An optimal spectral inequality for degenerate operators

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Abstract

In this paper we establish a Lebeau-Robbiano spectral inequality for a degenerate one dimensional elliptic operator, with an optimal dependency with the frequency parameter. The proof relies on a combination of uniform local Carleman estimates away from the degeneracy and a moment method adapted to a degenerate elliptic operator. We also provide an application to the null controllability on a measurable set in time for the associated degenerate heat equation.

Contents

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1 Introduction and main results

1.1 Introduction and state of the art

The purpose of this article is to prove a spectral inequality for a family of degenerate operators acting on the unit interval $(0,1)$. In arbitrary dimension, for a second-order symmetric elliptic operator $\mathcal{P}$ on a smooth bounded domain Ω with homogeneous Dirichlet or Neumann boundary conditions, the spectral inequality also called Lebeau-Robbiano estimate takes the form

$$\|u\|_{L^2(\Omega)} \leqslant ce^{c\sqrt{\Lambda}}\|u\|_{L^2(\omega)}, \quad \forall u \in \operatorname{span}\{\Phi_j; \lambda_j \leqslant \Lambda\}, \tag{1.1}$$

where $\omega \subset \Omega$ is an open subset and where the functions Φ_j form a Hilbert basis of $L^2(\Omega)$ of eigenfunctions of $\mathcal{P}$ associated with the nonnegative eigenvalues λ_j, $j \in \mathbb{N}$, counted with their multiplicities. In other words, the family of spectral projectors associated with $\mathcal{P}$ enjoys an observability inequality on a set $\omega \subset \Omega$ for low frequencies $\lambda_j \leqslant \Lambda$ with a constant cost as $ce^{c\sqrt{\Lambda}}$.

The use of such spectral inequalities to obtain the null-controllability of the associated parabolic equations goes back to [LR, JL, LZ]. The authors of [Mi] (see also [BPS]) used a direct observability strategy to prove that spectral inequalities of the form (1.1) implies observability inequality

$$\|e^{-T\mathcal{P}}y_0\|^2_{L^2(\Omega)} \leqslant ce^{c/T}\int_0^T \|e^{-t\mathcal{P}}y_0\|^2_{L^2(\omega)}dt,$$

which in turn is equivalent to null-controllability property [Zu, FI, Mi, Mi2].

In dimension one in space, the moment method is a powerful method to prove null-controllability properties of parabolic semigroups. One may cite for instance the pioneering work of [FR], and recent developments [AKBGBdT, BBM, CMV5, ABCU] and the reference therein.

It appeared that inequalities of the form (1.1) are in fact equivalent to one-point-in-time observability estimates of the form

$$\|e^{-T\mathcal{P}}y_0\|_{L^2(\Omega)} \leqslant ce^{c/T}\|e^{-T\mathcal{P}}y_0\|^{1-\theta}_{L^2(\omega)}\|y_0\|^{\theta}_{L^2(\Omega)}, \quad \theta \in (0,1);$$

see for instance [AEWZ], [BaP] and [BP]. These estimates quantifies the backward uniqueness property of solutions of parabolic equations [BaT], and implies in particular observability on measurable sets in time, bang-bang property for norm and time optimal control [PW], impulse controllability [PWX], rapid and finite time stabilization [PWX, BP].

The main technique to prove (1.1) relies on the derivation of local elliptic Carleman estimates in the interior and at the boundary of the domain, and this field of research has been extensively investigated, see e.g [LR, LZ, JL, L, LRL, LRLeR1, LRLeR2, Le, LL, Q, FQZ]. Elliptic Carleman estimates were introduced by Carleman in [C], and further developed in [Ca, H].

Here we work in the context of the following family of one-dimensional degenerate operators

$$-\frac{d}{dx}\left(x^\alpha \frac{d}{dx}\right), \text{ acting on functions of the unit interval } (0,1) \text{ with } \alpha \in [0,2),$$

and with proper boundary condition described below. It is now well-known [CMV, CMV2, CMV6] that $\alpha \in [0,2)$ is necessary and sufficient for the null-controllability property to hold from arbitrary subsets. Besides Carleman estimates and moment method, one may cite other techniques that have been successfully used to control degenerate parabolic equations: backstepping techniques [GLM, LM] or flatness approach [Mo, BLR]. By using Carleman estimates, [CMV, CMV2, CTY, BP] proved null-controllability results, but failed to be optimal, for instance with respect to small time null-controllability cost. Moreover, the dependency of the control cost with respect to $\alpha \in [0,2)$ was not made explicit. The main difficulty relies on the construction of the Carleman weight function which is adapted to the degeneracy. The moment method appeared to be a very nice technique to overcome this difficulty in this context [CMV3, CMV4] and allowed to sharply quantify the blow up of the control cost when $T \to 0^+$ and $\alpha \to 2^-$. It relies on precise bounds of the spectral gap of the eigenvalues of the family of degenerate operators and on the construction of biorthogonal families with proper estimate on its L^2 norms, whose dependency on the uniform spectral gap constant is made explicit.

In the present work, we prove a Lebeau-Robbiano spectral inequality for the family of degenerate operators (see Theorem 1.1). The main novelty of this result is twofold; on the one hand, it is sharp with respect to the dependency on the frequency cut parameter Λ, when $\Lambda \to +\infty$. Here, sharpness is to be understood as the constant $ce^{c\sqrt{\Lambda}}$ coincide with the one in [LR, LZ] in the case $\alpha = 0$ (that is, the flat Laplace operator), which is known to be optimal [JL]. On the other hand, we obtain a bound of the blow up of the constants when $\alpha \to 2^-$ (in the spirit of [CMV3, CMV4] in the parabolic context). In Theorem 1.2, we deduce a L^1-observability for the associated parabolic equation on measurable subsets in time. In the last years, a lot of works have been devoted to the observability on measurable sets (see e.g. [AE], [EMZ], [PW], [WZ], [LiZ]). Theorem 1.2, as an application of Theorem 1.1, is thus an improvement in this direction of controllability results of [CMV3, CMV4]. However, it seems that our blow up bound of the control cost when $\alpha \to 2^-$ is weaker than [CMV4]. It could be interesting to improve Theorem 1.2 by obtaining the same observability cost given by [CMV4].

Our method relies on a combination of both the moment method and the use of Carleman estimates. We first adapt the moment method to an elliptic problem, posed in the finite dimensional subspaces of $L^2(0,1)$ spanned by the first eigenfunctions of the degenerate operators, to obtain Proposition 2.3. Note that Proposition 2.3 is very related to the spectral inequalities obtained in [LR, Le, BT]. We emphasize that this key step allows us to observe the entire domain (including a neighborhood of the region where ellipticity degenerates) from any arbitrary subset away from the degeneracy. Then, one can use use uniformly elliptic Carleman estimates and Hölder-type interpolation inequalities of Proposition 2.2. to propagate the observability and obtain the spectral inequality of Theorem 1.1. This Carleman estimate is made uniform with respect to the degeneracy parameter $\alpha \in [0,2)$. The main theorem follows from the combination of these two results.

The article is organized as follows. In the remainder of this section, we introduce the problem and the main results. In Section 2, we introduce two propositions corresponding to the strategy described above, and combine them to deduce the main result. Section 3 is devoted to proving Proposition 2.3 using the moment method, whereas Section 4 is devoted to obtain Proposition 2.2 by the use of Carleman estimates. We prove some useful spectral properties in Section 5. Finally, we show the second main theorem in the last section.

1.2 Functional setting and spectral properties

We shall consider the linear unbounded operators $\mathcal{P}$ in $L^2(0,1)$, defined by

$$\begin{cases} \mathcal{P} = -\dfrac{d}{dx}\left(x^\alpha \dfrac{d}{dx}\right), \text{ with } \alpha \in [0,2), \\ D(\mathcal{P}) = \{\vartheta \in H^1_{\alpha,0}(0,1)\,;\, \mathcal{P}\vartheta \in L^2(0,1) \text{ and } \mathrm{BC}_\alpha(\vartheta) = 0\}, \end{cases}$$

where

$$H^1_{\alpha,0}(0,1) := \left\{\vartheta \in L^2(0,1);\ \vartheta \text{ is loc. absolutely continuous in } (0,1]\,,\ \int_0^1 x^\alpha |\vartheta'|^2 < \infty,\ \vartheta(1) = 0\right\},$$

and

$$\mathrm{BC}_\alpha(\vartheta) = \begin{cases} \vartheta_{|x=0}, & \text{for } \alpha \in [0,1), \\ (x^\alpha \vartheta')_{|x=0}, & \text{for } \alpha \in [1,2). \end{cases}$$

For any $\alpha \in [0,2)$, the operator $\mathcal{P}$ is a closed self-adjoint positive densely defined operator, with compact resolvent. As a consequence, for any $\alpha \in [0,2)$, there exists a countable family of eigenfunctions $\{\Phi_j\}_{j\geqslant 1}$, forming a Hilbert basis of $L^2(0,1)$, associated with a positive and increasing sequence of eigenvalues $\{\lambda_j\}_{j\geqslant 1}$ going to infinity, satisfying $\mathcal{P}\Phi_j = \lambda_j \Phi_j$, $j \geqslant 1$. We emphasize that the spectral eigenpairs depends on α, but we omit it to ease the notation. An explicit expression of the eigenvalues is given in [Gu] for the weakly degenerate case $\alpha \in (0,1)$, and in [Mo] for the strongly degenerate case $\alpha \in [1,2)$, and depends on the zeros of the Bessel functions of first kind (see [MM]). The eigenfunctions Φ_j also depends on the Bessel functions and are described in [Gu, CMV4]. Note that the eigenvalues are simple and more properties are emphasized by Cannarsa, Martinez and Vancostenoble: first, a uniform bound for the first eigenvalue

$$\exists c_1, c_2 > 0 \quad \forall \alpha \in [0,2) \quad c_1 \leqslant \lambda_1 \leqslant c_2 \tag{1.2}$$

(see [CMV3] (10) at page 176 and (34) at page 183 for $\alpha \in [0,1)$; see [CMV4] proposition 2.13 at page 10 and (3.8)-(3.9) at page 13 for $\alpha \in [1,2)$); secondly, a uniform spectral gap

$$\exists \gamma > 0 \quad \forall \alpha \in [0,2) \quad \forall k \geqslant 1 \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geqslant \gamma(2-\alpha) \tag{1.3}$$

(see [CMV3] (74) at page 198 for $\alpha \in [0,1)$; see [CMV4] at page 30 for $\alpha \in [1,2)$).

1.3 Main results

We are interested in the spectral inequality for the sum of eigenfunctions. Such Lebeau-Robbiano estimate is done with explicit dependence on $\alpha \in [0,2)$. Our main result is as follows.

Theorem 1.1. *Let ω be an open and nonempty subset of $(0,1)$. There exists a constant $C > 0$ such that*

$$\sum_{\lambda_j \leqslant \Lambda} |a_j|^2 \leqslant C e^{C \frac{1}{(2-\alpha)^2} \sqrt{\Lambda}} \int_\omega \left| \sum_{\lambda_j \leqslant \Lambda} a_j \Phi_j \right|^2 dx$$

for any $\alpha \in [0,2)$, $\{a_j\} \in \mathbb{R}$ and any $\Lambda > 0$.

