^2\int_{\Omega_r}|\nabla v|^2\,dy-4\int_{\Omega_r}|(y-y_o)\cdot\nabla v|^2\,dy\ ,
\end{aligned}
$$

and this is (C.6).

Consequently, from (C.5) and (C.6), when $\Delta_y v = 0$ in D and $v_{|\Gamma} = 0$, we have

$$\begin{array}{rl} D'(r) = \frac{N+1}{r} D(r) + \frac{4}{r}\int_{\Omega_r} \left|(y-y_o)\cdot\nabla v(y)\right|^2 dy & \\ \quad + \frac{1}{r}\int_{\Gamma\cap B_{y_o,r}} \left|\partial_\nu v\right|^2 \left(r^2 - \left|y-y_o\right|^2\right)(y-y_o)\cdot\nu d\sigma(y) \ . & \end{array} \tag{C.7}$$

The computation of the derivative of $\Phi(r) = \frac{D(r)}{H(r)}$ gives

$$\Phi'(r) = \frac{1}{H^2(r)}\left[D'(r)H(r) - D(r)H'(r)\right] \ ,$$

which implies from (C.4) and (C.7) that

$$\begin{array}{rl} H^2(r)\Phi'(r) = \frac{1}{r}\left(4\int_{\Omega_r}\left|(y-y_o)\cdot\nabla v(y)\right|^2 dy\ H(r) - D^2(r)\right) & \\ \quad + \frac{H(r)}{r}\int_{\Gamma\cap B_{y_o,r}}\left|\partial_\nu v\right|^2\left(r^2-\left|y-y_o\right|^2\right)(y-y_o)\cdot\nu d\sigma(y) & \end{array}$$

Thanks to (C.3) and Cauchy-Schwarz inequality, we obtain that

$$0 \leq 4\int_{\Omega_r}\left|(y-y_o)\cdot\nabla v(y)\right|^2 dy\ H(r) - D^2(r) \ .$$

The inequality $0 \leq (y-y_o)\cdot\nu$ on Γ holds when $B_{y_o,r}\cap D$ is star-shaped with respect to y_o for any $r \in (0, R_o)$. Therefore, we get the desired monotonicity for Φ which completes the proof of Lemma C.

4.2 Quantitative unique continuation property for the Laplacian

Let $D \subset \mathbb{R}^{N+1}$, $N \geq 1$, be a connected bounded open set with boundary ∂D. Let Γ be a non-empty Lipschitz open part of ∂D. We consider the Laplacian in D, with a homogeneous Dirichlet boundary condition on $\Gamma \subset \partial\Omega$:

$$\left\{\begin{array}{rl} \Delta_y v = 0 & \text{in } D \ , \\ v = 0 & \text{on } \Gamma \ , \\ v = v(y) \in H^2(D) \ . & \end{array}\right. \tag{D.1}$$

The goal of this section is to describe interpolation inequalities associated with solutions v of (D.1).

Theorem D. *Let ω be a non-empty open subset of D. Then, for any $D_1 \subset D$, $\partial D_1 \cap \partial D \Subset \Gamma$ and $\overline{D_1}\backslash(\Gamma\cap\partial D_1) \subset D$, there exist $C > 0$ and $\mu \in (0,1)$ such that for any v solution of* (D.1), *we have*

$$\int_{D_1}\left|v(y)\right|^2 dy \leq C\left(\int_\omega \left|v(y)\right|^2 dy\right)^\mu \left(\int_D \left|v(y)\right|^2 dy\right)^{1-\mu} \ .$$

Or in an equivalent way and by a minimization technique, there occur the following results:

Theorem D'. *Let ω be a non-empty open subset of D. Then, for any $D_1 \subset D$, $\partial D_1 \cap \partial D \Subset \Gamma$ and $\overline{D_1} \backslash (\Gamma \cap \partial D_1) \subset D$, there exist $C > 0$ and $\mu \in (0,1)$ such that for any v solution of* (D.1), *we have*

$$\int_{D_1} |v(y)|^2 \, dy \leq C \left(\frac{1}{\varepsilon} \right)^{\frac{1-\mu}{\mu}} \int_\omega |v(y)|^2 \, dy + \varepsilon \int_D |v(y)|^2 \, dy \quad \forall \varepsilon > 0 \, .$$

Proof of Theorem D. We divide the proof into two steps.

