EXPONENTIAL DECAY TOWARD EQUILIBRIUM VIA LOG CONVEXITY FOR A DEGENERATE REACTION-DIFFUSION SYSTEM

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Abstract. We consider a system of two reaction-diffusion equations coming out of reversible chemistry. When the reaction happens on the totality of the domain, it is known that exponential convergence to equilibrium holds (with explicit rate). We show in this paper that this exponential convergence also holds when the reaction happens only on a given open set of a ball, thanks to an observation estimate deduced by logarithmic convexity.

1. Introduction and main result

Considering as in [CDF] a general reversible chemical reaction between species $\mathbf{A}_1$,.., $\mathbf{A}_m$ diffusing in a chemical reactor:

$$\alpha_1 \mathbf{A}_1 + \cdots + \alpha_m \mathbf{A}_m \rightleftharpoons \beta_1 \mathbf{A}_1 + \cdots + \beta_m \mathbf{A}_m, \qquad \alpha_i, \beta_i \in \mathbb{N}, \tag{1}$$

and modeling the above reaction according to the *mass action law* with the stoichiometric coefficients $\alpha_i, \beta_i \in \mathbb{N}$, and with the reaction rates $l_1, l_2 > 0$, we end up with the following system for the concentrations a_i of the species $\mathbf{A}_i$:

$$\partial_t a_i - d_i\, \Delta_x a_i = (\beta_i - \alpha_i) \left(l_1 \prod_{j=1}^{m} a_j^{\alpha_j} - l_2 \prod_{j=1}^{m} a_j^{\beta_j} \right), \qquad i = 1, .., m, \tag{2}$$

where d_i is the diffusion rate of species $\mathbf{A}_i$. We refer to [BDS] for a formal derivation of this system (when $m = 2$) starting from kinetic theory. We also refer to [DFT] and the references therein for more complex systems, corresponding to networks of chemical reactions.

In [DF, DF3], exponential convergence to an homogeneous equilibrium is proven with an explicit rate thanks to entropy methods, when $d_i > 0$ $(i = 1, .., m)$ and $l_1, l_2 > 0$ are constant, for simple systems of 2, 3, or 4 equations (and for homogeneous Neumann boundary conditions). Those results are extended to a much larger class of systems in [DFT]. The case when one diffusion rate is zero, for simple systems, has also been studied in [DF2] (see also [BDS], Rmk. 3.7).

In this paper, we investigate an issue which has not been considered yet, namely the case when the reaction rates are proportional to the concentration of a catalyst which may be equal to 0 on a large part of the domain. More precisely, we suppose that $l_1 = l_2 := k(x,t)/2 \geq 0$, where k is strictly positive on a (possibly small) ball included in the domain.

We present for that a new method, based on an observation inequality, on a typical example of nonlinear reaction-diffusion systems coming out of reversible chemistry, namely

$$2\,\mathbf{A} \qquad \rightleftharpoons \qquad 2\,\mathbf{B},$$

where $\mathbf{A}$ and $\mathbf{B}$ are chemical species of respective concentrations $a := a(x,t) \geq 0$, $b := b(x,t) \geq 0$. This corresponds (with slightly different notations) to (1) when $m = 2$, $\alpha_1 = \beta_2 = 2$, $\alpha_2 = \beta_1 = 0$.

We suppose that the species $\mathbf{A}$ has a diffusion rate $d_1 > 0$ and that the species $\mathbf{B}$ has a diffusion rate $d_2 > 0$. We also assume that those species are confined in a chemical reactor represented by the ball $\Omega := B\left(0,R\right) := \{x \in \mathbb{R}^{\mathrm{n}}; |x| < R\}$, where n$\in \{1,2,3\}$, and $|\Omega| = 1$ (that is $R = \frac{1}{2}$ if n$= 1$, $R = \pi^{-1/2}$ if n$= 2$, and $R = (\frac{3}{4\pi})^{1/3}$ if n$= 3$), so that homogeneous Neumann boundary conditions are imposed. Finally and most importantly, the terms arising from the reaction process are given by the mass action law as described earlier, and $l_1 = l_2 := k(x,t)/2 \geq 0$. Recording (2), the corresponding system writes

$$\left\{\begin{array}{ll} \partial_t a - d_1 \Delta a = k(x,t)\ (b^2 - a^2)\ , & \text{in } \Omega \times (0,+\infty)\ , \\ \partial_t b - d_2 \Delta b = -k(x,t)\ (b^2 - a^2)\ , & \text{in } \Omega \times (0,+\infty)\ , \\ \partial_n a = \partial_n b = 0\ , & \text{on } \partial\Omega \times (0,+\infty)\ , \\ a\left(\cdot,0\right) = a_0\ ,\ b\left(\cdot,0\right) = b_0\ , & \text{in } \Omega\ , \end{array}\right. \tag{3}$$

where $\partial_n := n(x) \cdot \nabla$, and $n(x)$ is the unit outward normal vector at point $x \in \partial\Omega$.

We consider initial data $a_0, b_0 \in C^2\left(\overline{\Omega}\right)$ (compatible with the Neumann boundary condition) which satisfy the bound:

$$\forall x \in \overline{\Omega}\ , \quad 0 < B_0 \leq a_0\left(x\right) \quad \text{and} \quad 0 < B_0 \leq b_0\left(x\right), \tag{4}$$

for some constant $B_0 > 0$, and we suppose that

$$\int_\Omega [a_0(x) + b_0(x)]\, dx = 2 \ . \tag{5}$$

At this point, we remark that at the formal level, the following *a priori* estimates hold:

$$\frac{d}{dt}\int_\Omega (a+b) = 0 \ , \tag{6}$$

$$\frac{1}{2}\frac{d}{dt}\|(a,b)\|^2_{(L^2(\Omega))^2} + d_1 \int_\Omega |\nabla a|^2 + d_2 \int_\Omega |\nabla b|^2 + \int_\Omega k(x,t)\ (a+b)\,|a-b|^2 = 0 \ . \tag{7}$$

Because of the terms $d_1 \int_\Omega |\nabla a|^2$ and $d_2 \int_\Omega |\nabla b|^2$, we expect that $\lim_{t\to\infty} a(t,x) = a_\infty$, $\lim_{t\to\infty} b(t,x) = b_\infty$, for some constants $a_\infty \geq 0$ and $b_\infty \geq 0$. Moreover, as soon as k is strictly positive on some fixed open set of Ω, $a_\infty = b_\infty$ because of the term $\int_\Omega k\,(a+b)\,|a-b|^2$. Finally, estimate (6) ensures that $\int_\Omega (a_\infty + b_\infty) = \int_\Omega (a_0 + b_0) = 2$.

Remembering that $|\Omega| = 1$, we finally expect that the equilibrium is $(a_\infty, b_\infty) = (1,1)$.

Our main result shows that even if the catalyst has a concentration $k := k(x,t)$ which is strictly positive only on a small ball, then exponential convergence to equilibrium with an explicit rate still holds (we recall that this is indeed known when k is a strictly positive constant, cf. for example [DFT]).

Theorem 1.1. *We define $\Omega := B(0,R) := \{x \in \mathbb{R}^{\mathrm{n}}; |x| < R\}$, where $\mathrm{n} \in \{1,2,3\}$, the centered ball of $\mathbb{R}^{\mathrm{n}}$ of measure 1. We also assume that $d_1, d_2 > 0$ and $k \in C^2(\overline{\Omega}\times\mathbb{R}_+;\mathbb{R}_+)$. We consider initial data $a_0, b_0 \in C^2\left(\overline{\Omega}\right)$ (compatible with the Neumann boundary condition) which satisfy (4) and (5).*

We finally assume that there exists $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$ and $k(x,t) \geq k_0 > 0$ for any $(x,t) \in B(x_0,r) \times (0,+\infty)$.

Then there exists a unique smooth ($C^2(\overline{\Omega} \times [0,+\infty))$) solution to system (3), such that

$$\inf_{t\geq 0, x\in\overline{\Omega}} a(x,t) \geq B_0 \ , \qquad \inf_{t\geq 0, x\in\overline{\Omega}} b(x,t) \geq B_0 \ , \tag{8}$$

and for any $t \geq 0$,

$$\|a(\cdot,t) - 1\|^2_{L^2(\Omega)} + \|b(\cdot,t) - 1\|^2_{L^2(\Omega)} \leq \gamma\, e^{-\beta t} \left(\|a_0 - 1\|^2_{L^2(\Omega)} + \|b_0 - 1\|^2_{L^2(\Omega)}\right) \ , \tag{9}$$

where $\gamma > 0$ *and* $\beta > 0$ *can be explicitly estimated (from above for* γ *and from below for* β*) in terms of* $|x_0|$, r, $\|(a_0, b_0)\|_{L^3(\Omega)}$, B_0, k_0, $||k||_{L^\infty(\Omega\times\mathbb{R}_+)}$, d_1, d_2.

A first part of the proof is based on the entropy method (cf. [DFT]). We choose for the entropy the simplest possible functional, that is the square of the L^2 norm of $a-1$ added to the square of the L^2 norm of $b-1$. However, because of the degeneracy of k, it is not possible to directly relate the entropy dissipation to the entropy itself.

Our idea is therefore to introduce a new observation estimate (Prop. 2.1) for the system, which enables to perform a Gronwall lemma, and get the expected exponential decay. The proof of this observation estimate is based on another observation estimate, for two heat equations whose right-hand sides satisfy some specific inequality. More precisely, we show the

Proposition 1.2. *Let* $\Omega := B(0,R) := \{x \in \mathbb{R}^{\mathrm{n}}; |x| < R\}$*, when* $\mathrm{n} \in \{1,2,3\}$*, be the centered ball of* $\mathbb{R}^{\mathrm{n}}$ *of measure 1, and let* $d_1, d_2 > 0$*. We also consider* $x_0 \in \Omega$ *and* $r > 0$*, such that* $B(x_0, r) \subset \Omega$*. Let* (u_1, u_2, v_1, v_2) *be smooth (*$C^2(\overline{\Omega} \times [0, +\infty))$*) functions satisfying the system of two heat equations with unknowns* (u_1, u_2)*, together with homogeneous Neumann boundary conditions, and an outside force* (v_1, v_2)*:*

$$\left\{\begin{array}{ll} \partial_t u_1 - d_1 \Delta u_1 = v_1 \ , & \text{in } \Omega \times (0, +\infty) \ , \\ \partial_t u_2 - d_2 \Delta u_2 = v_2 \ , & \text{in } \Omega \times (0, +\infty) \ , \\ \partial_n u_1 = \partial_n u_2 = 0 \ , & \text{on } \partial\Omega \times (0, +\infty) \ . \end{array}\right. \tag{10}$$

We also assume that u_1, u_2*,* v_1, v_2 *satisfy the following bounds, for some* $K_0 > 0$*,*

$$\forall (x,t) \in \Omega \times \mathbb{R}_+ \ , \qquad |(v_1, v_2)(x,t)|^2 \leq K_0 \left(|(u_1, u_2)(x,t)|^2 + |(u_1, u_2)(x,t)|^4 \right) \ , \tag{11}$$

$$\forall i \in \{1,2\} \ , \forall t \in \mathbb{R}_+ \ , \qquad \|u_i(\cdot, t)\|^2_{L^3(\Omega)} \leq K_0 \ , \tag{12}$$

and

$$\forall \ 0 \leq t_1 < t_2 \ , \qquad \|(u_1, u_2)(\cdot, t_2)\|^2_{(L^2(\Omega))^2} \leq \|(u_1, u_2)(\cdot, t_1)\|^2_{(L^2(\Omega))^2} \ . \tag{13}$$

Then there exist $c > 1$ *and* $M > 1$ *(both depending on* K_0*,* $|x_0|$*,* r*, and* d_1*,* d_2*) such that for any* $T > 0$*,*

$$\begin{aligned} \left(\|(u_1, u_2)(\cdot, T)\|^2_{(L^2(\Omega))^2} \right)^{1+M} & \leq e^{c\left(1+\frac{1}{T}\right)} \|(u_1, u_2)(\cdot, T)\|^2_{(L^2(B(x_0,r)))^2} \\ & \quad \times \left(\|(u_1, u_2)(\cdot, 0)\|^2_{(L^2(\Omega))^2} \right)^M \ . \end{aligned}$$

This Proposition is based on the logarithmic convexity method introduced by [BT] and developed by [BP], [PW], [P] for linear parabolic problems with Dirichlet boundary conditions, and by [BuP] for linear parabolic problems with Neumann boundary conditions.

The main novelty here is the treatment of a nonlinear (because of assumption (11)) parabolic system with Neumann boundary conditions. Up to now, only scalar equations had been treated (cf. [BuP]). We also choose weights which are more explicit with respect to this last reference, this leads to a more explicit dependence of the rate of convergence towards equilibrium.

