ON THE EQUILIBRIATION OF CHEMICAL REACTION-DIFFUSION SYSTEMS WITH DEGENERATE REACTIONS

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Abstract. The trend to equilibrium for reaction-diffusion systems modelling chemical reaction networks is investigated, in the case when reaction processes happen on subsets of the domain. We prove the convergence to equilibrium by directly showing functional inequalities in terms of entropy method. Our approach allows us to deal with nonlinearities of arbitrary orders, for which only global renormalised solutions are known to globally exist. For bounded solutions, we also prove the convergence to equilibrium when the diffusion as well as the reaction are degenerate, that is both diffusion and reaction processes only act on specific subsets of the domain.

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Contents

1. Introduction and main results

Convergence to equilibrium for reaction-diffusion systems modelling chemical reactions has been studied since the eighties in e.g. [Grö83, GGH96, GH97] and has witnessed considerable progress recently, see e.g. [DF06, FL16, DFT17, MHM15, PSZ17, HHMM18, GS22, MS24] and references therein. Most of these works, if not all, assume a common condition: the diffusion and reactions in the system are non-degenerate, in the sense that all the chemical species diffuse and the diffusion and reactions take place everywhere in the spatial domain. When there is degeneracy, showing convergence to equilibrium is more challenging. The case of degenerate diffusion, i.e. one or some chemical species do not diffuse, has been considered for some special systems in [DF15, FLT18, EMT20]. To show the convergence to equilibrium in this situation, these works utilised the so-called *indirect diffusion effect*, which, roughly speaking, means that the combination of the diffusion of some of the species and of reversible reactions leads to certain "diffusion effect" on non-diffusive species. Extending this theory to general systems still remains as an open problem. The case of degenerate reaction is much less studied, and, up to our knowledge, this has been considered only in the recent work [DP22]. In [DP22], the reversible reaction $2\mathsf{S}_1 \leftrightarrows 2\mathsf{S}_2$ was investigated in the case when the reactions happen only in an open subset of the domain with positive measure. By utilising a technique stemming from controllability theory, namely log convexity, and the regularity of solutions, it was shown that solutions still converge exponentially to the chemical equilibrium. While the method therein is sophisticated, it seems difficult to generalise it to more general reaction networks. In this paper, we use a different approach based on proving directly entropy-entropy dissipation functional inequalities. Thanks to this, we can deal with a much larger class of systems, namely complex balanced systems with arbitrarily high orders of reactions. This approach is also sufficiently robust so that we can deal with various types of degeneracy, for instance when reactions happen in very rough domains, or when both reactions and diffusions are degenerate. The results in this work significantly extend the literature convergence to equilibrium for chemical reaction networks, cf. [GGH96, MHM15, DFT17, FT18], to the situation with degenerate reactions. To our knowledge, this is also the first work showing convergence to equilibrium when both reactions and diffusion can be degenerate.

1.1. **Chemical reaction-diffusion systems.** Consider m chemical species $\mathsf{S}_1, \ldots, \mathsf{S}_m$ reacting via the following R reactions

$$y_{r,1}\mathsf{S}_1 + \cdots + y_{r,m}\mathsf{S}_m \xrightarrow{k_r(x,t)} y'_{r,1}\mathsf{S}_1 + \cdots + y'_{r,m}\mathsf{S}_m, \quad r = 1, \ldots, R, \tag{1.1}$$

where $k_r(x,t)$ are the *reaction rate coefficients* whose value depends on the spatial variable $x \in \Omega$ and time $t \in \mathbb{R}_+$, and $y_{r,i}, y'_{r,i} \in \{0\} \cup [1, \infty)$ are stoichiometric coefficients. Denoting by $y_r = (y_{r,i})_{i=1,\ldots,m}$ and $y'_r = (y'_{r,i})_{i=1,\ldots,m}$, $r = 1, \ldots, R$, the vector of stoichiometric coefficients, we can rewrite the reactions in (1.1) as

$$y_r \xrightarrow{k_r(x,t)} y'_r, \quad r = 1, \ldots, R. \tag{1.2}$$

Assume that the reaction system takes place in a bounded vessel $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let $u_i := u_i(x,t)$ be the concentration of S_i at position $x \in \Omega$ at time $t > 0$. Assume that each species diffuses at a different rate. Then one can apply second Fick's law and the law of mass action to obtain the following reaction diffusion system for the vector of concentrations $u = (u_1, \ldots, u_m)$

$$\begin{cases} \partial_t u_i - \nabla \cdot (D_i(x,t)\nabla u_i) = R_i(x,t,u) := \displaystyle\sum_{r=1}^{R} k_r(x,t)(y'_{r,i} - y_{r,i})u^{y_r}, & x \in \Omega, \\ D_i(x,t)\nabla u_i \cdot \nu = 0, & x \in \partial\Omega, \\ u_i(x,0) = u_{i,0}(x), & x \in \Omega, \end{cases} \tag{1.3}$$