Step 1. We apply Lemma B, and use a standard argument (see e.g., [Ro]) which consists of constructing a sequence of balls chained along a curve. More precisely, we claim that for any non-empty compact sets in D, K_1 and K_2, meas $(K_1) > 0$, there exists $\mu \in (0,1)$ such that for any $v = v(y) \in H^2(D)$, solution of $\Delta_y v = 0$ in D, we have

$$\int_{K_2} |v(y)|^2 \, dy \leq \left(\int_{K_1} |v(y)|^2 \, dy \right)^\mu \left(\int_D |v(y)|^2 \, dy \right)^{1-\mu} \, . \qquad \text{(D.2)}$$

Step 2. We apply Lemma C, and choose y_o in a neighborhood of the part Γ such that the conditions *i, ii, iii* hold. Next, by an adequate partition of D, we deduce from (D.2) that for any $D_1 \subset D$, $\partial D_1 \cap \partial D \Subset \Gamma$ and $\overline{D_1} \backslash (\Gamma \cap \partial D_1) \subset D$, there exist $C > 0$ and $\mu \in (0,1)$ such that for any $v = v(y) \in H^2(D)$, $\Delta_y v = 0$ on D and $v = 0$ on Γ, we have

$$\int_{D_1} |v(y)|^2 \, dy \leq C \left(\int_\omega |v(y)|^2 \, dy \right)^\mu \left(\int_D |v(y)|^2 \, dy \right)^{1-\mu} \, .$$

This completes the proof.

4.3 Quantitative unique continuation property for the elliptic operator $\partial_t^2 + \Delta$

In this section, we present the following result.

Theorem E. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 1$, either convex or C^2 connected. We choose $T_2 > T_1$ and $\delta \in (0, (T_2 - T_1)/2)$. Let $f \in L^2(\Omega \times (T_1, T_2))$. We consider the elliptic operator of the second order in $\Omega \times (T_1, T_2)$ with a homogeneous Dirichlet boundary condition on $\partial\Omega \times (T_1, T_2)$,*

$$\begin{cases} \partial_t^2 w + \Delta w = f & \text{in } \Omega \times (T_1, T_2) \, , \\ w = 0 & \text{on } \partial\Omega \times (T_1, T_2) \, , \\ w = w(x,t) \in H^2(\Omega \times (T_1, T_2)) \, . \end{cases} \qquad \text{(E.1)}$$

Then, for any $\varphi \in C_0^\infty(\Omega \times (T_1, T_2))$, $\varphi \neq 0$, *there exist* $C > 0$ *and* $\mu \in (0,1)$ *such that for any* w *solution of (*E.1*), we have*

$$\begin{aligned} &\int_{T_1+\delta}^{T_2-\delta} \int_\Omega |w(x,t)|^2 \, dxdt \\ &\leq C \left(\int_{T_1}^{T_2} \int_\Omega |w(x,t)|^2 \, dxdt \right)^{1-\mu} \\ &\quad \left(\int_{T_1}^{T_2} \int_\Omega |\varphi w(x,t)|^2 \, dxdt + \int_{T_1}^{T_2} \int_\Omega |f(x,t)|^2 \, dxdt \right)^{\mu} . \end{aligned}$$

