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Impulse output rapid stabilization for heat equations

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Abstract

The main aim of this paper is to provide a new feedback law for the heat equations in a bounded domain Ω with Dirichlet boundary condition. Two constraints will be compulsory: First, the controls are active in a subdomain of Ω and at discrete time points; Second, the observations are made in another subdomain and at different discrete time points. Our strategy consists in linking an observation estimate at one time, minimal norm impulse control, approximate inverse source problem and rapid output stabilization.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain, with a C^2 boundary $\partial\Omega$. Let V be a function in $L^\infty(\Omega)$ with its norm $\|\cdot\|_\infty$. Define

$$A := \Delta - V, \text{ with } D(A) = H^2(\Omega) \cap H_0^1(\Omega) \ .$$

Write $\{e^{tA}, t \geq 0\}$ for the semigroup generated by A on $L^2(\Omega)$. It is well-known that when $V = 0$, the semigroup $\{e^{tA}, t \geq 0\}$ has the exponential decay with the rate α_1, which is the first eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition. The aim of this study is to build up, for each $\gamma > 0$, an output feedback law $\mathcal{F}_\gamma$ so that any solution to the closed-loop controlled heat equation, associated with A and $\mathcal{F}_\gamma$, has an exponential decay with the rate γ. Further, two constraints are imposed: We only have accesses to the system at time $\frac{1}{2}T + \overline{\mathbb{N}}T$ on an open subdomain $\omega_1 \subset \Omega$; We only can control at time $T + \overline{\mathbb{N}}T$ on another open subset $\omega_2 \subset \Omega$. Here, T is an arbitrarily fixed positive number and $\overline{\mathbb{N}} := \mathbb{N} \cup \{0\}$, with $\mathbb{N}$ the set of all positive natural numbers. Therefore, the closed-loop controlled equation under consideration reads:

$$\begin{cases} y'(t) - Ay(t) = 0\,, & t \in (0,\infty) \setminus \mathbb{N}T\ , \\ y(0) \in L^2(\Omega)\ , & \\ y(nT) = y(nT_-) + 1_{\omega_2}\mathcal{F}_\gamma\left(1_{\omega_1}^* y\left(\left(n - \frac{1}{2}\right)T\right)\right)\,, & \forall n \in \mathbb{N}\,. \end{cases} \tag{1.1}$$

Here, $y(nT_-)$ denotes the left limit of the function: $t \mapsto y(t)$ (from $[0,\infty)$ to $L^2(\Omega)$) at time nT; 1_{ω_2} denotes the zero-extension operator from $L^2(\omega_2)$ to $L^2(\Omega)$ (i.e., for each $f \in L^2(\omega_2)$, $1_{\omega_2}(f)$ is defined to be the zero-extension of f over Ω); $1_{\omega_1}^*$ stands for the adjoint operator of 1_{ω_1}; $\mathcal{F}_\gamma$ is a linear and bounded operator from $L^2(\omega_1)$ to $L^2(\omega_2)$. The operator $\mathcal{F}_\gamma$ is what we will build up. The evolution distributed system (1.1) is well-posed and can be understood as the coupling of a sequence of heat equations:

$$y(t) := \left| \begin{array}{ll} y^0(t)\ , & \text{if } t \in [0,T) \\ y^n(t)\ , & \text{if } t \in [nT,(n+1)T) \end{array} \right.$$

for any $n \in \mathbb{N}$, where

$$\begin{cases} \partial_t y^0 - \Delta y^0 + Vy^0 = 0\,, & \text{in } \Omega \times (0,T)\ , \\ y^0 = 0\,, & \text{on } \partial\Omega \times (0,T)\ , \\ y^0(0) = y(0) \in L^2(\Omega)\ , & \end{cases}$$

and

$$\begin{cases} \partial_t y^n - \Delta y^n + Vy^n = 0\,, & \text{in } \Omega \times (nT,(n+1)T)\ , \\ y^n = 0\,, & \text{on } \partial\Omega \times (nT,(n+1)T)\ , \\ y^n(nT) = y^{n-1}(nT) + 1_{\omega_2}\mathcal{F}_\gamma\left(1_{\omega_1}^* y^{n-1}\left(\left(n - \frac{1}{2}\right)T\right)\right) \in L^2(\Omega)\ . & \end{cases}$$

Throughout this paper, we denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product of $L^2(\Omega)$ respectively. The notation $\|\cdot\|_{\omega_1}$ and $\langle\cdot,\cdot\rangle_{\omega_1}$ will mean the norm and the inner product of $L^2(\omega_1)$

respectively. We write $\mathcal{L}(H_1, H_2)$ for the space consisting of all bounded linear operators from one Hilbert space H_1 to another Hilbert space H_2. Lastly, we introduce the set $\{\lambda_j\}_{j=1}^{\infty}$ for the family of all eigenvalues of $-A$ so that

$$\lambda_1 \le \lambda_2 \le \cdot\cdot \le \lambda_m \le 0 < \lambda_{m+1} \le \cdots \text{ and } \lim_{j\to\infty} \lambda_j = \infty\,, \tag{1.2}$$

and let $\{\xi_j\}_{j=1}^{\infty}$ be the family of the corresponding orthogonal normalized eigenfunctions.

The main theorem of this paper will be precisely presented in section 5. It can be simply stated as follows: For each $\gamma > 0$, there is $\mathcal{F}_\gamma \in \mathcal{L}\left(L^2(\omega_1), L^2(\omega_2)\right)$ and a positive constant C_γ (depending on γ but independent of t) so that each solution y to the equation (1.1) satisfies the inequality:

$$\|y(t)\| \le C_\gamma e^{-\gamma t} \|y(0)\| \qquad \text{for any } t \ge 0\,. \tag{1.3}$$

We now give two comments on this result. First, the aforementioned $\mathcal{F}_\gamma$ has the form:

$$\mathcal{F}_\gamma(p) = -\sum_{j=1}^{K_\gamma} e^{\lambda_j T/2} \langle g_j, p\rangle_{\omega_1} f_j(x) \qquad \text{for any } p \in L^2(\omega_1)\ . \tag{1.4}$$

Here, $K_\gamma \in \mathbb{N}$ is the number of all eigenvalues λ_j which are less that $\gamma + \frac{\ln 2}{T}$; g_j and f_j are vectors in $L^2(\omega_1)$ and $L^2(\omega_2)$ respectively. These vectors are minimal norm controls for a kind of minimal norm problems which can be given by constructive methods. Second, the operator norm of $\mathcal{F}_\gamma$ is bounded by $C_1 e^{C_2\gamma}$, with $C_1 > 0$ and $C_2 > 0$ independent of γ.

We next explain our strategy and key points to prove the above-mentioned results. First, we realize that if a solution y to the equation (1.1) satisfies the inequality:

$$\left\| y\left(\left(\tfrac{5}{4}+n\right)T\right)\right\| \le e^{-\gamma T}\left\| y\left(\left(\tfrac{1}{4}+n\right)T\right)\right\| \qquad \text{for all } n \in \overline{\mathbb{N}}\,,$$

then (1.3) holds. Next, by the time translation invariance of the equation, we can focus our study on the interval $\left[\left(\frac{1}{4}+n\right)T, \left(\frac{5}{4}+n\right)T\right)$. And the problem of stabilization is transferred into the following approximate controllability problem (of the system (1.1) over the above interval): Find a control in a feedback form driving the system from each initial datum $y\left(\left(\frac{1}{4}+n\right)T\right)$ to $y\left(\left(\frac{5}{4}+n\right)T\right)$ with the above estimate. When building up the feedback law, we propose a new method to reconstruct approximatively the initial data $y\left(\left(\frac{1}{4}+n\right)T\right)$ from the knowledge of $1_{\omega_1}^* y\left(\left(\frac{1}{2}+n\right)T\right)$. All along such process, we need to take care of the cost of $\mathcal{F}_\gamma$. Naturally, we may expect that smaller is $e^{-\gamma T}$, larger is the cost. It is worth mentioning, that by projecting the initial data $y\left(\left(\frac{1}{4}+n\right)T\right)$ into subspaces span$\{\xi_1, \cdots, \xi_{K_\gamma}\}$ and span$\{\xi_{K_\gamma+1}, \xi_{K_\gamma+2}, \cdots\}$ respectively, we have

$$\left\| P_{K_\gamma} y((n+1)T)\right\| \le e^{-\lambda_{K_\gamma+1} 3T/4}\left\| y\left(\left(\tfrac{1}{4}+n\right)T\right)\right\|$$

where P_{K_γ} denotes the orthogonal projection of $L^2(\Omega)$ onto the second subspace. Therefore, there is no interest to control the initial data $P_{K_\gamma} y\left(\left(\frac{1}{4}+n\right)T\right)$ when $\lambda_{K_\gamma+1} \gg \gamma$. This suggests the form of our feedback law.

In our analysis, a precise estimate is established and we will build the output stabilization law via some minimal norm impulse control problems. It requires to link the approximate impulse control and a quantitative unique continuation estimate called observation at one time. Our program follows the orientation described in [27] where stabilization, optimal control and exact controllability for hyperbolic systems are closely linked. Here, exact controllability for the wave equation is replaced by approximate impulse control for the heat equations.

Several notes are given in order.

1. Impulse control belongs to a class of important control and has wide applications (see, for instance, [4,7,36,40]). In many cases impulse control is an interesting alternative to deal with systems that cannot be acted on by means of continuous control inputs, for instance, relevant control for acting on a population of bacteria may be impulsive, so that the density of the bactericide may change instantaneously; indeed continuous control would be enhance drug resistance of bacteria (see [40] and [37]). In the book [40], the author systematically introduces both theory and applications for impulse controlled ODEs, and also presents ways about how to realize impulse controls.
   There are many studies on optimal control and controllability for impulse controlled equations (see, for instance, [6,35,7,28,42], [25, Chapter 8], [3,16,29,31,5] and references therein). However, we have not found any published paper on stabilization for impulse controlled equations. From this perspective, the problem studied in the current paper is new.
   In the system (1.1), we do not need to follow the rule: Make observation at each time and then add simultaneously control with a feedback form. (When systems have continuous feedback control inputs, one usually follows such a rule.) Instead of this, we only need to observe $1^*_{\omega_1} y$ at time points $\left(n-\frac{1}{2}\right)T$ (for all $n \in \mathbb{N}$) and then add the controls at time points nT. In this way, we can not only save observation and control time, but also allow the control time nT having a delay with respect to the observation time $\left(n-\frac{1}{2}\right)T$. Sometime, such time delay could be important in practical application.
2. Stabilization is one of the most important subject in control theory. In most studies of this subject, the aim of stabilization is to ask for a feedback law so that the closed loop equation decays exponentially. The current work aims to find, for each decay rate γ, a feedback law so that the closed loop equation has an exponential decay with the rate γ. Such kind of stabilization is called the rapid stabilization. About this subject, we would like to mention the works [21,38,10,39,9,14,13].
3. When observation region is not the whole Ω, the corresponding stabilization is a kind of output stabilization. Such stabilization is very useful in applications. Unfortunately, there is no systematic study on this subject, even for the simplest case when the controlled system is time-invariant linear ODE (see [8]). Most of publications on this subject focus on how to construct an output feedback law for some special equations (see, for instance, [19,15,41,12,30] and references therein). Our study also only provides an output feedback law for a special equation.

4. In this paper, we present a new way to build up the feedback law. In particular, the structure of our feedback law is not based on LQ theory or Lyapunov functions (see, for instance, [2] and [11]).
5. In the current study, one of the keys to build up our feedback law is the use of the unique continuation estimate at one time, established in [34] (see also [32] and [33]). Some new observations are made on it in this paper (see Theorem 2.1, Remark 2.2 and Remark 2.3).
6. The following extensions of the current work should be interesting: The first case is that V depends on both x and t variables; The second case is that the equation is a semi-linear heat equation; The third case is that the equation is other types of PDEs.