Our feeling is that our method can be generalized to much more general systems of the type (2), or even to systems appearing when networks of reactions are considered (cf. [DFT]), provided that the solutions to those systems are bounded uniformly in time (this is far from being always known, even when the reaction rates are constant). It could probably also be generalized to domains which are more general than a ball, though for general domains the explicit character of the rates of convergence could be lost.

We wish to point out that the case of a system of two reaction-diffusion equations in which one diffusion rate is zero and the reaction rate is also zero on a non-negligible set, is quite different, since the appearance of an homogeneous equilibrium is not expected in such a situation.

Since the proofs that we propose are long and technical, we present here an overview of these proofs, where we try to explain the ideas underlying them.

We start the proof of Theorem 1.1, described in Section 2, by an attempt to use the entropy method (cf. [DFT]). In the context of this method, we introduce a simple entropy (Lyapunov functional), that is $H(t) := ||a(\cdot,t)-1||^2_{L^2(\Omega)} + ||b(\cdot,t)-1||^2_{L^2(\Omega)}$. Because of the degeneracy of k, it is not possible to directly relate the entropy dissipation to the entropy H, but only to its localized version $H_{loc}(t) := ||a(\cdot,t)-1||^2_{L^2(B(x_0,r))} + ||b(\cdot,t)-1||^2_{L^2(B(x_0,r))}$. We use therefore an observation inequality for the system (3), stated in Proposition 2.1, which relates $H^{1+M}(t)$ and $H_{loc}(t)\ H^M(0)$, for some $M>1$. Proposition 2.1 is a direct consequence of Proposition 1.2 stated above, which is itself an observation inequality, but for a different system, made of two heat equations with source terms, obtained by freezing the right-hand sides in system (3) (what remains of the original nonlinear system is the assumed estimate (11), linking (v_1,v_2) and (u_1,u_2)). A slight modification of the entropy method enables then to conclude to the exponential convergence towards equilibrium stated in Theorem 1.1.

We are then left to prove Proposition 1.2, that is, to obtain an observation estimate for a system of two heat equations (with unknowns denoted by (u_1, u_2)) with Neumann boundary conditions, and a source term (v_1, v_2). The proof of this observation estimate is based on a variant of the logarithmic convexity method for linear and nonlinear parabolic systems introduced by [BT].

A first step (step 1 in Section 3) consists in a change of unknowns in the system of two heat equations. In order to focus on the time T and the ball $B(x_0, r)$ appearing in the observation estimate, we introduce $(f_1, f_2) = \left(u_1\, e^{\Phi_1/2}, u_2\, e^{\Phi_1/2}\right)$ with $\Phi_1(x,t) := \frac{s\,\varphi_1(x)}{T-t+h}$, for s, h well chosen positive parameters, and where φ_1 is a carefully selected non positive function which has a unique critical point at x_0. In order to treat the Neumann boundary condition, we also introduce $(f_3, f_4) := \left(u_1\, e^{\Phi_3/2}, u_2\, e^{\Phi_3/2}\right)$, with Φ_3 chosen in such a way that $\Phi_1 = \Phi_3$ and $\partial_n \Phi_1 + \partial_n \Phi_3$ at the boundary, so that the boundary terms in the integrations by parts disappear in the computations performed later. This change of variables can be described as a deformation of the system via a weight function, and is typical of Carleman techniques (see e.g. [Le B], [HZ]). Step 1 is concluded by a statement of the basic properties of the system satisfied by $f := (f_i)_{i=1,..4}$, written as $\partial_t f + \mathcal{S} f = \mathcal{A} f + F$, where $\mathcal{S}$, $\mathcal{A}$ are the symmetric and antisymmetric parts of the system, and F is the source term.

In step 2, we write down the energy estimate related to the system satisfied by f, using two quantities: the energy $E := \left|\left| f\,(\cdot, t)\right|\right|^2_{(L^2(\Omega))^4} = \langle f, f\rangle$ and the frequency function $\mathbf{N} := \frac{\langle \mathcal{S} f, f\rangle}{\langle f, f\rangle}$. The computation of $\frac{d}{dt}\mathbf{N}$ involves a quantity related to the Carleman commutator, defined as $\langle [\mathcal{S}, -\partial_t + \mathcal{A}]\, f, f\rangle$.

Then, step 3 is devoted to the establishment of a suitable bound for $\langle [\mathcal{S}, -\partial_t + \mathcal{A}]\, f, f\rangle$, and consequently for $\frac{d}{dt}\mathbf{N}$. Here the choice of the parameter s will play a key role.

Using a lemma proven in [BuP] (which can be viewed as a variant of estimates appearing in the logarithmic convexity method, in which one computes $(\ln E)''$) for solutions of the two differential inequalities obtained in the previous step, one obtains at the end of step 4 an Hölder stability estimate for E (estimate (45)), typical of what is provided by the logarithmic convexity method.

In step 5, we use this Hölder stability estimate in order to make the L^2 norm of (u_1, u_2) on the ball $B(x_0, r)$ appear in the computations, which enables to conclude the proof of Proposition 1.2 by an optimization with respect to the parameter h.

2. Proof of Theorem 1.1

We start here the

Proof of Theorem 1.1: Note first that under the assumptions of Theorem 1.1, the existence, uniqueness and smoothness of a solution to (3) is a consequence of standard theorems for parabolic equations (cf. for example [D] or [LSU]). The minimum principle estimate (8) can easily be seen at the formal level by considering, for a given time t, the point $x \in \overline{\Omega}$ where $\min(a, b)$ reaches its minimum. For a rigorous proof, we refer to [D], p.100-101 (where the proof is detailed for a slightly different system).

Note then that for this solution, the following estimate holds:

$$\frac{1}{3}\frac{d}{dt}\int_\Omega (a^3+b^3) = -2d_1\int_\Omega a\,|\nabla a|^2 - 2d_2\int_\Omega b\,|\nabla b|^2 - \int_\Omega k\,(b^2-a^2)^2 \leq 0\ .$$

As a consequence

$$\forall t \geq 0\ , \qquad \int_\Omega \left[a^3(\cdot,t) + b^3(\cdot,t)\right] \leq \int_\Omega \left[a_0^3(x) + b_0^3(x)\right] dx\ . \tag{14}$$

Then we write down the energy identity (7) for the quantities $U_1 := a-1$, $U_2 := b-1$:

$$\frac{1}{2}\frac{d}{dt}\left\|(U_1,U_2)\right\|^2_{(L^2(\Omega))^2} + d_1\int_\Omega |\nabla U_1|^2 + d_2\int_\Omega |\nabla U_2|^2 + \int_\Omega k\,(a+b)\,|U_2-U_1|^2 = 0\ , \tag{15}$$

and the identity

$$\forall t \geq 0\ , \qquad \int_\Omega [U_1(\cdot,t) + U_2(\cdot,t)] = 0\ , \tag{16}$$

which is a direct consequence of (5), (6).

The proof of Theorem 1.1 is then an application of the following observation estimate at one point in time:

Proposition 2.1. *Under the assumptions of Theorem 1.1, there exist* $c > 1$ *and* $M > 1$ *(both depending on* $|x_0|$, r, $\|(a_0,b_0)\|_{L^3(\Omega)}$, $||k||_{L^\infty(\Omega\times\mathbb{R}_+)}$, d_1, d_2*) such that for any* $t > t_1 \geq 0$,

$$\begin{aligned}\left(\left\|(a-1,b-1)(\cdot,t)\right\|^2_{(L^2(\Omega))^2}\right)^{1+M} &\leq e^{c\left(1+\frac{1}{t-t_1}\right)}\left\|(a-1,b-1)(\cdot,t)\right\|^2_{(L^2(B(x_0,r)))^2} \\ &\quad\times\left(\left\|(a-1,b-1)(\cdot,t_1)\right\|^2_{(L^2(\Omega))^2}\right)^M\ .\end{aligned}$$

Proof of Proposition 2.1: Under the assumptions of Proposition 2.1 (which are those of Theorem 1.1), we see that $U_1 := a-1$ and $U_2 := b-1$ are smooth ($C^2(\overline{\Omega}\times$

$[0, +\infty)))$ and satisfy the system

$$\begin{cases} \partial_t U_1 - d_1 \Delta U_1 = k\,(U_1 + U_2 + 2)\,(U_2 - U_1)\ , & \text{in } \Omega \times (0, +\infty)\ , \\ \partial_t U_2 - d_2 \Delta U_2 = -k\,(U_1 + U_2 + 2)\,(U_2 - U_1)\ , & \text{in } \Omega \times (0, +\infty)\ , \\ \partial_n U_1 = \partial_n U_2 = 0\ , & \text{on } \partial\Omega \times (0, +\infty)\ . \end{cases}$$

Considering $v_1 := k\,(U_1 + U_2 + 2)\,(U_2 - U_1)$, $v_2 := -k\,(U_1 + U_2 + 2)\,(U_2 - U_1)$, we see that v_1 and v_2 are smooth ($C^2(\overline{\Omega} \times [0, +\infty)))$ and that (13) holds (when (u_1, u_2) is replaced by (U_1, U_2)) thanks to (15).

Moreover,

$$|(v_1, v_2)|^2 = 2k^2\,(U_1 + U_2 + 2)^2\,(U_2 - U_1)^2 \leq 32\,\|k\|^2_{L^\infty(\Omega \times \mathbb{R}_+)}\left(|(U_1, U_2)|^2 + |(U_1, U_2)|^4\right)\ ,$$

and finally (using (14)),

$$\begin{cases} ||(U_1(\cdot, t)||^3_{L^3(\Omega)} = \displaystyle\int_\Omega |a(\cdot, t) - 1|^3 \leq 4 \int_\Omega a^3(\cdot, t) + 4 \leq 4 \int_\Omega \left(a_0^3 + b_0^3\right) + 4\ , \\ ||(U_2(\cdot, t)||^3_{L^3(\Omega)} = \displaystyle\int_\Omega |b(\cdot, t) - 1|^3 \leq 4 \int_\Omega b^3(\cdot, t) + 4 \leq 4 \int_\Omega \left(a_0^3 + b_0^3\right) + 4\ , \end{cases}$$

so that (11) and (12) hold (when (u_1, u_2) is replaced by (U_1, U_2)) with

$$K_0 := \max\left(\left[4 \int_\Omega \left(a_0^3 + b_0^3\right) + 4\right]^{2/3}, 32\,\|k\|^2_{L^\infty(\Omega \times \mathbb{R}_+)}\right)\ .$$

Then it is possible to use Proposition 1.2 (when (u_1, u_2) is replaced by (U_1, U_2)) and to get Proposition 2.1 after a simple time translation.

We proceed with the

End of the Proof of Theorem 1.1:
By Poincaré-Wirtinger's inequality applied to $U_1 + U_2$ (using identity (16), and denoting by C_p the corresponding constant), the assumption $k\,(\cdot, t) \geq k_0 > 0$ on $B\,(x_0, r)$, and remembering (8), we see that

$$\begin{aligned} 2\,\|(U_1, U_2)\|^2_{(L^2(B(x_0,r)))^2} &= \|U_1 + U_2\|^2_{L^2(B(x_0,r))} + \|U_1 - U_2\|^2_{L^2(B(x_0,r))} \\ &\leq \|U_1 + U_2\|^2_{L^2(\Omega)} + \frac{1}{2B_0 k_0} \int_\Omega k\,(a + b)\,|U_2 - U_1|^2 \\ &\leq C_p\,\|\nabla\,(U_1 + U_2)\|^2_{L^2(\Omega)} + \frac{1}{2B_0 k_0} \int_\Omega k\,(a + b)\,|U_2 - U_1|^2 \\ &\leq 4\beta_1 \left(d_1 \int_\Omega |\nabla U_1|^2 + d_2 \int_\Omega |\nabla U_2|^2 + \int_\Omega k\,(a + b)\,|U_2 - U_1|^2\right)\ , \end{aligned}$$

with $\beta_1 := \max\left(\frac{C_p}{2d_1}, \frac{C_p}{2d_2}, \frac{1}{8B_0k_0}\right)$.