for all $i = 1, \ldots, m$, where

$$u^{y_r} = \prod_{i=1}^{m} u_i^{y_{r,i}},$$

the diffusion coefficients $D_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $\nu(x)$ is unit outward normal vector at $x \in \partial\Omega$. Here the homogeneous Neumann boundary conditions indicate that the chemical system is isolated. Thanks to these conditions, there are possibly a number of conservation laws corresponding to (1.3). Indeed, denoting

$$\mathsf{W} = (y'_r - y_r)_{r=1,\ldots,R} \in \mathbb{R}^{m\times R},$$

and $K := \dim(\ker(\mathsf{W}^\top))$, we let $q_1, \ldots, q_K$ be the column vectors forming a basis of $\ker(\mathsf{W}^\top)$. Then from the system (1.3) we have, *formally*, for any $1 \le j \le K$

$$\frac{d}{dt}\int_\Omega q_j \cdot u dx = \sum_{r=1}^{R}\int_\Omega k_r(x,t)(q_j \cdot (y'_r - y_r))u^{y_r} dx = 0,$$

which lead to K linearly independent conservation laws

$$\int_\Omega q_j \cdot u(x,t)dx = \int_\Omega q_j \cdot u_0(x)dx, \quad \forall j = 1,\ldots,K, \tag{1.4}$$

where $u_0 := (u_{1,0},\ldots,u_{m,0})$. By the rescaling $x \mapsto |\Omega|^{-1/n}x$, we can assume that Ω has volume one, i.e. $|\Omega| = 1$, **which we will assume throughout this paper**.

1.2. **General systems with degenerate reactions.** We consider first the case when (only) the reactions happen in a subdomain of Ω. In order to set up the problem, we use a graph representation of the reaction network (1.1). Let $\mathsf{V} = \{y_r, y_r'\}_{r=1,\ldots,R} \subset \mathbb{R}_+^m$ be the set of vertices (a set of points in $\mathbb{R}_+^m$). The set of directed edges are the reactions in (1.1), $\mathsf{Edge} = \{y_r \to y_r' : r = 1,\ldots,R\}$. We remark that for convenience, if a letter, say y, denotes the reactant complex in a reaction, then the corresponding letter y' denotes the product complex. Then $\mathsf{G} = (\mathsf{V},\mathsf{Edge})$ forms a directed graph. A subset of vertices $\mathsf{U} \subset \mathsf{V}$ is called *strongly connected*, if for any $v_1 \neq v_2 \in \mathsf{U}$, there exists a sequences of vertices $v_1 =: w_1 \to w_2 \to \cdots \to w_r := v_2$, $r \geq 2$, where $w_j \to w_{j+1} \in \mathsf{Edge}$ for all $j = 1,\ldots,r-1$. A subset $\mathsf{U} \subset \mathsf{V}$ is called a *strongly connected component* if it is strongly connected, and $\mathsf{U} \subsetneq \mathsf{U}' \subset \mathsf{V}$ then U' is not strongly connected. A classical result in graph theory implies that G can be decomposed into strongly connected components. In this paper, we assume the following:

> There is no edge between any two different strongly connected components of the graph G. (**A**)

It is remarked that even though the components are disconnected, they do not possess their own decoupled dynamics since the chemical species can be present in all components (see Figure 1). Denote by $s \geq 1$ the number of strongly connected components of G, i.e. G consists of the components $\mathsf{C}_1,\ldots,\mathsf{C}_s$. Thanks to assumption (**A**), without loss of generality, we can re-label the vertices of G such that there exist $L_0 = 0 < L_1 < \ldots < L_{s-1} < L_s = R$ with the property: for any $1 \leq l \leq s$ the reactions $y_j \to y_j'$ for $L_{l-1}+1 \leq j \leq L_l$ form the l-th component.

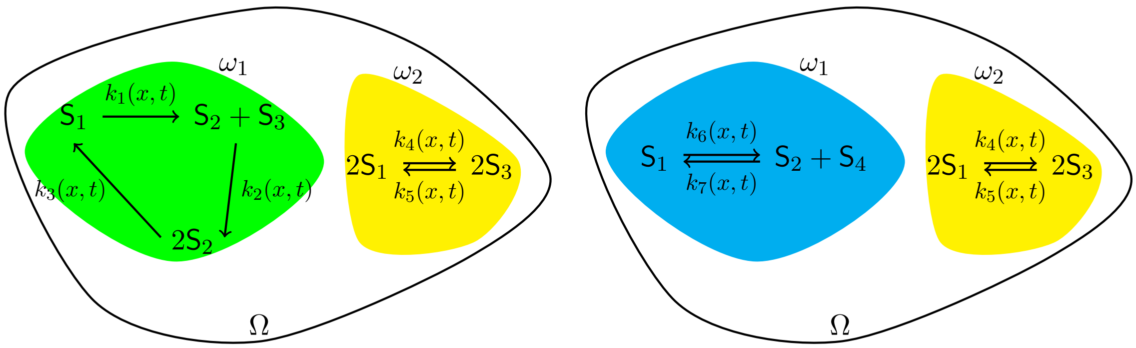
A subset $A \subset \mathbb{R}^n$ is said to satisfy an assumption (**P**) if the Poincaré-Wirtinger inequality in A holds, i.e. there exists a constant C_A depending only on A such that