Proof. First, by a difference quotient technique and a standard extension at $\Omega \times \{T_1, T_2\}$, we check the existence of a solution $u \in H^2(\Omega \times (T_1, T_2))$ solving

$$\begin{cases} \partial_t^2 u + \Delta u = f & \text{in } \Omega \times (T_1, T_2) \ , \\ u = 0 & \text{on } \partial\Omega \times (T_1, T_2) \cup \Omega \times \{T_1, T_2\} \ , \end{cases}$$

such that

$$\|u\|_{H^2(\Omega \times (T_1,T_2))} \leq c \, \|f\|_{L^2(\Omega \times (T_1,T_2))} \ ,$$

for some $c > 0$ only depending on (Ω, T_1, T_2). Next, we apply Theorem D with $D = \Omega \times (T_1, T_2)$, $\Omega \times (T_1 + \delta, T_2 - \delta) \subset D_1$, $y = (x,t)$, $\Delta_y = \partial_t^2 + \Delta$, and $v = w - u$.

4.4 Application to the wave equation

From the idea of L. Robbiano [Ro2] which consists of using an interpolation inequality of Hölder type for the elliptic operator $\partial_t^2 + \Delta$ and the Fourier-Bros-Iagolnitzer transform introduced by G. Lebeau and L. Robbiano [LR], we obtain the following estimate of logarithmic type.

Theorem F. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$, $n \geq 1$, *either convex or* C^2 *connected. Let* ω *be a non-empty open subset in* Ω. *Then, for any* $\beta \in (0,1)$ *and* $k \in \mathbb{N}^*$, *there exist* $C > 0$ *and* $T > 0$ *such that for any solution* u *of*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \ , \end{cases}$$

with non-identically zero initial data $(u_0, u_1) \in D(A^{k-1})$, *we have*

$$\|(u_0,u_1)\|_{D(A^{k-1})} \leq C e^{\left(C \frac{\|(u_0,u_1)\|_{D(A^{k-1})}}{\|(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}} \right)^{1/(\beta k)}} \|u\|_{L^2(\omega \times (0,T))} \ .$$

Proof. First, recall that with a standard energy method, we have that

$$\forall t \in \mathbb{R} \qquad \|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} = \int_\Omega \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx \ , \tag{F.1}$$

and there exists a constant $c > 0$ such that for all $T \geq 1$,

$$T \left\|(u_0, u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq c \int_0^T \int_\Omega |u(x,t)|^2 \, dx \ . \tag{F.2}$$

Next, let $\beta \in (0,1)$, $k \in \mathbb{N}^*$, and choose $N \in \mathbb{N}^*$ such that $0 < \beta + \frac{1}{2N} < 1$ and $2N > k$. Put $\gamma = 1 - \frac{1}{2N}$. For any $\lambda \geq 1$, the function $F_\lambda(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau} e^{-\left(\frac{\tau}{\lambda^\gamma}\right)^{2N}} d\tau$ is holomorphic in $\mathbb{C}$, and there exists four positive constants C_o, c_0, c_1 and c_2 (independent of λ) such that

$$\begin{cases} \forall z \in \mathbb{C} \quad |F_\lambda(z)| \leq C_o \lambda^\gamma e^{c_0 \lambda |\mathrm{Im} z|^{1/\gamma}} \ , \\ |\mathrm{Im} z| \leq c_2 \left|\mathrm{Re} z\right| \Rightarrow |F_\lambda(z)| \leq C_o \lambda^\gamma e^{-c_1 \lambda |\mathrm{Re} z|^{1/\gamma}} \ , \end{cases} \tag{F.3}$$

(see [LR]).

Now, let $s, \ell_o \in \mathbb{R}$, we introduce the following Fourier-Bros-Iagolnitzer transformation in [LR]:

$$W_{\ell_o,\lambda}(x,s) = \int_{\mathbb{R}} F_\lambda(\ell_o + is - \ell)\Phi(\ell)u(x,\ell)d\ell \ , \tag{F.4}$$

where $\Phi \in C_0^\infty(\mathbb{R})$. As u is solution of the wave equation, $W_{\ell_o,\lambda}$ satisfies:

$$\begin{cases} \partial_s^2 W_{\ell_o,\lambda}(x,s) + \Delta W_{\ell_o,\lambda}(x,s) \\ \quad = \int_{\mathbb{R}} -F_\lambda(\ell_o + is - \ell)\left[\Phi''(\ell)u(x,\ell) + 2\Phi'(\ell)\partial_t u(x,\ell)\right] d\ell \ , \\ W_{\ell_o,\lambda}(x,s) = 0 \quad \text{for } x \in \partial\Omega \ , \\ W_{\ell_o,\lambda}(x,0) = \left(F_\lambda * \Phi u(x,\cdot)\right)(\ell_o) \quad \text{for } x \in \Omega \ . \end{cases} \tag{F.5}$$