Combining the above estimate with (15) and Proposition 2.1, we deduce that

$$\beta_1 \frac{d}{dt} \left\| (U_1, U_2)(\cdot, t) \right\|^2_{(L^2(\Omega))^2} + \frac{1}{e^{c\left(1+\frac{1}{t-t_1}\right)}} \frac{\left(\left\| (U_1, U_2)(\cdot, t) \right\|^2_{(L^2(\Omega))^2} \right)^{1+M}}{\left(\left\| (U_1, U_2)(\cdot, t_1) \right\|^2_{(L^2(\Omega))^2} \right)^{M}} \leq 0 \ ,$$

which can be rewritten

$$\frac{e^{-c\left(1+\frac{1}{t-t_1}\right)} \frac{M}{\beta_1}}{\left(\left\| (U_1, U_2)(\cdot, t_1) \right\|^2_{(L^2(\Omega))^2} \right)^{M}} \leq \frac{d}{dt} \left[\left(\left\| (U_1, U_2)(\cdot, t) \right\|^2_{(L^2(\Omega))^2} \right)^{-M} \right] \ . \tag{17}$$

Integrating (17) over (t_1+1, t_2) with $t_2 > t_1 + 1 \geq 1$ and using $\frac{1}{t-t_1} \leq 1$, so that $e^{-c\left(1+\frac{1}{t-t_1}\right)} \geq e^{-2c}$, we obtain

$$\frac{e^{-2c\frac{M}{\beta_1}} (t_2 - t_1 - 1)}{\left\| (U_1, U_2)(\cdot, t_1) \right\|^{2M}_{(L^2(\Omega))^2}} \leq \frac{1}{\left\| (U_1, U_2)(\cdot, t_2) \right\|^{2M}_{(L^2(\Omega))^2}} - \frac{1}{\left\| (U_1, U_2)(\cdot, t_1+1) \right\|^{2M}_{(L^2(\Omega))^2}} \ .$$

But $\left\| (U_1, U_2)(\cdot, t_1+1) \right\|^2_{(L^2(\Omega))^2} \leq \left\| (U_1, U_2)(\cdot, t_1) \right\|^2_{(L^2(\Omega))^2}$ thanks to (15). Therefore,

$$\left\| (U_1, U_2)(\cdot, t_2) \right\|^2_{(L^2(\Omega))^2} \leq \left(\frac{1}{1 + e^{-2c\frac{M}{\beta_1}} (t_2 - t_1 - 1)} \right)^{\frac{1}{M}} \left\| (U_1, U_2)(\cdot, t_1) \right\|^2_{(L^2(\Omega))^2} \ . \tag{18}$$

Now, choose $t_1 = 2m$ and $t_2 = 2(m+1)$, where $m \in \mathbb{N}$ (so that $t_2 > t_1 + 1 \geq 1$). Then estimate (18) becomes

$$\left\| (U_1, U_2)(\cdot, 2(m+1)) \right\|^2_{(L^2(\Omega))^2} \leq \theta \left\| (U_1, U_2)(\cdot, 2m) \right\|^2_{(L^2(\Omega))^2} \ ,$$

where $\theta := \left(\frac{1}{1+e^{-2c} M/\beta_1} \right)^{1/M} \in (0,1)$. A direct induction shows that

$$\left\| (U_1, U_2)(\cdot, 2m) \right\|^2_{(L^2(\Omega))^2} \leq \theta^m \left\| (U_1, U_2)(\cdot, 0) \right\|^2_{(L^2(\Omega))^2} \ .$$

Choosing $2m \leq t < 2(m+1)$, we obtain thanks to (15) that

$$\left\| (U_1, U_2)(\cdot, t) \right\|^2_{(L^2(\Omega))^2} \leq \frac{1}{\theta} e^{-t\frac{|\ln\theta|}{2}} \left\| (U_1, U_2)(\cdot, 0) \right\|^2_{(L^2(\Omega))^2} \ .$$

We conclude the proof of Theorem 1.1 by taking $\gamma := 1/\theta$, $\beta := |\ln\theta|/2$.

3. Heat systems with Neumann boundary conditions

The object of this section is the

Proof of Proposition 1.2: It starts here and is divided into 5 steps, described in subsections 3.1 to 3.5.

3.1. Step 1: Change of functions.

We recall that Ω is the centered ball of measure 1 of $\mathbb{R}^{\mathrm{n}}$, with n$= 1,2,3$. Without loss of generality, we suppose that $x_0 = (|x_0|, 0, .., 0)$. Then we consider

$$\psi\left(x\right) := \psi(x_1, .., x_{\mathrm{n}}) = \left(R^2 - |x|^2\right)\left(\frac{2|x_0|\,R}{|x_0|^2 + R^2 - 2|x_0|x_1}\right) \ , \tag{19}$$

which is well defined and is C^∞ on an open ball containing $\overline{\Omega}$. Moreover $\psi > 0$ on Ω, and $\psi = 0$ on $\partial\Omega$. In particular $\partial_n\psi \leq 0$ on $\partial\Omega$. One can check that

$$\frac{\partial\psi}{\partial x_k}(x) = \frac{-4|x_0|R\,x_k}{|x_0|^2 + R^2 - 2|x_0|x_1} \qquad \text{if} \qquad k \neq 1 \ ,$$

$$\frac{\partial\psi}{\partial x_1}(x) = \frac{-4|x_0|R\,x_1}{|x_0|^2 + R^2 - 2|x_0|x_1} + (R^2 - |x|^2)\frac{4|x_0|^2R}{(|x_0|^2 + R^2 - 2|x_0|x_1)^2} \ .$$

Then it is easy to see that ψ has a unique critical point at x_0 on $\overline{\Omega}$, which is a global nondegenerate maximum. Indeed (for $j, k = 1, ..,$n),

$$\frac{\partial^2\psi}{\partial x_j\partial x_k}(x_0) = -\frac{4|x_0|R}{R^2 - |x_0|^2}\,\delta_{jk} \ .$$

Further, there exist $c_{01}, c_{02} > 0$, depending only on $|x_0|$, such that for any x in a small neighborhood (also only depending on $|x_0|$) of x_0, the following estimate holds:

$$c_{01}\left|\nabla\psi\left(x\right)\right|^2 \leq \psi\left(x_0\right) - \psi\left(x\right) \leq c_{02}\left|\nabla\psi\left(x\right)\right|^2 \ . \tag{20}$$

We introduce $f := (f_i)_{1\leq i\leq 4}$, where $f_i := u_i\,e^{\Phi_i/2}$, and $\Phi_i\left(x,t\right) := \frac{s\varphi_i(x)}{\Gamma(t)}$, $s \in (0,1]$, $h \in (0,1]$,

$$\left\{\begin{array}{ll} \Gamma\left(t\right) := T - t + h \ , & \text{for any } t \in [0,T] \ , \\ \varphi_1\left(x\right) := \psi\left(x\right) - \psi\left(x_0\right) \ , & \text{for any } x \in \overline{\Omega} \ , \\ \varphi_2 := \varphi_1 \ , & \text{on } \overline{\Omega} \ , \\ \varphi_3\left(x\right) := -\psi\left(x\right) - \psi\left(x_0\right) \ , & \text{for any } x \in \overline{\Omega} \ , \\ \varphi_4 := \varphi_3 \ , & \text{on } \overline{\Omega} \ , \end{array}\right. \tag{21}$$

and

$$u_3 := u_1 \ , \qquad u_4 := u_2 \ . \tag{22}$$

Clearly,

$$\Phi_2 = \Phi_1 \qquad \text{and} \qquad \Phi_4 = \Phi_3 \ . \tag{23}$$

Notice that

$$\begin{cases} \varphi_1 = \varphi_3 \ , & \text{on } \partial\Omega \ , \\ \partial_n\varphi_1 + \partial_n\varphi_3 = 0 \ , & \text{on } \partial\Omega \ , \end{cases}$$

so that,

$$\begin{cases} \Phi_1 = \Phi_3 \ , & \text{on } \partial\Omega \times (0,T) \ , \\ \partial_n\Phi_1 + \partial_n\Phi_3 = 0 \ , & \text{on } \partial\Omega \times (0,T) \ . \end{cases} \tag{24}$$

We look for the equation solved by f_i by computing $e^{\Phi_i/2}\left(\partial_t - d_i\Delta\right)\left(e^{-\Phi_i/2}f_i\right)$, where

$$d_3 := d_1 \qquad \text{and} \qquad d_4 := d_2 \ . \tag{25}$$

We introduce for that purpose the operators

$$\begin{cases} \mathcal{A}_i f_i := -d_i\nabla\Phi_i\cdot\nabla f_i - \frac{1}{2}d_i\Delta\Phi_i f_i \ , \\ \mathcal{S}_i f_i := -d_i\Delta f_i - \eta_i f_i \ , \end{cases} \tag{26}$$

where $i = 1,..,4$ and

$$\eta_i := \frac{1}{2}\partial_t\Phi_i + \frac{1}{4}d_i\left|\nabla\Phi_i\right|^2 \ . \tag{27}$$

We also define $\mathcal{S}f := \left(\mathcal{S}_i f_i\right)_{1\le i\le 4}$, $\mathcal{A}f := \left(\mathcal{A}_i f_i\right)_{1\le i\le 4}$, and $F := \left(v_i e^{\Phi_i/2}\right)_{1\le i\le 4}$ where

$$v_3 := v_1 \ , \qquad v_4 := v_2 \ . \tag{28}$$

After this change of functions and the introduction of the new notations, the system (10) rewrites

$$\begin{cases} \partial_t f + \mathcal{S}f = \mathcal{A}f + F \text{ in } \Omega\times(0,T) \ , \\ \partial_n f_i - \frac{1}{2}\partial_n\Phi_i f_i = 0 \text{ on } \partial\Omega\times(0,T) \ , \qquad i = 1,..,4 \ . \end{cases} \tag{29}$$

As pointed out in the introduction, We have deformed the original solution (u_1, u_2) by a weight function $e^{\Phi_1/2}$, in order to focus on the information around x_0. Our choice to add the functions $(f_3, f_4) = \left(u_1 e^{\Phi_3/2}, u_2 e^{\Phi_3/2}\right)$ to the system is motivated by the problems related to the boundary conditions. Namely, it enables to get the identities (24) on the boundary, which will kill the boundary terms in the future integrations by parts.

Let now $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $\left(L^2(\Omega)\right)^4$, and $\|\cdot\|$ be its corresponding norm. We regroup some useful identities in the following:

Lemma 3.1. *For any (smooth enough, $\mathbb{R}^4$-valued) functions $f := f(x,t) = (f_i)_{i=1,..,4}$, any constants $d_i > 0$, and any (smooth enough) functions $\Phi_i := \Phi_i(x,t)$, $u_i = f_i\, e^{-\Phi_i/2}$ $(i = 1,..,4)$, the following identities hold as soon as (21) – (29) hold:*

$$\left\{ \begin{array}{l} \langle \mathcal{A} f, f \rangle = 0 \ , \\ \langle \mathcal{S} f, f \rangle = \displaystyle\sum_{i=1,..,4} \left[d_i \int_\Omega \left|\nabla f_i\right|^2 - \int_\Omega \eta_i \left|f_i\right|^2 \right] \ , \\ \dfrac{d}{dt} \langle \mathcal{S} f, f \rangle = \displaystyle\sum_{i=1,..,4} \int_\Omega \left(-\partial_t \eta_i\right) \left|f_i\right|^2 + 2 \langle \mathcal{S} f, \partial_t f \rangle := \langle \mathcal{S}' f, f \rangle + 2 \langle \mathcal{S} f, \partial_t f \rangle \ . \end{array} \right. \tag{30}$$

Proof of Lemma 3.1 . Thanks to an integration by parts,

$$\begin{aligned} &\langle \mathcal{A} f, f \rangle := \sum_{i=1,..,4} \int_\Omega \left(-d_i \nabla \Phi_i \cdot \nabla f_i - \frac{1}{2} d_i \Delta \Phi_i f_i \right) f_i \\ &= - \sum_{i=1,..,4} \int_\Omega \left[d_i \nabla \Phi_i \cdot \nabla \left(\frac{1}{2} \left|f_i\right|^2 \right) + \frac{1}{2} d_i \Delta \Phi_i \left|f_i\right|^2 \right] \\ &= - \int_{\partial\Omega} \left[d_1 \partial_n \Phi_1 \left(\frac{1}{2} \left|f_1\right|^2 \right) + d_2 \partial_n \Phi_2 \left(\frac{1}{2} \left|f_2\right|^2 \right) + d_3 \partial_n \Phi_3 \left(\frac{1}{2} \left|f_3\right|^2 \right) + d_4 \partial_n \Phi_4 \left(\frac{1}{2} \left|f_4\right|^2 \right) \right] \\ &= - \int_{\partial\Omega} d_1 \left(\partial_n \Phi_1 e^{\Phi_1} + \partial_n \Phi_3 e^{\Phi_3} \right) \left(\frac{1}{2} \left|u_1\right|^2 \right) - \int_{\partial\Omega} d_2 \left(\partial_n \Phi_1 e^{\Phi_1} + \partial_n \Phi_3 e^{\Phi_3} \right) \left(\frac{1}{2} \left|u_2\right|^2 \right) \end{aligned}$$

using (22), (23), (25) and recalling that $f_i = u_i e^{\Phi_i/2}$.