We emphasize that the constant $C > 0$ appearing in Theorem 1.1 is independent on $\alpha \in [0,2)$. Moreover, the blow-up rate when Λ goes to infinity is sharpened comparing with [BP]. This is equivalent to the existence of $C > 0$ such that

$$\sum_{j=1,\cdot\cdot,N} |a_j|^2 \leqslant C e^{C \frac{1}{(2-\alpha)^2} \sqrt{\lambda_N}} \int_\omega \left| \sum_{j=1,\cdot\cdot,N} a_j \Phi_j \right|^2 dx$$

for any $\alpha \in [0,2)$, $\{a_j\} \in \mathbb{R}$ and any $N > 0$.

Here, our approach is based on a combination of both Carleman techniques and the moment method for an elliptic equation. In one hand, it seems difficult to find the appropriate weight function in Carleman techniques or logarithmic convexity methods for getting directly the desired spectral inequality (1.1). On the other hand, the moment method [CMV3, CMV4] is an appropriate tool to get the cost of controllability for the one-dimensional degenerate parabolic operator.

As a consequence of Theorem 1.1, we have the following observability estimate from a measurable set in time for the one-dimensional degenerate parabolic operator.

Theorem 1.2. *Let ω be an open and nonempty subset of $(0,1)$ and $E \subset (0,T)$ be a measurable set of positive measure. There exists a constant $C > 0$ such that*

$$\| e^{-T\mathcal{P}} y_0 \|_{L^2(0,1)} \leqslant C e^{C \frac{1}{(2-\alpha)^4}} \int_{\omega \times E} | e^{-t\ \mathcal{P}} y_0 | dx dt$$

for any $\alpha \in [0,2)$ and any $y_0 \in L^2(0,1)$.

Note that this is equivalent to

$$\|y(\cdot,T)\|_{L^2(0,1)} \leqslant Ce^{C\frac{1}{(2-\alpha)^4}} \int_{\omega\times E} |y(x,t)|dxdt$$

for any $\alpha \in [0,2)$ and any $y_0 \in L^2(0,1)$ where y is the weak solution of the degenerate heat equation

$$\begin{cases} \partial_t y - \partial_x(x^\alpha \partial_x y) = 0, & \text{in } (0,1)\times(0,T), \\ \mathrm{BC}_\alpha(y) = 0, & \text{on } \{0\}\times(0,T), \\ y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0, & x \in (0,1). \end{cases}$$

A few remarks can be made. First, it could be of interest to quantify the dependency of the constant $C > 0$ with respect to the measure of E when $E \neq (0,T)$. Second, note that from [CMV4], one may expect a constant of the form $Ce^{C/(2-\alpha)^2}$ in Theorem 1.2. Thus it could be interesting to improve that blow-up rate. In particular, we don't known if the blow up $Ce^{C\frac{1}{(2-\alpha)^2}\sqrt{\Lambda}}$ is sharp with respect to α in Theorem 1.1.
Finally, it is worth noticing that in addition to Theorem 1.2, applications to impulse control and finite-time stabilization can be obtained from Theorem 1.1 by using the abstract results of [BP].

2 Partial elliptic observability estimates and proof of Theorem 1.1

In this section, our aim is to prove Theorem 1.1 and we start with presenting the following three results that are central in our article. We end this section with the proof of Theorem 1.1 by combining these three propositions.

2.1 Partial elliptic observability estimates

First we have an uniform observability estimate for a single eigenfunction.

Proposition 2.1. *For any ω open and nonempty subset of $(0,1)$,*

$$\exists \rho > 0 \quad \forall \alpha \in [0,2) \quad \forall j \geqslant 1 \quad \int_\omega |\Phi_j|^2 dx \geqslant \rho(2-\alpha).$$

This result can be found in [CMV4] in the case $\alpha \in [1,2)$. Here, we need to extend it to the case $\alpha \in [0,1)$, and the proof is postponed in Section 5. Given $T > 0$ arbitrary, we now consider the following homogeneous elliptic problem:

$$\begin{cases} \partial_t^2 \varphi - \mathcal{P}\varphi = 0, & \text{in } (0,1)\times(0,T), \\ \mathrm{BC}_\alpha(\varphi) = 0, & \text{on } \{0\}\times(0,T), \\ \varphi = 0, & \text{on } \{1\}\times(0,T), \\ \varphi(\cdot,0) = \varphi_0, & \text{in } (0,1), \\ \partial_t\varphi(\cdot,0) = \varphi_1, & \text{in } (0,1), \end{cases} \tag{2.1}$$

where φ_0 and φ_1 belong to span $\{\Phi_j; 1 \leqslant j \leqslant N\}$.
Next, we establishes a quantitative Hölder type estimate for an elliptic equation away from the degeneracy.

Proposition 2.2. *Let $0 < a < b < 1$ and $T > 0$. There exist $c > 0$ and $\delta \in (0,1)$ such that for all $\alpha \in [0,2)$, and for all $N \in \mathbb{N}\setminus\{0\}$, the solution φ of (2.1) satisfies*

$$\|\varphi\|_{H^1\left(\left(\frac{2a+b}{3},\frac{a+2b}{3}\right)\times(0,T/4)\right)} \leqslant c\|\varphi\|^{1-\delta}_{H^1((a,b)\times(0,T))}\left(\|\varphi_0\|_{H^1(a,b)} + \|\varphi_1\|_{L^2(a,b)}\right)^\delta .$$

Note that this estimate is independent on $\alpha \in [0,2)$. This is due to the fact that we work on localized regions where $\mathcal{P}$ is uniformly elliptic. Finally we shall use the following uniform observability estimate for the elliptic equation near the degeneracy.

Proposition 2.3. *Let ω be an open and nonempty subset of $(0,1)$. For any $N \geqslant 1$, $T>0$, and any $\alpha \in [0,2)$, the solution φ of (2.1) satisfies*

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)} \leqslant \frac{C(1+\lambda_N)}{\rho^2(2-\alpha)^2}\left(1+\frac{1}{T}\right)e^{C\sqrt{\lambda_N}\left(T+\frac{1}{T\gamma^2(2-\alpha)^2}\right)}\int_0^T\int_\omega |\varphi(x,t)|^2 dxdt,$$

where $C>0$ is independent of $N, T>0$ and $\alpha \in [0,2)$. Here ρ is given by Proposition 2.1 and γ comes from (1.3).

This estimate deals with the observability of the degeneracy. The constant here depends on α, mainly due to the fact that the spectral gap (1.3) and the bound from below of Proposition 2.1 vanishes as α goes to 2.

The proofs of propositions 2.1, 2.2 and 2.3 are postponed in Section 5, Section 4 and Section 3, respectively.

Now, we are able to present the proof of Theorem 1.1, and our strategy is as follows. We shall use Proposition 2.3 and we observe the whole domain (including the region where the ellipticity degenerates) from one region where the operator $\partial_t^2 - \mathcal{P}$ is uniformly elliptic; there, we use classical global Carleman techniques to observe from the boundary $(a,b)\times\{0\}$ with Proposition 2.2. That observation region provides precisely the right hand side of Theorem 1.1.

2.2 Proof of Theorem 1.1

We consider the above homogeneous elliptic problem with $\varphi_0 = 0$ and $\varphi_1 = \sum_{j=1,\cdot\cdot,N} a_j\Phi_j$ where $\{a_j\} \in \mathbb{R}$. Recall that φ can be explicitly written by its spectral decomposition

$$\varphi(\cdot,t) = \sum_{j=1,\cdot\cdot,N} \frac{1}{\sqrt{\lambda_j}}\sinh(\sqrt{\lambda_j}t)a_j\Phi_j.$$

Let $0<a<b<1$ and set $\omega=(a,b)$ and $\widetilde{\omega} = ((2a+b)/3, (a+2b)/3)$. We have, for some constants $C, C_1, C_2 > 0$ independent on N and α,

$$\begin{aligned}
\sum_{j=1,\cdot\cdot,N}|a_j|^2 &\leqslant \sum_{j=1,\cdot\cdot,N}|a_j|^2\frac{1}{\lambda_j}\sinh^2(\sqrt{\lambda_j}/4)Ce^{C\sqrt{\lambda_N}} && \text{by (1.2)}\\
&= Ce^{C\sqrt{\lambda_N}}\|\varphi(\cdot,1/4)\|^2_{L^2(0,1)} && \text{by orthogonality of the eigenfunctions}\\
&\leqslant C_1 e^{C_1\frac{1}{(2-\alpha)^2}\sqrt{\lambda_N}}\int_0^{1/4}\int_{\widetilde{\omega}}|\varphi(x,t)|^2dxdt && \text{by Proposition 2.3 applied to } \widetilde{\omega}\times(0,1/4)\\
&\leqslant C_2 e^{C_2\frac{1}{(2-\alpha)^2}\sqrt{\lambda_N}}\|\varphi\|^{2(1-\delta)}_{H^1(\omega\times(0,1))}\|\varphi_1\|^{2\delta}_{L^2(\omega)} && \text{by Proposition 2.2.}
\end{aligned}$$

But,

$$\|\varphi_1\|^2_{L^2(\omega)} = \int_\omega\left|\sum_{j=1,\cdot\cdot,N}a_j\Phi_j\right|^2dx,$$

and for some constants $c_1, c_2 > 0$ independent on N and α, it holds

$$\begin{aligned}
\|\varphi\|^2_{H^1(\omega\times(0,1))} &= \|\varphi\|^2_{L^2(\omega\times(0,1))} + \|\partial_x\varphi\|^2_{L^2(\omega\times(0,1))} + \|\partial_t\varphi\|^2_{L^2(\omega\times(0,1))}\\
&\leqslant \|\varphi\|^2_{L^2((0,1)^2)} + c_1\|x^{\alpha/2}\partial_x\varphi\|^2_{L^2((0,1)^2)} + \|\partial_t\varphi\|^2_{L^2((0,1)^2)}\\
&\leqslant c_2e^{c_2\sqrt{\lambda_N}}\sum_{j=1,\cdot\cdot,N}|a_j|^2.
\end{aligned}$$

Note that we are allowed to introduce the factor $x^{\alpha/2}$ in front of $\partial_x\varphi$ from the first line to the second line, as in $\omega=(a,b)$ we have $1 \leqslant a^{-\alpha/2}x^{\alpha/2} \leqslant a^{-1}x^{\alpha/2}$. Combining the above estimates completes the proof of Theorem 1.1.

3 Elliptic observability near the degeneracy by the moment method (proof of Proposition 2.3)

In this section, we shall prove Proposition 2.3.

3.1 Setting of an elliptic control problem

We first introduce a notation.