On the other hand, we also have for any $T > 0$,

$$\begin{aligned} \left\|\Phi u(x,\cdot)\right\|_{L^2\left(\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\right)} &\leq \left\|\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\right\|_{L^2\left(\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\right)} \\ &\quad + \left\|F_\lambda * \Phi u(x,\cdot)\right\|_{L^2\left(\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\right)} \\ &\leq \left\|\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\right\|_{L^2(\mathbb{R})} \\ &\quad + \left(\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)} \left|W_{t,\lambda}(x,0)\right|^2 dt\right)^{1/2} . \end{aligned} \tag{F.6}$$

Denoting $\mathcal{F}(f)$ the Fourier transform of f, by using Parseval equality and $\mathcal{F}(F_\lambda)(\tau) = e^{-\left(\frac{\tau}{\lambda^\gamma}\right)^{2N}}$, one obtains

$$\begin{aligned} &\left\|\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\right\|_{L^2(\mathbb{R})} \\ &= \tfrac{1}{\sqrt{2\pi}} \left\|\mathcal{F}\left(\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\right)\right\|_{L^2(\mathbb{R})} \\ &= \tfrac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} \left|\left(1 - e^{-\left(\frac{\tau}{\lambda^\gamma}\right)^{2N}}\right) \mathcal{F}\left(\Phi u(x,\cdot)\right)(\tau)\right|^2 d\tau\right)^{1/2} \\ &\leq C \left(\int_{\mathbb{R}} \left|\left(\tfrac{\tau}{\lambda^\gamma}\right)^k \mathcal{F}\left(\Phi u(x,\cdot)\right)(\tau)\right|^2 d\tau\right)^{1/2} \quad \text{because } k < 2N \\ &\leq C \tfrac{1}{\lambda^{\beta k}} \left(\int_{\mathbb{R}} \left|\mathcal{F}\left(\partial_t^k\left(\Phi u(x,\cdot)\right)\right)(\tau)\right|^2 d\tau\right)^{1/2} \quad \text{because } \beta < \gamma \\ &\leq C \tfrac{1}{\lambda^{\beta k}} \left\|\partial_t^k\left(\Phi u(x,\cdot)\right)\right\|_{L^2(\mathbb{R})} \ . \end{aligned} \tag{F.7}$$

Therefore, from (F.6) and (F.7), one gets

$$\begin{array}{l}\left\|\Phi u\left(x,\cdot\right)\right\|_{L^2\left(\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\right)}\\ \leq C\frac{1}{\lambda^{\beta k}}\left\|\partial_t^k\left(\Phi u(x,\cdot)\right)\right\|_{L^2(\mathbb{R})}+\left(\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|W_{t,\lambda}(x,0)\right|^2dt\right)^{1/2}\ .\end{array}\quad\text{(F.8)}$$

Now, recall that from the Cauchy's theorem we have:

Proposition 1. *Let f be a holomorphic function in a domain $D\subset\mathbb{C}$. Let $a,b>0$, $z\in\mathbb{C}$. We suppose that*

$$D_o=\left\{(x,y)\in\mathbb{R}^2\simeq\mathbb{C}\setminus\left|x-\mathrm{Re}z\right|\leq a,\ \left|y-\mathrm{Im}z\right|\leq b\right\}\subset D\ ,$$

then

$$f\left(z\right)=\frac{1}{\pi ab}\int\int_{\left|\frac{x-\mathrm{Re}z}{a}\right|^2+\left|\frac{y-\mathrm{Im}z}{b}\right|^2\leq 1}f\left(x+iy\right)dxdy\ .$$