$$\|\nabla u\|_{L^2(A)}^2 \geq C_A \left\| u - \frac{1}{|A|}\int_A u(x)dx \right\|_{L^2(A)}^2 \qquad \forall u \in H^1(A). \tag{$\mathbf{P}$}$$

The next assumption is concerning the case when the partial reactions take place in positive measured sets.

> For each $l \in \{1,\ldots,s\}$, there is a function $\alpha_l : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $k_j(x,t) = \beta_j \alpha_l(x,t)$ for some $\beta_j > 0$ and for $j = L_{l-1}+1,\ldots,L_l$. Moreover, there exists a subset $\omega_l \subset \Omega$ with $|\omega_l| > 0$ satisfying (**P**), and a positive number $\underline{\alpha_l} > 0$ such that $\alpha_l(x,t) \geq \underline{\alpha_l}$ for a.e. $x \in \omega_l$. (**B**)

Roughly speaking, assumption (**B**) means in particular that for each strongly connected component C_l, all of its reaction rate coefficients scale with a function α_l. This is important to define a complex balanced equilibrium to (1.3) (see Definition 1.1). The lower bound assumption of α_l means that there is a positively measured set ω_l, which satisfies (**P**), where all reactions of the component C_l happen. A specific example for (**B**) is when the sets $\{x \in \Omega : \alpha_l(x) \geq \underline{\alpha_l}\}$ are open for all $l = 1,\ldots,s$. We present in Figure 1 (a) an example where (**A**) and (**B**) are satisfied. Of particular physical relevance is the case when each component consists of a reversible reaction. This happens, for instance, when each of these reversible reactions requires a certain catalysis, which is present only in a subset of the medium, see Figure 1 (b) for such a situation.



(a) (b)

FIGURE 1. Example of a complex balanced network satisfying (**A**) and (**B**).
• In (a), there are two strongly connected components $\mathsf{C}_1 = \{\mathsf{S}_1, \mathsf{S}_2 + \mathsf{S}_3, 2\mathsf{S}_2\}$ and $\mathsf{C}_2 = \{2\mathsf{S}_1, 2\mathsf{S}_3\}$ together with corresponding reactions, and the reactions in these components happen in open sets ω_1 and ω_2, respectively. Here for $i \in \{1,2,3\}$, $k_i(x,t) = \beta_i\alpha_1(x,t)$ with $\alpha_1(x,t) \geq \underline{\alpha}_1 > 0$ in ω_1, and for $j \in \{4,5\}$, $k_j(x,t) = \beta_j\alpha_2(x,t)$ with $\alpha_2(x,t) \geq \underline{\alpha}_2 > 0$ in ω_2. Note that the chemicals S_1 and S_3 appear both on ω_1 and ω_2, and therefore the dynamics of the whole system couples the reactions in both of these subdomains.
• In (b) each component has a single reversible reaction. This could be physically relevant, for instance, when the reversible reactions $\mathsf{S}_1 \leftrightarrows \mathsf{S}_2 + \mathsf{S}_4$ and $2\mathsf{S}_1 \leftrightarrows 2\mathsf{S}_3$ require certain catalysts to happen and these catalysts are only present in ω_1 and ω_2 respectively.

Definition 1.1. *A spatially homogeneous state* $u_\infty = (u_{1,\infty}, \ldots, u_{m,\infty}) \in \mathbb{R}^m_{\geq 0}$ *is called a* ***complex balanced equilibrium (CBE for short)*** *for the system* (1.3) *if for any* $l = 1, \ldots, s$ *and any* $y \in \mathsf{C}_l$*, the following equality holds*

$$u_\infty^y \sum_{\substack{L_{l-1}+1 \leq j \leq L_l \\ y_j = y}} \beta_j = \sum_{\substack{L_{l-1}+1 \leq j \leq L_l \\ y_j = y}} \beta_j u_\infty^{y_j} = \sum_{\substack{L_{l-1}+1 \leq k \leq L_l \\ y_k' = y}} \beta_k u_\infty^{y_k}. \tag{1.5}$$

If u_∞ *is a CBE and* $u_\infty \in \partial\mathbb{R}^m_{\geq 0}$*, then it is called a* ***boundary equilibrium****.*

Thanks to assumptions (**A**) and (**B**), it follows that $R_i(u_\infty) = 0$ for all $i = 1, \ldots, m$, which means that u_∞ a spatially homogeneous steady state of system (1.3).