Definition 3.1. *Let $N \in \mathbb{N} \setminus \{0\}$. We define $\Pi_N L^2 = \operatorname{span}\{\Phi_j ; 1 \leqslant j \leqslant N\}$. The space $\Pi_N L^2$ endowed with the $L^2(\Omega)$ norm is a finite dimensional Hilbert space.*

Let ω be an open and nonempty subset of $(0,1)$. Given $T > 0$ arbitrary, we consider the following non-homogeneous elliptic problem:

$$\begin{cases} \partial_t^2 u - \mathcal{P}u = h, & \text{in } (0,1)\times(0,T), \\ \mathrm{BC}_\alpha(u) = 0, & \text{on } \{0\}\times(0,T), \\ u = 0, & \text{on } \{1\}\times(0,T), \\ u(\cdot,0) = u_0, & \text{in } (0,1), \\ \partial_t u(\cdot,0) = u_1, & \text{in } (0,1), \end{cases} \tag{3.1}$$

where denoting $(\cdot,\cdot)_{L^2(\omega)}$ the standard scalar product of $L^2(\omega)$,

$$\begin{cases} h(\cdot,t) = \displaystyle\sum_{j=1,\cdot\cdot,N} \sum_{k=1,\cdot\cdot,N} g_k(t) \left(\Phi_j,\Phi_k\right)_{L^2(\omega)} \Phi_j \text{ with } g(\cdot,t) = \sum_{k=1,\cdot\cdot,N} g_k(t)\Phi_k, \\ u_0 = \displaystyle\sum_{j=1,\cdot\cdot,N} a_j \Phi_j \in \Pi_N L^2, \\ u_1 = \displaystyle\sum_{j=1,\cdot\cdot,N} b_j \Phi_j \in \Pi_N L^2. \end{cases} \tag{3.2}$$

3.2 Well-posedness property and duality between null-controllability and observability.

It is well-known that when $g_j \in L^2(0,T)$, the unique solution of (3.1) verifies $u \in H^2(0,T;\Pi_N L^2)$ and is given by the Duhamel formula

$$u(\cdot,t) = \sum_{j=1,\cdot\cdot,N} \cosh(\sqrt{\lambda_j}t) a_j \Phi_j + \sum_{j=1,\cdot\cdot,N} \frac{\sinh(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}} b_j \Phi_j + \sum_{j=1,\cdot\cdot,N} \sum_{k=1,\cdot\cdot,N} \left(\Phi_j,\Phi_k\right)_{L^2(\omega)} \int_0^t \frac{\sinh(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} g_k(s) ds \Phi_j.$$

Definition 3.2. *We say that system (3.1) is controllable at time T if for any $(u_0,u_1) \in (\Pi_N L^2)^2$ there is $g \in L^2(0,T;\Pi_N L^2)$ as in (3.2) such that*

$$u(\cdot,T) = \partial_t u(\cdot,T) = 0.$$

By classical duality, if system (3.1) is controllable, one can deduce an observability inequality for the adjoint system.

Lemma 3.3. *We say that* (3.1) *is controllable in time $T > 0$ if and only if, for any $(u_0,u_1) \in (\Pi_N L^2)^2$ there is $g \in L^2(0,T;\Pi_N L^2)$ as in (3.2) such that the following relation holds*

$$-\int_0^1 u_1(x)\varphi(x,T)dx - \int_0^1 u_0(x)\partial_t\varphi(x,T)dx = \int_0^T \int_\omega g(x,t)\varphi(x,T-t)dxdt \tag{3.3}$$

for any $(\varphi_0, \varphi_1) \in (\Pi_N L^2)^2$, where φ is the solution of (2.1). Further, if the system (3.1) is controllable at time T with a control $g \in L^2(0,T;\Pi_N L^2)$ satisfying the bound

$$\|g\|^2_{L^2((0,1)\times(0,T))} := \sum_{j=1,\cdot\cdot,N} \int_0^T |g_j(t)|^2 dt \leqslant K\|(u_0,u_1)\|^2_{(L^2(0,1))^2} := K \sum_{j=1,\cdot\cdot,N} (a_j^2 + b_j^2)$$

for some $K > 0$, then the solution φ of (2.1) satisfies

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)} + \|\partial_t \varphi(\cdot,T)\|^2_{L^2(0,1)} \leqslant K \int_0^T \int_\omega |\varphi(x,t)|^2 dx dt.$$

Proof of Lemma 3.3. Let $g \in L^2(0,T;\Pi_N L^2)$ be arbitrary and u be the solution of (3.1). Given φ the solution of (2.1) then, by multiplying (3.1) by $\varphi(x, T-t)$ and by integrating by parts we obtain that

$$\int_0^1 \partial_t u(x,T)\varphi_0(x)dx + \int_0^1 u(x,T)\varphi_1(x)dx - \int_0^1 u_1(x)\varphi(x,T)dx - \int_0^1 u_0(x)\partial_t\varphi(x,T)dx$$
$$= \int_0^T \int_0^1 h(x,t)\varphi(x,T-t)dxdt$$

and

$$\int_0^T \int_0^1 h(x,t)\varphi(x,T-t)dxdt = \int_0^T \int_\omega g(x,t)\varphi(x,T-t)dxdt.$$

Now, if (3.3) is verified, it follows that

$$\int_0^1 \partial_t u(x,T)\varphi_0(x)dx + \int_0^1 u(x,T)\varphi_1(x)dx = 0$$

for any $(\varphi_0, \varphi_1) \in (\Pi_N L^2)^2$ which implies that $u(\cdot,T) = \partial_t u(\cdot,T) = 0$. Hence, the solution is controllable at time T and g is a control for (3.1). Reciprocally, if $g \in L^2(0,T;\Pi_N L^2)$ is a control for (3.1), we have that $u(\cdot,T) = \partial_t u(\cdot,T) = 0$. It implies that (3.3) holds. Finally, one can choose $(u_0,u_1) = (\partial_t\varphi(\cdot,T), \varphi(\cdot,T))$ and apply (3.3) to get the desired estimate thanks to Cauchy-Schwarz inequality and the proof finishes.

□

We now proceed with the proof of Proposition 2.3. It is divided into two steps corresponding to the two subsections below. First, we construct a control by adapting the moment method of [CMV3, CMV4] to the elliptic context of system (3.1); second, we deduce its control cost and Proposition 2.3 follows by applying Lemma 3.3.

3.3 Construction of the control

Our aim is to construct a control g given by $g(\cdot,t) = \sum_{k=1,\cdot\cdot,N} g_k(t)\Phi_k$ such that (3.3) holds. Let

$$\varphi_0 = \sum_{j=1,\cdot\cdot,N} c_j \Phi_j \in \Pi_N L^2, \quad \varphi_1 = \sum_{j=1,\cdot\cdot,N} d_j \Phi_j \in \Pi_N L^2$$

be the initial data of (2.1). Then, recall that φ can be explicitly written by its spectral decomposition,

$$\varphi(\cdot,t) = \sum_{j=1,\cdot\cdot,N} \left(e^{\sqrt{\lambda_j}t}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right) + e^{-\sqrt{\lambda_j}t}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)\right)\Phi_j.$$

First, let us clarify the expression $-\int_0^1 u_1(x)\varphi(x,T)dx - \int_0^1 u_0(x)\partial_t\varphi(x,T)dx$:

$$
\begin{aligned}
&-\int_0^1 u_1(x)\varphi(x,T)dx - \int_0^1 u_0(x)\partial_t\varphi(x,T)dx \\
&= \sum_{j=1,\cdot\cdot,N} e^{\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right)\int_0^1(-u_1(x) - \sqrt{\lambda_j}u_0(x))\Phi_j(x)dx \\
&\quad + \sum_{j=1,\cdot\cdot,N} e^{-\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)\int_0^1(-u_1(x) + \sqrt{\lambda_j}u_0(x))\Phi_j(x)dx.
\end{aligned}
\tag{3.4}
$$

Next, let us clarify the expression $\int_0^T\int_\omega g(x,t)\varphi(x,T-t)dxdt$, that is $\int_0^T\int_0^1 h(x,t)\varphi(x,T-t)dxdt$:

$$
\begin{aligned}
&\int_0^T\int_\omega g(x,t)\varphi(x,T-t)dxdt \\
&= \sum_{k=1,\cdot\cdot,N}\sum_{j=1,\cdot\cdot,N} e^{\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right)(\Phi_j,\Phi_k)_{L^2(\omega)}\int_0^T g_k(t)e^{-\sqrt{\lambda_j}t}dt \\
&\quad + \sum_{k=1,\cdot\cdot,N}\sum_{j=1,\cdot\cdot,N} e^{-\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)(\Phi_j,\Phi_k)_{L^2(\omega)}\int_0^T g_k(t)e^{\sqrt{\lambda_j}t}dt.
\end{aligned}
$$

Now, suppose that $g_k(t) = \alpha_k\sigma_k^0(t) + \beta_k\sigma_k^1(t)$ where σ_k^0, σ_k^1 belong to $L^2(0,T)$ and that the following moment formula holds:

$$
\begin{cases}
\int_0^T \sigma_k^0(t)e^{-\sqrt{\lambda_j}t}dt = 0 \text{ and } \int_0^T \sigma_k^1(t)e^{-\sqrt{\lambda_j}t}dt = \delta_{jk}; \\
\int_0^T \sigma_k^0(t)e^{\sqrt{\lambda_j}t}dt = \delta_{jk} \text{ and } \int_0^T \sigma_k^1(t)e^{\sqrt{\lambda_j}t}dt = 0.
\end{cases}
\tag{3.5}
$$

Then, we obtain

$$
\begin{aligned}
\int_0^T\int_\omega g(x,t)\varphi(x,T-t)dxdt = &\sum_{j=1,\cdot\cdot,N} e^{\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right)\beta_j\int_\omega|\Phi_j(x)|^2dx \\
&+ \sum_{j=1,\cdot\cdot,N} e^{-\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)\alpha_j\int_\omega|\Phi_j(x)|^2dx.
\end{aligned}
\tag{3.6}
$$

By comparing the identities (3.4) and (3.6), one can deduce that if

$$
\int_0^1(-u_1(x) - \sqrt{\lambda_j}u_0(x))\Phi_j(x)dx = \beta_j\int_\omega|\Phi_j(x)|^2dx
$$

$$
\text{and } \int_0^1(-u_1(x) + \sqrt{\lambda_j}u_0(x))\Phi_j(x)dx = \alpha_j\int_\omega|\Phi_j(x)|^2dx
$$

for any $j = 1,\cdot\cdot,N$, then (3.3) holds for any (φ_0,φ_1) which implies by Lemma 3.3 that (3.1) is controllable in time T. Therefore, one can conclude that the control given by $g(\cdot,t) := \sum_{j=1,\cdot\cdot,N}[\alpha_j\sigma_j^0(t) + \beta_j\sigma_j^1(t)]\Phi_j$ where

$$
\alpha_j := \frac{\int_0^1(-u_1(x) + \sqrt{\lambda_j}u_0(x))\Phi_j(x)dx}{\int_\omega|\Phi_j(x)|^2dx} = \frac{-b_j + \sqrt{\lambda_j}a_j}{\int_\omega|\Phi_j(x)|^2dx}
$$

$$
\text{and } \beta_j := \frac{\int_0^1(-u_1(x) - \sqrt{\lambda_j}u_0(x))\Phi_j(x)dx}{\int_\omega|\Phi_j(x)|^2dx} = \frac{-b_j - \sqrt{\lambda_j}a_j}{\int_\omega|\Phi_j(x)|^2dx}
$$

is an appropriate candidate. Notice that by Proposition 2.1, $\displaystyle\int_\omega |\Phi_j(x)|^2 dx \neq 0$. It remains to construct the sequence of functions $(\sigma_k^0, \sigma_k^1)_{k\geqslant 1}$ in $(L^2(0,T))^2$ such that (3.5) holds. Such property is called biorthogonality of the family $(\sigma_k^0, \sigma_k^1)_{k\geqslant 1}$. To do so, we apply the following result from Cannarsa, Martinez and Vancostenoble (see [CMV3] Theorem 2.4 at page 179).