Choosing $z=t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\subset\mathbb{R}$ and $x+iy=\ell_o+is$, we deduce that

$$\begin{array}{ll}\left|W_{t,\lambda}(x,0)\right| & \leq\frac{1}{\pi ab}\int_{|\ell_o-t|\leq a}\int_{|s|\leq b}\left|W_{\ell_o+is,\lambda}(x,0)\right|d\ell_o ds\\ & \leq\frac{1}{\pi ab}\int_{|\ell_o-t|\leq a}\int_{|s|\leq b}\left|W_{\ell_o,\lambda}(x,s)\right|dsd\ell_o\\ & \leq\frac{2}{\pi\sqrt{ab}}\left(\int_{|\ell_o-t|\leq a}\int_{|s|\leq b}\left|W_{\ell_o,\lambda}(x,s)\right|^2dsd\ell_o\right)^{1/2}\ ,\end{array}\quad\text{(F.9)}$$

and with $a=2b=1$,

$$\begin{array}{l}\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|W_{t,\lambda}(x,0)\right|^2dt\\ \leq\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left(\int_{|\ell_o-t|\leq 1}\int_{|s|\leq 1/2}\left|W_{\ell_o,\lambda}(x,s)\right|^2dsd\ell_o\right)dt\\ \leq\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\int_{|s|\leq 1/2}\left|W_{\ell_o,\lambda}(x,s)\right|^2dsd\ell_o dt\\ \leq 2\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\int_{|s|\leq 1/2}\left|W_{\ell_o,\lambda}(x,s)\right|^2dsd\ell_o\ .\end{array}\quad\text{(F.10)}$$

Consequently, from (F.8), (F.10) and integrating over Ω, we get the existence of $C>0$ such that

$$\begin{array}{l}\int_\Omega\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|\Phi(t)u(x,t)\right|^2dtdx\\ \leq C\frac{1}{\lambda^{2\beta k}}\int_\Omega\int_{\mathbb{R}}\left|\partial_t^k\left(\Phi\left(t\right)u(x,t)\right)\right|^2dtdx\\ \quad +4\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_\Omega\int_{|s|\leq 1/2}\left|W_{\ell_o,\lambda}(x,s)\right|^2dsdx\right)d\ell_o\ .\end{array}\quad\text{(F.11)}$$

Now recall the following quantification result for unique continuation of elliptic equation with Dirichlet boundary condition (Theorem E applied to $T_1=-1$, $T_2=1$, $\delta=1/2$, $\varphi\in C_0^\infty\left(\omega\times(-1,1)\right)$):

Proposition 2. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n\geq 1$, either convex or C^2 connected. Let ω be a non-empty open subset in Ω. Let*

$f = f(x,s) \in L^2(\Omega \times (-1,1))$. *Then there exists* $\widetilde{c} > 0$ *such that for all* $w = w(x,s) \in H^2(\Omega \times (-1,1))$ *solution of*

$$\begin{cases} \partial_s^2 w + \Delta w = f & in\ \Omega \times (-1,1)\ , \\ w = 0 & on\ \partial\Omega \times (-1,1)\ , \end{cases}$$

for all $\varepsilon > 0$, *we have :*

$$\begin{array}{l} \int_{|s|\leq 1/2} \int_\Omega |w(x,s)|^2\, dxds \\ \leq \widetilde{c} e^{\widetilde{c}/\varepsilon} \left(\int_{|s|\leq 1} \int_\omega |w(x,s)|^2\, dxds + \int_{|s|\leq 1} \int_\Omega |f(x,s)|^2\, dxds \right) \\ \quad + e^{-4c_0/\varepsilon} \int_{|s|\leq 1} \int_\Omega |w(x,s)|^2\, dxds\ . \end{array}$$