Remark 1.2. *Consider the reversible reaction* $\mathsf{S}_1 \underset{k_1(x)}{\overset{k_2(x)}{\leftrightarrows}} \mathsf{S}_2$*, where the reaction rate coefficients* k_1, k_2 *depend only on* $x \in \Omega$*, which results in the reaction-diffusion system*

$$\begin{cases} \partial_t u_1 - \nabla \cdot (D_1(x,t)\nabla u_1) = -k_1(x)u_1 + k_2(x)u_2, \\ \partial_t u_2 - \nabla \cdot (D_2(x,t)\nabla u_2) = k_1(x)u_1 - k_2(x)u_2, \\ D_i(x,t)\nabla u_i \cdot \nu = 0, \quad i = 1,2, \\ u_i(x,0) = u_{i,0}(x), \quad i = 1,2. \end{cases} \tag{1.6}$$

When k_1 *and* k_2 *are strictly positive constants, then the network is obviously complex balanced and there is a unique positive equilibrium for each positive initial total mass. However, if* $supp(k_1) \cap supp(k_2) = \emptyset$*, then* (**B**) *is violated and the only spatially homogeneous steady state of* (1.6) *is the zero state* $(0,0)$*.*

It is also remarked that in the case when the functions k_r are constants, the CBE u_∞ in Definition 1.1 coincides with the classical definition in chemical reaction network theory, see e.g. [Fei19]. It can be also seen from Definition 1.1 that the set of CBE forms a manifold (possibly with singularities) in $\mathbb{R}_+^m$. To uniquely determine u_∞, we need the conservation laws (1.4).

Lemma 1.3. [Fei19] *Assume assumptions* ($\mathbf{A}$) *and* ($\mathbf{B}$). *If there exists a CBE u_∞ as in Definition 1.1, then any spatially homogeneous steady state of* (1.3) *is complex balanced. Moreover, for any non-negative initial data $u_0 \in L^1_+(\Omega)^m$, there exists a unique **strictly positive** CBE $u_\infty \in (0,\infty)^m$ satisfying* (1.5) *and the conservation laws*

$$q_j \cdot u_\infty = \int_\Omega q_j \cdot u_0(x)dx, \quad \forall j = 1, \ldots, K$$

where $(q_j)_{j=1,\ldots,K}$ is defined in (1.4). *It is remarked that there might exist (possibly infinitely) many boundary equilibria.*

Due to Lemma 1.3 we will refer to the strictly positive CBE simply as CBE. Our first main result of this paper is the exponential convergence to equilibrium for the system (1.3) under assumptions ($\mathbf{A}$), ($\mathbf{B}$), and the fact that there are no boundary equilibria.

Theorem 1.4. *Assume the following*

(i) ($\mathbf{A}$) *and* ($\mathbf{B}$);

(ii) *the diffusion matrices are symmetric and bounded, i.e. $D_i \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}}))$, and*

$$\xi^\top D_i(x,t)\xi \geq \underline{D}_i|\xi|^2, \quad \forall \xi \in \mathbb{R}^m, \text{ a.e. } (x,t) \in \Omega \times \mathbb{R}_+$$

for some $\underline{D}_i > 0$, for all $i = 1, \ldots, m$;

(iii) *there exists a CBE to* (1.3) *as defined in Definition 1.1;*

(iv) *there are no boundary equilibria.*

Then for any non-negative initial data $u_0 \in L^1_+(\Omega)^m$ such that $\sum_{i=1}^m \int_\Omega u_{i,0}|\log u_{i,0}|dx < +\infty$, there exists a global renormalised solution to (1.3) *as in Definition 2.1 below. Moreover, all renormalised solutions converge exponentially to CBE with an exponential rate, i.e.*

$$\sum_{i=1}^m \|u_i(t) - u_{i,\infty}\|_{L^1(\Omega)} \leq Ce^{-\lambda t}, \quad \forall t \geq 0.$$