The rest of this paper is organized as follows. Section 2 provides several inequalities which are equivalent to the unique continuation estimate at one time. Such observation estimates are used in Section 3 in which we deal with the impulse control problem. In Section 4, we link the impulse control problem with an approximate inverse source problem. Finally, Section 5 presents the main result, as well as its proof.

2. Observation at one time

In this section, we present several equivalent inequalities. One of them is the unique continuation estimate at one time built up in [34] (see also [32] and [33]).

Theorem 2.1. *Let ω be an open and nonempty subset of Ω. Then the following propositions are equivalent and are true:*

(*i*) *There are two constants $C_1 > 0$ and $\beta \in (0, 1)$, which depend only on Ω and ω, so that for all $t > 0$ and $\Phi \in L^2(\Omega)$,*

$$\left\| e^{tA}\Phi \right\| \leq e^{C_1\left(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \|\Phi\|^\beta \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega^{1-\beta} .$$

(*ii*) *There is a positive constant C_2, depending only on Ω and ω, so that for each $\lambda \geq 0$ and each sequence of real numbers $\{a_j\} \subset \mathbb{R}$,*

$$\sum_{\lambda_j<\lambda} \left|a_j\right|^2 \leq e^{C_2\left(1+\|V\|_\infty^{2/3}+\sqrt{\lambda}\right)} \int_\omega \left| \sum_{\lambda_j<\lambda} a_j \xi_j \right|^2 dx .$$

(*iii*) *There is a positive constant C_3, depending only on Ω and ω, so that for all $\theta \in (0, 1)$, $t > 0$ and $\Phi \in L^2(\Omega)$,*

$$\left\| e^{tA}\Phi \right\| \leq e^{C_3\left(1+\frac{1}{\theta t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \|\Phi\|^\theta \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega^{1-\theta} .$$

(*iv*) *There is a positive constant C_3, depending only on Ω and ω, so that for all $\varepsilon, \beta > 0$, $t > 0$ and $\Phi \in L^2(\Omega)$,*

$$\left\| e^{tA}\Phi \right\| \leq \frac{1}{\varepsilon^\beta} e^{C_3(1+\beta)\left(1+\frac{1+\beta}{\beta t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega + \varepsilon \|\Phi\| .$$

(v) *There is a positive constant c, depending only on Ω and ω, so that for all $\varepsilon > 0$, $t > 0$ and $\Phi \in L^2(\Omega)$,*

$$\left\| e^{tA}\Phi \right\| \leq e^{c\left(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \exp\left(\sqrt{\frac{c}{t}\ln^+\frac{1}{\varepsilon}}\right) \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega + \varepsilon \left\|\Phi\right\| .$$

Here $\ln^+\frac{1}{\varepsilon} := \max\{\ln\frac{1}{\varepsilon}, 0\}$. *Moreover, constants C_3 in (iii) and (iv) can be chosen as the same number, and $c = 4C_3$ in (v).*

Remark 2.2. We would like to give several notes on Theorem 2.1.

1. The inequality in (i) of Theorem 2.1 implies a quantitative version of the unique continuation property of heat equations built up in [26] (see also [17]). Indeed, if $\left\| 1_\omega^* e^{tA}\Phi \right\|_\omega = 0$ for some $\Phi \in L^2(\Omega)$, then $\left\| e^{tA}\Phi \right\| = 0$, which, together with the backward uniqueness for heat equations, implies that $\Phi = 0$. Moreover, such inequality is equivalent to a kind of impulse control, with a bound on its cost, which will be explained in the next section. Further, such controllability is the base for our study on the stabilization.
2. In the following studies of this paper, we will essentially use the inequality in (iii) of Theorem 2.1. However, other inequalities seem to be interesting independently. For instance, when $V = 0$, the inequality in (ii) of Theorem 2.1 is exactly the Lebeau–Robbiano spectral inequality (see [22], [20] or [23]). Here, we get, in the case that $V \neq 0$, the same inequality, and find how the constant (on the right hand side of the inequality) depends on $\|V\|_\infty$.
3. It deserves to mention that all constant terms on the right hand side of inequalities in Theorem 2.1 have explicit expressions in terms of the norm of V and time, but not Ω and ω.
4. It was realized that when $V = 0$, the inequality in (i) of Theorem 2.1 can imply the inequality (ii) of Theorem 2.1 (see Remark 1 in [1]).
5. The key ingredient why the inequality in (i) and the one in (ii) of Theorem 2.1 are equivalent is that the evolution e^{tA} is time-invariant. We would like to mention that the inequality in (i) of Theorem 2.1 was proved in [34] (see also [32] and [33]) for the case when $V = V(x,t)$ (i.e., for the evolutions with time-varying coefficients). However, for the time-varying evolutions, we do not know whether (iii), (iv) and (v) in Theorem 2.1 are still valid and what should be the right alternative of the spectral inequality in (ii) of Theorem 2.1. These should be interesting open problems.

Remark 2.3. When $V = 0$, the function $\exp\left(\sqrt{\ln^+\frac{1}{\varepsilon}}\right)$ is optimal in the inequality in (v) of Theorem 2.1 in the following sense: For any function $\varepsilon \mapsto f(\varepsilon)$, $\varepsilon > 0$, with $\overline{\lim_{\varepsilon\to 0^+}} \frac{f(\varepsilon)}{\sqrt{\ln^+\frac{1}{\varepsilon}}} = 0$, the following inequality is not true:

$$\left\| e^{t\Delta}\Phi \right\| \leq f(\varepsilon)\, e^{f(\varepsilon)} \left\| 1_\omega^* e^{t\Delta}\Phi \right\|_\omega + \varepsilon \left\|\Phi\right\| \quad \text{for all } \Phi \in L^2(\Omega) \text{ and } \varepsilon > 0 .$$

We now explain why the optimality in Remark 2.3 is true. According to Proposition 5.5 in [24], there exist $C > 0$ and $n_0 > 0$ so that for each $n \geq n_0$, there is $\left\{a_{n,j}\right\}_{j=1}^{\infty} \subset \ell^2 \setminus \{0\}$ satisfying that

$$\left\| \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\| \ge C e^{C\sqrt{n}} \left\| 1_\omega^* \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\|_\omega .$$

Define, for each $n \in \mathbb{N}$,

$$\Phi_n := \sum_{\lambda_j \le n} e^{\lambda_j t} a_{n,j} \xi_j \text{ and } \delta_n := \frac{1}{2} e^{-nt} .$$

Therefore, we find that for each $n \ge n_0$,

$$\begin{aligned} \left\| e^{t\Delta} \Phi_n \right\| &= \left\| \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\| = \left(1 - \delta_n e^{nt}\right) \left\| \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\| + \delta_n e^{nt} \left\| \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\| \\ &\ge \frac{1}{2} \left\| \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\| + \delta_n \left\| \sum_{\lambda_j \le n} e^{\lambda_j t} a_{n,j} \xi_j \right\| \\ &\ge \frac{1}{2} C e^{C\sqrt{n}} \left\| 1_\omega^* \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\|_\omega + \delta_n \left\| \Phi_n \right\| . \end{aligned}$$

Meanwhile, it follows from the definition of δ_n that for each $n \in \mathbb{N}$,

$$\sqrt{\ln \frac{1}{\delta_n}} = \sqrt{\ln 2 + nt} \le 1 + \sqrt{n}\sqrt{t} .$$

Gathering all the previous estimates, we see that

$$\left\| e^{t\Delta} \Phi_n \right\| \ge \frac{1}{2} C e^{-\frac{C}{\sqrt{t}}} e^{\frac{C}{\sqrt{t}} \sqrt{\ln \frac{1}{\delta_n}}} \left\| 1_\omega^* \sum_{\lambda_j \le n} a_{n,j} \xi_j \right\|_\omega + \delta_n \left\| \Phi_n \right\| ,$$

which leads to the following property: There exists $c > 0$, $\{\Phi_n\} \subset L^2(\Omega) \setminus \{0\}$ and $\{\delta_n\} \subset (0, 1)$, with $\lim_{n \to \infty} \delta_n = 0$, so that

$$\left\| e^{t\Delta} \Phi_n \right\| \ge c e^{c\sqrt{\ln \frac{1}{\delta_n}}} \left\| 1_\omega^* e^{t\Delta} \Phi_n \right\|_\omega + \delta_n \left\| \Phi_n \right\| \quad \text{for all } n .$$

The rest of this section is devoted to the proof of Theorem 2.1.

Proof. We organize the proof of Theorem 2.1 by several steps.
Step 1: On the proposition (i).
The conclusion (i) has been proved in [34] (see also [32] and [33]).
Step 2: To prove that $(i) \Rightarrow (ii)$.
Let $C_1 > 0$ and $\beta \in (0, 1)$ be given by (i). Arbitrarily fix $\lambda \ge 0$ and $\{a_j\} \subset \mathbb{R}$. By applying the inequality in (i), with $\Phi = \sum_{\lambda_j < \lambda} a_j e^{\lambda_j t} \xi_j$, we get that

$$\sum_{\lambda_j<\lambda} \left|a_j\right|^2 \le e^{2C_1\left(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \left(\sum_{\lambda_j<\lambda} \left|a_j e^{\lambda_j t}\right|^2\right)^{\beta} \left(\int_\omega \left|\sum_{\lambda_j<\lambda} a_j\xi_j\right|^2 dx\right)^{1-\beta} ,$$

which implies that

$$\sum_{\lambda_j<\lambda} |a_j|^2 \le e^{\frac{2}{1-\beta}C_1\left(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)} e^{\frac{2\beta}{1-\beta}\lambda t} \int_\omega \left|\sum_{\lambda_j<\lambda} a_j\xi_j\right|^2 dx \quad \text{for each } t>0 . \tag{2.1}$$

Meanwhile, since $\|V\|_\infty^{1/2} \le 1+\|V\|_\infty^{2/3}$ and $\beta \in (0,1)$, we see that

$$\begin{aligned}
&\inf_{t>0}\left[C_1\left(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)+\beta\lambda t\right]\\
&=\inf_{t>0}\left[C_1(1+\|V\|_\infty^{2/3})+\frac{C_1}{t}+(C_1\|V\|_\infty+\beta\lambda)\,t\right]\\
&=C_1\left(1+\|V\|_\infty^{2/3}\right)+2\sqrt{C_1\,(C_1\|V\|_\infty+\beta\lambda)}\\
&\le C_1\left(1+\|V\|_\infty^{2/3}+2\|V\|_\infty^{1/2}\right)+2\sqrt{C_1}\sqrt{\beta\lambda} \le \max\left\{3C_1, 2\sqrt{C_1}\right\}\left(1+\|V\|_\infty^{2/3}+\sqrt{\lambda}\right) .
\end{aligned}$$

This, along with (2.1), leads to the conclusion (*ii*), with $C_2=\max\left\{\frac{6C_1}{1-\beta}, \frac{4\sqrt{C_1}}{1-\beta}\right\}$.

Step 3: To show that (*ii*) $\Rightarrow$ (*iii*).