Theorem 3.4 (existence of a suitable biorthogonal family and upper bounds)**.** *Assume that*

$$\forall n > 0,\ \mu_n \geqslant 0$$

and that there is some $r > 0$ such that

$$\forall n > 0,\ \sqrt{\mu_{n+1}} - \sqrt{\mu_n} \geqslant r.$$

Then there exists a family $(\theta_m)_{m>0}$ which is biorthogonal to the family $(e^{\mu_n t})_{n>0}$ in $L^2(0,T)$:

$$\forall m, n > 0,\ \int_0^T \theta_m(t) e^{\mu_n t} dt = \delta_{mn}.$$

Moreover, it satisfies: there is some universal constant c independent of T, r and m such that, for all $m > 0$, we have

$$\|\theta_m\|^2_{L^2(0,T)} \leqslant c e^{-2\mu_m T} e^{c\frac{1}{r}\sqrt{\mu_m}} B(T,r)$$

with

$$B(T,r) = \begin{cases} \left(\dfrac{1}{T} + \dfrac{1}{T^2 r^2}\right) e^{c\frac{1}{Tr^2}} & \text{if } T \leqslant \dfrac{1}{r^2}, \\ cr^2 & \text{if } T \geqslant \dfrac{1}{r^2}. \end{cases}$$

Now, define the increasing sequence of non negative real numbers $(\mu_n)_{n\geqslant 1}$ as follows:

$$\mu_n = \begin{cases} \sqrt{\lambda_N} - \sqrt{\lambda_{N-(n-1)}} & \text{if } 1 \leqslant n \leqslant N, \\ \sqrt{\lambda_N} + \sqrt{\lambda_{n-N}} & \text{if } N+1 \leqslant n \leqslant 2N, \\ \left(\sqrt{\mu_{n-1}} + \gamma(\lambda_N)^{-1/4}\right)^2 & \text{if } n \geqslant 2N+1. \end{cases}$$

We need to check that such sequence fulfills the assumption of Theorem 3.4 thanks to the fact that $\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geqslant \gamma(2-\alpha)$ given by (1.3). Indeed, for any $1 \leqslant n \leqslant N-1$,

$$\sqrt{\mu_{n+1}} - \sqrt{\mu_n} = \frac{\sqrt{\lambda_{N-(n-1)}} - \sqrt{\lambda_{N-n}}}{\sqrt{\sqrt{\lambda_N} - \sqrt{\lambda_{N-n}}} + \sqrt{\sqrt{\lambda_N} - \sqrt{\lambda_{N-(n-1)}}}} \geqslant \frac{\gamma(2-\alpha)}{2(\lambda_N)^{1/4}};$$

for any $N+1 \leqslant n \leqslant 2N-1$,

$$\sqrt{\mu_{n+1}} - \sqrt{\mu_n} = \frac{\sqrt{\lambda_{n+1-N}} - \sqrt{\lambda_{n-N}}}{\sqrt{\sqrt{\lambda_N} + \sqrt{\lambda_{n+1-N}}} + \sqrt{\sqrt{\lambda_N} + \sqrt{\lambda_{n-N}}}} \geqslant \frac{\gamma(2-\alpha)}{2\sqrt{2}(\lambda_N)^{1/4}};$$

for any $n \geqslant 2N$, $\sqrt{\mu_{n+1}} - \sqrt{\mu_n} = \gamma(\lambda_N)^{-1/4}$ and

$$\sqrt{\mu_{N+1}} - \sqrt{\mu_N} = \frac{2\sqrt{\lambda_1}}{\sqrt{\sqrt{\lambda_N} + \sqrt{\lambda_1}} + \sqrt{\sqrt{\lambda_N} - \sqrt{\lambda_1}}} \geqslant \frac{2\sqrt{\lambda_1}}{(1+\sqrt{2})(\lambda_N)^{1/4}}.$$

Consequently, it fulfills by a straightforward computation the assumptions of the above Theorem 3.4: precisely,

$$\forall n > 0,\ \mu_n \geqslant 0 \text{ and } \sqrt{\mu_{n+1}} - \sqrt{\mu_n} \geqslant r$$

with

$$r = \frac{\varsigma}{(\lambda_N)^{1/4}}, \text{ and } \quad \varsigma = \min\left(\frac{\gamma(2-\alpha)}{2\sqrt{2}}, \frac{2\sqrt{\lambda_1}}{1+\sqrt{2}}\right). \tag{3.7}$$

By Theorem 3.4, we have a family $(\theta_m)_{m>0}$ which is biorthogonal to the family $(e^{\mu_n t})_{n>0}$ in $L^2(0,T)$:

$$\forall m, n > 0, \quad \int_0^T \theta_m(t) e^{\mu_n t} dt = \delta_{mn}.$$

Therefore,

$$\text{if } 1 \leqslant n \leqslant N, \text{ then } \int_0^T \theta_m(t) e^{\sqrt{\lambda_N} t} e^{-\sqrt{\lambda_{N-(n-1)}} t} dt = \delta_{mn};$$

$$\text{if } N+1 \leqslant n \leqslant 2N, \text{ then } \int_0^T \theta_m(t) e^{\sqrt{\lambda_N} t} e^{\sqrt{\lambda_{n-N}} t} dt = \delta_{mn}.$$

That is, for any $j = 1, \cdot\cdot, N$,

$$\int_0^T \theta_{N-(j-1)}(t) e^{\sqrt{\lambda_N} t} e^{-\sqrt{\lambda_j} t} dt = 1; \quad \int_0^T \theta_m(t) e^{\sqrt{\lambda_N} t} e^{-\sqrt{\lambda_j} t} dt = 0 \text{ when } m \neq N-(j-1); \tag{3.8}$$

$$\int_0^T \theta_{N+j}(t) e^{\sqrt{\lambda_N} t} e^{\sqrt{\lambda_j} t} dt = 1; \quad \int_0^T \theta_m(t) e^{\sqrt{\lambda_N} t} e^{\sqrt{\lambda_j} t} dt = 0 \text{ when } m \neq N+j. \tag{3.9}$$

Finally, we set for any $k = 1, \cdot\cdot, N$,

$$\sigma_k^0(t) = \theta_{N+k}(t) e^{\sqrt{\lambda_N} t} \text{ and } \sigma_k^1(t) = \theta_{N-(k-1)}(t) e^{\sqrt{\lambda_N} t}$$

in order that by (3.8), for $k, j = 1, \dots, N$,

$$\int_0^T \sigma_k^0(t) e^{-\sqrt{\lambda_j} t} dt = 0 \text{ and } \int_0^T \sigma_k^1(t) e^{-\sqrt{\lambda_j} t} dt = \delta_{jk}$$

and by (3.9)

$$\int_0^T \sigma_k^0(t) e^{\sqrt{\lambda_j} t} dt = \delta_{jk} \text{ and } \int_0^T \sigma_k^1(t) e^{\sqrt{\lambda_j} t} dt = 0.$$

Further, it holds that for any $k = 1, \cdot\cdot, N$,

$$\|\sigma_k^0\|^2_{L^2(0,T)} \leqslant e^{2\sqrt{\lambda_N} T} \|\theta_{N+k}\|^2_{L^2(0,T)} \text{ and } \|\sigma_k^1\|^2_{L^2(0,T)} \leqslant e^{2\sqrt{\lambda_N} T} \|\theta_{N-(k-1)}\|^2_{L^2(0,T)}. \tag{3.10}$$

This completes the construction of our control given by $g(\cdot, t) := \sum_{j=1,\cdot\cdot,N} [\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)] \Phi_j$.

3.4 Cost of the control

Theorem 3.4 along with (3.7) implies that there is some universal constant c independent of T and N such that for any $m = 1, \cdot\cdot, 2N$,

$$\begin{aligned} \|\theta_m\|^2_{L^2(0,T)} &\leqslant ce^{c\frac{1}{r}\sqrt{\mu_m}} B(T, r) := ce^{c\frac{(\lambda_N)^{1/4}}{\varsigma}\sqrt{\mu_m}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right) \\ &\leqslant ce^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right) \end{aligned}$$

because $\sqrt{\mu_m} \leqslant \sqrt{2}(\lambda_N)^{1/4}$, $\forall m \in \{1, \cdot\cdot, 2N\}$. Therefore, by (3.10) we have

$$\sup_{k=1,\cdot\cdot,N} \left(\|\sigma_k^0\|^2_{L^2(0,T)} + \|\sigma_k^1\|^2_{L^2(0,T)}\right) \leqslant 2ce^{2\sqrt{\lambda_N} T} e^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right). \tag{3.11}$$

Our control given by $g(\cdot,t) := \sum_{j=1,\cdot\cdot,N} [\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)]\Phi_j$ where

$$\alpha_j := \frac{\int_0^1 (-u_1(x) + \sqrt{\lambda_j} u_0(x))\Phi_j(x)dx}{\int_\omega |\Phi_j(x)|^2 dx} = \frac{-b_j + \sqrt{\lambda_j} a_j}{\int_\omega |\Phi_j(x)|^2 dx}$$

$$\text{and } \beta_j := \frac{\int_0^1 (-u_1(x) - \sqrt{\lambda_j} u_0(x))\Phi_j(x)dx}{\int_\omega |\Phi_j(x)|^2 dx} = \frac{-b_j - \sqrt{\lambda_j} a_j}{\int_\omega |\Phi_j(x)|^2 dx}$$

satisfies

$$\sum_{j=1,\cdot\cdot,N} (\alpha_j^2 + \beta_j^2) = 2 \sum_{j=1,\cdot\cdot,N} \frac{(\lambda_j a_j^2 + b_j^2)}{\left(\int_\omega |\Phi_j(x)|^2 dx\right)^2} \leqslant \frac{2(1+\lambda_N)}{\left(\inf_{j=1,\cdot\cdot,N} \int_\omega |\Phi_j(x)|^2 dx\right)^2} \sum_{j=1,\cdot\cdot,N} (a_j^2 + b_j^2). \tag{3.12}$$