It is emphasised that, in general, the global existence theory for (1.3) is highly non-trivial due to the possible arbitrarily high orders of the nonlinearities, see e.g. [Pie10] for an extensive survey. If (1.3) possesses an entropic dissipation structure, which is a consequence of having a CBE, the only known concept of global solution to (1.3) is *renormalised solutions*, see e.g. [Fis15], which has minimal regularity, which in turns makes the study of their dynamics highly challenging. In order to prove Theorem 1.4, we use the entropy method, which was widely used in kinetic theory and other fields in the 90s (cf. for example [DV00]), and later extended to chemical reaction-diffusion systems [DF06, DF07, MHM15, MM18, HHMM18]. An important feature of this method is that it relies on functional inequalities and consequently requires minimal regularity of solutions, see e.g. [FT18]. This is in contrast to that of [DP22] and therefore it allows us to show the equilibration of *all* renormalised solutions. More precisely, we consider the following relative entropy

$$\mathcal{E}(u|u_\infty) = \sum_{i=1}^m \int_\Omega \left(u_i \log\frac{u_i}{u_{i,\infty}} - u_i + u_{i,\infty}\right)dx,$$

and the corresponding entropy dissipation

$$\mathcal{D}(u) = \sum_{i=1}^m \int_\Omega D_i(x,t)\nabla u_i \cdot \frac{\nabla u_i}{u_i}dx + \sum_{r=1}^R \int_\Omega k_r(x,t)u_\infty^{y_r}\Psi\left(\frac{u^{y_r}}{u_\infty^{y_r}}; \frac{u^{y_r'}}{u_\infty^{y_r'}}\right)dx,$$

where the function Ψ is defined as

$$\Psi(w;z) = w\log\frac{w}{z} - w + z.$$

Formally, one expects the entropy-entropy dissipation law, see [DFT17, Proposition 2.1]

$$\frac{d}{dt}\mathcal{E}(u|u_\infty) = -\mathcal{D}(u). \tag{1.7}$$

The cornerstone of the entropy method is to show the following functional inequality

$$\boxed{\mathcal{D}(u) \gtrsim \mathcal{E}(u|u_\infty)} \quad \forall u : \Omega \to \mathbb{R}_+^m \text{ satisfying the conservation laws (1.4).}$$

To overcome the difficulty stemming from degeneracy of the reactions, our key idea is to control the reaction terms in the entropy dissipation by their partial averages in corresponding subdomains where reactions happen, and then to estimate the differences by using the diffusion of all species.

1.3. **A specific case of even more degenerate situations.** In the proof of Theorem 1.4, it is of importance that the diffusion is non-degenerate and the reaction happens in a subdomain which has certain regularity, e.g. Lipschitz boundary, or contains an open domain. The latter allows us to apply the Poincaré inequality (**P**) in (a subset of) the subdomain, which then combines with the reaction to drive the trajectory eventually to the spatially homogeneous equilibrium. Due to the low regularity of renormalised solutions, relaxing or weakening these assumptions seem difficult. However, if the solution is known to be bounded uniformly in time, it might be possible to handle degenerate diffusion as well as reaction in much rougher subdomains, namely subsets of Ω which are only *measurable* with positive measure. A key idea in these situations is that using the uniform boundedness of solutions, we can estimate many quantities, e.g. the relative entropy, *pointwise* rather just through integrals. We illustrate this by studying the following reversible reactions

$$\mathsf{S}_1 \underset{k_1(x,t)}{\overset{k_1(x,t)}{\leftrightarrows}} 2\mathsf{S}_2, \quad \mathsf{S}_2 \underset{k_2(x,t)}{\overset{k_2(x,t)}{\leftrightarrows}} 2\mathsf{S}_3,$$

which result in the reaction-diffusion system

$$\begin{cases} \partial_t u_1 - \nabla\cdot(d_1(x,t)\nabla u_1) = k_1(x,t)\left(u_2^2 - u_1\right), & \text{in } \Omega\times\mathbb{R}_+,\\ \partial_t u_2 - \nabla\cdot(d_2(x,t)\nabla u_2) = -2k_1(x,t)\left(u_2^2 - u_1\right) + k_2(x,t)\left(u_3^2 - u_2\right), & \text{in } \Omega\times\mathbb{R}_+,\\ \partial_t u_3 - \nabla\cdot(d_3(x,t)\nabla u_3) = -2k_2(x,t)\left(u_3^2 - u_2\right), & \text{in } \Omega\times\mathbb{R}_+,\\ d_1(x,t)\nabla u_1\cdot\nu = d_2(x,t)\nabla u_2\cdot\nu = d_3(x,t)\nabla u_3\cdot\nu = 0, & \text{on } \partial\Omega\times\mathbb{R}_+,\\ u_1(\cdot,0) = u_{1,0}, \quad u_2(\cdot,0) = u_{2,0}, \quad u_3(\cdot,0) = u_{3,0}, & \text{in } \Omega. \end{cases} \tag{1.8}$$

Here $0 \le k_1, k_2 \in L^\infty(\Omega\times(0,+\infty))$ are reaction rate coefficients. It is easy to check that solutions to (1.8) satisfy the conservation law

$$\int_\Omega (4u_1(x,t) + 2u_2(x,t) + u_3(x,t))dx = \int_\Omega (4u_{1,0}(x) + 2u_{2,0}(x) + u_{3,0}(x))dx =: M, \tag{1.9}$$

for all t where the solutions exist. It is easy to see that for any positive initial mass M defined in (1.9) there exists a unique strictly positive equilibrium $u_\infty = (u_{1,\infty}, u_{2,\infty}, u_{3,\infty})$ which solves

$$\begin{cases} u_{2,\infty}^2 = u_{1,\infty},\\ u_{3,\infty}^2 = u_{2,\infty},\\ 4u_{1,\infty} + 2u_{2,\infty} + u_{3,\infty} = M, \end{cases} \tag{1.10}$$

since $x \mapsto 4x^4 + 2x^2 + x$ is strictly increasing on $\mathbb{R}_+$.