Arbitrarily fix $\lambda \ge 0$, $t>0$ and $\Phi=\sum_{j\ge 1} a_j\xi_j$ with $\{a_j\}\subset \ell^2$. Write

$$e^{tA}\Phi=\sum_{\lambda_j<\lambda} a_j e^{-\lambda_j t}\xi_j+\sum_{\lambda_j\ge\lambda} a_j e^{-\lambda_j t}\xi_j .$$

Then by (*ii*), we find that

$$\begin{aligned}
\left\|e^{tA}\Phi\right\| &\le \left\|\sum_{\lambda_j<\lambda} a_j e^{-\lambda_j t}\xi_j\right\|+\left\|\sum_{\lambda_j\ge\lambda} a_j e^{-\lambda_j t}\xi_j\right\|\\
&\le \left(\sum_{\lambda_j<\lambda}\left|a_j e^{-\lambda_j t}\right|^2\right)^{1/2}+e^{-\lambda t}\|\Phi\|\\
&\le \left(e^{C_2\left(1+\|V\|_\infty^{2/3}+\sqrt{\lambda}\right)}\int_\omega\left|\sum_{\lambda_j<\lambda} a_j e^{-\lambda_j t}\xi_j\right|^2 dx\right)^{1/2}+e^{-\lambda t}\|\Phi\| .
\end{aligned}$$

This, along with the triangle inequality for the norm $\|\cdot\|_\omega$, yields that

$$\begin{aligned}\left\|e^{tA}\Phi\right\| &\leq \left(e^{C_2\left(1+\|V\|_\infty^{2/3}+\sqrt{\lambda}\right)}\int_\omega\left|\sum_{j\geq 1}a_je^{-\lambda_jt}\xi_j\right|^2dx\right)^{1/2}\\ &\quad+\left(e^{C_2\left(1+\|V\|_\infty^{2/3}+\sqrt{\lambda}\right)}\int_\omega\left|\sum_{\lambda_j\geq\lambda}a_je^{-\lambda_jt}\xi_j\right|^2dx\right)^{1/2}+e^{-\lambda t}\|\Phi\|\ .\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\left\|e^{tA}\Phi\right\| &\leq e^{\frac{C_2}{2}\left(1+\|V\|_\infty^{2/3}+\sqrt{\lambda}\right)}\left\|1_\omega^*e^{tA}\Phi\right\|_\omega+e^{\frac{C_2}{2}\left(1+\|V\|_\infty^{2/3}+\sqrt{\lambda}\right)}e^{-\lambda t}\|\Phi\|+e^{-\lambda t}\|\Phi\|\\ &\leq 2e^{\frac{C_2}{2}\left(1+\|V\|_\infty^{2/3}\right)}e^{\frac{C_2}{2}\sqrt{\lambda}}\left(\left\|1_\omega^*e^{tA}\Phi\right\|_\omega+e^{-\lambda t}\|\Phi\|\right)\ .\end{aligned}$$

Combining the above estimate with the following inequality:

$$\frac{C_2}{2}\sqrt{\lambda}\leq\frac{\rho}{2}\lambda t+\frac{1}{2t\rho}\left(\frac{C_2}{2}\right)^2\quad\text{for any }\rho>0\ ,$$

we have that for all $\rho\in(0,2)$ and $\lambda\geq 0$,

$$\left\|e^{tA}\Phi\right\|\leq 2e^{\frac{C_2}{2}\left(1+\|V\|_\infty^{2/3}\right)}e^{\frac{1}{2t\rho}\left(\frac{C_2}{2}\right)^2}\left(e^{\frac{\rho}{2}\lambda t}\left\|1_\omega^*e^{tA}\Phi\right\|_\omega+e^{-\frac{2-\rho}{2}\lambda t}\|\Phi\|\right)\ .$$

Since λ was arbitrarily taken from $[0,\infty)$, we choose

$$\lambda=\frac{1}{t}\ln\left(\frac{e^{t\|V\|_\infty}\|\Phi\|}{\left\|1_\omega^*e^{tA}\Phi\right\|_\omega}\right)$$

to get

$$\left\|e^{tA}\Phi\right\|\leq 2e^{\frac{C_2}{2}\left(1+\|V\|_\infty^{2/3}\right)}e^{\frac{1}{2t\rho}\left(\frac{C_2}{2}\right)^2}\left(2e^{t\|V\|_\infty}\left\|1_\omega^*e^{tA}\Phi\right\|_\omega^{1-\frac{\rho}{2}}\|\Phi\|^{\frac{\rho}{2}}\right)$$

which is the inequality in (*iii*) with $\theta=\rho/2$.
Step 4: To show that (*iii*) $\Rightarrow$ (*iv*).

We write the inequality in (*iii*) in the following way:

$$\left\|e^{tA}\Phi\right\|\leq\|\Phi\|^\theta\left(e^{\frac{C_3}{1-\theta}\left(1+\frac{1}{\theta t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)}\left\|1_\omega^*e^{tA}\Phi\right\|_\omega\right)^{1-\theta}\ .$$

Notice that for any real numbers $E, B, D>0$ and $\theta\in(0,1)$

$$\begin{aligned}&E\leq B^\theta D^{1-\theta}\\ \Leftrightarrow\quad &E\leq\varepsilon B+(1-\theta)\,\theta^{\frac{\theta}{1-\theta}}\frac{1}{\varepsilon^{\frac{\theta}{1-\theta}}}D\quad\forall\varepsilon>0\ .\end{aligned}$$

Then by taking $\beta = \frac{\theta}{1-\theta}$ in the last inequality, we are led to the inequality in (iv) with the same constant C_3 as that in (iii). (The above equivalence can be easily verified by using the Young inequality and by choosing $\varepsilon = \theta \left(\frac{D}{B}\right)^{1-\theta}$.)

Step 5: To show that $(iv) \Rightarrow (v)$.

We write the inequality in (iv) in the following way:

$$\left\| e^{tA}\Phi \right\| \leq \exp\left(\Lambda + \Upsilon + \beta\left(\ln\frac{1}{\varepsilon} + \Lambda\right) + \frac{1}{\beta}\Upsilon\right) \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega + \varepsilon \left\|\Phi\right\| ,$$

with $\Lambda = C_3\left(1 + \frac{1}{t} + t\left\|V\right\|_\infty + \left\|V\right\|_\infty^{2/3}\right)$ and $\Upsilon = \frac{C_3}{t}$. Next, we optimize the above inequality with respect to $\beta > 0$ by choosing $\beta = \sqrt{\frac{\Upsilon}{\ln^+\frac{1}{\varepsilon} + \Lambda}}$ to get

$$\begin{aligned}
\left\| e^{tA}\Phi \right\| &\leq \exp\left(\Lambda + \Upsilon + 2\sqrt{\Upsilon\left(\ln^+\tfrac{1}{\varepsilon} + \Lambda\right)}\right) \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega + \varepsilon \left\|\Phi\right\| \\
&\leq \exp\left(\Lambda + \Upsilon + 2\sqrt{\Upsilon\Lambda} + 2\sqrt{\Upsilon \ln^+\tfrac{1}{\varepsilon}}\right) \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega + \varepsilon \left\|\Phi\right\| \\
&\leq \exp\left(4\Lambda + 2\sqrt{\Upsilon \ln^+\tfrac{1}{\varepsilon}}\right) \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega + \varepsilon \left\|\Phi\right\| .
\end{aligned}$$

This implies the inequality in (v), with $c = 4C_3$.

Step 6: to show that $(v) \Rightarrow (i)$.

Since

$$\sqrt{\frac{c}{t}\ln^+\frac{1}{\varepsilon}} \leq \frac{c}{\alpha t} + \alpha \ln\left(e + \frac{1}{\varepsilon}\right) \quad \forall \alpha > 0 ,$$

the inequality in (v) becomes

$$\left\| e^{tA}\Phi \right\| \leq e^{c\left(1+\frac{1}{t}+t\left\|V\right\|_\infty + \left\|V\right\|_\infty^{2/3}\right)} e^{\frac{c}{\alpha t}} \left(e + \frac{1}{\varepsilon}\right)^\alpha \left\| 1_\omega^* e^{tA}\Phi \right\|_\omega + \varepsilon \left\|\Phi\right\| .$$

Next, we choose

$$\varepsilon = \frac{1}{2}\frac{\left\| e^{tA}\Phi \right\|}{\left\|\Phi\right\|}$$

and we use the fact that $\left\| e^{tA}\Phi \right\| \leq e^{t\left\|V\right\|_\infty} \left\|\Phi\right\|$ to deduce the inequality in (i).

This ends the proof.

3. Impulse control

In this section, we first state our key result on impulse approximate controllability.

Theorem 3.1. *Let* $0 \leq T_1 < T_2 < T_3$. *Let* $\varepsilon > 0$ *and* $z \in L^2(\Omega)$. *Then the following conclusions are true:*

(i) *There exists* $f \in L^2(\omega)$ *such that the unique solution* y *to the equation:*

$$\begin{cases} y'(t) - Ay(t) = 0\,, & t \in (T_1, T_3) \setminus \{T_2\}\,, \\ y(T_1) = z\,, \\ y(T_2) = y(T_{2-}) + 1_\omega f\,, \end{cases}$$

satisfies

$$\|y(T_3)\| \leq \varepsilon \|z\|\ .$$

Moreover, it holds that

$$\|f\|_\omega \leq \mathcal{C}_\varepsilon (T_3 - T_2, T_2 - T_1) \|z\|$$

where $\mathcal{C}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is given by*

$$\mathcal{C}_\varepsilon(t, s) = e^{4s\|V\|_\infty} e^{c\left(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \exp\left(\sqrt{\frac{c}{t}\ln^+\frac{1}{\varepsilon}}\right)\,,$$

and $c > 0$ *is the same constant than in* (v) *of Theorem 2.1.*

(ii) *There is a unique* f *solving the following problem:*

$$\inf\left\{\|h\|_\omega\,; h \in L^2(\omega) \text{ such that } \|y(T_3)\| \leq \varepsilon \|z\| \text{ with } f \text{ replaced by } h \text{ in } (i)\right\}\ .$$

The proof of Theorem 3.1 will be given in Subsections 3.1–3.2. We now apply Theorem 3.1 to the eigenfunctions $\{\xi_j\}_{j=1}^\infty$. More precisely, by choosing $z = \xi_j$ in Theorem 3.1, we have the following corollary:

Corollary 3.2. *For any* $0 \leq T_1 < T_2 < T_3$ *and any* $\varepsilon > 0$, $j \in \mathbb{N}$, *there is a pair* (y_j, f_j) *such that*

$$\begin{cases} y_j'(t) - Ay_j(t) = 0\,, & t \in (T_1, T_3) \setminus \{T_2\}\,, \\ y_j(T_1) = \xi_j\,, \\ y_j(T_2) = y_j(T_{2-}) + 1_\omega f_j\,, \end{cases}$$

and

$$\begin{cases} \left\|y_j(T_3)\right\| \leq \varepsilon\,, \\ \left\|f_j\right\|_\omega \leq \mathcal{C}_\varepsilon(T_3 - T_2, T_2 - T_1)\,, \end{cases}$$

where $\mathcal{C}_\varepsilon$ *is given in Theorem 3.1. Further, the control function* f_j *can be taken as the unique solution of the problem*

$$\inf\left\{ \begin{array}{c} \|h\|_\omega\,; h \in L^2(\omega) \text{ and the property } (i) \text{ of Theorem 3.1} \\ \text{holds with } (z, f) \text{ replaced by } (\xi_j, h) \end{array} \right\}\ .$$

Further we will apply Theorem 3.1 to a finite combination of eigenfunctions. More precisely, we have the following consequence:

Theorem 3.3. *Let ω_2 be a non-empty open subset of Ω and $K \in \mathbb{N}$. Let $0 \leq T_1 < T_2 < T_3$ and $\varepsilon > 0$. Then for any $b = (b_j)_{j=1,\cdot\cdot,K}$, there is a pair $(\widetilde{y}, \widetilde{f})$ such that*

$$\begin{cases} \widetilde{y}'(t) - A\widetilde{y}(t) = 0\,, & t \in (T_1, T_3) \setminus \{T_2\}\,, \\ \widetilde{y}(T_1) = \sum_{j=1,\cdot\cdot,K} b_j \xi_j\,, \\ \widetilde{y}(T_2) = \widetilde{y}(T_{2-}) + 1_{\omega_2} \widetilde{f}\,, \end{cases}$$

and

$$\begin{cases} \|\widetilde{y}(T_3)\| \leq \varepsilon\sqrt{K}\, \|b\|_{\ell^2}\,, \\ \widetilde{f} = \sum_{j=1,\cdot\cdot,K} b_j f_j\,, \end{cases}$$

where f_j is given by Corollary 3.2 with $\omega = \omega_2$ and satisfies

$$\left\| f_j \right\|_{\omega_2} \leq \mathcal{C}_\varepsilon\,(T_3 - T_2, T_2 - T_1)\ .$$

In the study of our stabilization, we will use Corollary 3.2 and Theorem 3.3. The rest of this section is devoted to the proof of Theorem 3.1, and the studies on some minimal norm control problem.