Combining the above estimates (3.11) and (3.12), there is some universal constant c independent of T such that for any $N \geqslant 1$

$$\begin{aligned} \|g\|^2_{L^2((0,1)\times(0,T))} &= \sum_{j=1,\cdot\cdot,N} \int_0^T |\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)|^2 dt \\ &\leqslant 2 \sum_{j=1,\cdot\cdot,N} (\alpha_j^2 + \beta_j^2) \sup_{k=1,\cdot\cdot,N} \left(\|\sigma_k^0\|^2_{L^2(0,T)} + \|\sigma_k^1\|^2_{L^2(0,T)} \right) \\ &\leqslant \frac{8(1+\lambda_N)}{\left(\inf_{j=1,\cdot\cdot,N} \int_\omega |\Phi_j(x)|^2 dx\right)^2} c e^{2\sqrt{\lambda_N} T} e^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right) \sum_{j=1,\cdot\cdot,N} (a_j^2 + b_j^2). \end{aligned} \tag{3.13}$$

Recall that the bound

$$\|g\|^2_{L^2((0,1)\times(0,T))} := \sum_{j=1,\cdot\cdot,N} \int_0^T |\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)|^2 dt \leqslant K \|(u_0,u_1)\|^2_{(L^2(0,1))^2} := K \sum_{j=1,\cdot\cdot,N} (a_j^2 + b_j^2)$$

will imply that the solution φ of (2.1) satisfies

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)} + \|\partial_t \varphi(\cdot,T)\|^2_{L^2(0,1)} \leqslant K \int_0^T \int_\omega |\varphi(x,t)|^2 dx dt.$$

Now our aim is to bound the quantity

$$\frac{8(1+\lambda_N)}{\left(\inf_{j=1,\cdot\cdot,N} \int_\omega |\Phi_j(x)|^2 dx\right)^2} c e^{2\sqrt{\lambda_N} T} e^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right)$$

appearing in (3.13) in order to get the cost K.

First, by Proposition 2.1, $\left(\inf_{j=1,\cdot\cdot,N} \int_\omega |\Phi_j(x)|^2 dx\right)^{-2} \leqslant \frac{1}{\rho^2(2-\alpha)^2}$. Second, recall that $\varsigma = \min\left(\frac{\gamma(2-\alpha)}{2\sqrt{2}}, \frac{2\sqrt{\lambda_1}}{1+\sqrt{2}}\right)$ and since $\alpha \in [0,2)$ with (1.2), we have that $\underline{c}\gamma(2-\alpha) \leqslant \varsigma \leqslant \frac{1}{\underline{c}}$ where $\underline{c}$ is a positive constant independent on $\alpha \in [0,2)$. Finally, the estimate of $B(T,r)$ in Theorem 3.4

$$B(T,r) = \begin{cases} \left(\frac{1}{T} + \frac{1}{T^2 r^2}\right) e^{c\frac{1}{Tr^2}} & \text{if } T \leqslant \frac{1}{r^2} \\ cr^2 & \text{if } T \geqslant \frac{1}{r^2} \end{cases} \leqslant \begin{cases} \left(1 + \frac{1}{c}\right) \frac{1}{T} e^{2c\frac{1}{Tr^2}} & \text{if } T \leqslant \frac{1}{r^2} \\ cr^2 & \text{if } T \geqslant \frac{1}{r^2} \end{cases} \leqslant \left((1+\frac{1}{c})\frac{1}{T} + cr^2\right) e^{2c\frac{1}{Tr^2}}$$

leads to the bound

$$\begin{aligned}B(T,\varsigma\,(\lambda_N)^{-1/4}) &\leqslant \left((1+\frac{1}{c})\frac{1}{T}+c\frac{\varsigma^2}{\sqrt{\lambda_N}}\right)e^{2c\frac{\sqrt{\lambda_N}}{T\varsigma^2}}\\ &\leqslant C(1+\frac{1}{T})e^{C\frac{\sqrt{\lambda_N}}{T(2-\alpha)^2}}\end{aligned}$$

for some $C>0$ independent on $N>0$, $\alpha\in[0,2)$ and $T>0$. Therefore, by (3.13) one can conclude that

$$\|g\|^2_{L^2((0,1)\times(0,T))}\leqslant\frac{C(1+\lambda_N)}{\rho^2(2-\alpha)^2}e^{C\sqrt{\lambda_N}T}e^{C\frac{\sqrt{\lambda_N}}{\gamma(2-\alpha)}}C(1+\frac{1}{T})e^{C\frac{\sqrt{\lambda_N}}{T\gamma^2(2-\alpha)^2}}\sum_{j=1,\cdot\cdot,N}(a_j^2+b_j^2),$$

which gives, using $\frac{1}{\gamma(2-\alpha)}\leqslant T+\frac{1}{T\gamma^2(2-\alpha)^2}$, that

$$\|g\|^2_{L^2((0,1)\times(0,T))}\leqslant\frac{C\,(1+\lambda_N)}{\rho^2(2-\alpha)^2}\left(1+\frac{1}{T}\right)e^{C\sqrt{\lambda_N}\left(T+\frac{1}{T\gamma^2(2-\alpha)^2}\right)}\sum_{j=1,\cdot\cdot,N}\left(a_j^2+b_j^2\right).$$

By the cost estimate in Lemma 3.3, we obtain that for any φ solution of (2.1) and any $N\geqslant 1$

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)}+\|\partial_t\varphi(\cdot,T)\|^2_{L^2(0,1)}\leqslant\frac{(1+\lambda_N)}{\rho^2(2-\alpha)^2}\left(1+\frac{1}{T}\right)e^{C\sqrt{\lambda_N}\left(T+\frac{1}{T\gamma^2(2-\alpha)^2}\right)}\int_0^T\int_\omega|\varphi(x,t)|^2dxdt,$$

where $C>0$ does not depend on (N,T,α). This completes the proof of Proposition 2.3.

4 Elliptic observation away from the degeneracy by Carleman techniques (proof of Proposition 2.2)

In this section, we shall prove Proposition 2.2. This kind of interpolation estimate is classical for elliptic operators [R, LR], but we need to prove that the constants appearing in Proposition 2.2 are independent on $\alpha\in[0,2)$. Moreover, we propose a proof using a Carleman estimate with a global weight function. Let $0<a<b<1$ and $\Omega=(a,b)\times(0,T)$. We set $(x,t)=(x_1,x_2)\in\Omega$, and for $\alpha\in[0,2)$, introduce

$$Q=-\partial_t^2-\mathcal{P}=-\nabla\cdot\left(A(x_1,x_2)\nabla\cdot\right),\quad A(x_1,x_2)=\begin{pmatrix}x_1^\alpha & 0\\ 0 & 1\end{pmatrix},\quad \nabla=\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\end{pmatrix}.$$

Note that there exists $C_0>0$ such that

$$\|A\|_{W^{3,\infty}(\Omega)}\leqslant C_0,\quad A(x_1,x_2)\xi\cdot\xi\geqslant\frac{1}{C_0}|\xi|^2,\quad\forall\xi\in\mathbb{R}^2,\forall(x_1,x_2)\in\Omega,\tag{4.1}$$

where $C_0>0$ is independent on $\alpha\in[0,2)$. We set

$$v=e^{\tau\phi}\chi z,$$

where $\tau>0$, $z\in H^2\left(\Omega\right)$, $\chi\left(x_1,x_2\right)=\chi_1\left(x_1\right)\chi_2\left(x_2\right)$ with

$$\begin{cases}\chi_1\in C_0^\infty\left(a,b\right),\quad 0\leqslant\chi_1\leqslant 1,\quad \chi_1=1\text{ on }\left(\dfrac{3a+b}{4},\dfrac{a+3b}{4}\right),\\ \chi_2\in C^\infty\left(0,T\right),\quad 0\leqslant\chi_2\leqslant 1,\quad \chi_2=1\text{ on }\left(0,\dfrac{T}{3}\right)\text{ and }\chi_2=0\text{ on }\left(\dfrac{2T}{3},T\right),\end{cases}$$

and we shall consider weight functions $\phi\in C^\infty(\overline{\Omega})$ of the form

$$\phi(x_1,x_2)=e^{\lambda\psi(x_1,x_2)},\quad\lambda>0,\quad\psi\in C^\infty(\overline{\Omega}),\quad\nabla\psi\neq 0\text{ on }\overline{\Omega}.\tag{4.2}$$

Here, we give explicitly ψ as follows:

$$\psi(x_1,x_2) = -(x_1-x_0)^{2k} - \beta^{2k}(x_2+1)^{2k}, \tag{4.3}$$

where $x_0 = \dfrac{a+b}{2}$, $\beta = \dfrac{2}{3}\left(\dfrac{b-a}{T+4}\right)$ and $k = \max(\ln 2/\ln((4T+12)/(3T+12)); \ln 2/\ln(3/2))$.

We set

$$Q_\phi = e^{\tau\phi} Q e^{-\tau\phi}.$$

We have $Q_\phi v = \mathcal{S}v + \mathcal{A}v + \mathcal{R}v$ with

$$\mathcal{S}v = -\nabla\cdot(A\nabla v) - \tau^2 A\nabla\phi\cdot\nabla\phi v, \quad \mathcal{A}v = 2\tau A\nabla\phi\cdot\nabla v + 2\tau\nabla\cdot(A\nabla\phi)v, \quad \mathcal{R}v = -\tau\nabla\cdot(A\nabla\phi)v,$$

which gives $\|Q_\phi v - \mathcal{R}v\|^2_{L^2(\Omega)} = \|\mathcal{S}v\|^2_{L^2(\Omega)} + \|\mathcal{A}v\|^2_{L^2(\Omega)} + 2(\mathcal{S}v,\mathcal{A}v)_{L^2(\Omega)}$. Note that $0 \leqslant \|Q_\phi v - \mathcal{R}v\|^2_{L^2(\Omega)} - 2(\mathcal{S}v,\mathcal{A}v)_{L^2(\Omega)}$ implies

$$(\mathcal{S}v,\mathcal{A}v)_{L^2(\Omega)} \leqslant \|Q_\phi v\|^2_{L^2(\Omega)} + \|\mathcal{R}v\|^2_{L^2(\Omega)}. \tag{4.4}$$