Let ω_1 and ω_2 be non-empty subsets of Ω. To study the convergence to equilibrium for (1.8), we assume that there is some $\kappa > 0$ such that

$$\begin{aligned} k_1(x,t) &\geq \kappa \quad \forall (x,t) \in \omega_1 \times \mathbb{R}_+, \\ k_2(x,t) &\geq \kappa \quad \forall (x,t) \in \omega_2 \times \mathbb{R}_+. \end{aligned} \tag{1.11}$$

For the (scalar) diffusion coefficients d_i, we assume that

$$d_i \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\Omega)), \quad i = 1,2,3, \tag{1.12}$$

and that there is some $\delta > 0$ such that

$$d_1(x,t) \geq \delta \quad \text{and} \quad d_2(x,t) \geq \delta, \quad \forall (x,t) \in \Omega \times \mathbb{R}_+. \tag{1.13}$$

We also assume that there exists $\delta_0 \geq 0$ with

$$d_3(x,t) \geq \delta_0, \quad \text{a.e. } (x,t) \in \Omega \times \mathbb{R}_+. \tag{1.14}$$

In the first case, we consider $\delta_0 > 0$, meaning that $\mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3$ have full diffusion, but the sets ω_1 and ω_2 where reactions happen are *only measurable* with positive measures.

Theorem 1.5. *Assume* (1.11)*,* (1.12)*,* (1.13) *and* (1.14) *with* $\delta_0 > 0$ *and* ω_1, ω_2 *are* measurable sets *with positive measures. Then for any non-negative, bounded initial data* $u_0 \in L^\infty_+(\Omega)^3$*, there exists a unique non-negative, weak solution to* (1.8)*, which is bounded uniformly in time, i.e.*

$$\sup_{t\geq 0} \sup_{i=1,\ldots,3} \|u_i(t)\|_{L^\infty(\Omega)} \leq \mathcal{C}_0 < +\infty. \tag{1.15}$$

Moreover, this solution converges exponentially fast, i.e. there are explicitly computable *constants* $C, \lambda > 0$ *such that*

$$\sum_{i=1}^3 \|u_i(t) - u_{i,\infty}\|^2_{L^1(\Omega)} \leq Ce^{-\lambda t}, \quad \forall t > 0,$$

where u_∞ *solves* (1.10)*.*

Remark 1.6.

- *The convergence rate to equilibrium depends on* ω_1 *and* ω_2 *in the following way*

$$\frac{1}{\lambda} = C\left(\frac{1}{|\omega_1|} + \frac{1}{|\omega_2|}\right),$$

  where the constant $C > 0$ *is independent of* ω_1 *and* ω_2*. It is clear that with this relation,* $\lambda \to 0$ *if either* $|\omega_1| \to 0$ *or* $|\omega_2| \to 0$*.*
- *By interpolating the exponential convergence with the* $L^\infty(\Omega)$*-bound* (1.15)*, one can immediately get exponential convergence to equilibrium in* $L^p(\Omega)$ *for any* $1 < p < \infty$*. Convergence in stronger norms is possible to obtain when the functions* d_i *and* k_i *are sufficiently smooth.*

If the diffusion of S_3 is completely degenerate, i.e. $d_3(x,t) = 0$ for all $(x,t) \in \Omega \times \mathbb{R}_+$, the global existence of solutions can be obtained with bounded initial data, see e.g. [EMT20] or [BET22]. However, if $\{(x,t) : d_3(x,t) = 0\}$ has strictly positive measure but is not zero on $\Omega \times \mathbb{R}_+$, the global existence of solutions to (1.8) is nontrivial, see e.g. [DFPV07]. In our second example, we prove the global existence and convergence to equilibrium in the case when the diffusion of S_3 is not "too" degenerate, i.e. d_3 depends only on $x \in \Omega$ and vanishes only on a zero measure set, that is

$$|\{x \in \Omega : d_3(x) = 0\}| = 0. \tag{1.16}$$

Theorem 1.7. *Suppose that* (1.11)*,* (1.12)*,* (1.13) *and* (1.16) *hold. Moreover, assume that* $d_3 \in W^{1,q}(\Omega)$ *for some* $q > \max(n,2)$*. Then for any non-negative, bounded initial data* $u_0 \in L^\infty_+(\Omega)^3$*,* *there exists a unique global non-negative weak solution to* (1.8)*, which is bounded uniformly in time, i.e.*

$$\sup_{t\geq 0}\sup_{i=1,\ldots,3}\|u_i(t)\|_{L^\infty(\Omega)} \leq \mathcal{C}_0 < +\infty. \tag{1.17}$$