3.1. Existence of impulse control functions and its cost

The aim of this subsection is to prove the conclusion (i) of Theorem 3.1. It deserves mentioning what follows: The existence of controls with (i) of Theorem 3.1 is indeed the existence of impulse controls (with a cost) driving the solution of the equation in (i) of Theorem 3.1 from the initial state z to the closed ball in $L^2(\Omega)$, centered at the origin and of radius $\varepsilon \|z\|$, at the ending time T_3.

To prove the conclusion (i) of Theorem 3.1, we let $\varepsilon > 0$ and $z \in L^2(\Omega)$. Denote $\hbar = \varepsilon^2$ and $k = (\mathcal{C}_\varepsilon\,(T_3 - T_2, T_2 - T_1))^2$. Consider the strictly convex C^1 functional $\mathcal{F}$ defined on $L^2(\Omega)$ given by

$$\mathcal{F}(\Phi) := \frac{k}{2} \left\| 1_\omega^* e^{(T_3-T_2)A} \Phi \right\|_\omega^2 + \frac{\hbar}{2} \|\Phi\|^2 + \left\langle z, e^{(T_3-T_1)A} \Phi \right\rangle .$$

Notice that $\mathcal{F}$ is coercive and therefore $\mathcal{F}$ has a unique minimizer $w \in L^2(\Omega)$, i.e. $\mathcal{F}(w) = \min_{\Phi \in L^2(\Omega)} \mathcal{F}(\Phi)$. Since $\mathcal{F}'(w)\Phi = 0$ for any $\Phi \in L^2(\Omega)$, we have

$$k \left\langle 1_\omega^* e^{(T_3-T_2)A} w, 1_\omega^* e^{(T_3-T_2)A} \Phi \right\rangle_\omega + \hbar \langle w, \Phi \rangle - \left\langle z, e^{(T_3-T_1)A} \Phi \right\rangle = 0 \quad \forall \Phi \in L^2(\Omega)\ . \quad (3.1.1)$$

But by multiplying by $e^{(T_3-t)A}\Phi$ the system solved by y, one gets

$$\langle y(T_3), \Phi\rangle = \left\langle z, e^{(T_3-T_1)A}\Phi\right\rangle + \left\langle 1_\omega f, e^{(T_3-T_2)A}\Phi\right\rangle \quad \forall \Phi \in L^2(\Omega) \ .$$

By choosing $f = -k1_\omega^* e^{(T_3-T_2)A}w$, the above two equalities yield that

$$y(T_3) = \hbar w \ .$$

Further, one can deduce that

$$\frac{1}{k}\|f\|_\omega^2 + \frac{1}{\hbar}\|y(T_3)\|^2 = k\left\|1_\omega^* e^{(T_3-T_2)A}w\right\|_\omega^2 + \hbar\|w\|^2 \ .$$

Next, notice that by taking $\Phi = w$ in (3.1.1) and by Cauchy–Schwarz inequality, we have

$$k\left\|1_\omega^* e^{(T_3-T_2)A}w\right\|_\omega^2 + \hbar\|w\|^2 = \left\langle z, e^{(T_3-T_1)A}w\right\rangle \leq \frac{1}{2}\|z\|^2 + \frac{1}{2}\left\|e^{(T_3-T_1)A}w\right\|^2 \ .$$

Now, we claim that

$$\left\|e^{(T_3-T_1)A}w\right\|^2 \leq k\left\|1_\omega^* e^{(T_3-T_2)A}w\right\|_\omega^2 + \hbar\|w\|^2 \ .$$

When it is proved, we can gather the previous three estimates to yield

$$\frac{1}{k}\|f\|_\omega^2 + \frac{1}{\hbar}\|y(T_3)\|^2 \leq \|z\|^2 \ .$$

As a consequence of our choice of $(\hbar, k)$, we conclude that

$$\|y(T_3)\| \leq \sqrt{\hbar}\,\|z\| = \varepsilon\,\|z\|$$

and

$$\|f\|_\omega \leq \mathcal{C}_\varepsilon(T_3 - T_2, T_2 - T_1)\,\|z\| \, .$$

From these, we see that the above f satisfies two properties in (i) of Theorem 3.1.

It remains to prove the above claim. To this end, we write the inequality in Theorem 2.1 (iii) as follows.

$$\left\|e^{tA}\Phi\right\|^2 \leq \left(\|\Phi\|^2\right)^\theta \left(e^{\frac{2C_3}{1-\theta}\left(1+\frac{1}{\theta t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)}\left\|1_\omega^* e^{tA}\Phi\right\|_\omega^2\right)^{1-\theta} \ .$$

Since $\left\|e^{LA}\Phi\right\| \leq e^{(L-t)\|V\|_\infty}\left\|e^{tA}\Phi\right\|$ for $L \geq t$, it holds that

$$\left\|e^{LA}\Phi\right\|^2 \leq \left(\|\Phi\|^2\right)^\theta \left(e^{\frac{2}{1-\theta}(L-t)\|V\|_\infty} e^{\frac{2C_3}{1-\theta}\left(1+\frac{1}{\theta t}+t\|V\|_\infty+\|V\|_\infty^{2/3}\right)}\left\|1_\omega^* e^{tA}\Phi\right\|_\omega^2\right)^{1-\theta} \ .$$

Following the same technique as that used in the proof of Theorem 2.1 (step 5), using the Young inequality, we have that for any $\varepsilon > 0$ and any $\Phi \in L^2(\Omega)$,

$$\left\|e^{LA}\Phi\right\|^2 \leq \exp\left(2\Lambda_L + 2\Upsilon + \beta\left(\ln\frac{1}{\varepsilon} + 2\Lambda_L\right) + \frac{1}{\beta}2\Upsilon\right)\left\|1_\omega^* e^{tA}\Phi\right\|_\omega^2 + \varepsilon\left\|\Phi\right\|^2 ,$$

that is equivalent to

$$\left\|e^{LA}\Phi\right\|^2 \leq \exp\left(2\left(\Lambda_L + \Upsilon + \beta\left(\ln\frac{1}{\varepsilon} + \Lambda_L\right) + \frac{1}{\beta}\Upsilon\right)\right)\left\|1_\omega^* e^{tA}\Phi\right\|_\omega^2 + \varepsilon^2\left\|\Phi\right\|^2 ,$$

with $\Lambda_L = (L-t)\left\|V\right\|_\infty + C_3\left(1 + \frac{1}{t} + t\left\|V\right\|_\infty + \left\|V\right\|_\infty^{2/3}\right)$ and $\Upsilon = \frac{C_3}{t}$. Next, we choose $\beta = \sqrt{\frac{\Upsilon}{\ln^+\frac{1}{\varepsilon} + \Lambda_L}}$ to get that

$$\left\|e^{LA}\Phi\right\|^2 \leq \exp\left(8\Lambda_L + 4\sqrt{\Upsilon\ln^+\frac{1}{\varepsilon}}\right)\left\|1_\omega^* e^{tA}\Phi\right\|_\omega^2 + \varepsilon^2\left\|\Phi\right\|^2 ,$$

for any $\varepsilon > 0$, $L \geq t > 0$ and $\Phi \in L^2(\Omega)$. Setting $c = 4C_3$ (here $c > 0$ is the same constant than in (v) of Theorem 2.1) and applying the above, with the choices $L = T_3 - T_1$, $t = T_3 - T_2$, and $\Phi = w$, give the desired claim.

This completes the proof of the conclusion (i) in Theorem 3.1.

3.2. *Uniqueness of minimal norm impulse control and its construction*

The aim of this subsection is to study a minimal norm problem, which is indeed given in (ii) of Theorem 3.1. We will present some properties on this problem (see Theorem 3.4), and give the proof of (ii) of Theorem 3.1 (see (a) of Remark 3.5).

Arbitrarily fix $z \in L^2(\Omega)\setminus\{0\}$ and $\varepsilon > 0$. Recall the following impulse controlled equation over $[T_1, T_3]$:

$$\begin{cases} y'(t) - Ay(t) = 0\,, & t \in (T_1, T_3)\setminus\{T_2\}\,, \\ y(T_1) = z\,, \\ y(T_2) = y(T_{2-}) + 1_\omega f\,. \end{cases} \tag{3.2.1}$$

In this subsection, we discuss the following minimal norm impulse control problem $(\mathcal{P})$:

$$\mathcal{N}_z := \inf\left\{\left\|f\right\|_\omega;\, f \in L^2(\omega) \text{ and } \left\|y(T_3)\right\| \leq \varepsilon\left\|z\right\|\right\}. \tag{3.2.2}$$

This problem is to ask for a control which has the minimal norm among all controls (in $L^2(\omega)$) driving solutions of equation (3.2.1) from the initial state z to the closed ball in $L^2(\Omega)$, centered at the origin and of radius $\varepsilon\left\|z\right\|$, at the ending time T_3. In this problem, $\mathcal{N}_z$ is called the minimal norm, while $f^* \in L^2(\omega)$ is called a minimal norm control, if the solution y of (3.2.1) with $f = f^*$ satisfies

$$\left\|y(T_3)\right\| \leq \varepsilon\left\|z\right\| \text{ and } \left\|f^*\right\|_\omega = \mathcal{N}_z\,.$$

The main result of this subsection is as:

Theorem 3.4. *The following conclusions are true:*

(i) *The problem* ($\mathcal{P}$) *has a unique minimal norm control.*
(ii) *The minimal norm control* f^* *to* ($\mathcal{P}$) *satisfies that*

$$f^* = 0 \text{ if and only if the solution } y^0 \text{ of (3.2.1) with } f = 0 \text{ satisfies } \left\| y^0 (T_3) \right\| \leq \varepsilon \|z\| \ .$$

(iii) *The minimal norm control* f^* *to* ($\mathcal{P}$) *is given by*

$$f^* = 1^*_\omega e^{(T_3 - T_2)} w$$

where w *is the unique minimizer to* $J : L^2(\Omega) \to \mathbb{R}$ *defined by*

$$J(\Phi) := \frac{1}{2} \left\| 1^*_\omega e^{(T_3 - T_2)A} \Phi \right\|^2_\omega + \left\langle z, e^{(T_3 - T_1)A} \Phi \right\rangle + \varepsilon \|z\| \|\Phi\| \ .$$

Remark 3.5. (a) The conclusion (i) of Theorem 3.4 clearly gives the uniqueness in (ii) of Theorem 3.1. This, along with (i) of Theorem 3.1, shows the conclusion (ii) in Theorem 3.1. (b) The construction of the control in (iii) of Theorem 3.4 is inspired by a standard duality strategy used in [18] for the distributed controlled heat equations with a control in $L^2(\omega \times (T_1, T_3))$.

Proof of Theorem 3.4.
Proof of (i)*:* Write

$$\mathcal{F}_{ad} := \left\{ f \in L^2(\omega) ; \|y(T_3)\| \leq \varepsilon \|z\| \right\} \ .$$

By the previous subsection, we see that $\mathcal{F}_{ad} \neq \emptyset$. Meanwhile, one can easily check that $\mathcal{F}_{ad}$ is weakly closed in $L^2(\omega)$. From these, it follows that ($\mathcal{P}$) has a minimal norm control.

Suppose that f_1 and f_2 are two minimal norm controls to ($\mathcal{P}$). Then we have that

$$0 \leq \|f_1\|_\omega = \|f_2\|_\omega = \mathcal{N}_z < \infty \ ,$$

where $\mathcal{N}_z$ is given by (3.2.2). Meanwhile, one can easily check that $(f_1 + f_2)/2$ is also a minimal norm control to ($\mathcal{P}$). This, along with the Parallelogram Law, yields that

$$(\mathcal{N}_z)^2 = \|(f_1 + f_2)/2\|^2_\omega = \frac{1}{2} \left(\|f_1\|^2_\omega + \|f_2\|^2_\omega \right) - \|(f_1 - f_2)/2\|^2_\omega \ .$$

From the two above identities on $\mathcal{N}_z$, we find that $f_1 = f_2$. Thus, the minimal norm control to ($\mathcal{P}$) is unique. This ends the proof.
Proof of (ii)*:* The second conclusion in Theorem 3.4 follows from the definition of Problem ($\mathcal{P}$) (see (3.2.2)) at once.
Proof of (iii)*:* To prove the last conclusion in Theorem 3.4, we need the next Lemma 3.6 whose proof will be given at the end of the proof of Theorem 3.4.