Now we compute $(\mathcal{S}v,\mathcal{A}v)_{L^2(\Omega)}$: by integration by parts, one has with standard summation notations and $A = (A_{ij})_{1\leqslant i,j\leqslant 2}$,

$$\begin{aligned}
(\mathcal{S}v,\mathcal{A}v)_{L^2(\Omega)} &= 2\tau\int_\Omega A\nabla^2 v A\nabla\phi\cdot\nabla v\ dx_1dx_2 + 2\tau\int_\Omega A\nabla^2\phi A\nabla v\cdot\nabla v\ dx_1dx_2 \\
&+ 2\tau\int_\Omega A_{ij}\partial_{x_i}v\partial_{x_\ell}v\partial_{x_j}A_{k\ell}\partial_{x_k}\phi\ dx_1dx_2 \\
&+ 2\tau\int_\Omega (A\nabla v\cdot\nabla v)\,\nabla\cdot(A\nabla\phi)\ dx_1dx_2 + 2\tau\int_\Omega A\nabla v\cdot\nabla(\nabla\cdot(A\nabla\phi))\,v\ dx_1dx_2 \\
&+ \tau^3\int_\Omega \left[A\nabla\phi\cdot\nabla(A\nabla\phi\cdot\nabla\phi) - (A\nabla\phi\cdot\nabla\phi)(\nabla\cdot(A\nabla\phi))\right]|v|^2\ dx_1dx_2 \\
&+ 2\tau\int_{\partial\Omega}(A\nabla v\cdot n)(A\nabla\phi\cdot\nabla v + (\nabla\cdot(A\nabla\phi))v)\ d\sigma - \tau^3\int_{\partial\Omega}(A\nabla\phi\cdot\nabla\phi)(A\nabla\phi\cdot n)|v|^2\ d\sigma,
\end{aligned}$$

where $d\sigma$ denotes the measure of the boundary $\partial\Omega$. But by one integration by parts

$$\begin{aligned}
\int_\Omega A\nabla^2 v A\nabla\phi\cdot\nabla v\ dx_1dx_2 &= \frac{1}{2}\int_{\partial\Omega}(A\nabla v\cdot\nabla v)(A\nabla\phi\cdot n)\ dx_1dx_2 - \frac{1}{2}\int_\Omega \partial_{x_\ell}A_{ij}\partial_{x_j}v\partial_{x_i}vA_{k\ell}\partial_{x_k}\phi\ dx_1dx_2 \\
&-\frac{1}{2}\int_\Omega (A\nabla v\cdot\nabla v)\,\nabla\cdot(A\nabla\phi)\ dx_1dx_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\mathcal{S}v,\mathcal{A}v)_{L^2(\Omega)} &= 2\tau\int_\Omega A\nabla^2\phi A\nabla v\cdot\nabla v\ dx_1dx_2 + \tau\int_\Omega (A\nabla v\cdot\nabla v)\,\nabla\cdot(A\nabla\phi)\ dx_1dx_2 \\
&+ \tau^3\int_\Omega \left[A\nabla\phi\cdot\nabla(A\nabla\phi\cdot\nabla\phi) - (A\nabla\phi\cdot\nabla\phi)(\nabla\cdot(A\nabla\phi))\right]|v|^2\ dx_1dx_2 \\
&+ R_1 + R_2
\end{aligned}$$

with

$$\begin{aligned}
R_1 &= 2\tau\int_\Omega A_{ij}\partial_{x_i}v\partial_{x_\ell}v\partial_{x_j}A_{k\ell}\partial_{x_k}\phi\ dx_1dx_2 - \tau\int_\Omega \partial_{x_\ell}A_{ij}\partial_{x_j}v\partial_{x_i}vA_{k\ell}\partial_{x_k}\phi\ dx_1dx_2 \\
&+ 2\tau\int_\Omega A\nabla v\cdot\nabla(\nabla\cdot(A\nabla\phi))\,v\ dx_1dx_2, \\
R_2 &= -2\tau\int_{\partial\Omega}(A\nabla v\cdot n)(A\nabla\phi\cdot\nabla v)\ d\sigma + \tau\int_{\partial\Omega}(A\nabla v\cdot\nabla v)(A\nabla\phi\cdot n)\ d\sigma \\
&- 2\tau\int_{\partial\Omega}(A\nabla v\cdot n)(\nabla\cdot(A\nabla\phi))\,v\ d\sigma - \tau^3\int_{\partial\Omega}(A\nabla\phi\cdot\nabla\phi)(A\nabla\phi\cdot n)|v|^2\ d\sigma,
\end{aligned}$$

where n is the outward normal vector to $\partial\Omega$. Notice that from the form of A and ϕ given by (4.1) and (4.2), we have the existence of $C_1 > 0$ independent on $\alpha \in [0,2)$ such that for $\tau > 0$ sufficiently large

$$|R_1| \leqslant C_1 \left((\tau^{1/2}\lambda^2 + \tau\lambda)\|\phi^{1/2}\nabla v\|^2_{L^2(\Omega)} + \tau^{3/2}\lambda^4\|\phi^{1/2}v\|^2_{L^2(\Omega)} \right).$$

Note also that from the form of A and ϕ given by (4.1) and (4.2), we have

$$A\nabla^2\phi A\nabla v \cdot \nabla v = \lambda^2\phi(A\nabla\psi\cdot\nabla v)^2 + \lambda\phi A\nabla^2\psi A\nabla v\cdot\nabla v \geqslant -C_2\lambda\phi|\nabla v|^2,$$

and

$$\begin{aligned}\tau\int_\Omega (A\nabla v\cdot\nabla v)\nabla\cdot(A\nabla\phi)\,dx_1dx_2 &= \tau\int_\Omega (A\nabla v\cdot\nabla v)\phi(\lambda\nabla\cdot(A\nabla\psi) + \lambda^2 A\nabla\psi\cdot\nabla\psi))\,dx_1dx_2\\ &\geqslant C_2\tau\lambda^2\|\phi^{1/2}\nabla v\|^2_{L^2(\Omega)} - C_3\tau\lambda\|\phi^{1/2}\nabla v\|^2_{L^2(\Omega)}\\ &\geqslant C_4\tau\lambda^2\|\phi^{1/2}\nabla v\|^2_{L^2(\Omega)},\end{aligned}$$

for $\lambda > 0$ chosen sufficiently large (independently on $\alpha \in [0,2)$, and where the constants $C_2, C_3, C_4 > 0$ are independent on $\alpha \in [0,2)$. Arguing in the same way, there exist constants $C_5 > 0$ and $\lambda_0 > 0$ such that for all $\alpha \in [0,2)$ and for all $\lambda > \lambda_0$,

$$\tau^3\int_\Omega [A\nabla\phi\cdot\nabla(A\nabla\phi\cdot\nabla\phi) - (A\nabla\phi\cdot\nabla\phi)(\nabla\cdot(A\nabla\phi))]\,|v|^2dx_1dx_2 \geqslant C_5\tau^3\lambda^4\|\phi^{3/2}v\|^2_{L^2(\Omega)}.$$

Summing up, (4.4) becomes

$$\begin{aligned}&C_5\tau^3\lambda^4\|\phi^{3/2}v\|^2_{L^2(\Omega)} + C_4\tau\lambda^2\|\phi^{1/2}\nabla v\|^2_{L^2(\Omega)} + R_2\\ &\qquad\leqslant C_1\left((\tau^{1/2}\lambda^2+\tau\lambda)\|\phi^{1/2}\nabla v\|^2_{L^2(\Omega)} + \tau^{3/2}\lambda^4\|\phi^{1/2}v\|^2_{L^2(\Omega)}\right) + \|Q_\phi v\|^2_{L^2(\Omega)} + \|\mathcal{R}v\|^2_{L^2(\Omega)},\end{aligned}$$

where the constants are independent on $\alpha \in [0,2)$. Fixing $\lambda > \lambda_0$ large, and then taking $\tau > \tau_0$ sufficiently large (constants may depend on λ from now), we obtain the existence of $C_6 > 0$ such that

$$C_6\tau^3\|v\|^2_{L^2(\Omega)} + C_6\tau\|\nabla v\|^2_{L^2(\Omega)} + R_2 \leqslant \|Q_\phi v\|^2_{L^2(\Omega)} + \|\mathcal{R}v\|^2_{L^2(\Omega)}.$$

Next, one can see that from the form of A and ϕ, there is $C_7 > 0$ such that for all $\alpha \in [0,2)$,

$$\|\mathcal{R}v\|^2_{L^2(\Omega)} \leqslant C_7\tau^2\|v\|^2_{L^2(\Omega)}.$$

Therefore, taking $\tau > 0$ sufficiently large yields the existence of $C_8 > 0$ such that

$$C_8\left(\tau^3\|v\|^2_{L^2(\Omega)} + \tau\|\nabla v\|^2_{L^2(\Omega)}\right) + R_2 \leqslant \|Q_\phi v\|^2_{L^2(\Omega)}. \tag{4.5}$$

Now we treat the boundary term R_2: since $v = A\nabla v\cdot n = 0$ on $\partial\Omega\setminus\Gamma$ where $\Gamma = \{(x_1,0)\,;x_1\in(a,b)\}$, one can deduce that

$$\begin{aligned}R_2 &= \tau\int_a^b \partial_2\phi|\partial_2 v(x_1,0)|^2dx_1\\ &\quad + 2\tau\int_a^b x_1^\alpha\partial_1\phi\partial_1 v(x_1,0)\partial_2 v(x_1,0)dx_1 - \tau\int_a^b x_1^\alpha\partial_2\phi|\partial_1 v(x_1,0)|^2dx_1\\ &\quad + 2\tau\int_a^b (\nabla\cdot(A\nabla\phi))v(x_1,0)\partial_2 v(x_1,0)dx_1 + \tau^3\int_a^b (A\nabla\phi\cdot\nabla\phi)\partial_2\phi\,|v(x_1,0)|^2\,dx_1,\end{aligned}$$

which gives the existence of $C_9 > 0$ independent on $\alpha \in [0,2)$ such that for any $\tau > 0$ sufficiently large

$$|R_2| \leqslant C_9\left(\tau\|\partial_2 v(\cdot,0)\|^2_{L^2(a,b)} + \tau^3\|v(\cdot,0)\|^2_{H^1(a,b)}\right).$$

Finally, by (4.5) we have for any $\tau > \tau_0$ with $\tau_0 > 1$, the following inequality:

$$C_8\left(\tau^3\|v\|^2_{L^2(\Omega)} + \tau\|\nabla v\|^2_{L^2(\Omega)}\right) \leqslant \|Q_\phi v\|^2_{L^2(\Omega)} + C_9\left(\tau\|\partial_2 v(\cdot,0)\|^2_{L^2(a,b)} + \tau^3\|v(\cdot,0)\|^2_{H^1(a,b)}\right). \tag{4.6}$$

Let $U = \left(\dfrac{2a+b}{3}, \dfrac{a+2b}{3}\right) \times \left(0, \dfrac{T}{4}\right)$, $W_1 = \left(\left[a, \dfrac{3a+b}{4}\right] \cup \left[\dfrac{a+3b}{4}, b\right]\right) \times \left[0, \dfrac{2T}{3}\right]$, $W_2 = [a,b] \times \left[\dfrac{T}{3}, \dfrac{2T}{3}\right]$ and $W = W_1 \cup W_2$. We have $\mathrm{supp}\nabla\chi = W$ and $\chi = 1$ in U.