Now, (24) implies that

$$\partial_n \Phi_1 e^{\Phi_1} + \partial_n \Phi_3 e^{\Phi_3} = 0 \text{ on } \partial\Omega \times (0,T) \ . \tag{31}$$

This completes the proof of the identity $\langle \mathcal{A} f, f \rangle = 0$.

Then, we observe that

$$\begin{aligned}
&\langle \mathcal{S}f, f\rangle - \left[\sum_{i=1,..,4} d_i \int_\Omega |\nabla f_i|^2 - \sum_{i=1,..,4}\int_\Omega \eta_i |f_i|^2\right] \\
&= -\sum_{i=1,..,4}\int_{\partial\Omega} d_i f_i \partial_n f_i \\
&= -\sum_{i=1,..,4}\int_{\partial\Omega} d_i \left(\frac{1}{2}\partial_n \Phi_i |f_i|^2\right) \text{ because of the boundary condition in (29)} \\
&= 0 \text{ similarly as for } \langle \mathcal{A}f, f\rangle \ .
\end{aligned}$$

We finally compute $\frac{d}{dt}\langle \mathcal{S}f, f\rangle := \frac{d}{dt}\left(\sum_{i=1,..,4}\left[d_i\int_\Omega |\nabla f_i|^2 - \int_\Omega \eta_i |f_i|^2\right]\right)$. By an integration by parts,

$$\begin{aligned}
\frac{d}{dt}\langle \mathcal{S}f, f\rangle &= \sum_{i=1,..,4}\left[d_i\int_\Omega 2\nabla f_i\cdot\nabla\partial_t f_i - \int_\Omega \partial_t\eta_i |f_i|^2 - \int_\Omega 2\eta_i f_i\partial_t f_i\right] \\
&= -\sum_{i=1,..,4} d_i\int_\Omega 2\Delta f_i\partial_t f_i + \sum_{i=1,..,4}\left[d_i\int_{\partial\Omega} 2\partial_n f_i\partial_t f_i - \int_\Omega \partial_t\eta_i |f_i|^2 - \int_\Omega 2\eta_i f_i\partial_t f_i\right] \\
&= \sum_{i=1,..,4} 2\int_{\partial\Omega} d_i\partial_n f_i\partial_t f_i + \sum_{i=1,..,4}\int_\Omega (-\partial_t\eta_i)|f_i|^2 + 2\langle \mathcal{S}f, \partial_t f\rangle \ .
\end{aligned}$$

But, by the boundary condition in (29), it holds

$$\begin{aligned}
\sum_{i=1,..,4}\int_{\partial\Omega} d_i\partial_n f_i\partial_t f_i &= \sum_{i=1,..,4}\int_{\partial\Omega} d_i\left(\frac{1}{2}\partial_n\Phi_i f_i\partial_t f_i\right) \\
&= \sum_{i=1,..,4}\frac{1}{2}\int_{\partial\Omega} d_i\partial_n\Phi_i u_i e^{\Phi_i/2}\left(\partial_t u_i e^{\Phi_i/2} + u_i\frac{1}{2}\partial_t\Phi_i e^{\Phi_i/2}\right) \ .
\end{aligned}$$

The first contribution is proportional to

$$\begin{aligned}
&\int_{\partial\Omega}\left[d_1\partial_n\Phi_1 u_1\partial_t u_1 e^{\Phi_1} + d_2\partial_n\Phi_2 u_2\partial_t u_2 e^{\Phi_2} + d_3\partial_n\Phi_3 u_3\partial_t u_3 e^{\Phi_3} + d_4\partial_n\Phi_4 u_4\partial_t u_4 e^{\Phi_4}\right] \\
&= \int_{\partial\Omega}\left[d_1\left(\partial_n\Phi_1 e^{\Phi_1} + \partial_n\Phi_3 e^{\Phi_3}\right)u_1\partial_t u_1 + d_2\left(\partial_n\Phi_1 e^{\Phi_1} + \partial_n\Phi_3 e^{\Phi_3}\right)u_2\partial_t u_2\right]
\end{aligned}$$

using (22), (23), (25). Thanks to (31), we see that $\sum_{i=1,..,4}\int_{\partial\Omega} d_i\partial_n\Phi_i u_i\partial_t u_i e^{\Phi_i} = 0$.

The last contribution is proportional to

$$\begin{aligned}
&\sum_{i=1,..,4} \int_{\partial\Omega} d_i \partial_n \Phi_i \partial_t \Phi_i \left|u_i\right|^2 e^{\Phi_i} \\
&= \int_{\partial\Omega} \left[d_1 \left(\partial_n \Phi_1 \partial_t \Phi_1 e^{\Phi_1} + \partial_n \Phi_3 \partial_t \Phi_3 e^{\Phi_3} \right) \left|u_1\right|^2 + d_2 \left(\partial_n \Phi_1 \partial_t \Phi_1 e^{\Phi_1} + \partial_n \Phi_3 \partial_t \Phi_3 e^{\Phi_3} \right) \left|u_2\right|^2 \right] \\
&= 0
\end{aligned}$$

where in the second line, we used (22), (23), (25). In the third line, we used the identity $\partial_t \Phi_1 = \partial_t \Phi_3$ on $\partial\Omega \times (0, T)$, which is a consequence of (24).

This completes the proof of Lemma 3.1.

3.2. **Step 2: Energy estimates.** Thanks to a standard energy method (taking the scalar product against f and integrating by parts), we get, starting from eq. (29) and using the first identity of Lemma 3.1:

$$\frac{1}{2} \frac{d}{dt} \left\|f\right\|^2 + \langle \mathcal{S} f, f \rangle = \langle \digamma, f \rangle \quad . \tag{32}$$

Introducing the frequency function

$$\mathbf{N} := \mathbf{N}\left(t\right) = \frac{\langle \mathcal{S} f, f \rangle}{\left\|f\right\|^2} \ , \tag{33}$$

we see that, thanks to (29), to the third part of Lemma 3.1, and to identity (32),

$$\begin{aligned}
\tfrac{d}{dt}\mathbf{N}\left(t\right)\left(\left\|f\right\|^2\right)^2 &= \left(\langle \mathcal{S}' f, f \rangle + 2 \langle \mathcal{S} f, \partial_t f \rangle\right) \left\|f\right\|^2 - \langle \mathcal{S} f, f \rangle \left(-2 \langle \mathcal{S} f, f \rangle + 2 \langle \digamma, f \rangle\right) \\
&= \left(\langle \mathcal{S}' f, f \rangle + 2 \langle \mathcal{S} f, \mathcal{A} f \rangle\right) \left\|f\right\|^2 - 2 \left\|\mathcal{S} f\right\|^2 \left\|f\right\|^2 + 2 \langle \mathcal{S} f, \digamma \rangle \left\|f\right\|^2 \\
&\quad + 2 \langle \mathcal{S} f, f \rangle^2 - 2 \langle \mathcal{S} f, f \rangle \langle \digamma, f \rangle \\
&= \left(\langle \mathcal{S}' f, f \rangle + 2 \langle \mathcal{S} f, \mathcal{A} f \rangle\right) \left\|f\right\|^2 - 2 \left\|\mathcal{S} f - \tfrac{1}{2} \digamma\right\|^2 \left\|f\right\|^2 + \tfrac{1}{2} \left\|\digamma\right\|^2 \left\|f\right\|^2 \\
&\quad + 2 \left\langle \mathcal{S} f - \tfrac{1}{2} \digamma, f \right\rangle^2 - \tfrac{1}{2} \langle \digamma, f \rangle^2 \quad .
\end{aligned}$$

Thanks to Cauchy-Schwarz inequality, we obtain the following estimate for $\frac{d}{dt}\mathbf{N}\left(t\right)$:

$$\frac{d}{dt}\mathbf{N}\left(t\right) \left\|f\right\|^2 \leq \langle \mathcal{S}' f, f \rangle + 2 \langle \mathcal{S} f, \mathcal{A} f \rangle + \left\|\digamma\right\|^2 \quad . \tag{34}$$

Note that the computation above is similar to the one of Bardos-Tartar (cf. [BT]), in which however $\mathcal{A} = 0$. Note also that the vocabulary "logarithmic convexity method" is related to the fact that in the case when $\digamma = 0$, one has $\frac{d^2}{dt^2} \ln\left(\left\|f\right\|^2\right) = -2 \frac{d}{dt}\mathbf{N}$.

3.3. **Step 3: Carleman commutator estimates.** The next ingredient in the proof of Proposition 1.2 is the following:

Proposition 3.2. *Under the assumptions of Proposition 1.2 (and using the notations (21) – (28)), there exist $s_0 \in (0,1]$, $C_0 \in (0,1)$, $C_1 > 1$ depending only on K_0, $|x_0|$, d_1, d_2, such that when $s \in (0, s_0]$, $h \in (0,1]$,*

(i) $\quad \eta_i \leq 0 \quad$ *and* $\quad \langle \mathcal{S} f, f \rangle \geq 0,$

(ii) $\quad \left\| F \right\|^2 \leq C_1 \left(\left\| f \right\|^2 + \langle \mathcal{S} f, f \rangle \right) ,$

(iii) $\quad \langle \mathcal{S}' f, f \rangle + 2 \langle \mathcal{S} f, \mathcal{A} f \rangle \leq \dfrac{1 + C_0}{\Gamma} \langle \mathcal{S} f, f \rangle + \dfrac{C_1}{h^2} \left\| f \right\|^2 .$

We briefly explain here why we call the Proposition 3.2 above a "Carleman commutator estimate". Using the standard definition for a commutator $[A, B] = AB - BA$, one can check that the quantity $\langle [\mathcal{S}, -\partial_t + \mathcal{A}] f, f \rangle$, called Carleman commutator, satisfies the identity (for $f \in C^1([0,T]; C_0^\infty(\Omega))$): $\langle [\mathcal{S}, -\partial_t + \mathcal{A}] f, f \rangle = \langle \mathcal{S}' f, f \rangle + 2 \langle \mathcal{S} f, \mathcal{A} f \rangle$, and this last quantity is the one which is estimated in Proposition 3.2, (iii).

Proof of Proposition 3.2: We recall that

$$\left\{ \begin{array}{ll} \varphi_1(x) = \psi(x) - \psi(x_0) \ , & \text{for any } x \in \overline{\Omega} \ , \\ \varphi_3(x) = -\psi(x) - \psi(x_0) \ , & \text{for any } x \in \overline{\Omega} \ , \end{array} \right.$$

and start with the

Lemma 3.3. *We define $\vartheta := \{ \rho \leq |x| < R \} \subset \Omega = B(0,R)$, selecting $\rho > 0$ in such a way that $x_0 \notin \overline{\vartheta}$. Then there exists $c_1, c_2, c_3 > 0$ depending only on $|x_0|$ and ρ, such that:*

(i) *For any $x \in \Omega$,*

$$|\nabla \varphi_1(x)|^2 \leq c_1 |\varphi_1(x)| \ \text{ and } \ |\nabla \varphi_3(x)|^2 \leq c_1 |\varphi_3(x)| \ ;$$

(ii) *For any $x \in \vartheta$,*

$$|\varphi_1(x)| \leq c_2 |\nabla \varphi_1(x)|^2 \ \text{ and } \ |\varphi_3(x)| \leq c_2 |\nabla \varphi_3(x)|^2 \ ;$$

(iii) *For any $x \in \Omega \backslash \vartheta$,*

$$|\varphi_1(x)| \leq c_2 |\nabla \varphi_1(x)|^2 \ \text{ and } \varphi_3(x) - \varphi_1(x) \leq -c_3 \ .$$

Proof of Lemma 3.3: Thanks to estimate (20), and noticing that $|\nabla \psi(x)|^2 = |\nabla \varphi_1(x)|^2$ and $\psi(x_0) - \psi(x) = |\varphi_1(x)|$, we conclude that $(i) - (ii) - (iii)$ holds for φ_1. For φ_3, we get $(i) - (ii)$ because $|\varphi_3(x)| > 0$ for any $x \in \overline{\Omega}$ and $|\nabla \varphi_3(x)| > 0$ for any $x \in \overline{\vartheta}$. Finally, $\varphi_3(x) - \varphi_1(x) = -2\psi(x) < 0$ for any $x \in \Omega\backslash\vartheta$, which enables to complete the proof of Lemma 3.3.