Lemma 3.6. *The functional J in Theorem 3.4 has the following properties:*

(*a*) *It satisfies that*

$$\lim_{q\to\infty}\inf_{\|\Phi\|=q}\frac{J(\Phi)}{\|\Phi\|}\geq\varepsilon\|z\| \ . \tag{3.2.3}$$

(*b*) *It has a unique minimizer over $L^2(\Omega)$.*
(*c*) *Write w for its minimizer. Then*

$$w=0 \text{ if and only if the solution } y^0 \text{ of (3.2.1) with } f=0 \text{ satisfies } \left\|y^0(T_3)\right\|\leq\varepsilon\|z\| \ .$$

We now show the third conclusion of Theorem 3.4. Notice that when $\|y^0(T_3)\|\leq\varepsilon\|z\|$ where y^0 is the solution of (3.2.1) with $f=0$, it follows respectively from the second conclusion of Theorem 3.4 and the conclusion (c) of Lemma 3.6 that $f^*=0$ and $w=0$. Hence, the third conclusion of Theorem 3.4 is true in this particular case.

We now consider the case where

$$\left\|y^0(T_3)\right\|>\varepsilon\|z\| \tag{3.2.4}$$

where y^0 is the solution of (3.2.1) with $f=0$. Let $w\in L^2(\Omega)$ be the minimizer of the functional J (see (b) of Lemma 3.6). Write

$$\widehat{f}=1_\omega^* e^{(T_3-T_2)A}w \ . \tag{3.2.5}$$

We first claim that

$$\widehat{f}\in\mathcal{F}_{ad}:=\left\{f\in L^2(\omega)\,;\,\|y(T_3)\|\leq\varepsilon\|z\|\right\} \ . \tag{3.2.6}$$

In fact, by (3.2.4) and (c) of Lemma 3.6, we find that $w\neq 0$. Then the Euler–Lagrange equation associated to w reads:

$$e^{(T_3-T_2)A}\chi_\omega e^{(T_3-T_2)A}w+e^{(T_3-T_1)A}z+\varepsilon\|z\|\frac{w}{\|w\|}=0\,. \tag{3.2.7}$$

Meanwhile, since $1_\omega 1_\omega^*=\chi_\omega$, it follows from (3.2.5) that the solution y of (3.2.1) with $f=\widehat{f}$ satisfies

$$y(T_3)=e^{(T_3-T_1)A}z+e^{(T_3-T_2)A}1_\omega\widehat{f}=e^{(T_3-T_2)A}\chi_\omega e^{(T_3-T_2)A}w+e^{(T_3-T_1)A}z \ .$$

This, together with (3.2.7), indicates that

$$y(T_3)=-\varepsilon\|z\|\frac{w}{\|w\|} \ ,$$

from which, (3.2.6) follows at once.

We next claim that

$$\left\|\widehat{f}\right\|_{\omega} \leq \|f\|_{\omega} \quad \text{for all } f \in \mathcal{F}_{ad}\,. \tag{3.2.8}$$

To this end, we arbitrarily fix an $f \in \mathcal{F}_{ad}$. Then we have that the solution y of (3.2.1) satisfies

$$\|y\,(T_3)\| \leq \varepsilon\,\|z\|\;. \tag{3.2.9}$$

Since $\widehat{f} := 1_{\omega}^{*} e^{(T_3-T_2)A} w$,

$$\begin{aligned}
\left\|\widehat{f}\right\|_{\omega}^{2} &= \left\|1_{\omega}^{*} e^{(T_3-T_2)A} w\right\|_{\omega}^{2} \\
&= \left\|1_{\omega}^{*} e^{(T_3-T_2)A} w\right\|_{\omega}^{2} + \varepsilon\,\|z\|\,\|w\| - \varepsilon\,\|z\|\,\|w\| \\
&= -\varepsilon\,\|z\|\,\|w\| + \left\langle e^{(T_3-T_2)A}\chi_{\omega} e^{(T_3-T_2)A} w + \varepsilon\,\|z\|\,\tfrac{w}{\|w\|},\, w\right\rangle \\
&= -\varepsilon\,\|z\|\,\|w\| - \left\langle e^{(T_3-T_1)A} z,\, w\right\rangle \\
&\leq \langle y\,(T_3)\,,\, w\rangle - \left\langle e^{(T_3-T_1)A} z,\, w\right\rangle = \left\langle e^{(T_3-T_2)A} 1_{\omega} f,\, w\right\rangle \\
&= \left\langle 1_{\omega} f,\, \chi_{\omega} e^{(T_3-T_2)A} w\right\rangle = \left\langle 1_{\omega} f,\, 1_{\omega}\widehat{f}\right\rangle \leq \tfrac{1}{2}\,\|f\|_{\omega}^{2} + \tfrac{1}{2}\left\|\widehat{f}\right\|_{\omega}^{2}\;.
\end{aligned} \tag{3.2.10}$$

Notice that we used $1_{\omega}1_{\omega}^{*} = \chi_{\omega}$, $\chi_{\omega}1_{\omega} = 1_{\omega}$ and Cauchy–Schwarz inequality in the last line in (3.2.10); the equality (3.2.7) is applied in the fourth equality of (3.2.10); the inequality (3.2.9) and Cauchy–Schwarz, as well as the formula $y\,(T_3) = e^{(T_3-T_1)A} z + e^{(T_3-T_2)A} 1_{\omega} f$, are used in the fifth line of (3.2.10). Now, (3.2.10) clearly leads to (3.2.8).

From (3.2.6) and (3.2.8), we find that $\widehat{f}$ is a minimal norm control to $(\mathcal{P})$. Since the minimal norm control of $(\mathcal{P})$ is unique, we have that $\widehat{f} = f^{*}$. So the third conclusion of Theorem 3.4 is true.

Finally, we are ready to prove Lemma 3.6.

Proof of Lemma 3.6. We will prove conclusions (a), (b), (c) one by one.
Proof of (a): By contradiction, suppose that (3.2.3) was not true. Then there would be an $\sigma \in (0, \varepsilon)$ and a sequence $\{\Phi_n\}_{n=1}^{\infty}$ in $L^2(\Omega)$ so that

$$\lim_{n\to\infty} \|\Phi_n\| = \infty \tag{3.2.11}$$

and

$$\frac{J\,(\Phi_n)}{\|\Phi_n\|} \leq (\varepsilon - \sigma)\,\|z\| \quad \text{for all } n \in \mathbb{N}\,. \tag{3.2.12}$$

From (3.2.11), we can assume, without loss of generality, that $\Phi_n \neq 0$ for all n. Thus we can set

$$\varphi_n = \frac{\Phi_n}{\|\Phi_n\|} \quad \text{for all } n \in \mathbb{N}\,. \tag{3.2.13}$$

From (3.2.13), we see that $\left\{e^{(T_3-T_1)A}\varphi_n\right\}_{n=1}^{\infty}$ is bounded in $L^2(\Omega)$. Then, from the definition of J in Theorem 3.4, (3.2.13), (3.2.11) and (3.2.12), we find that

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\frac{1}{2}\left\|1_\omega^* e^{(T_3-T_2)A}\varphi_n\right\|_\omega^2\\&=\overline{\lim_{n\to\infty}}\frac{1}{\|\Phi_n\|}\left[\frac{J(\Phi_n)}{\|\Phi_n\|}-\left\langle z, e^{(T_3-T_1)A}\varphi_n\right\rangle-\varepsilon\|z\|\right]\\&\leq\overline{\lim_{n\to\infty}}\frac{-\sigma\|z\|}{\|\Phi_n\|}+\overline{\lim_{n\to\infty}}\frac{-\left\langle z, e^{(T_3-T_1)A}\varphi_n\right\rangle}{\|\Phi_n\|}=0\,.\end{aligned}\tag{3.2.14}$$

Meanwhile, by (3.2.13), there is a subsequence of $\{\varphi_n\}$, denoted in the same manner, so that

$$\varphi_n\to\varphi\quad\text{weakly in } L^2(\Omega)\ ,$$

for some $\varphi\in L^2(\Omega)$. Since the semigroup $\left\{e^{tA}\right\}_{t\geq 0}$ is compact, the above convergence leads to

$$e^{(T_3-T_2)A}\varphi_n\to e^{(T_3-T_2)A}\varphi\quad\text{strongly in } L^2(\Omega)\tag{3.2.15}$$

and

$$1_\omega^* e^{(T_3-T_2)A}\varphi_n\to 1_\omega^* e^{(T_3-T_2)A}\varphi\quad\text{strongly in } L^2(\omega)\ .\tag{3.2.16}$$

From (3.2.14) and the convergence in (3.2.16), we find that

$$1_\omega^* e^{(T_3-T_2)A}\varphi=0\,,$$

which, along with the unique continuation property of heat equations (see (1) of Remark 2.2) yields that $\varphi=0$. Then from the definition of J in Theorem 3.4 and the convergence in (3.2.15), we see that

$$\begin{aligned}\underline{\lim_{n\to\infty}}\frac{J(\Phi_n)}{\|\Phi_n\|}&\geq\underline{\lim_{n\to\infty}}\left[\left\langle z, e^{(T_3-T_1)A}\varphi_n\right\rangle+\varepsilon\|z\|\right]\\&=\left\langle z, e^{(T_3-T_1)A}\varphi\right\rangle+\varepsilon\|z\|=\varepsilon\|z\|\ .\end{aligned}$$

This, along with (3.2.12), leads to a contradiction. Therefore, (3.2.3) is true.

Proof of (b)*:* From (3.2.3), we see that the functional J is coercive on $L^2(\Omega)$. Further J is continuous and convex on $L^2(\Omega)$. Thus, it has a minimizer on $L^2(\Omega)$.

Next, we show the uniqueness of the minimizer. It suffices to prove that the functional J is strictly convex. For this purpose, we arbitrarily fix $\Phi_1,\Phi_2\in L^2(\Omega)\setminus\{0\}$, with $\Phi_1\neq\Phi_2$. There are only three possibilities: (a) $\Phi_1\neq\mu\Phi_2$ for any $\mu\in\mathbb{R}$; (b) $\Phi_1=-\mu_0\Phi_2$ for some $\mu_0>0$; (c) $\Phi_1=\mu_0\Phi_2$ for some $\mu_0>0$. In the cases (a) and (b), one can easily check that

$$\|\lambda\Phi_1+(1-\lambda)\Phi_2\|<\lambda\|\Phi_1\|+(1-\lambda)\|\Phi_2\|\quad\text{for all }\lambda\in(0,1)\ .\tag{3.2.17}$$

In the case (c), we let

$$H(\lambda)=J(\lambda\Phi_2)\ ,\quad\lambda>0\,.$$

Since $\Phi_2 \neq 0$ in $L^2(\Omega)$, it follows by the unique continuation property of heat equations (see (1) of Remark 2.2) that $\left\| 1_\omega^* e^{(T_3-T_2)A}\Phi_2 \right\|_\omega \neq 0$. Thus, H is a quadratic function with a positive leading coefficient. Hence, H is strictly convex. This, along with (3.2.17), yields the strict convexity of J.

Proof of (c)*:* Let y^0 be the solution of (3.2.1) with $f = 0$. We first show that

$$\left\| y^0(T_3) \right\| \leq \varepsilon \|z\| \Rightarrow w = 0 . \tag{3.2.18}$$

In fact, by multiplying by $e^{(T_3-t)A}\Phi$ the system solved by y^0, we see that

$$\left\langle z, e^{(T_3-T_1)A}\Phi \right\rangle = \left\langle y^0(T_3), \Phi \right\rangle \quad \text{for all } z, \Phi \in L^2(\Omega) .$$

This, along with the definition of J in Theorem 3.4 and the inequality on the left hand side of (3.2.18), yields that for all $\Phi \in L^2(\Omega)$

$$J(\Phi) \geq \left\langle y^0(T_3), \Phi \right\rangle + \varepsilon \|z\| \|\Phi\| \geq 0 = J(0) .$$

This implies the equality on the right hand side of (3.2.18). Hence, (3.2.18) is true.