Moreover, assume that ω_1 is measurable with $|\omega_1| > 0$, $\omega_2 \subset \Omega$ is open with Lipschitz boundary, and

$$\{x \in \overline{\Omega} : d_3(x) = 0\} \subset \omega_2. \tag{1.18}$$

Then the weak solution to (1.8) *converges to the equilibrium exponentially fast, i.e. there exist* explicitly computable *strictly positive constants $C, \lambda > 0$ such that*

$$\sum_{i=1}^{3}\|u_i(t) - u_{i,\infty}\|_{L^1(\Omega)} \leq Ce^{-\lambda t}, \quad \forall t \geq 0, \tag{1.19}$$

where u_∞ solves (1.10).

Remark 1.8.

- *Assumption* (1.18) *and the fact that ω_2 is open imply that there is an open set where a strictly positive diffusion of S_3 and the reaction $\mathsf{S}_2 \leftrightarrows 2\mathsf{S}_3$ are both present. This will be used crucially in our proof.*
- *When* (1.16) *is not satisfied, i.e. d_3 can be zero on a set of positive measure, the global existence of solutions to* (1.8) *is unclear, see e.g.* [DFPV07]*. Nevertheless, by replacing d_3 by $d_3 + \varepsilon$, we can use the same arguments in the proof of Theorem 1.7 to show that the solution to* (1.8) *(with d_3 replaced by $d_3 + \varepsilon$) converges to equilibrium exponentially with rates and constants* independent of $\varepsilon > 0$*. The global existence of solutions and convergence to equilibrium for* (1.8) *in case $|\{x \in \Omega : d_3(x) = 0\}| > 0$ remains as an interesting open problem.*

Notation. We use the following notation in this paper:

- for a measurable set A with positive measure, $[u]_A$ denotes the spatial average of u over A,
$$[u]_A := \frac{1}{|A|}\int_A u(x)dx;$$
- we use capital letters to denote the square roots of the corresponding letters, e.g.
$$U_i = \sqrt{u_i}, \quad U_{i,\infty} = \sqrt{u_{i,\infty}};$$
- the notation $X \lesssim Y$ means that there exists $C > 0$ independent of X and Y such that $X \leq CY$. Occasionally, we write
$$X \lesssim_{\alpha,\beta,\ldots} Y$$
to emphasise the dependence of the inequality on the parameters $\alpha, \beta, \ldots$
- for a positive vector $u \in (0,\infty)^m$ and $y \in \mathbb{R}^m$,
$$u^y := \prod_{i=1}^{m} u_i^{y_i}.$$

Organization of the paper. In the next section, we will prove Theorem 1.4 for degenerate reaction. The convergence to equilibrium with reactions happening in measurable sets (Theorem 1.5) will be shown in Section 3. Finally, Section 4 considers (1.8) with both degenerate diffusion and reactions as stated in Theorem 1.7.

2. Degenerate reactions - Proof of Theorem 1.4

We start with the definition of renormalised solutions to (1.3).

Definition 2.1. *A vector of concentration $u : \Omega\times\mathbb{R}_+ \to \mathbb{R}_+^m$ is called a global renormalised solution to* (1.3) *if for any $T > 0$, $u_i \log u_i \in L^\infty(0,T;L^1(\Omega))$, $\sqrt{u_i} \in L^2(0,T;H^1(\Omega))$ and for any smooth function $\xi : \mathbb{R}_+^m \to \mathbb{R}$ with compactly supported derivative $\nabla\xi$ and every $\psi \in C^\infty(\overline{\Omega}\times\mathbb{R}_+)$, there holds*

$$\begin{aligned}&\int_\Omega \xi(u(\cdot,T))\psi(\cdot,T)dx - \int_\Omega \xi(u_0)\psi(\cdot,0)dx - \int_0^T\int_\Omega \xi(u)\partial_t\psi dxdt\\&= -\sum_{i,j=1}^m\int_0^T\int_\Omega \psi\partial_i\partial_j\xi(u)(D_i(x,t)\nabla u_i)\cdot\nabla u_j dxdt\\&\quad -\sum_{i=1}^m\int_0^T\int_\Omega \partial_i\xi(u)(D_i(x,t)\nabla u_i)\cdot\nabla\psi dxdt + \sum_{i=1}^m\int_0^T\int_\Omega \partial_i\xi(u)R_i(x,t,u)\psi dxdt.\end{aligned}$$

Proposition 2.2. *Assume (i)–(iv) in Theorem 1.4. Then for any non-negative initial data $u_0 = (u_{i,0}) \in L^1_+(\Omega)^m$ with $\sum_{i=1}^m\int_\Omega u_{i,0}\log u_{i,0}dx < +\infty$, there exists a global renormalised solution to* (1.3).