We come back to the proof of Proposition 3.2. We observe that

$$\eta_i = \frac{1}{2}\partial_t \Phi_i + \frac{1}{4} d_i \left|\nabla \Phi_i\right|^2 = \frac{s}{\Gamma^2}\left(-\frac{1}{2}\left|\varphi_i\right| + \frac{1}{4} d_i s \left|\nabla\varphi_i\right|^2\right) \ ,$$

so that using Lemma 3.3 (i), there exists $s_1 \in (0,1]$ depending only on d_1, d_2, and $|x_0|$ such that when $s \in (0, s_1]$,

$$\eta_i(x,t) \leq 0 \quad \text{ for any } \ (x,t) \in \Omega \times [0,T] \ . \tag{35}$$

As a consequence, using the second part of identity (30), one can deduce that for any $s \in (0, s_1]$, the estimate $\langle \mathcal{S} f, f\rangle \geq 0$ holds.

Now, one can prove part $ii)$ of Proposition 3.2. Indeed, we see that

$$\begin{aligned}
\|F\|^2 &= \sum_{i=1,..,4} \int_\Omega |v_i|^2 e^{\Phi_i} \\
&\leq 2\int_\Omega (|v_1|^2 + |v_2|^2) e^{\Phi_1} \text{ using } \Phi_3 \leq \Phi_1,\ \Phi_2 = \Phi_1,\ \Phi_4 = \Phi_3,\ v_3 = v_1,\ v_4 = v_2 \\
&\leq 2K_0 \int_\Omega \left[|u_1|^2 + |u_2|^2 + (|u_1|^2 + |u_2|^2)^2\right] e^{\Phi_1} \text{ using (11)} \\
&\leq 4K_0 \left(\|f\|^2 + \int_\Omega |u_1|^4 e^{\Phi_1} + \int_\Omega |u_2|^4 e^{\Phi_2}\right) \text{ because } \Phi_1 = \Phi_2 \ .
\end{aligned}$$

But thanks to Hölder and Sobolev inequalities (and denoting by C_{Sob} the constant in this last inequality), for any $i \in \{1,2\}$,

$$\begin{aligned}
\int_\Omega |u_i|^4 e^{\Phi_i} &\leq \|u_i^2\|_{L^p(\Omega)} \left\|u_i^2 e^{\Phi_i}\right\|_{L^q(\Omega)} \text{ whenever } \tfrac{1}{p} + \tfrac{1}{q} = 1 \\
&\leq \left(\int_\Omega |u_i|^{2p}\right)^{\frac{1}{p}} \left(\int_\Omega |f_i|^{2q}\right)^{\frac{1}{q}} \text{ with } p = \tfrac{3}{2} \text{ and } q = 3 \\
&\leq K_0\, C_{Sob} \left(\int_\Omega |f_i|^2 + \int_\Omega |\nabla f_i|^2\right) \ ,
\end{aligned}$$

using assumption (12) in the last inequality, and remembering that $H^1(\Omega) \subset L^6(\Omega)$ for n$\leq$ 3. Since $\eta_i \leq 0$ (see (35)) and therefore $d_1 \int_\Omega |\nabla f_1|^2 + d_2 \int_\Omega |\nabla f_2|^2 \leq \langle \mathcal{S} f, f\rangle$ by the second part of identity (30), this gives the desired estimate when

$$C_1 \geq 4K_0 \max\left(1 + K_0 C_{Sob}, (\min_i d_i)^{-1} K_0 C_{Sob}\right) \ .$$

It remains to prove part *iii*) of Proposition 3.2.

We recall that

$$\langle \mathcal{S}' f, f\rangle := \sum_{i=1,..,4} \int_\Omega (-\partial_t \eta_i)\, |f_i|^2 \ .$$

Moreover, using the definition of $\mathcal{S}f$ and $\mathcal{A}f$, the bracket $2\langle \mathcal{S}f, \mathcal{A}f\rangle$ writes

$$2\langle \mathcal{S}f, \mathcal{A}f\rangle = 2\sum_{i=1,..,4} \int_\Omega (d_i \Delta f_i + \eta_i f_i)\left(d_i \nabla\Phi_i \cdot \nabla f_i + \frac{1}{2} d_i \Delta\Phi_i f_i \right) \ .$$

Then, four integrations by parts give

$$\begin{aligned} 2\langle \mathcal{S}f, \mathcal{A}f\rangle \ &= \sum_{i=1,..,4} \left(-2d_i^2 \int_\Omega \nabla f_i \nabla^2 \Phi_i \nabla f_i - d_i^2 \int_\Omega \nabla f_i \Delta \nabla \Phi_i f_i - d_i \int_\Omega \nabla \eta_i \cdot \nabla \Phi_i \, |f_i|^2 \right) \\ &\quad + 2\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n f_i \nabla\Phi_i \cdot \nabla f_i - \sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n \Phi_i \, |\nabla f_i|^2 \\ &\quad + \sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n f_i \Delta \Phi_i f_i + \sum_{i=1,..,4} d_i \int_{\partial\Omega} \eta_i \partial_n \Phi_i \, |f_i|^2 \ . \end{aligned}$$

Indeed,

$$\int_\Omega \Delta f_i \nabla \Phi_i \cdot \nabla f_i = \int_{\partial\Omega} \partial_n f_i \nabla\Phi_i \cdot \nabla f_i - \int_\Omega \nabla f_i \nabla^2 \Phi_i \nabla f_i - \int_\Omega \nabla f_i \nabla^2 f_i \nabla \Phi_i \ ,$$

and

$$\int_\Omega \nabla f_i \nabla^2 f_i \nabla\Phi_i = \frac{1}{2}\int_{\partial\Omega} \partial_n \Phi_i \, |\nabla f_i|^2 - \frac{1}{2}\int_\Omega \Delta\Phi_i \, |\nabla f_i|^2 \ .$$

Second,

$$\int_\Omega \Delta f_i \Delta \Phi_i f_i = \int_{\partial\Omega} \partial_n f_i \Delta\Phi_i f_i - \int_\Omega \nabla f_i \Delta \nabla \Phi_i f_i - \int_\Omega \Delta\Phi_i \, |\nabla f_i|^2 \ .$$

Third,

$$2\int_\Omega \eta_i f_i \nabla\Phi_i \cdot \nabla f_i = \int_{\partial\Omega} \eta_i \partial_n \Phi_i \, |f_i|^2 - \int_\Omega \nabla\eta_i \cdot \nabla\Phi_i \, |f_i|^2 - \int_\Omega \eta_i \Delta\Phi_i \, |f_i|^2 \ .$$

Then, we see that

$$\begin{aligned}\langle \mathcal{S}' f, f\rangle + 2\langle \mathcal{S} f, \mathcal{A} f\rangle &= \sum_{i=1,..,4}\left(-2d_i^2\int_\Omega \nabla f_i \nabla^2\Phi_i \nabla f_i - d_i^2\int_\Omega \nabla f_i \Delta\nabla\Phi_i f_i\right)\\ &\quad+\sum_{i=1,..,4}\int_\Omega\left(-\partial_t\eta_i - d_i\nabla\eta_i\cdot\nabla\Phi_i\right)\left|f_i\right|^2\\ &\quad+2\sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n f_i\nabla\Phi_i\cdot\nabla f_i - \sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n\Phi_i\left|\nabla f_i\right|^2\\ &\quad+\sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n f_i\Delta\Phi_i f_i + \sum_{i=1,..,4} d_i\int_{\partial\Omega}\eta_i\partial_n\Phi_i\left|f_i\right|^2 \ .\end{aligned}$$

Next, the computation of $\partial_t\eta_i + d_i\nabla\eta_i\cdot\nabla\Phi_i$ gives

$$\begin{aligned}\partial_t\eta_i + d_i\nabla\eta_i\cdot\nabla\Phi_i &= \frac{1}{2}\partial_t^2\Phi_i + d_i\nabla\partial_t\Phi_i\cdot\nabla\Phi_i + \frac{1}{2}d_i^2\nabla\Phi_i\nabla^2\Phi_i\nabla\Phi_i\\ &= \frac{1}{\Gamma}\partial_t\Phi_i + \frac{1}{\Gamma}d_i\left|\nabla\Phi_i\right|^2 + \frac{s}{2\Gamma}d_i^2\nabla\Phi_i\nabla^2\varphi_i\nabla\Phi_i\\ &= \frac{2}{\Gamma}\left(\frac{1}{2}\partial_t\Phi_i + \frac{1}{4}d_i\left|\nabla\Phi_i\right|^2\right) + \frac{1}{2\Gamma}d_i\left|\nabla\Phi_i\right|^2 + \frac{s}{2\Gamma}d_i^2\nabla\Phi_i\nabla^2\varphi_i\nabla\Phi_i,\end{aligned}$$

since $\partial_t^2\Phi_i = \frac{2}{\Gamma}\partial_t\Phi_i$ and $\partial_t\Phi_i = \frac{1}{\Gamma}\Phi_i$. Therefore, by definition (27) of η_i,

$$-\partial_t\eta_i - d_i\nabla\eta_i\cdot\nabla\Phi_i = -\frac{2}{\Gamma}\eta_i - \frac{1}{2\Gamma}d_i\left|\nabla\Phi_i\right|^2 - \frac{s}{2\Gamma}d_i^2\nabla\Phi_i\nabla^2\varphi_i\nabla\Phi_i\,,$$

and one can conclude that

$$\begin{aligned}&\langle \mathcal{S}' f, f\rangle + 2\langle \mathcal{S} f, \mathcal{A} f\rangle\\ &= \sum_{i=1,..,4}\left(-2d_i^2\int_\Omega \nabla f_i \nabla^2\Phi_i \nabla f_i - d_i^2\int_\Omega \nabla f_i \Delta\nabla\Phi_i f_i\right)\\ &\quad+\frac{1}{\Gamma}\sum_{i=1,..,4}\int_\Omega\left(-2\eta_i - \frac{1}{2}d_i\left|\nabla\Phi_i\right|^2 - \frac{s}{2}d_i^2\nabla\Phi_i\nabla^2\varphi_i\nabla\Phi_i\right)\left|f_i\right|^2\\ &\quad+2\sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n f_i\nabla\Phi_i\cdot\nabla f_i - \sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n\Phi_i\left|\nabla f_i\right|^2\\ &\quad+\sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n f_i\Delta\Phi_i f_i + \sum_{i=1,..,4} d_i\int_{\partial\Omega}\eta_i\partial_n\Phi_i\left|f_i\right|^2 \ .\end{aligned} \tag{36}$$

First we estimate the contribution of the gradient terms. We observe that (since ψ is smooth on $\overline{\Omega}$ (see (19)),

$$\left|d_i\nabla^2\Phi_i\right| + |d_i\Delta\nabla\Phi_i| \leq \frac{C_2\, s}{\Gamma}\ ,$$

where C_2 only depends on d_1, d_2, and $|x_0|$.

As a consequence, using Young's inequality,

$$\begin{aligned} &\sum_{i=1,..,4}\left(-2d_i^2\int_\Omega \nabla f_i\nabla^2\Phi_i\nabla f_i - d_i^2\int_\Omega \nabla f_i\Delta\nabla\Phi_i f_i\right)\\ &\leq \frac{C_3 s}{\Gamma}\sum_{i=1,..,4} d_i\int_\Omega |\nabla f_i|^2 + \frac{C_3 s}{\Gamma}\|f\|^2 \leq \frac{C_3 s}{\Gamma}\sum_{i=1,..,4} d_i\int_\Omega |\nabla f_i|^2 + \frac{C_3}{h}\|f\|^2\ , \end{aligned} \tag{37}$$

where we used in the last line the inequalities $\frac{1}{\Gamma} \leq \frac{1}{h}$ and $s \in (0,1]$. Here, C_3 only depends on d_1, d_2, and $|x_0|$.