We next show that

$$w = 0 \Rightarrow \left\| y^0(T_3) \right\| \leq \varepsilon \|z\| . \tag{3.2.19}$$

By contradiction, suppose that (3.2.19) were not true. Then we would have that

$$\left\| y^0(T_3) \right\| > \varepsilon \|z\| \text{ and } w = 0 . \tag{3.2.20}$$

Set $\psi := -y^0(T_3)$, which clearly belongs to $L^2(\Omega) \backslash \{0\}$. Then we have that

$$\left\langle z, e^{(T_3-T_1)A}\psi \right\rangle = \left\langle y^0(T_3), \psi \right\rangle = -\left\| y^0(T_3) \right\| \|\psi\| .$$

This, along with the first inequality in (3.2.20), yields that

$$\left\langle z, e^{(T_3-T_1)A}\psi \right\rangle + \varepsilon \|z\| \|\psi\| < 0 .$$

Thus, there is an $\sigma > 0$ so that

$$J(\sigma\psi) = \sigma^2 \frac{1}{2} \left\| 1_\omega^* e^{(T_3-T_2)A}\psi \right\|_\omega^2 + \sigma \left(\left\langle z, e^{(T_3-T_1)A}\psi \right\rangle + \varepsilon \|z\| \|\psi\| \right) < 0 .$$

This, along with the second equation in (3.2.20), indicates that

$$0 = J(0) = \min_{\Phi \in L^2(\Omega)} J(\Phi) < 0 ,$$

which leads to a contradiction. So we have proved (3.2.19). Finally, the conclusion (c) of Lemma 3.6 follows from (3.2.18) and (3.2.19) at once.

This ends the proof of Lemma 3.6 and completes the proof of Theorem 3.4.

3.3. Best connection between Theorem 3.4 and Theorem 2.1

What we study in this subsection will not have influence on the study of our stabilization. However, it is independently interesting. Consider the following problem $(\mathcal{NP})$ (with arbitrarily fixed $\varepsilon > 0$. Recall that y is the solution of (3.2.1) associated with the initial datum z and control f):

$$\mathcal{N} := \sup_{\|z\|\leq 1} \mathcal{N}_z = \sup_{\|z\|\leq 1} \inf \left\{ \|f\|_\omega ; f \in L^2(\omega) \text{ and } \|y(T_3)\| \leq \varepsilon \|z\| \right\} . \tag{3.3.1}$$

The quantity $\mathcal{N}$ is called the value of the problem $(\mathcal{NP})$. Next, let $C > 0$ and introduce the following property $(\mathcal{Q}_C)$: For any $z \in L^2(\Omega)$, there is a control $f \in L^2(\omega)$ so that

$$\max \left\{ \frac{1}{C} \|f\|_\omega , \frac{1}{\varepsilon} \|y(T_3)\| \right\} \leq \|z\| . \tag{3.3.2}$$

We would like to mention that the property $(\mathcal{Q}_C)$ may not hold for some $C > 0$ and $\varepsilon > 0$. However, we have seen in Theorem 3.1 that given $\varepsilon > 0$, there is $C = \mathcal{C}_\varepsilon (T_3 - T_2, T_2 - T_1) > 0$ so that the property $(\mathcal{Q}_C)$ is true.

The main result of this subsection is as follows: The value $\mathcal{N}$ is the optimal coefficient C so that

$$\left\| e^{(T_3-T_1)A} \Phi \right\| \leq C \left\| 1_\omega^* e^{(T_3-T_2)A} \Phi \right\|_\omega + \varepsilon \|\Phi\| \quad \text{for any } \Phi \in L^2(\Omega) . \tag{3.3.3}$$

Precisely, we have the following result:

Theorem 3.7. *Let $\varepsilon > 0$. It holds that*

$$\inf \{C > 0; C \text{ satisfies (3.3.3)}\} = \mathcal{N} .$$

Further the connections among the problem $(\mathcal{NP})$, the property $(\mathcal{Q}_C)$ and the observation inequalities in Theorem 2.1 are presented in the next Theorem 3.8, which will be used in the proof of the above Theorem 3.7.

Theorem 3.8. *Let $\varepsilon > 0$ and $C > 0$. The following statements are equivalent:*

(i) Let $\mathcal{N}$ be given by (3.3.1). Then $\mathcal{N} \leq C$.
(ii) The property $(\mathcal{Q}_C)$ defined by (3.3.2) is true.
(iii) For any $\Phi \in L^2(\Omega)$, the following estimate holds:

$$\left\| e^{(T_3-T_1)A} \Phi \right\| \leq C \left\| 1_\omega^* e^{(T_3-T_2)A} \Phi \right\|_\omega + \varepsilon \|\Phi\| .$$

Proof of Theorem 3.8. We organize the proof by three steps as follows:
Step 1. To show that $(i) \Leftrightarrow (ii)$.

We first prove that $(i) \Rightarrow (ii)$. Assume that (i) is true. When $z = 0$ in $L^2(\Omega)$, we find that (3.3.2) holds for $f = 0$. Thus, it suffices to show (ii) with an arbitrarily fixed $z \in L^2(\Omega) \setminus \{0\}$.

For this purpose, we write $\widehat{z} = z/\|z\|$. Let $\widehat{f}$ be the solution to $(\mathcal{P})$ associated to $\widehat{z}$. Then the solution $\widehat{y}$ of (3.2.1) associated with initial data $\widehat{z}$ and control $\widehat{f}$ satisfies that

$$\|\widehat{y}(T_3)\| \le \varepsilon \|\widehat{z}\| = \varepsilon .$$

Setting $f = \|z\| \widehat{f}$, the solution y of (3.2.1) have the following property:

$$\|y(T_3)\| = \|z\| \|\widehat{y}(T_3)\| \le \varepsilon \|z\| .$$

Thus, to show that the above f satisfies (3.3.2), we only need to prove that $\|f\|_\omega \le C\|z\|$. This will be done in what follows: Since $\widehat{f}$ is the solution to $(\mathcal{P})$ associated to $\widehat{z}$, we have that $\|\widehat{f}\|_\omega = \mathcal{N}_{\widehat{z}}$. This, along with (3.3.1) and (i) of Theorem 3.8, yields that

$$\|f\|_\omega = \|z\| \left\|\widehat{f}\right\|_\omega = \|z\| \mathcal{N}_{\widehat{z}} \le \|z\| \mathcal{N} \le C\|z\| .$$

Hence, (ii) is true.

We next show that $(ii) \Rightarrow (i)$. Assume that (ii) is true. By contradiction, suppose that (i) were false. Then there would be $z \ne 0$ with $\|z\| \le 1$ so that $\mathcal{N}_z > C$. Let $\widehat{z} = z/\|z\|$. Then we have that

$$\mathcal{N}_{\widehat{z}} = \frac{1}{\|z\|}\mathcal{N}_z \ge \mathcal{N}_z > C .$$

Therefore, we see that there is no $f \in L^2(\Omega)$ so that the solution y of (3.2.1), associated with the initial datum $\widehat{z}$ and the control f, has the property:

$$\|y(T_3)\| \le \varepsilon \|\widehat{z}\| \text{ and } \|f\|_\omega \le C = C\|\widehat{z}\| .$$

This contradicts (ii). Hence, (i) stands.

Step 2. To show that $(ii) \Rightarrow (iii)$.

Suppose that (ii) holds. Then, given $z \in L^2(\Omega)$, there is $f \in L^2(\Omega)$ so that (3.3.2) holds. Meanwhile, by multiplying by $e^{(T_3-t)A}\Phi$ the system solved by y, one gets

$$\langle y(T_3), \Phi\rangle - \left\langle z, e^{(T_3-T_1)A}\Phi\right\rangle = \left\langle 1_\omega f, e^{(T_3-T_2)A}\Phi\right\rangle \quad \forall \Phi \in L^2(\Omega) .$$

This, along with the inequality (3.3.2), yields that for each $\Phi \in L^2(\Omega)$,

$$\begin{aligned}
\left\|e^{(T_3-T_1)A}\Phi\right\| &= \sup_{\|z\|\le 1} \left\langle e^{(T_3-T_1)A}\Phi, z\right\rangle \\
&= \sup_{\|z\|\le 1} \left[\langle y(T_3), \Phi\rangle - \left\langle 1_\omega f, e^{(T_3-T_2)A}\Phi\right\rangle\right] \\
&\le \sup_{\|z\|\le 1} \left[\|y(T_3)\| \|\Phi\| + \|f\|_\omega \left\|1_\omega^* e^{(T_3-T_2)A}\Phi\right\|_\omega\right] \\
&\le \sup_{\|z\|\le 1} \left[\left(\varepsilon\|\Phi\| + C\left\|1_\omega^* e^{(T_3-T_2)A}\Phi\right\|_\omega\right)\|z\|\right] \\
&= C\left\|1_\omega^* e^{(T_3-T_2)A}\Phi\right\|_\omega + \varepsilon\|\Phi\| ,
\end{aligned}$$

which leads to the desired observation estimate. Hence, (iii) is true.

Step 3. To show that (*iii*) ⇒ (*ii*).

Suppose that (*iii*) is true. Arbitrarily fix $z \in L^2(\Omega)$. Denote by y^0 the solution of (3.2.1) with $f = 0$. In the case that $\left\| y^0(T_3) \right\| \le \varepsilon \|z\|$, (3.3.2) holds for $f = 0$. Thus, we only need to consider the case that

$$\left\| y^0(T_3) \right\| > \varepsilon \|z\| \ . \tag{3.3.4}$$

In this case, we let $f := 1_\omega^* e^{(T_3-T_2)A} w$, where w is the unique minimizer of the functional J, which is given in Theorem 3.4. Then according to (*iii*) of Theorem 3.4, f is the minimal norm control to $(\mathcal{P})$. By Lemma 3.6 and (3.3.4), we see that $w \neq 0$. Then using the Euler–Lagrange equation (3.2.7) and noticing that $\chi_\omega = 1_\omega 1_\omega^*$, we find that

$$\begin{aligned}
&\left\langle z, e^{(T_3-T_1)A} w \right\rangle = \left\langle e^{(T_3-T_1)A} z, w \right\rangle \\
&= -\left\langle e^{(T_3-T_2)A} \chi_\omega e^{(T_3-T_2)A} w + \varepsilon \|z\| \tfrac{w}{\|w\|}, w \right\rangle \\
&= -\left\| 1_\omega^* e^{(T_3-T_2)A} w \right\|_\omega^2 - \varepsilon \|z\| \|w\| \ .
\end{aligned}$$

Since $f = 1_\omega^* e^{(T_3-T_2)A} w$, the above equality, along with the definition of J in Theorem 3.4, shows that

$$J(w) = \frac{1}{2} \left\| 1_\omega^* e^{(T_3-T_2)A} w \right\|_\omega^2 + \left\langle z, e^{(T_3-T_1)A} w \right\rangle + \varepsilon \|z\| \|w\| = -\frac{1}{2} \|f\|_\omega^2 \ . \tag{3.3.5}$$

Meanwhile, it follows from the above and the observation estimate in (*iii*) in Theorem 3.8 that

$$\begin{aligned}
J(w) &\ge \tfrac{1}{2} \left\| 1_\omega^* e^{(T_3-T_2)A} w \right\|_\omega^2 + \varepsilon \|z\| \|w\| - \left\| e^{(T_3-T_1)A} w \right\| \|z\| \\
&\ge \tfrac{1}{2} \left\| 1_\omega^* e^{(T_3-T_2)A} w \right\|_\omega^2 + \varepsilon \|z\| \|w\| - \left(C \left\| 1_\omega^* e^{(T_3-T_2)A} w \right\|_\omega + \varepsilon \|w\| \right) \|z\| \\
&\ge \tfrac{1}{2} \|f\|_\omega^2 - C \|f\|_\omega \|z\| \ .
\end{aligned} \tag{3.3.6}$$

From (3.3.5) and (3.3.6), it follows that

$$\|f\|_\omega \le C \|z\| \ . \tag{3.3.7}$$

On the other hand, since f is the minimal norm control to $(\mathcal{P})$, it holds that $\|y(T_3)\| \le \varepsilon \|z\|$. From this and (3.3.7), we find that (3.3.2) is true. Hence, (*ii*) stands.