Applying to $W_{\ell_0,\lambda}$, from (F.5) we deduce that for all $\varepsilon > 0$,

$$\begin{array}{l} \int_{|s|\leq 1/2} \int_\Omega |W_{\ell_0,\lambda}(x,s)|^2\, dxds \\ \leq e^{-4c_0/\varepsilon} \int_{|s|\leq 1} \int_\Omega |W_{\ell_0,\lambda}(x,s)|^2\, dxds \\ \quad + \widetilde{c} e^{\widetilde{c}/\varepsilon} \int_{|s|\leq 1} \int_\omega |W_{\ell_0,\lambda}(x,s)|^2\, dxds \\ \quad + \widetilde{c} e^{\widetilde{c}/\varepsilon} \int_{|s|\leq 1} \int_\Omega \Big| \int_{\mathbb{R}} -F_\lambda(\ell_0 + is - \ell) \\ \qquad\qquad [\Phi''(\ell)u(x,\ell) + 2\Phi'(\ell)\partial_t u(x,\ell)]\, d\ell|^2\, dxds \Big|^2\, dxds\ . \end{array} \tag{F.12}$$

Consequently, from (F.11) and (F.12), there exists a constant $C > 0$, such that for all $\varepsilon > 0$,

$$\begin{array}{l} \int_\Omega \int_{t\in(\frac{T}{2}-1,\frac{T}{2}+1)} |\Phi(t)u(x,t)|^2\, dtdx \\ \leq C \frac{1}{\lambda^{2\beta k}} \int_\Omega \int_{\mathbb{R}} \left|\partial_t^k (\Phi(t)\, u(x,t)))\right|^2 dtdx \\ \quad + 4e^{-4c_0/\varepsilon} \int_{\ell_o\in(\frac{T}{2}-2,\frac{T}{2}+2)} \left(\int_{|s|\leq 1} \int_\Omega |W_{\ell_0,\lambda}(x,s)|^2\, dxds \right) d\ell_o \\ \quad + 4Ce^{C/\varepsilon} \int_{\ell_o\in(\frac{T}{2}-2,\frac{T}{2}+2)} \left(\int_{|s|\leq 1} \int_\omega |W_{\ell_0,\lambda}(x,s)|^2\, dxds \right) d\ell_o \\ \quad + 4Ce^{\widetilde{c}/\varepsilon} \int_{\ell_o\in(\frac{T}{2}-2,\frac{T}{2}+2)} \Big(\int_{|s|\leq 1} \int_\Omega \Big| \int_{\mathbb{R}} -F_\lambda(\ell_0 + is - \ell) \\ \qquad\qquad [\Phi''(\ell)u(x,\ell) + 2\Phi'(\ell)\partial_t u(x,\ell)]\, d\ell|^2\, dxds \Big)\, d\ell_o\ . \end{array} \tag{F.13}$$

Let us define $\Phi \in C_0^\infty(\mathbb{R})$ more precisely now: we choose $\Phi \in C_0^\infty((0,T))$, $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on $\left(\frac{T}{4}, \frac{3T}{4}\right)$. Furthermore, let $K = \left[0, \frac{T}{4}\right] \cup \left[\frac{3T}{4}, T\right]$ such that $\text{supp}(\Phi') = K$ and $\text{supp}(\Phi'') \subset K$.

Let $K_0 = \left[\frac{3T}{8}, \frac{5T}{8}\right]$. Particularly, dist $(K, K_o) = \frac{T}{8}$. Let us define $T > 0$ more precisely now: we choose $T > 16\max(1, 1/c_2)$ in order that $\left(\frac{T}{2} - 2, \frac{T}{2} + 2\right) \subset K_0$ and $\text{dist}(K, K_o) \geq \frac{2}{c_2}$.