Lemma 3.4. *There exists a constant $C_4 > 0$ depending only on $|x_0|$ and d_1, d_2 such that for any $s \in (0,1]$,*

$$\begin{aligned} &2\sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n f_i\nabla\Phi_i\cdot\nabla f_i - \sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n\Phi_i\,|\nabla f_i|^2\\ &+\sum_{i=1,..,4} d_i^2\int_{\partial\Omega}\partial_n f_i\Delta\Phi_i f_i + \sum_{i=1,..,4} d_i\int_{\partial\Omega}\eta_i\partial_n\Phi_i\,|f_i|^2\\ &\leq \frac{C_4 s}{\Gamma}\sum_{i=1,..,4} d_i\int_\Omega |\nabla f_i|^2 + \frac{C_4 s}{\Gamma}\sum_{i=1,2} d_i\int_\Omega |\nabla\Phi_i|^2\,|f_i|^2 + \frac{C_4}{h^2}\|f\|^2\ . \end{aligned}$$

Proof of Lemma 3.4: We claim that $\displaystyle\sum_{i=1,..,4} d_i\int_{\partial\Omega}\eta_i\partial_n\Phi_i\,|f_i|^2 = 0.$

We first observe that thanks to (24), the following extra identities hold:

$$\partial_t\Phi_1 = \partial_t\Phi_3\ , \qquad |\nabla\Phi_1| = |\nabla\Phi_3| \quad \text{on } \partial\Omega\times(0,T)\ . \tag{38}$$

Then, since $\eta_i = \frac{1}{2}\partial_t\Phi_i + \frac{1}{4}d_i\left|\nabla\Phi_i\right|^2$,

$$\begin{aligned}\sum_{i=1,..,4} d_i \int_{\partial\Omega} \eta_i \partial_n \Phi_i \left|f_i\right|^2 &= \sum_{i=1,..,4} d_i \int_{\partial\Omega} \left(\frac{1}{2}\partial_t\Phi_i + \frac{1}{4}d_i\left|\nabla\Phi_i\right|^2\right)\partial_n\Phi_i\left|u_i\right|^2 e^{\Phi_i} \\ &= d_1 \int_{\partial\Omega} \left(\frac{1}{2}\partial_t\Phi_1 + \frac{1}{4}d_1\left|\nabla\Phi_1\right|^2\right)\partial_n\Phi_1\left|u_1\right|^2 e^{\Phi_1} \\ &\quad + d_2 \int_{\partial\Omega} \left(\frac{1}{2}\partial_t\Phi_1 + \frac{1}{4}d_2\left|\nabla\Phi_1\right|^2\right)\partial_n\Phi_1\left|u_2\right|^2 e^{\Phi_1} \\ &\quad + d_1 \int_{\partial\Omega} \left(\frac{1}{2}\partial_t\Phi_3 + \frac{1}{4}d_1\left|\nabla\Phi_3\right|^2\right)\partial_n\Phi_3\left|u_1\right|^2 e^{\Phi_3} \\ &\quad + d_2 \int_{\partial\Omega} \left(\frac{1}{2}\partial_t\Phi_3 + \frac{1}{4}d_2\left|\nabla\Phi_3\right|^2\right)\partial_n\Phi_3\left|u_2\right|^2 e^{\Phi_3} ,\end{aligned}$$

where in the second line, we used identities (22), (23), (25). This completes the claim thanks to identities (24), (38).

We then observe that

$$2\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n f_i \nabla\Phi_i \cdot \nabla f_i - \sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n\Phi_i \left|\nabla f_i\right|^2 = 0 . \tag{39}$$

Indeed, since $\nabla\Phi_i = \partial_n\Phi_i \overrightarrow{n}$ on $\partial\Omega\times(0,T)$, we see first that

$$\begin{aligned} 2\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n f_i \nabla\Phi_i \cdot \nabla f_i &= 2\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n\Phi_i \left|\partial_n f_i\right|^2 \\ &= 2\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n\Phi_i \left|\frac{1}{2}\partial_n\Phi_i f_i\right|^2 \text{ because } \partial_n f_i = \frac{1}{2}\partial_n\Phi_i f_i \\ &= 0 ,\end{aligned}$$

thanks to identities (22), (23), (25) and (24).

We then observe that on $\partial\Omega\times(0,T)$,

$$\begin{aligned}\left|\nabla f_i\right|^2 &= \left|\nabla u_i e^{\Phi_i/2} + u_i \tfrac{1}{2}\nabla\Phi_i e^{\Phi_i/2}\right|^2 \\ &= \left|\partial_\tau u_i \overrightarrow{\tau} + u_i\tfrac{1}{2}\partial_n\Phi_i\overrightarrow{n}\right|^2 e^{\Phi_i} \text{ because } \partial_n u_i = 0 \text{ and } \partial_\tau\Phi_i\left|_{\partial\Omega}\right. = 0 \\ &= \left(\left|\partial_\tau u_i\right|^2 + \left|\tfrac{1}{2}u_i\partial_n\Phi_i\right|^2\right)e^{\Phi_i} .\end{aligned}$$

As a consequence, $-\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n\Phi_i \left|\nabla f_i\right|^2 = 0$, where we used identities (22), (23), (24), (25).

We complete in this way the proof of identity (39).

Next, it remains to treat the contribution of $\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n f_i \Delta\Phi_i f_i$. We introduce $C_5 := \max_{\overline{\Omega}} |\Delta\psi|$. Note that C_5 only depends on $|x_0|$.

$$\begin{aligned}
\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n f_i \Delta\Phi_i f_i &= \sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \frac{1}{2}\partial_n \Phi_i \Delta\Phi_i \left|f_i\right|^2 \text{ because } \partial_n f_i = \frac{1}{2}\partial_n \Phi_i f_i \\
&\leq \frac{C_5 s}{2\Gamma} \sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \left|\partial_n \Phi_i\right| \left|f_i\right|^2 \text{ because } \left|\Delta\Phi_i\right| = \frac{s}{\Gamma}\left|\Delta\varphi_i\right| \leq \frac{C_5 s}{\Gamma} \\
&\leq \frac{C_5 s}{\Gamma} \sum_{i=1,2} d_i^2 \int_{\partial\Omega} \left|\partial_n \Phi_i\right| \left|f_i\right|^2 \\
&= \frac{C_5 s}{\Gamma} \sum_{i=1,2} d_i^2 \int_{\partial\Omega} \left(-\partial_n \Phi_i\right) \left|f_i\right|^2 \text{ because } \partial_n \Phi_1 \leq 0 \text{ and } \Phi_2 = \Phi_1 \ .
\end{aligned}$$

In the third line we used $d_3^2 \int_{\partial\Omega} \left|\partial_n \Phi_3\right| \left|f_3\right|^2 = d_1^2 \int_{\partial\Omega} \left|\partial_n \Phi_1\right| \left|f_1\right|^2$, which holds thanks to identities (25), (24), and $f_3 = f_1$ on $\partial\Omega \times (0,T)$. Similar computations hold for $d_4^2 \int_{\partial\Omega} \left|\partial_n \Phi_4\right| \left|f_4\right|^2$.

Then, thanks to an integration by parts,

$$\begin{aligned}
\int_{\partial\Omega} \left(-\partial_n \Phi_i\right) \left|f_i\right|^2 &= -2 \int_\Omega \nabla f_i \cdot \nabla\Phi_i f_i - \int_\Omega \Delta\Phi_i \left|f_i\right|^2 \\
&\leq \int_\Omega \left|\nabla f_i\right|^2 + \int_\Omega \left|\nabla\Phi_i\right|^2 \left|f_i\right|^2 + \frac{C_5 s}{h} \left\|f\right\|^2 \ ,
\end{aligned}$$

using Young's inequality and the estimate $\left|\Delta\Phi_i\right| = \frac{s}{\Gamma}\left|\Delta\varphi_i\right| \leq \frac{C_5 s}{h}$. Therefore, one can conclude that for any $s \in (0,1]$,

$$\begin{aligned}
\sum_{i=1,..,4} d_i^2 \int_{\partial\Omega} \partial_n f_i \Delta\Phi_i f_i &\leq \max_i d_i \frac{C_5\, s}{\Gamma} \sum_{i=1,..,4} d_i \int_\Omega \left|\nabla f_i\right|^2 \\
&\quad + \max_i d_i \frac{C_5\, s}{\Gamma} \sum_{i=1,2} d_i \int_\Omega \left|\nabla\Phi_i\right|^2 \left|f_i\right|^2 + (\max_i d_i)^2 \frac{C_5^2}{h^2} \left\|f\right\|^2 \ .
\end{aligned}$$

This completes the proof of Lemma 3.4.

Finally, we estimate the contribution of

$$\frac{1}{\Gamma} \int_\Omega \left(-2\eta_i - \frac{1}{2} d_i \left|\nabla\Phi_i\right|^2 - \frac{s}{2} d_i^2 \nabla\Phi_i \nabla^2\varphi_i \nabla\Phi_i \right) \left|f_i\right|^2 + \frac{C_4 s}{\Gamma} d_i \int_\Omega \left|\nabla\Phi_i\right|^2 \left|f_i\right|^2 \ ,$$

where C_4 is the constant appearing in Lemma 3.4.

Lemma 3.5. *There exists $s_2 \in (0,1]$ and $C_6 > 0$, both depending only on $|x_0|$ and d_1, d_2 such that when $s \in (0, s_2]$,*

$$\frac{1}{\Gamma} \sum_{i=1,..,4} \int_\Omega \left(-2\eta_i - \frac{1}{2} d_i \left| \nabla \Phi_i \right|^2 - \frac{s}{2} d_i^2 \nabla \Phi_i \nabla^2 \varphi_i \nabla \Phi_i \right) \left| f_i \right|^2 + \frac{C_4 s}{\Gamma} \sum_{i=1,2} d_i \int_\Omega \left| \nabla \Phi_i \right|^2 \left| f_i \right|^2$$
$$\leq \frac{1}{\Gamma} \sum_{i=1,..,4} \left(2 - \frac{1}{4} d_i \frac{s}{c_2} \right) \int_\Omega \left(-\eta_i \right) \left| f_i \right|^2 + \frac{C_6}{h} \left\| f \right\|^2 \ .$$

Proof of Lemma 3.5: First observe that

$$\begin{aligned} &\left(-2\eta_i - \frac{1}{2} d_i \left| \nabla \Phi_i \right|^2 - \frac{s}{2} d_i^2 \nabla \Phi_i \nabla^2 \varphi_i \nabla \Phi_i \right) + C_4 s d_i \left| \nabla \Phi_i \right|^2 \\ &\leq -2\eta_i + \left(-\frac{1}{2} + s \left(\frac{1}{2} \max_i d_i \max_{\overline{\Omega}} \left| \nabla^2 \psi \right| + C_4 \right) \right) d_i \left| \nabla \Phi_i \right|^2 \\ &\leq -\left(2\eta_i + \frac{1}{8} d_i \left| \nabla \Phi_i \right|^2 \right) \end{aligned}$$

for any $s \in (0, s_2]$ if $s_2 > 0$ is well chosen. Indeed, it is sufficient to take $s_2 := \frac{3}{8} (\frac{1}{2} \max_i d_i \max_{\overline{\Omega}} |\nabla^2 \psi| + C_4)^{-1}$.

Now, from

$$\sum_{i=1,..,4} \int_\Omega \left(-2\eta_i - \frac{1}{2} d_i \left| \nabla \Phi_i \right|^2 - \frac{s}{2} d_i^2 \nabla \Phi_i \nabla^2 \varphi_i \nabla \Phi_i \right) \left| f_i \right|^2 + C_4 \, s \sum_{i=1,2} d_i \int_\Omega \left| \nabla \Phi_i \right|^2 \left| f_i \right|^2$$
$$\leq \sum_{i=1,..,4} \left(2 \int_\Omega \left(-\eta_i \right) \left| f_i \right|^2 - \frac{1}{8} d_i \int_\Omega \left| \nabla \Phi_i \right|^2 \left| f_i \right|^2 \right) \ ,$$

we want to achieve

$$\sum_{i=1,..,4} \int_\Omega \left(-2\eta_i - \frac{1}{2} d_i \left| \nabla \Phi_i \right|^2 - \frac{s}{2} d_i^2 \nabla \Phi_i \nabla^2 \varphi_i \nabla \Phi_i \right) \left| f_i \right|^2 + C_4 \ s \sum_{i=1,2} d_i \int_\Omega \left| \nabla \Phi_i \right|^2 \left| f_i \right|^2$$
$$\leq \sum_{i=1,..,4} \left(2 - \frac{1}{4} d_i \frac{s}{c_2} \right) \int_\Omega \left(-\eta_i \right) \left| f_i \right|^2 + C_6 \left\| f \right\|^2 \ .$$

We will treat separately the case $i = 1$ (which is similar to the case $i = 2$ since $\Phi_2 = \Phi_1$) and the case $i = 3$ (which is similar to the case $i = 4$ since $\Phi_4 = \Phi_3$).