In summary, we complete the proof of Theorem 3.8.

The rest of this subsection is devoted to the proof of Theorem 3.7.

Proof of Theorem 3.7. Define

$$C^* := \inf\{C > 0; C \text{ satisfies } (3.3.2)\} \ .$$

Proof of $\mathcal{N} \le C^*$: It directly follows from Theorem 3.8.
Proof of $C^* \le \mathcal{N}$: It suffices to prove that the property $(\mathcal{Q}_\mathcal{N})$ holds. Indeed, by making use of the proof of "(*ii*) ⇒ (*iii*)" of Theorem 3.8, we find that

$$C^* \leq \mathcal{N} .$$

Therefore, the remainder is that for each $z \in L^2(\Omega)$, there is a control $f \in L^2(\omega)$ satisfying that

$$\max\left\{\frac{1}{\mathcal{N}}\|f\|_\omega , \frac{1}{\varepsilon}\|y(T_3)\|\right\} \leq \|z\| . \tag{3.3.8}$$

When $z = 0$, we can easily get (3.3.8) by taking $f = 0$. So it suffices to prove (3.3.8) for an arbitrarily fixed $z \in L^2(\Omega)\setminus\{0\}$. To this end, we let $\widehat{z} = z/\|z\|$. Denote y the solution of (3.2.1) associated with the initial datum $\widehat{z}$. It follows from (3.3.1) that

$$\inf\left\{\|f\|_\omega ; f \in L^2(\omega) \text{ and } \|y(T_3)\| \leq \varepsilon\|\widehat{z}\|\right\} \leq \mathcal{N} .$$

Because the infimum on the left hand side of the above inequality can be reached, there is $\widehat{f} \in L^2(\omega)$ so that

$$\|\widehat{y}(T_3)\| \leq \delta\|\widehat{z}\| \quad \text{with } \left\|\widehat{f}\right\|_\omega \leq \mathcal{N} ,$$

where $\widehat{y}$ the solution of (3.2.1) associated with the initial datum $\widehat{z}$ and control $\widehat{f}$. From these, we see that (3.3.8) holds for $f = \|z\|\,\widehat{f}$. This ends the proof.

4. Inverse source problem

This section concerns an inverse source problem: Suppose that we have a solution φ of $\varphi' - A\varphi = 0$ with a priori bound on the initial data in $L^2(\Omega)$. The question is how to recover approximatively the initial data from the knowledge of the solution φ in the future. This can be done as follows thanks to the impulse control.

Theorem 4.1. *Let ω_1 be a non-empty open subset of Ω and $K \in \mathbb{N}$. Let $0 \leq T_1 < T_2 < T_3$ and let φ be a solution of*

$$\begin{cases} \varphi'(t) - A\varphi(t) = 0 , & t \in (T_1, T_3) , \\ \varphi(T_1) \in L^2(\Omega) . \end{cases}$$

Then for any $\varepsilon > 0$, there exists $\left\{g_j\right\}_{j=1,\cdot\cdot,K} \in L^2(\omega_1)$ such that for any $j = 1, \cdot\cdot, K$,

$$\left|\langle\varphi(T_1), \xi_j\rangle + e^{(T_3-T_1)\lambda_j}\langle g_j, 1^*_{\omega_1}\varphi(T_1+T_3-T_2)\rangle_{\omega_1}\right| \leq e^{(T_3-T_1)\lambda_j}\varepsilon\|\varphi(T_1)\|$$

and

$$\left\|g_j\right\|_{\omega_1} \leq \mathcal{C}_\varepsilon(T_3-T_2, T_2-T_1)$$

where $\mathcal{C}_\varepsilon$ is given in Theorem 3.1. Further g_j is the control function given in Corollary 3.2 with $\omega = \omega_1$.

Proof.

Step 1: We apply Corollary 3.2 with $\omega = \omega_1$ and get the existence of $\left(y_j, g_j\right)$ such that

$$\begin{cases} y_j'(t) - Ay_j(t) = 0\,, & t \in (T_1, T_3) \setminus \{T_2\}\,, \\ y_j(T_1) = \xi_j\,, & \\ y_j(T_2) = y_j(T_{2-}) + 1_{\omega_1} g_j\,, & \end{cases}$$

and $\left\| y_j(T_3) \right\| \leq \varepsilon$ where g_j has the desired bound.

Step 2: Write $\varphi(T_1) = \sum_{i=1,\cdot\cdot,+\infty} a_i \xi_i$ with $a_i = \langle \varphi(T_1), \xi_i \rangle$. Then we have that

$$\varphi(T_3) = \sum_{i=1,\cdot\cdot,+\infty} a_i e^{-(T_3-T_1)\lambda_j} \xi_i\,.$$

Hence, $\langle y_j(T_1), \varphi(T_3)\rangle = \langle \xi_j, \varphi(T_3)\rangle = a_j e^{-(T_3-T_1)\lambda_j}$.

Step 3: Multiply the equation solved by y_j by the solution $\varphi(T_1 + T_3 - t)$, with $t \in [T_1, T_3]$, to get

$$\langle y_j(T_3), \varphi(T_1)\rangle = \langle y_j(T_1), \varphi(T_3)\rangle + \langle g_j, 1^*_{\omega_1} \varphi(T_1 + T_3 - T_2)\rangle_{\omega_1}\,.$$

Therefore by step 2, it holds that

$$\left| a_j + e^{(T_3-T_1)\lambda_j} \langle g_j, 1^*_{\omega_1} \varphi(T_1 + T_3 - T_2)\rangle_{\omega_1} \right| = e^{(T_3-T_1)\lambda_j} \left| \langle y_j(T_3), \varphi(T_1)\rangle \right|\,.$$

This, along with the Cauchy–Schwarz inequality and step 1, leads to the desired result.

5. Main result

This section presents the main result of this paper, as well as its proof. We first recall that ω_1 and ω_2 are two arbitrarily fixed open and non-empty subsets of Ω. We next recall that $\{\lambda_j\}_{j=1}^{\infty}$ is the family of all eigenvalues of $-A$ so that (1.2) holds and that $\{\xi_j\}_{j=1}^{\infty}$ is the family of the corresponding normalized eigenfunctions. For each $\gamma > 0$, we define a natural number K in the following manner:

$$K := \text{card}\left\{ j \in \mathbb{N},\ \lambda_j < \gamma + \frac{\ln 2}{T} \right\}\,. \tag{5.1}$$

Next, we define

$$\varepsilon := \frac{1}{6(1+K)} e^{-\gamma T} e^{-\|V\|_\infty T} e^{-\lambda_K T/2}\,. \tag{5.2}$$

Denote by $\left\{f_j\right\}_{j=1,\cdot\cdot,K} \in L^2(\omega_2)$ the minimal norm control functions obtained by applying Corollary 3.2 with $T_1 = \frac{T}{4}$, $T_2 = T$, $T_3 = \frac{5T}{4}$ and $\omega = \omega_2$. Denote by $\left\{g_j\right\}_{j=1,\cdot\cdot,K} \in L^2(\omega_1)$ the minimal norm control functions obtained by applying Corollary 3.2 with $T_1 = \frac{T}{4}$, $T_2 = \frac{T}{2}$,

$T_3 = \frac{3T}{4}$ and $\omega = \omega_1$. Now for each $\gamma > 0$, we define a linear bounded operator $\mathcal{F}$ from $L^2(\omega_1)$ into $L^2(\omega_2)$ in the following manner:

$$\mathcal{F}(p) := -\sum_{j=1,\cdot\cdot,K} e^{\lambda_j T/2} \langle g_j, p\rangle_{\omega_1} f_j(x) \quad \text{for each } p \in L^2(\omega_1)\,. \tag{5.3}$$

The closed-loop equation under consideration reads:

$$\begin{cases} y'(t) - Ay(t) = 0\,, & \text{in } (0, +\infty)\setminus \mathbb{N}T\,, \\ y(0) \in L^2(\Omega)\,, & \\ y((n+1)T) = y((n+1)T_-) + 1_{\omega_2}\mathcal{F}\left(1^*_{\omega_1} y\left(\left(n+\frac{1}{2}\right)T\right)\right)\,, & \text{for } n \in \overline{\mathbb{N}}\,. \end{cases} \tag{5.4}$$

The main result of this paper is the following theorem:

Theorem 5.1. *For each $\gamma > 0$, let $\mathcal{F}$ be given by (5.3). Then the following conclusions are true:*

(i) *Each solution y to the equation (5.4) satisfies that*

$$\|y(t)\| \le e^{T(\gamma+\|V\|_\infty)}\left(1 + \|\mathcal{F}\|_{\mathcal{L}(L^2(\omega_1),L^2(\omega_2))}\right)e^{-\gamma t}\|y(0)\| \quad \textit{for all } t \ge 0\,.$$

(ii) *The operator $\mathcal{F}$ satisfies the estimate:*

$$\|\mathcal{F}\|_{\mathcal{L}(L^2(\omega_1),L^2(\omega_2))} \le Ce^{C\gamma}\,,$$

where C is a positive constant independent of γ, depending on Ω, ω_1, ω_2, d, T and $\|V\|_\infty$. Moreover, the manner how it depends on T, d, $\|V\|_\infty$ is explicitly given.

Proof.
Step 1:
Set $L_n = nT + \frac{T}{4}$. In order to have the conclusion (i) in the theorem, it suffices to prove that the solution y of (5.4) satisfies

$$\|y(L_{n+1})\| \le e^{-\gamma T}\|y(L_n)\|$$

for any $n \ge 0$. Indeed, thanks to the above inequality, we find that when $t \in [L_n, (n+1)T]$,

$$\begin{aligned} \|y(t)\| &\le e^{(t-L_n)\|V\|_\infty}\|y(L_n)\| \le e^{(t-L_n)\|V\|_\infty}e^{-n\gamma T}\|y(0)\| \\ &\le e^{T\|V\|_\infty}e^{-n\gamma T}\|y(0)\| \le e^{T(\gamma+\|V\|_\infty)}e^{-\gamma t}\|y(0)\|\,, \end{aligned}$$

and when $t \in \left[(n+1)T, L_{n+1}\right]$,

$$\begin{aligned} \|y(t)\| &\le e^{(t-(n+1)T)\|V\|_\infty}\|y((n+1)T)\| \\ &\le e^{(t-(n+1)T)\|V\|_\infty}\|y((n+1)T)_-\| + e^{(t-(n+1)T)\|V\|_\infty}\left\|\mathcal{F}\left(1^*_{\omega_1} y\left(\left(n+\frac{1}{2}\right)T\right)\right)\right\|_{\omega_2} \\ &\le e^{(t-(n+1)T+3T/4)\|V\|_\infty}\|y(L_n)\| + \|\mathcal{F}\|_{\mathcal{L}(L^2(\omega_1),L^2(\omega_2))}e^{(t-(n+1)T+T/4)\|V\|_\infty}\|y(L_n)\| \\ &\le e^{T(\gamma+\|V\|_\infty)}\left(1 + \|\mathcal{F}\|_{\mathcal{L}(L^2(\omega_1),L^2(\omega_2))}\right)e^{-\gamma t}\|y(0)\|\,. \end{aligned}$$

From these and the time translation invariance of the equation (5.4), we see that the conclusion (i) in Theorem 5.1 is true for any $t \geq \frac{T}{4}$. But the case $t \leq \frac{T}{4}$ is trivial.