Now, we will choose $\ell_0 \in \left(\frac{T}{2} - 2, \frac{T}{2} + 2\right) \subset K_0$ and $s \in [-1,1]$. Consequently, for any $\ell \in K$, $|\ell_0 - \ell| \geq \frac{2}{c_2} \geq \frac{1}{c_2}|s|$ and it will imply from the second line of (F.3) that

$$\forall \ell \in K \quad |F_\lambda(\ell_o + is - \ell)| \leq A\lambda^\gamma e^{-c_1\lambda\left(\frac{T}{8}\right)^{1/\gamma}}\ . \tag{F.14}$$

Till the end of the proof, C and C_T will denote a generic positive constant independent of ε and λ but dependent on Ω and respectively (Ω, T), whose value may change all along the line.

The first term on the right hand side of (F.13) becomes, using (F.1),

$$\frac{1}{\lambda^{2\beta k}} \int_\Omega \int_{\mathbb{R}} \left|\partial_t^k \left(\Phi\left(t\right) u(x,t)\right)\right)\right|^2 dt dx \leq C_T \frac{1}{\lambda^{2\beta k}} \left\|(u_0, u_1)\right\|^2_{D(A^{k-1})} \ . \tag{F.15}$$

The second term on the right hand side of (F.13) becomes, using the first line of (F.3),

$$\begin{array}{l} e^{-4/\varepsilon} \int_{\ell_o \in (\frac{T}{2}-2, \frac{T}{2}+2)} \left(\int_{|s|\leq 1} \int_\Omega \left|W_{\ell_0,\lambda}(x,s)\right|^2 dxds\right) d\ell_o \\ \leq \left(C_o \lambda^\gamma e^{\lambda c_0}\right)^2 e^{-4c_0/\varepsilon} \int_{\ell_o \in (\frac{T}{2}-2, \frac{T}{2}+2)} \left[\int_{|s|\leq 1} \int_\Omega \left|\int_0^T \left|u(x,\ell)\right| d\ell\right|^2 dxds\right] \\ \leq C_T \lambda^{2\gamma} e^{2\lambda c_0} e^{-4c_0/\varepsilon} \left\|(u_0, u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \ . \end{array} \tag{F.16}$$

The third term on the right hand side of (F.13) becomes, using the first line of (F.3),

$$\begin{array}{l} e^{C/\varepsilon} \int_{\ell_o \in (\frac{T}{2}-2, \frac{T}{2}+2)} \left(\int_{|s|\leq 1} \int_\omega \left|W_{\ell_0,\lambda}(x,s)\right|^2 dxds\right) d\ell_o \\ \leq \left(C_o \lambda^\gamma e^{\lambda c_0}\right)^2 e^{C/\varepsilon} \int_{\ell_o \in (\frac{T}{2}-2, \frac{T}{2}+2)} \left[\int_{|s|\leq 1} \int_\omega \left|\int_0^T \left|u(x,\ell)\right| d\ell\right|^2 dxds\right] d\ell_o \\ \leq C \lambda^{2\gamma} e^{C\lambda} e^{C/\varepsilon} \int_\omega \int_0^T \left|u(x,t)\right|^2 dtdx \ . \end{array} \tag{F.17}$$

The fourth term on the right hand side of (F.13) becomes, using (F.14) and the choice of Φ,

$$\begin{array}{l} e^{\widetilde{c}/\varepsilon} \int_{\ell_o \in (\frac{T}{2}-2, \frac{T}{2}+2)} \left(\int_{|s|\leq 1} \int_\Omega \left|\int_{\mathbb{R}} -F_\lambda(\ell_0 + is - \ell)\right.\right. \\ \qquad \left.\left. \left[\Phi''(\ell) u(x,\ell) + 2\Phi'(\ell)\partial_t u(x,\ell)\right] d\ell\right|^2 dxds\right) d\ell_o \\ \leq C \left(A\lambda^\gamma e^{-c_1 \lambda \left(\frac{T}{8}\right)^{1/\gamma}}\right)^2 e^{\widetilde{c}/\varepsilon} \int_\Omega \left|\int_K \left(\left|u(x,\ell)\right| + \left|\partial_t u(x,\ell)\right|\right) d\ell\right|^2 dx \\ \leq C \lambda^{2\gamma} e^{-2c_1 \lambda \left(\frac{T}{8}\right)^{1/\gamma}} e^{\widetilde{c}/\varepsilon} \left\|(u_0, u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \ . \end{array} \tag{F.18}$$