Recall that

$$\eta_i = \frac{s}{\Gamma^2} \left(-\frac{1}{2} \left| \varphi_i \right| + \frac{1}{4} d_i s \left| \nabla \varphi_i \right|^2 \right) \ .$$

Thanks to Lemma 3.3 $(ii)-(iii)$, for any $x\in\Omega$, $|\varphi_1(x)|\le c_2|\nabla\varphi_1(x)|^2$. This implies

$$-|\nabla\Phi_1|^2=-\frac{s^2}{\Gamma^2}|\nabla\varphi_1|^2\le-\frac{s^2}{c_2\Gamma^2}|\varphi_1|=\frac{2s}{c_2}\left(-\frac{s}{2\Gamma^2}|\varphi_1|\right)\le\frac{2s}{c_2}\eta_1\ .$$

Therefore,

$$-\frac{1}{8}d_1\int_\Omega|\nabla\Phi_1|^2|f_1|^2\le\frac{1}{4}d_1\frac{s}{c_2}\int_\Omega\eta_1|f_1|^2$$

and similarly for $i=2$, one has $-\frac{1}{8}d_2\int_\Omega|\nabla\Phi_2|^2|f_2|^2\le\frac{1}{4}d_2\frac{s}{c_2}\int_\Omega\eta_2|f_2|^2$. Consequently,

$$\sum_{i=1,2}\left(-\frac{1}{8}d_i\int_\Omega|\nabla\Phi_i|^2|f_i|^2\right)\le\sum_{i=1,2}\frac{1}{4}d_i\frac{s}{c_2}\int_\Omega\eta_i|f_i|^2=\sum_{i=1,2}\left(-\frac{1}{4}d_i\frac{s}{c_2}\right)\int_\Omega(-\eta_i)|f_i|^2\,.$$

Thanks again to Lemma 3.3 $(ii)-(iii)$, the properties of φ_3 require to treat separately the cases when $x\in\vartheta$ and $x\in\Omega\backslash\vartheta$, where $\vartheta=\{\rho\le|x|<R\}$, with $\rho>0$ such that $x_0\notin\overline{\vartheta}$. We take here $\rho=\frac{|x_0|+R}{2}$. We first observe that when $x\in\vartheta$,

$$-|\nabla\Phi_3(x,\cdot)|^2=-\frac{s^2}{\Gamma^2}|\nabla\varphi_3(x)|^2\le-\frac{s^2}{c_2\Gamma^2}|\varphi_3(x)|\le\frac{2s}{c_2}\eta_3(x)\ . \tag{40}$$

We see that

$$\begin{aligned}\sum_{i=3,4}\left(-\frac{1}{8}d_i\int_\Omega|\nabla\Phi_i|^2|f_i|^2\right)&\le\sum_{i=3,4}\left(-\frac{1}{8}d_i\int_\vartheta|\nabla\Phi_i|^2|f_i|^2\right)\text{ since }\vartheta\subset\Omega\\&\le\sum_{i=3,4}\frac{1}{4}d_i\frac{s}{c_2}\int_\vartheta\eta_i|f_i|^2\text{ thanks to estimate (40)}\\&=\sum_{i=3,4}\frac{1}{4}d_i\frac{s}{c_2}\int_\Omega\eta_i|f_i|^2-\sum_{i=3,4}\frac{1}{4}d_i\frac{s}{c_2}\int_{\Omega\backslash\vartheta}\eta_i|f_i|^2\\&\le\sum_{i=3,4}\left(-\frac{1}{4}d_i\frac{s}{c_2}\right)\int_\Omega(-\eta_i)|f_i|^2+C_6\sum_{i=1,2}\int_\Omega|f_i|^2\end{aligned}$$

where in the last line, we defined

$$C_6:=\max_i d_i\frac{C_7}{c_2\,c_3^2}\ ,\qquad C_7:=\max_{\overline{\Omega}}|\psi|+\max_i d_i\max_{\overline{\Omega}}|\nabla\psi|^2\ ,$$

and noticed that

$$\begin{aligned}
-\frac{1}{4}d_3\frac{s}{c_2}\int_{\Omega\backslash\vartheta}\eta_3\left|f_3\right|^2 &\leq \frac{1}{4}d_3\frac{s}{c_2}\int_{\Omega\backslash\vartheta}\left(\frac{C_7 s}{\Gamma^2}\right)\left|u_3\right|^2 e^{s\frac{1}{\Gamma}\varphi_3} \text{ since } \left|\eta_3\right|\leq\frac{C_7 s}{\Gamma^2} \\
&= \frac{1}{4}d_3\frac{s}{c_2}\int_{\Omega\backslash\vartheta}\left(\frac{C_7 s}{\Gamma^2}\right)\left|u_3\right|^2 e^{s\frac{1}{\Gamma}\varphi_1}e^{s\frac{1}{\Gamma}(\varphi_3-\varphi_1)} \\
&\leq \frac{1}{4}d_3\frac{s}{c_2}\int_{\Omega\backslash\vartheta}\left(\frac{C_7 s}{\Gamma^2}\right)\left|u_3\right|^2 e^{s\frac{1}{\Gamma}\varphi_1}e^{-s\frac{1}{\Gamma}c_3} \text{ by Lemma 3.3}(iii) \\
&\leq C_6\int_{\Omega\backslash\vartheta}\left|u_3\right|^2 e^{s\frac{1}{\Gamma}\varphi_1} = C_6\int_{\Omega\backslash\vartheta}\left|u_1\right|^2 e^{s\frac{1}{\Gamma}\varphi_1} \text{ because } u_3=u_1 \\
&\leq C_6\int_\Omega\left|f_1\right|^2 .
\end{aligned}$$

We proceed similarly for $i=4$ and get $-\frac{1}{4}d_4\frac{s}{c_2}\int_{\Omega\backslash\vartheta}\eta_4\left|f_4\right|^2\leq C_6\int_\Omega\left|f_2\right|^2$. Finally, we see that

$$\sum_{i=1,..,4}\left(-\frac{1}{8}d_i\int_\Omega\left|\nabla\Phi_i\right|^2\left|f_i\right|^2\right)\leq\sum_{i=1,..,4}\left(-\frac{1}{4}d_i\frac{s}{c_2}\right)\int_\Omega(-\eta_i)\left|f_i\right|^2+C_6\sum_{i=1,2}\int_\Omega\left|f_i\right|^2 .$$

The fact that $\frac{1}{\Gamma}\leq\frac{1}{h}$ completes the proof of Lemma 3.5.

Consequently, by (36), (37), Lemma 3.4 and Lemma 3.5, for any $s\in(0,s_1]\cap(0,s_2]$ and any $h\in(0,1]$, we see that

$$\begin{aligned}
\langle\mathcal{S}'f,f\rangle+2\langle\mathcal{S}f,\mathcal{A}f\rangle &\leq (C_3+C_4)\frac{s}{\Gamma}\sum_{i=1,..,4}d_i\int_\Omega\left|\nabla f_i\right|^2+(C_3+C_4+C_6)\frac{1}{h^2}\left\|f\right\|^2 \\
&\quad+\frac{1}{\Gamma}\sum_{i=1,..,4}\left(2-\frac{d_i s}{4c_2}\right)\int_\Omega(-\eta_i)\left|f_i\right|^2 \\
&\leq (C_3+C_4)\frac{s}{\Gamma}\sum_{i=1,..,4}d_i\int_\Omega\left|\nabla f_i\right|^2+(C_3+C_4+C_6)\frac{1}{h^2}\left\|f\right\|^2 \\
&\quad+\left(2-\min_i d_i\frac{s}{4c_2}\right)\frac{1}{\Gamma}\sum_{i=1,..,4}\int_\Omega(-\eta_i)\left|f_i\right|^2 .
\end{aligned}$$

For $s\in(0,s_0]$, where $s_0:=\min(s_1,s_2,(C_3+C_4)^{-1},c_2\,(\min_i d_i)^{-1})$, we see that

$$\langle\mathcal{S}'f,f\rangle+2\langle\mathcal{S}f,\mathcal{A}f\rangle\leq\frac{1+C_0}{\Gamma}\langle\mathcal{S}f,f\rangle+\frac{C_1}{h^2}\left\|f\right\|^2 ,$$

with $C_0:=1-\min_i d_i\,\frac{s_0}{4c_2}\in(0,1)$ and $C_1\geq C_3+C_4+C_6$.

Taking

$$C_1 := \max(1, C_3 + C_4 + C_6, 4K_0\,(1 + K_0\,C_{Sob}), 4K_0^2\,C_{Sob}(\min_i d_i)^{-1})\ ,$$

we complete the proof of Proposition 3.2.

Let us summarize the inequalities for the energy $\|f\|^2$ and the frequency function $\mathbf{N}$, that we got so far: We first observe that thanks to (32), Young's inequality and Proposition 3.2(ii), when $h \in (0,1]$, we get (for all $s \in (0, s_0]$)

$$\left|\frac{1}{2}\frac{d}{dt}\|f\|^2 + \langle \mathcal{S}f, f\rangle\right| = |\langle F, f\rangle| \le \frac{1}{2C_1}\|F\|^2 + \frac{C_1}{2}\|f\|^2 \le \frac{1}{2}\langle \mathcal{S}f, f\rangle + \frac{C_1}{h}\|f\|^2\ . \tag{41}$$

Moreover, thanks to (34), Proposition 3.2(ii) and (iii), we also get (when $h \in (0,1]$ and for all $s \in (0, s_0]$) for some $C_0 \in (0,1)$:

$$\frac{d}{dt}\mathbf{N} \le \frac{\langle \mathcal{S}'f, f\rangle + 2\langle \mathcal{S}f, \mathcal{A}f\rangle + \|F\|^2}{\|f\|^2} \le \left(\frac{1+C_0}{\Gamma} + C_1\right)\mathbf{N} + \frac{2C_1}{h^2}\ . \tag{42}$$

From now on, we take $s := s_0$ given by Proposition 3.2, and recall that s_0 only depends on K_0, $|x_0|$ and d_1, d_2.

3.4. **Step 4: Use of a differential inequality.** We now state the following Lemma:

Lemma 3.6. *Let $h > 0$, $T > 0$ and $F_1, F_2 \in C([0,T])$. Consider two nonnegative functions $E, N \in C^1([0,T])$ such that (when $t \in [0,T)$):*

$$\begin{cases} \left|\frac{1}{2}\frac{d}{dt}E(t) + N(t)E(t)\right| \le \left(\frac{1}{2}N(t) + \frac{C_0}{T-t+h} + C_1\right)E(t) + F_1(t)E(t)\ , \\ \frac{d}{dt}N(t) \le \left(\frac{1+C_0}{T-t+h} + C_1\right)N(t) + F_2(t)\ , \end{cases}$$

where $C_0, C_1 \ge 0$.

Then for any $0 \le t_1 < t_2 < t_3 \le T$, the following estimate holds:

$$E(t_2)^{1+M} \le e^D\left(\frac{T-t_1+h}{T-t_3+h}\right)^{3C_0(1+M)} E(t_3)\,E(t_1)^M\ , \tag{43}$$

with

$$M := 3\frac{\displaystyle\int_{t_2}^{t_3} \frac{e^{tC_1}}{(T-t+h)^{1+C_0}}dt}{\displaystyle\int_{t_1}^{t_2} \frac{e^{tC_1}}{(T-t+h)^{1+C_0}}dt}$$

and

$$D := 3\left(1+M\right)\left[(t_3-t_1)\left(C_1+\int_{t_1}^{t_3}\left|F_2\right|dt\right)+\int_{t_1}^{t_3}\left|F_1\right|dt\right] \ .$$

The proof of Lemma 3.6 can be found in [BP].

We now come back to the proof of Proposition 1.2.

Consider $h \in (0,1]$ and $\ell > 1$ such that $\ell h <\min(1/2, T/4)$. Taking $t_3 := T$, $t_2 := T-\ell h$, and $t_1 := T-2\ell h$, in Lemma 3.6, estimate (43) becomes

$$E\left(T-\ell h\right)^{1+M_\ell} \leq e^{D_\ell}\left(2\ell+1\right)^{3C_0(1+M_\ell)}\ E\left(T\right)E\left(T-2\ell h\right)^{M_\ell} \ ,$$

where

$$D_\ell := 3\left(1+M_\ell\right)\left(C_1+\int_{T-2\ell h}^{T}\left(\left|F_1\right|+2\ell h\left|F_2\right|\right)dt\right) \ ,$$

and

$$M_\ell := 3\frac{\displaystyle\int_{T-\ell h}^{T} \frac{e^{tC_1}}{(T-t+h)^{1+C_0}}dt}{\displaystyle\int_{T-2\ell h}^{T-\ell h} \frac{e^{tC_1}}{(T-t+h)^{1+C_0}}dt} \ . \tag{44}$$

We now observe that thanks to inequalities (41), (42), the assumptions of Lemma 3.6 are fulfilled when $E\left(t\right) := \left\|f\left(\cdot,t\right)\right\|^2$, N is the frequency function given by (33), $h \in (0,1]$, $F_1(t) := \dfrac{C_1}{h}$, $F_2(t) := \dfrac{2C_1}{h^2}$ and C_0, C_1 are the constants appearing in Proposition 3.2.