Step 2:

Denote $y(L_n) = \sum_{j=1,\cdot\cdot,+\infty} a_j \xi_j$ and $a = \left(a_j\right)_{j=1,\cdot\cdot,+\infty}$. Then one deduces that $\langle y(L_n), \xi_j \rangle = a_j$ and $\|a\|_{\ell^2} = \|y(L_n)\|$.

For the rest of the proof, recall that K and ε are given by (5.1) and (5.2) respectively.

Step 3:

Notice that the solution y of (5.4) evolves freely without a control function between in $\left[L_n, nT + \frac{3T}{4}\right]$. Thus, we can apply Theorem 4.1 with the choice $T_1 = L_n$, $T_2 = nT + \frac{T}{2}$, $T_3 = nT + \frac{3T}{4}$ and $\varphi = y$ to get $\left\{g_j\right\}_{j=1,\cdot\cdot,K} \in L^2(\omega_1)$ such that for any $j = 1, \cdot\cdot, K$,

$$\begin{aligned} \left| a_j + e^{\lambda_j T/2} \langle g_j, 1^*_{\omega_1} y\left(\left(n + \tfrac{1}{2}\right) T\right) \rangle_{\omega_1} \right| & \leq e^{\lambda_j T/2} \varepsilon \left\| y(L_n) \right\| \\ & \leq e^{\lambda_K T/2} \varepsilon \left\| y(L_n) \right\| \end{aligned}$$

and

$$\left\| g_j \right\|_{\omega_1} \leq \mathcal{C}_\varepsilon (T/4, T/4)$$

where $\mathcal{C}_\varepsilon$ is given in Theorem 3.1. Further such g_j is given in Corollary 3.2 with $\omega = \omega_1$. By the time translation invariance of the equation (5.4), $\left\{g_j\right\}_{j=1,\cdot\cdot,K} \in L^2(\omega_1)$ is the control function obtained by applying Corollary 3.2 with $T_1 = \frac{T}{4}$, $T_2 = \frac{T}{2}$, $T_3 = \frac{3T}{4}$ and $\omega = \omega_1$.

Step 4:

Denote $b_j := -e^{\lambda_j T/2} \langle g_j, 1^*_{\omega_1} y\left(\left(n + \frac{1}{2}\right) T\right) \rangle_{\omega_1}$ for $j = 1, \cdot\cdot, K$, and $b = \left(b_j\right)_{j=1,\cdot\cdot,K}$. Then by step 3 and step 2, we have

$$\begin{aligned} \|b\|_{\ell^2} & \leq \left\| \left(a_j + e^{\lambda_j T/2} \langle g_j, 1^*_{\omega_1} y\left(\left(n + \tfrac{1}{2}\right) T\right) \rangle_{\omega_1} \right)_{j=1,\cdot\cdot,K} \right\|_{\ell^2} + \|a\|_{\ell^2} \\ & \leq \left(\sqrt{K} e^{\lambda_K T/2} \varepsilon + 1 \right) \left\| y(L_n) \right\| \ . \end{aligned}$$

Step 5:

We apply Theorem 3.3 with $T_1 = L_n$, $T_2 = (n+1)T$, $T_3 = L_{n+1}$ and the above choice of b. Then there is a solution $\left(\widetilde{y}, \widetilde{f}\right)$ such that

$$\begin{cases} \widetilde{y}'(t) - A\widetilde{y}(t) = 0\,, & t \in (L_n, L_{n+1}) \setminus \{(n+1)T\}\ , \\ \widetilde{y}(L_n) = \sum_{j=1,\cdot\cdot,K} b_j \xi_j\,, & \\ \widetilde{y}((n+1)T) = \widetilde{y}((n+1)T_-) + 1_{\omega_2} \widetilde{f}\,, & \end{cases}$$

and

$$\begin{cases} \|\widetilde{y}(L_{n+1})\| \leq \varepsilon \sqrt{K} \, \|b\|_{\ell^2}\ , \\ \widetilde{f} = \sum_{j=1,\cdot\cdot,K} b_j f_j := - \sum_{j=1,\cdot\cdot,K} e^{\lambda_j T/2} \langle g_j, 1^*_{\omega_1} y((n + \frac{1}{2})T) \rangle_{\omega_1} f_j\ , \end{cases}$$

where f_j is given by Corollary 3.2 with $\omega=\omega_2$ and satisfies

$$\left\| f_j \right\|_{\omega_2} \leq \mathcal{C}_\varepsilon\left(T/4, 3T/4\right)$$

with $\mathcal{C}_\varepsilon$ given in Theorem 3.1. By the time translation invariance of the equation (5.4), $\left\{f_j\right\}_{j=1,\cdot\cdot,K} \in L^2\left(\omega_2\right)$ is the control function obtained by applying Corollary 3.2 with $T_1=\frac{T}{4}$, $T_2=T$, $T_3=\frac{5T}{4}$ and $\omega=\omega_2$.

Step 6:

One can check that $\widetilde{f}=\mathcal{F}\left(1^*_{\omega_1} y\left(\left(n+\frac{1}{2}\right)T\right)\right)$ and $y=\widetilde{y}+\widehat{y}+\overline{y}$ where $\widetilde{y}$ is given in step 5, $\widehat{y}$ solves

$$\begin{cases} \widehat{y}'\left(t\right)-A\widehat{y}\left(t\right)=0\,, & t\in\left(L_n,L_{n+1}\right)\setminus\left\{\left(n+1\right)T\right\}\,, \\ \widehat{y}\left(L_n\right)=\displaystyle\sum_{j=1,\cdot\cdot,K}\left(a_j-b_j\right)\xi_j\,, & \\ \widehat{y}\left(\left(n+1\right)T\right)=\widehat{y}\left(\left(n+1\right)T_-\right)\,, & \end{cases}$$

and $\overline{y}$ satisfies

$$\begin{cases} \overline{y}'\left(t\right)-A\overline{y}\left(t\right)=0\,, & t\in\left(L_n,L_{n+1}\right)\setminus\left\{\left(n+1\right)T\right\}\,, \\ \overline{y}\left(L_n\right)=\displaystyle\sum_{j>K}a_j\xi_j\,, & \\ \overline{y}\left(\left(n+1\right)T\right)=\overline{y}\left(\left(n+1\right)T_-\right)\,. & \end{cases}$$

Step 7: We estimate $\left\|y\left(L_{n+1}\right)\right\|=\left\|\widetilde{y}\left(L_{n+1}\right)+\widehat{y}\left(L_{n+1}\right)+\overline{y}\left(L_{n+1}\right)\right\|$ as follows.

First, by step 5 and step 4,

$$\left\|\widetilde{y}\left(L_{n+1}\right)\right\| \leq \varepsilon\sqrt{K}\left\|b\right\|_{\ell^2} \leq \varepsilon\sqrt{K}\left(\sqrt{K}e^{\lambda_K T/2}\varepsilon+1\right)\left\|y\left(L_n\right)\right\|\,.$$

Second, by step 3

$$\left\|\widehat{y}\left(L_{n+1}\right)\right\| \leq e^{\left\|V\right\|_\infty T}\left\|\widehat{y}\left(L_n\right)\right\| \leq e^{\left\|V\right\|_\infty T}\sqrt{K}e^{\lambda_K T/2}\varepsilon\left\|y\left(L_n\right)\right\|\,.$$

Third,

$$\left\|\overline{y}\left(L_{n+1}\right)\right\|=\left(\sum_{j>K}\left|a_j e^{-\lambda_j\left(L_{n+1}-L_n\right)}\right|^2\right)^{1/2} \leq e^{-\lambda_{K+1}T}\left\|y\left(L_n\right)\right\|\,.$$

Gathering all the previous estimates, one concludes that

$$\left\|y\left(L_{n+1}\right)\right\| \leq \left(e^{-\lambda_{K+1}T}+3e^{\left\|V\right\|_\infty T}e^{\lambda_K T/2}\left(1+K\right)\varepsilon\right)\left\|y\left(L_n\right)\right\|$$

with $\varepsilon\in(0,1)$. Finally, the choice of K (see (5.1)) gives $e^{-\lambda_{K+1}T} \leq \frac{1}{2}e^{-\gamma T}$, and the choice of $\varepsilon\in(0,1)$ (see (5.2)) gives $3e^{\left\|V\right\|_\infty T}e^{\lambda_K T/2}\left(1+K\right)\varepsilon=\frac{1}{2}e^{-\gamma T}$, which implies the desired estimate for $\left\|y\left(L_{n+1}\right)\right\|$, that is $\left\|y\left(L_{n+1}\right)\right\| \leq e^{-\gamma T}\left\|y\left(L_n\right)\right\|$.

Step 8: We treat the boundedness of $\mathcal{F}$ as follows:

$$\begin{aligned}\|\mathcal{F}(w)\|_{\omega_2}^2 &= \int_{\omega_2}\left|\sum_{j=1,\cdot\cdot,K} e^{\lambda_j T/2}\langle g_j, w\rangle_{\omega_1} f_j(x)\right|^2 dx \\ &\leq \sum_{j=1,\cdot\cdot,K}\left|e^{\lambda_j T/2}\langle g_j, w\rangle_{\omega_1}\right|^2 \sum_{j=1,\cdot\cdot,K}\int_{\omega_2}\left|f_j(x)\right|^2 dx \\ &\leq \|w\|_{\omega_1}^2 e^{\lambda_K T}\sum_{j=1,\cdot\cdot,K}\|g_j\|_{\omega_1}^2\sum_{j=1,\cdot\cdot,K}\|f_j\|_{\omega_2}^2 \\ &\leq \|w\|_{\omega_1}^2 e^{\lambda_K T} K^2\left[\mathcal{C}_\varepsilon(T/4, T/4)\,\mathcal{C}_\varepsilon(T/4, 3T/4)\right]^2\end{aligned}$$

which implies

$$\|\mathcal{F}\|_{\mathcal{L}(L^2(\omega_1), L^2(\omega_2))} \leq e^{\lambda_K T/2} K\mathcal{C}_\varepsilon(T/4, T/4)\,\mathcal{C}_\varepsilon(T/4, 3T/4)\ .$$

Step 9: We estimate $1/\varepsilon$ and $\mathcal{C}_\varepsilon$:

By the Weyl's asymptotic law for the Dirichlet eigenvalues α_j, there is a constant $\overline{C} > 0$ (depending only on Ω and d) such that for any $\mu > 0$,

$$\text{card}\left\{j \in \mathbb{N},\ \alpha_j < \mu\right\} \leq \overline{C}\left(1 + \mu^{d/2}\right)\ .$$

By the min–max formula, one has

$$-\|V\|_\infty + \alpha_j \leq \lambda_j \leq \alpha_j + \|V\|_\infty\ .$$

One deduces that there is a constant $\overline{C} > 0$ (depending only on Ω and d) such that for any $\gamma > 0$

$$K := \text{card}\left\{j \in \mathbb{N},\ \lambda_j < \gamma + \frac{\ln 2}{T}\right\} \leq \overline{C}\left(1 + \|V\|_\infty^{d/2} + \left(\frac{\ln 2}{T}\right)^{d/2} + \gamma^{d/2}\right)\ .$$

Further, for some constant $\overline{C} > 0$ (depending only on Ω and d), we have

$$\frac{1}{\varepsilon} := 6e^{\gamma T} e^{\|V\|_\infty T} e^{\lambda_K T/2}(1 + K) \leq \overline{C}\left(\gamma^{d/2} + \|V\|_\infty^{d/2} + \frac{1}{T^{d/2}}\right) e^{\|V\|_\infty T} e^{2\gamma T}\ .$$

We finish the proof by gathering the previous estimates with the definition of $\mathcal{C}_\varepsilon$, that is $\mathcal{C}_\varepsilon(t, s) := e^{4s\|V\|_\infty} e^{c\left(1 + \frac{1}{t} + t\|V\|_\infty + \|V\|_\infty^{2/3}\right)} \exp\left(\sqrt{\frac{c}{t}\ln^+\frac{1}{\varepsilon}}\right)$.

Hence, we complete the proof of Theorem 5.1.

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