We finally obtain from (F.15), (F.16), (F.17), (F.18) and (F.13) that

$$\begin{array}{l} \int_\Omega \int_{t \in (\frac{T}{2}-1, \frac{T}{2}+1)} \left|\Phi(t) u(x,t)\right|^2 dtdx \\ \leq C_T \frac{1}{\lambda^{2\beta k}} \left\|(u_0, u_1)\right\|^2_{D(A^{k-1})} \\ \quad + C_T \lambda^{2\gamma} e^{2\lambda c_0} e^{-4c_0/\varepsilon} \left\|(u_0, u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \\ \quad + C \lambda^{2\gamma} e^{C\lambda} e^{C/\varepsilon} \int_\omega \int_0^T \left|u(x,t)\right|^2 dtdx \\ \quad + C \lambda^{2\gamma} e^{-2c_1 \lambda \left(\frac{T}{8}\right)^{1/\gamma}} e^{\widetilde{c}/\varepsilon} \left\|(u_0, u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \ . \end{array} \tag{F.19}$$

We begin to choose $\lambda = \frac{1}{\varepsilon}$ in order that

$$\begin{aligned}
&\int_\Omega \int_{t\in(\frac{T}{2}-1,\frac{T}{2}+1)} |\Phi(t)u(x,t)|^2 \, dtdx \\
&\leq \varepsilon^{2\beta k} C_T \|(u_0,u_1)\|^2_{D(A^{k-1})} \\
&\quad + e^{-2c_0/\varepsilon} \frac{1}{\varepsilon^{2\gamma}} C_T \|(u_0,u_1)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \\
&\quad + e^{C/\varepsilon} C \int_\omega \int_0^T |u(x,t)|^2 \, dtdx \\
&\quad + C \frac{1}{\varepsilon^{2\gamma}} \exp\left(\left(-2c_1 \left(\tfrac{T}{8}\right)^{1/\gamma} + \widetilde{c}\right) \frac{1}{\varepsilon}\right) \|(u_0,u_1)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \ .
\end{aligned} \tag{F.20}$$

We finally need to choose $T > 16\max(1, 1/c_2)$ large enough such that $\left(-2c_1 \left(\frac{T}{8}\right)^{1/\gamma} + \widetilde{c}\right) \leq -1$ that is $8\left(\frac{1+\widetilde{c}}{2c_1}\right)^\gamma \leq T$, to deduce the existence of $C > 0$ such that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
\int_\Omega \int_{t\in(\frac{T}{2}-1,\frac{T}{2}+1)} |u(x,t)|^2 \, dtdx &\leq \int_\Omega \int_{t\in(\frac{T}{2}-1,\frac{T}{2}+1)} |\Phi(t)u(x,t)|^2 \, dtdx \\
&\leq C\varepsilon^{2\beta k} \|(u_0,u_1)\|^2_{D(A^{k-1})} \\
&\quad + Ce^{C/\varepsilon} \int_\omega \int_0^T |u(x,t)|^2 \, dtdx \ .
\end{aligned} \tag{F.21}$$

Now we conclude from (F.2) that there exist a constant $c > 0$ and a time $T > 0$ large enough such that for all $\varepsilon > 0$ we have

$$\begin{aligned}
&\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \\
&\leq e^{c/\varepsilon} \int_\omega \int_0^T |u(x,t)|^2 \, dtdx + \varepsilon^{2\beta k} \|(u_0,u_1)\|^2_{D(A^{k-1})} \ .
\end{aligned} \tag{F.22}$$

Finally, we choose

$$\varepsilon = \left(\frac{\|(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}}{\|(u_0,u_1)\|_{D(A^{k-1})}}\right)^{1/(\beta k)} .$$

Theorem 1.1 is deduced by applying Theorem F to $\partial_t u$.

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