Coming back to the function z where $v = e^{\tau\phi}\chi z$, $Q_\phi v = e^{\tau\phi}Q(\chi z) = e^{\tau\phi}(\chi Q z + [Q,\chi]z)$ where the bracket $[Q,\chi] = -\partial_t^2\chi - 2(\partial_t\chi)\partial_t - x^\alpha(\partial_x^2\chi) - 2(\partial_x\chi)x^\alpha\partial_x - \alpha(\partial_x\chi)x^{\alpha-1}$ is a differential operator of order one, supported in W, which is away from a neighborhood of the degeneracy $\{x=0\}$. From (4.6) and taking any τ sufficiently large yields

$$\begin{aligned} \tau^3\|e^{\tau\phi}z\|^2_{L^2(U)} + \tau\|e^{\tau\phi}\nabla z\|^2_{L^2(U)} \leqslant\ & C\left(\|e^{\tau\phi}\chi Q z\|^2_{L^2(\Omega)} + \tau\|e^{\tau\phi}z\|^2_{L^2(W)} + \|e^{\tau\phi}\nabla z\|^2_{L^2(W)}\right. \\ & \left. + \tau\|e^{\tau\phi(\cdot,0)}\partial_t z(\cdot,0)\|^2_{L^2(a,b)} + \tau^5\|e^{\tau\phi(\cdot,0)}z(\cdot,0)\|^2_{H^1(a,b)}\right). \end{aligned}$$

Let $D = \max\limits_{\Omega}\phi$, $D_W = \max\limits_{W}\phi$, $D_0 = \max\limits_{(a,b)}\phi(\cdot,0)$ and $D_U = \min\limits_{U}\phi$. We have for any $\tau > \tau_0$ sufficiently large

$$\begin{aligned} e^{2\tau D_U}\left(\|z\|^2_{L^2(U)} + \|\nabla z\|^2_{L^2(U)}\right) \leqslant\ & Ce^{2\tau D}\|Qz\|^2_{L^2(\Omega)} + Ce^{2\tau D_K}\left(\tau\|z\|^2_{L^2(W)} + \|\nabla z\|^2_{L^2(W)}\right) \\ & + Ce^{2\tau D_0}\left(\tau\|\partial_t z(\cdot,0)\|^2_{L^2(a,b)} + \tau^5\|z(\cdot,0)\|^2_{H^1(a,b)}\right). \end{aligned}$$

Our choice of ψ given by (4.3) allows to get $D > D_U$ and $D_0 > D_U > D_K$. Indeed, by a straightforward computation,

$$\begin{cases} \max\limits_{W_1}\psi - \min\limits_{U}\psi & \leqslant -\left(\dfrac{b-a}{4}\right)^{2k} - \beta^{2k} + \left(\dfrac{b-a}{6}\right)^{2k} + \beta^{2k}\left(\dfrac{T}{4}+1\right)^{2k} \\ & = \beta^{2k}\left(-1 + \left(\dfrac{3}{8}(T+4)\right)^{2k}\left(-1 + 2\left(\dfrac{2}{3}\right)^{2k}\right)\right) < 0, \\ \max\limits_{W_2}\psi - \min\limits_{U}\psi & \leqslant -\beta^{2k}\left(\dfrac{T}{3}+1\right)^{2k} + \left(\dfrac{b-a}{6}\right)^{2k} + \beta^{2k}\left(\dfrac{T}{4}+1\right)^{2k} \\ & = \beta^{2k}\left(\dfrac{1}{3}(T+4)\right)^{2k}\left(-1 + 2\left(\dfrac{3T+12}{4T+12}\right)^{2k}\right) < 0. \end{cases}$$

Using $W \subset \overline{\Omega}$ and optimizing with respect to τ yield the desired interpolation estimate (see e.g. [R] or [LRLeR1, Lemma 5.4, page 189]). This completes the proof of

$$\|\varphi\|_{H^1\left(\left(\frac{2a+b}{3}, \frac{a+2b}{3}\right)\times\left(0,\frac{T}{4}\right)\right)} \leqslant c\|\varphi\|^{1-\delta}_{H^1((a,b)\times(0,T))}\left(\|\varphi_0\|_{H^1(a,b)} + \|\varphi_1\|_{L^2(a,b)}\right)^\delta,$$

since $Q\varphi = 0$.

5 Observability estimate for the eigenfunctions (proof of Proposition 2.1)

In this section we aim to prove Proposition 2.1. Given $0 < a < b < 1$, we shall use the notation $X \lesssim Y$, or $Y \gtrsim X$ to denote the bound $|X| \leqslant cY$ for some constant $c > 0$ only dependent on (a,b).

5.1 Proof of Proposition 2.1

Cannarsa, Martinez and Vancostenoble proved (see [CMV4] proposition 2.15 at page 10) that

$$\forall \alpha \in [1,2) \qquad \forall j \geqslant 1 \qquad ||\Phi_j||^2_{L^2(a,b)} \gtrsim 2-\alpha.$$

In this section, we extend this result to $\alpha \in [0,2)$. To this end, we focus on the case $\alpha \in [0,1)$ and apply the following observability estimate, which proof is given in Section 5.2.

Proposition 5.1. *For all $\sigma \in \mathbb{R}$, for all $\alpha \in [0,1)$, for all $\vartheta \in D(\mathcal{P})$*

$$\sigma^2\|\vartheta\|^2_{L^2(0,1)} + \|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} \lesssim \left(\|(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + (1+\sigma^2)\|\vartheta\|^2_{L^2(a,b)}\right).$$

Since $\Phi_j \in D(\mathcal{P})$ is the normalized eigenfunctions of $\mathcal{P}$ associated with an eigenvalue λ_j, $j \in \mathbb{N}\setminus\{0\}$. Applying Proposition 5.1 with $\vartheta = \Phi_j$ and $\sigma^2 = \lambda_j$, we obtain

$$\frac{\lambda_j}{1+\lambda_j} \lesssim \|\Phi_j\|^2_{L^2(a,b)}.$$

Using $\dfrac{\lambda_1}{1+\lambda_1} \leqslant \dfrac{\lambda_j}{1+\lambda_j}$ and (1.2), one can deduce that

$$\forall \alpha \in [0,1) \qquad \forall j \geqslant 1 \qquad \|\Phi_j\|^2_{L^2(a,b)} \gtrsim 1 \geqslant \frac{1}{2}(2-\alpha).$$

This completes the proof of Proposition 2.1.

5.2 Proof of an intermediate result and technical lemmas

Now, we prove Proposition 5.1. Before proceeding to the proof we need two lemmas.

Lemma 5.2. *There exists $C>0$ such that for all $\sigma \in \mathbb{R}$, for all $\alpha \in [0,1)$,*

$$\sigma^2\|\vartheta\|^2_{L^2(0,1)} + \|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} \leqslant C\left(\|(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + |\vartheta'(1)|^2\right),$$

for all $\vartheta \in D(\mathcal{P})$.

Lemma 5.3. *There exists $C>0$ such that for all $\sigma \in \mathbb{R}$, for all $\alpha \in [0,1)$,*

$$|\vartheta'(1)|^2 \leqslant C\left(\|(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + \|\vartheta\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)}\right),$$

for all $\vartheta \in D(\mathcal{P})$.

Proof of Proposition 5.1. By Lemmas 5.2 and 5.3,

$$\sigma^2\|\vartheta\|^2_{L^2(0,1)} + \|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} \lesssim C\left(\|(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + \|\vartheta\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)}\right).$$

Let $\chi \in C_0^\infty(a,b)$ such that $0 \leqslant \chi \leqslant 1$ and $\chi = 1$ on $[\dfrac{3a+b}{4}, \dfrac{a+3b}{4}]$. We have

$$\begin{aligned}
\|\vartheta\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)} = \|\chi\vartheta\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)} &\lesssim \|\vartheta\|^2_{L^2(a,b)} + \left|\int_0^1 \chi^2 \mathcal{P}\vartheta\vartheta dx\right| \\
&\lesssim \|\vartheta\|^2_{L^2(a,b)} + \left|\int_0^1 \chi^2\left(\mathcal{P}-\sigma^2\right)\vartheta\vartheta dx\right| + \sigma^2\|\vartheta\|^2_{L^2(a,b)} \\
&\lesssim \|(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + (1+\sigma^2)\|\vartheta\|^2_{L^2(a,b)}
\end{aligned}$$

by Cauchy-Schwarz. Combining the above estimates ends the proof Proposition 5.1. □

Proof of Lemma 5.2. Let us consider $\phi(x) = x^{2-\alpha}$ and $v = e^{\phi}\vartheta$. Note that for $\alpha \in [0,1)$, $v \in D(\mathcal{P})$ because $\vartheta \in D(\mathcal{P})$. We set

$$P_\phi = e^{\phi}\mathcal{P}e^{-\phi} - \sigma^2$$

with

$$\mathcal{S} = -\frac{d}{dx}\left(x^\alpha \frac{d}{dx}\right) - (2-\alpha)^2 x^{2-\alpha} - \sigma^2, \quad \mathcal{A} = 2(2-\alpha)x\frac{d}{dx} + (2-\alpha),$$

in order that $P_\phi v = e^{\phi}\left(\mathcal{P} - \sigma^2\right)\vartheta$ and $P_\phi v = \mathcal{S}v + \mathcal{A}v$ which gives $\|P_\phi v\|^2_{L^2(0,1)} = \|\mathcal{S}v\|^2_{L^2(0,1)} + \|\mathcal{A}v\|^2_{L^2(0,1)} + 2(\mathcal{S}v, \mathcal{A}v)_{L^2(0,1)}$. Classical computations lead to

$$\begin{aligned}(\mathcal{S}v, \mathcal{A}v)_{L^2(0,1)} &= (2-\alpha)^2\|x^{\alpha/2}v'\|^2_{L^2(0,1)} + (2-\alpha)^4\|x^{(2-\alpha)/2}v\|^2_{L^2(0,1)} - (2-\alpha)|v'(1)|^2 \\ &\quad + (2-\alpha)\lim_{x\to 0^+}\left[x^{1+\alpha}|v'(x)|^2 + x^\alpha v'(x)v(x) + (2-\alpha)^2x^{3-\alpha}|v(x)|^2 + \sigma^2 x|v(x)|^2\right].\end{aligned}$$

The above limit vanishes from the boundary conditions and the regularity of v. Therefore, the fact that $0 \leqslant \|P_\phi v\|^2_{L^2(0,1)} - 2(\mathcal{S}v, \mathcal{A}v)_{L^2(0,1)}$ implies

$$\begin{aligned}2(2-\alpha)^2\|x^{\alpha/2}v'\|^2_{L^2(0,1)} + 2(2-\alpha)^4\|x^{(2-\alpha)/2}v\|^2_{L^2(0,1)} &\leqslant \|P_\phi v\|^2_{L^2(0,1)} + 2(2-\alpha)|v'(1)|^2 \\ &= \|e^{\phi}(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + 2(2-\alpha)|\vartheta'(1)|^2.\end{aligned}$$

Since $x^{\alpha/2}\vartheta' = e^{-\phi}(x^{\alpha/2}v' - (2-\alpha)x^{(2-\alpha)/2}v)$,

$$\|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} \leqslant 2\|x^{\alpha/2}v'\|^2_{L^2(0,1)} + 2(2-\alpha)^2\|x^{(2-\alpha)/2}v\|^2_{L^2(0,1)}.$$