Therefore, thanks to Lemma 3.6, for all $h \in (0,1]$ and $\ell > 1$ such that $\ell h <\min(1/2, T/4)$, the following estimate holds (with $s = s_0$):

$$\left(\left\|f\left(\cdot,T-\ell h\right)\right\|^2\right)^{1+M_\ell} \leq K_\ell\ \left(\left\|f\left(\cdot,T\right)\right\|^2\right)\left(\left\|f\left(\cdot,T-2\ell h\right)\right\|^2\right)^{M_\ell} \ , \tag{45}$$

where $K_\ell := e^{D_\ell}\left(2\ell+1\right)^{3C_0(1+M_\ell)}$, with $D_\ell = 3C_1\ \left(1+M_\ell\right)(1+2\ell+8\ell^2)$, and M_ℓ given by (44).

Note that (when $\ell h <\min(1/2, T/4)$),

$$
\begin{aligned}
M_\ell &\leq 3\frac{e^{C_1 T}}{e^{C_1(T-2\ell h)}} \frac{\displaystyle\int_{T-\ell h}^{T} \frac{1}{(T-t+h)^{1+C_0}} dt}{\displaystyle\int_{T-2\ell h}^{T-\ell h} \frac{1}{(T-t+h)^{1+C_0}} dt} \\
&\leq 3e^{2C_1 \ell h} \frac{[(\ell+1)h]^{-C_0} - h^{-C_0}}{[(2\ell+1)h]^{-C_0} - [(\ell+1)h]^{-C_0}} \\
&\leq 3e^{C_1} \frac{(\ell+1)^{C_0}}{1-\left(\frac{\ell+1}{2\ell+1}\right)^{C_0}} \ .
\end{aligned} \tag{46}
$$

3.5. **Step 5: Introducing $B(x_0, r)$.** Observe that

$$\|(f_1, f_2)\|^2_{(L^2(\Omega))^2} \leq \|f\|^2 \leq 2\,\|(f_1, f_2)\|^2_{(L^2(\Omega))^2} \ .$$

Indeed, the fact that $\varphi_3 \leq \varphi_1$ and $\frac{s_0}{\Gamma} > 0$, implies $\|(f_3, f_4)\|^2_{(L^2(\Omega))^2} \leq \|(f_1, f_2)\|^2_{(L^2(\Omega))^2}$ by (21) and (22), which gives the desired estimates. Therefore, thanks to (45),

$$
\begin{aligned}
&\left(\|(f_1, f_2)(\cdot, T-\ell h)\|^2_{(L^2(\Omega))^2}\right)^{1+M_\ell} \\
&\leq K_\ell \left(2\,\|(f_1, f_2)(\cdot, T)\|^2_{(L^2(\Omega))^2}\right)\left(2\,\|(f_1, f_2)(\cdot, T-2\ell h)\|^2_{(L^2(\Omega))^2}\right)^{M_\ell} \ .
\end{aligned} \tag{47}
$$

On the other hand, the fact that $\varphi_2 = \varphi_1$, $\varphi_1 \leq 0$ and $\frac{s_0}{\Gamma} > 0$, implies

$$\|(f_1, f_2)\|^2_{(L^2(\Omega))^2} \leq \|(u_1, u_2)\|^2_{(L^2(\Omega))^2} \ .$$

Combining the above with the non-increasing property (13) of the energy, one can first get

$$\|(f_1, f_2)(\cdot, T-2\ell h)\|^2_{(L^2(\Omega))^2} \leq \|(u_1, u_2)(\cdot, 0)\|^2_{(L^2(\Omega))^2} \ . \tag{48}$$

Second, we make $B(x_0, r)$ appear out of $\|(f_1, f_2)(\cdot, T)\|^2_{(L^2(\Omega))^2}$ as follows:

$$
\begin{aligned}
&\|(f_1, f_2)(\cdot, T)\|^2_{(L^2(\Omega))^2} \\
&= \int_{B(x_0,r)} |(u_1, u_2)(\cdot, T)|^2\, e^{\frac{s_0}{h}\varphi_1} + \int_{\Omega \backslash B(x_0,r)} |(u_1, u_2)(\cdot, T)|^2\, e^{\frac{s_0}{h}\varphi_1} \\
&\leq \|(u_1, u_2)(\cdot, T)\|^2_{(L^2(B(x_0,r)))^2} + e^{-\frac{s_0\mu_0}{h}}\, \|(u_1, u_2)(\cdot, 0)\|^2_{(L^2(\Omega))^2}
\end{aligned} \tag{49}
$$

thanks to estimate (13), $\varphi_2 = \varphi_1$, and the fact that on $\Omega \backslash B(x_0, r)$, $\varphi_1 \leq -\mu_0$ for some $\mu_0 > 0$ depending only on $|x_0|$ and r.

Third, thanks to estimate (13) and $\varphi_1 = \varphi_2$,

$$\begin{aligned} \left\| (u_1,u_2)(\cdot,T) \right\|^2_{(L^2(\Omega))^2} &\leq \int_\Omega \left| (u_1,u_2)(\cdot,T-\ell h) \right|^2 e^{\frac{s_0}{(\ell+1)h}\varphi_1} e^{-\frac{s_0}{(\ell+1)h}\varphi_1} \\ &\leq e^{\frac{s_0\mu_1}{(\ell+1)h}} \left\| (f_1,f_2)(\cdot,T-\ell h) \right\|^2_{(L^2(\Omega))^2} , \end{aligned} \tag{50}$$

where $\mu_1 := \sup_{x\in\overline{\Omega}}(-\varphi_1(x))$ only depends on $|x_0|$.

Combining estimates (47) to (50), we can deduce

$$\begin{aligned} &\left(\left\| (u_1,u_2)(\cdot,T) \right\|^2_{(L^2(\Omega))^2} \right)^{1+M_\ell} \\ &\leq e^{\frac{s_0\mu_1(1+M_\ell)}{(\ell+1)h}} \left(\left\| (f_1,f_2)(\cdot,T-\ell h) \right\|^2_{(L^2(\Omega))^2} \right)^{1+M_\ell} \\ &\leq K_\ell\, e^{\frac{s_0\mu_1(1+M_\ell)}{(\ell+1)h}} \left(2\left\| (f_1,f_2)(\cdot,T) \right\|^2_{(L^2(\Omega))^2} \right) \left(2\left\| (f_1,f_2)(\cdot,T-2\ell h) \right\|^2_{(L^2(\Omega))^2} \right)^{M_\ell} \\ &\leq 2K_\ell\, e^{\frac{s_0\mu_1(1+M_\ell)}{(\ell+1)h}} \left(2\left\| (u_1,u_2)(\cdot,0) \right\|^2_{(L^2(\Omega))^2} \right)^{M_\ell} \\ &\quad \times \left(\left\| (u_1,u_2)(\cdot,T) \right\|^2_{(L^2(B(x_0,r)))^2} + e^{-\frac{s_0\mu_0}{h}} \left\| (u_1,u_2)(\cdot,0) \right\|^2_{(L^2(\Omega))^2} \right) . \end{aligned} \tag{51}$$

This estimate is true for any $\ell > 1$ and any $h \in (0,1]$ satisfying $\ell h <\min(1/2, T/4)$. Recall that s_0 only depends on K_0, $|x_0|$ and d_1, d_2.

We now will choose $\ell > 1$ sufficiently large to fulfill the inequality $\frac{\mu_1(1+M_\ell)}{(\ell+1)} \leq \frac{\mu_0}{2}$, so that $\frac{s_0\mu_1(1+M_\ell)}{(\ell+1)h} - \frac{s_0\mu_0}{h} \leq -\frac{s_0\mu_0}{2h}$. This is possible since $\lim_{\ell\to+\infty} \frac{\mu_1(1+M_\ell)}{(\ell+1)} = 0$ because (see (46)) $M_\ell \leq 3e^{C_1} \frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$, with $C_0 \in (0,1)$. Note that chosen in this way, ℓ depends on $|x_0|$, r and d_1, d_2.

Thus, (51) becomes

$$\begin{aligned} &\left(\left\| (u_1,u_2)(\cdot,T) \right\|^2_{(L^2(\Omega))^2} \right)^{1+M_\ell} \\ &\leq 2K_\ell \left(2\left\| (u_1,u_2)(\cdot,0) \right\|^2_{(L^2(\Omega))^2} \right)^{M_\ell} \\ &\quad \times \left(e^{\frac{s_0\mu_0}{2h}} \left\| (u_1,u_2)(\cdot,T) \right\|^2_{(L^2(B(x_0,r)))^2} + e^{-\frac{s_0\mu_0}{2h}} \left\| (u_1,u_2)(\cdot,0) \right\|^2_{(L^2(\Omega))^2} \right) . \end{aligned} \tag{52}$$

This estimate is true for any $h \in (0,1]$ satisfying $\ell h <\min(1/2, T/4)$. Recall that $1/(2\ell) \leq 1$, therefore the interpolation inequality (52) holds for any $h > 0$ satisfying $h <\min(1/(2\ell), T/(4\ell))$.

Since now $\ell > 1$ is large but fixed, we denote $M := M_\ell$ and $K := K_\ell$. Then, for any $h > 0$ satisfying $h <\min(1/(2\ell), T/(4\ell))$, estimate (52) becomes

$$\begin{aligned} & \left(\|(u_1,u_2)(\cdot,T)\|^2_{(L^2(\Omega))^2}\right)^{1+M} \\ & \leq 2^{1+M} K \left(\|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right)^{M} \\ & \quad \times \left(e^{\frac{s_0\mu_0}{2h}} \|(u_1,u_2)(\cdot,T)\|^2_{(L^2(B(x_0,r)))^2} + e^{-\frac{s_0\mu_0}{2h}} \|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right) . \end{aligned} \tag{53}$$

On the other hand, for any $h > 0$ satisfying $h \geq \min(1/(2\ell), T/(4\ell))$, $1 = e^{-\frac{s_0\mu_0}{2h}} e^{\frac{s_0\mu_0}{2h}} \leq e^{-\frac{s_0\mu_0}{2h}} e^{\frac{s_0\mu_0}{2}\left(2\ell+\frac{4\ell}{T}\right)}$, which implies thanks to (13):

$$\begin{aligned} & \left(\|(u_1,u_2)(\cdot,T)\|^2_{(L^2(\Omega))^2}\right)^{1+M} \\ & \leq \left(\|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right)^{1+M} e^{-\frac{s_0\mu_0}{2h}} e^{\frac{s_0\mu_0}{2}\left(2\ell+\frac{4\ell}{T}\right)} \\ & \leq e^{\mu_2\left(1+\frac{1}{T}\right)} \left(\|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right)^{M} \\ & \quad \times \left(e^{\frac{s_0\mu_0}{2h}} \|(u_1,u_2)(\cdot,T)\|^2_{(L^2(B(x_0,r)))^2} + e^{-\frac{s_0\mu_0}{2h}} \|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right) , \end{aligned} \tag{54}$$

where $\mu_2 := 2s_0\mu_0\ell$ only depends on K_0, $|x_0|$, r and d_1, d_2.

Consequently, combining (53) and (54), one can get an estimate which holds for any $h > 0$: Precisely, there exists $\mu_3 := \mu_2 + 1 + M + K$ depending only on K_0, $|x_0|$, r and d_1, d_2, such that for any $h > 0$,

$$\begin{aligned} & \left(\|(u_1,u_2)(\cdot,T)\|^2_{(L^2(\Omega))^2}\right)^{1+M} \\ & \leq e^{\mu_3\left(1+\frac{1}{T}\right)} \left(\|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right)^{M} \\ & \quad \times \left(e^{\frac{s_0\mu_0}{2h}} \|(u_1,u_2)(\cdot,T)\|^2_{(L^2(B(x_0,r)))^2} + e^{-\frac{s_0\mu_0}{2h}} \|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right) . \end{aligned}$$

Now, one can minimize with respect to h in $(0, +\infty)$ or simply choose $h > 0$ such that

$$e^{-\frac{s_0\mu_0}{2h}} = \frac{\|(u_1,u_2)(\cdot,T)\|_{(L^2(B(x_0,r)))^2}}{\|(u_1,u_2)(\cdot,0)\|_{(L^2(\Omega))^2}} ,$$

in order to obtain the desired estimate under the form

$$\begin{aligned} \left(\|(u_1,u_2)(\cdot,T)\|^2_{(L^2(\Omega))^2}\right)^{1+(1+2M)} & \leq 4e^{2\mu_3\left(1+\frac{1}{T}\right)} \|(u_1,u_2)(\cdot,T)\|^2_{(L^2(B(x_0,r)))^2} \\ & \quad \times \left(\|(u_1,u_2)(\cdot,0)\|^2_{(L^2(\Omega))^2}\right)^{(1+2M)} . \end{aligned}$$

This concludes the proof of Proposition 1.2.

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