Combining the two above inequalities, we get, for $\alpha \in [0,1)$,

$$\begin{aligned}\|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} &\leqslant \frac{1}{(2-\alpha)^2}\left(\|e^{\phi}(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + 2(2-\alpha)|\vartheta'(1)|^2\right) \\ &\lesssim \|(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)} + |\vartheta'(1)|^2.\end{aligned}$$

It remains to bound $\sigma^2\,\|\vartheta\|^2_{L^2(0,1)}$. By Cauchy-Schwarz,

$$\begin{aligned}\sigma^2\|\vartheta\|^2_{L^2(0,1)} &= \int_0^1 \mathcal{P}\vartheta\vartheta dx - \int_0^1(\mathcal{P}-\sigma^2)\vartheta\vartheta dx = \|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} - \int_0^1(\mathcal{P}-\sigma^2)\vartheta\vartheta dx \\ &\leqslant \|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} + \|\vartheta\|_{L^2(0,1)}\|(\mathcal{P}-\sigma^2)\vartheta\|_{L^2(0,1)} \\ &\leqslant \|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} + 2\|x\vartheta'\|_{L^2(0,1)}\|(\mathcal{P}-\sigma^2)\vartheta\|_{L^2(0,1)} \\ &\lesssim \|x^{\alpha/2}\vartheta'\|^2_{L^2(0,1)} + \|(\mathcal{P}-\sigma^2)\vartheta\|^2_{L^2(0,1)},\end{aligned}$$

where $\|\vartheta\|^2_{L^2(0,1)} \leqslant 4\|x\vartheta'\|^2_{L^2(0,1)}$ comes from one integration by parts. This ends the proof of Lemma 5.2. □

Proof of Lemma 5.3 . Denote $\widetilde{a} = \dfrac{3a+b}{4}$ and $\widetilde{b} = \dfrac{a+3b}{4}$ in order that $0 < a < \widetilde{a} < \widetilde{b} < b < 1$. Let us consider $\phi(x) = e^{\lambda\psi}$, with $\lambda > 0$, $\psi \in C^\infty(0,1)$, $\psi' \neq 0$ on $[\widetilde{a}, 1]$ and $\psi'(1) < 0$. Let $\chi \in C^\infty(0,1)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 0$ on $[0, \widetilde{a}]$ and $\chi = 1$ on $[\widetilde{b}, 1]$, and let $v = e^{\tau\phi}\chi\vartheta$ with $\tau > 0$. We set

$$P_\phi = e^{\tau\phi}\mathcal{P}e^{-\tau\phi} - \sigma^2$$

with

$$\mathcal{S} = -\frac{d}{dx}\left(x^\alpha\frac{d}{dx}\right) - \tau^2x^\alpha|\phi'|^2 - \sigma^2, \quad \mathcal{A} = 2\tau x^\alpha\phi'\frac{d}{dx} + \tau(x^\alpha\phi')',$$

in order that $P_\phi v = e^{\tau\phi}(\mathcal{P}-\sigma^2)(\chi\vartheta)$ and $P_\phi v = \mathcal{S}v+\mathcal{A}v$ which gives $\|P_\phi v\|^2_{L^2(0,1)} = \|\mathcal{S}v\|^2_{L^2(0,1)} + \|\mathcal{A}v\|^2_{L^2(0,1)} + 2(\mathcal{S}v,\mathcal{A}v)_{L^2(0,1)}$.

Classical computations lead to

$$\begin{aligned}(\mathcal{S}v,\mathcal{A}v)_{L^2(0,1)} &= 2\tau\int_0^1 x^{2\alpha}\phi''|v'|^2dx + \tau\alpha\int_0^1 x^{2\alpha-1}\phi'|v'|^2dx \\ &-\frac{\tau}{2}\int_0^1 (\mathcal{P}^2\phi)|v|^2dx + 2\tau^3\int_0^1 x^{2\alpha}\phi''(\phi')^2|v|^2dx + \alpha\tau^3\int_0^1 x^{2\alpha-1}(\phi')^3|v|^2dx - \tau\phi'(1)|v'(1)|^2.\end{aligned}$$

But, using $\phi = e^{\lambda\psi}$ with ψ having a non-vanishing gradient, there exist five constants $C_0, C_1, C_2, C_3, C_4 > 0$ independent on $\alpha \in [0,1)$ such that

$$\begin{aligned}(\mathcal{S}v,\mathcal{A}v)_{L^2(0,1)} \geqslant \tau\lambda^2 C_0\int_0^1 \phi|v'|^2dx + \tau^3\lambda^4 C_1\int_0^1 \phi^3|v|^2dx - \tau\lambda C_2\int_0^1 \phi|v'|^2dx \\ - \tau\lambda^4 C_3\int_0^1 \phi|v|^2dx - \tau^3\lambda^3 C_4\int_0^1 \phi^3|v|^2dx + \tau|\phi'(1)||v'(1)|^2.\end{aligned}$$

Therefore, the fact that $0 \leqslant \|P_\phi v\|^2_{L^2(0,1)} - 2(\mathcal{S}v,\mathcal{A}v)_{L^2(0,1)}$ implies by taking $\lambda > 0$ sufficiently large, and $\tau > 0$ sufficiently large the following inequality

$$\|v\|^2_{H^1(0,1)} + |v'(1)|^2 \lesssim \|P_\phi v\|^2_{L^2(0,1)}.$$

Taking the weights off the integrals and using commutators, we have

$$|\vartheta'(1)|^2 \lesssim \|P_\phi \vartheta\|^2_{L^2(0,1)} + \|\vartheta\|^2_{H^1(\widetilde{a},\widetilde{b})}.$$

This ends the proof of Lemma 5.3.

□

6 Observability estimate for the degenerate heat equation (proof of Theorem 1.2)

In this section, we prove that the refine observability from measurable set of Theorem 1.2 is a corollary of the spectral Lebeau-Robbiano inequality of Theorem 1.1.

Let $\widetilde{\omega} \Subset \omega$ and $\chi \in C_0^\infty(\omega)$ be such that $0 \leqslant \chi \leqslant 1$ and $\chi = 1$ in $\widetilde{\omega}$. We start with Theorem 3.1 of [BP, page 1142] stating that (i) implies (ii) where

(i) $\exists C_1 > 0$, $\forall \{a_j\} \in \mathbb{R}$, $\forall \Lambda > 0$

$$\sum_{\lambda_j \leqslant \Lambda} |a_j|^2 \leqslant e^{C_1\left(1+\sqrt{\Lambda}\right)}\int_{\widetilde{\omega}}\left|\sum_{\lambda_j\leqslant\Lambda} a_j\Phi_j\right|^2 dx;$$

(ii) $\forall t > 0$, $\forall \varepsilon \in (0,2)$, $\forall y_0 \in L^2(0,1)$

$$\|e^{-t\mathcal{P}}y_0\|_{L^2(0,1)} \leqslant 4e^{\frac{C_1}{2}}e^{\frac{C_1^2}{2\varepsilon t}}\|e^{-t\mathcal{P}}y_0\|^{1-\varepsilon/2}_{L^2(\widetilde{\omega})}\|y_0\|^{\varepsilon/2}_{L^2(0,1)}.$$

Therefore, by Theorem 1.1 we know that (ii) holds with $C_1 = C\dfrac{1}{(2-\alpha)^2} > 1$. By Nash inequality and regularizing effect, we get for some constants $c > 1$ and $\theta \in (0,1)$ independent on (y_0, t) and $\alpha \in [0,2)$

$$\|e^{-t\mathcal{P}}y_0\|_{L^2(\widetilde{\omega})} \leqslant c\left(1+\frac{1}{\sqrt{t}}\right)^\theta \|e^{-t\mathcal{P}}y_0\|^{1-\theta}_{L^1(\omega)}\|y_0\|^\theta_{L^2(0,1)}.$$

Therefore, since $1+\frac{1}{\sqrt{t}} \leqslant 4e^{\frac{C_1^2}{2\varepsilon t}}$, with $C_2 = 16ce^{\frac{C_1}{2}}e^{\frac{C_1^2}{\varepsilon t}} \geqslant 4e^{\frac{C_1}{2}}e^{\frac{C_1^2}{2\varepsilon t}}\left(c\left(1+\frac{1}{\sqrt{t}}\right)^{\theta}\right)^{1-\varepsilon/2}$ and $\mu = 1-(1-\theta)\left(1-\frac{\varepsilon}{2}\right)$, we obtain

$$\|e^{-t\mathcal{P}}y_0\|_{L^2(0,1)} \leqslant C_2\|e^{-t\mathcal{P}}y_0\|_{L^1(\omega)}^{1-\mu}\|y_0\|_{L^2(0,1)}^{\mu},$$

which implies by Young inequality

$$\begin{aligned}\|e^{-t\mathcal{P}}y_0\|_{L^2(0,1)} &\leqslant s\|y_0\|_{L^2(0,1)} + \frac{1}{s^{\frac{\mu}{1-\mu}}}C_2^{\frac{1}{1-\mu}}\|e^{-t\mathcal{P}}y_0\|_{L^1(\omega)} \\ &\leqslant s\|y_0\|_{L^2(0,1)} + \frac{1}{s^{\frac{\mu}{1-\mu}}}\left(16ce^{\frac{C_1}{2}}\right)^{\frac{1}{1-\mu}}e^{\frac{C_1^2}{\varepsilon t}\frac{1}{1-\mu}}\|e^{-t\mathcal{P}}y_0\|_{L^1(\omega)}.\end{aligned}$$

Reproducing the proof of Theorem 1.1 of [PW, page 684], we have for our system that (iii) implies (iv) where

(iii) $\exists K_1, K_2, \ell > 0$, $\forall s > 0$

$$\|e^{-t\mathcal{P}}y_0\|_{L^2(0,1)} \leqslant s\|y_0\|_{L^2(0,1)} + \frac{1}{s^{\ell}}K_1e^{\frac{K_2}{t}}\|e^{-t\mathcal{P}}y_0\|_{L^1(\omega)};$$

(iv) $\forall y_0 \in L^2(0,1)$

$$\|e^{-T\mathcal{P}}y_0\|_{L^2(0,1)} \leqslant K_3\int_{\omega\times E}\left|e^{-t\mathcal{P}}y_0\right| dxdt$$

with

$$K_3 = \begin{cases} c\frac{K_1}{K_2}e^{cK_2} & \text{when } E\subset(0,T) \text{ is a measurable set of positive measure,} \\ \kappa\frac{K_1}{K_2}e^{\kappa\frac{K_2}{T}} & \text{when } E=(0,T) \text{ for some } \kappa>0 \text{ independent on } T.\end{cases}$$

Therefore, with $K_1 = \left(16ce^{\frac{C_1}{2}}\right)^{\frac{1}{(1-\theta)\left(1-\frac{\varepsilon}{2}\right)}}$ and $K_2 = \frac{1}{\varepsilon(1-\theta)\left(1-\frac{\varepsilon}{2}\right)}C_1^2$ we have $K_3 \leqslant Ce^{C\frac{1}{(2-\alpha)^4}}$. This completes the proof of Theorem 1.2.

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