Proof. We will apply [Fis15, Theorem 1]. In order to do that, it is sufficient to check that

$$\sum_{i=1}^m R_i(x,t,u)\log\frac{u_i}{u_{i,\infty}} \le 0.$$

We use the ideas in [DFT17, Proposition 2.1]. We rewrite, by using $R(x,t,u) := (R_i(x,t,u))$ and $\log(u/u_\infty) := (\log(u_i/u_{i,\infty}))$ for $i = 1,..,m$,

$$\begin{aligned}&\sum_{i=1}^m R_i(x,t,u)\log\frac{u_i}{u_{i,\infty}} = R(x,t,u)\cdot\log\frac{u}{u_\infty} = \sum_{r=1}^R k_r(x,t)u^{y_r}(y_r'-y_r)\cdot\log\frac{u}{u_\infty}\\&= -\sum_{r=1}^R k_r(x,t)u^{y_r}\log\frac{u^{y_r-y_r'}}{u_\infty^{y_r-y_r'}}\\&= -\sum_{r=1}^R k_r(x,t)u_\infty^{y_r}\left[\frac{u^{y_r}}{u_\infty^{y_r}}\log\left(\frac{u^{y_r}}{u_\infty^{y_r}}\Big/\frac{u^{y_r'}}{u_\infty^{y_r'}}\right) - \frac{u^{y_r}}{u_\infty^{y_r}} + \frac{u^{y_r'}}{u_\infty^{y_r'}}\right] - \sum_{r=1}^R\left(k_r(x,t)u^{y_r} - k_r(x,t)u^{y_r'}\frac{u_\infty^{y_r}}{u_\infty^{y_r'}}\right)\\&\le -\sum_{r=1}^R\left(k_r(x,t)u^{y_r} - k_r(x,t)u^{y_r'}\frac{u_\infty^{y_r}}{u_\infty^{y_r'}}\right).\end{aligned}$$

It remains to show that the last sum vanishes. Using assumption (**A**), we can write

$$\sum_{r=1}^R\left(k_r(x,t)u^{y_r} - k_r(x,t)u^{y_r'}\frac{u_\infty^{y_r}}{u_\infty^{y_r'}}\right) = \sum_{l=1}^s\sum_{j=L_{l-1}+1}^{L_l}\left(k_j(x,t)u^{y_j} - k_j(x,t)u^{y_j'}\frac{u_\infty^{y_j}}{u_\infty^{y_j'}}\right).$$

Now thanks to assumption (**B**), for each $l \in \{1,\ldots,s\}$,

$$\sum_{j=L_{l-1}+1}^{L_l}\left(k_j(x,t)u^{y_j} - k_j(x,t)u^{y_j'}\frac{u_\infty^{y_j}}{u_\infty^{y_j'}}\right) = \alpha_l(x,t)\sum_{j=L_{l-1}+1}^{L_l}\left(\beta_j u^{y_j} - \beta_j u^{y_j'}\frac{u_\infty^{y_j}}{u_\infty^{y_j'}}\right)$$

$$
\begin{aligned}
&= \alpha_l(x,t)\sum_{y\in \mathsf{C}_l}\left[\sum_{\substack{L_{l-1}+1\le j\le L_l\\ y_j=y}}\beta_j u^{y_j} - \sum_{\substack{L_{l-1}+1\le k\le L_l\\ y_k'=y}}\beta_k u^{y_k'}\frac{u_\infty^{y_k}}{u_\infty^{y_k'}}\right]\\
&= \alpha_l(x,t)\sum_{y\in \mathsf{C}_l}\left[u^y\sum_{\substack{L_{l-1}+1\le j\le L_l\\ y_j=y}}\beta_j - \frac{u^y}{u_\infty^y}\sum_{\substack{L_{l-1}+1\le k\le L_l\\ y_k'=y}}\beta_k u_\infty^{y_k}\right]\\
&= \alpha_l(x,t)\sum_{y\in \mathsf{C}_l}\frac{u^y}{u_\infty^y}\left[\sum_{\substack{L_{l-1}+1\le j\le L_l\\ y_j=y}}\beta_j u_\infty^{y_j} - \sum_{\substack{L_{l-1}+1\le k\le L_l\\ y_k'=y}}\beta_k u_\infty^{y_k}\right]\\
&= 0,
\end{aligned}
$$

thanks to the definition of u_∞ in Definition 1.1. □

Due to the low regularity of renormalised solution, we can only prove a weak version of the entropy-entropy dissipation relation (1.7). Moreover, it can also be shown that renormalised solutions satisfy the conservation laws (1.4).

Lemma 2.3. *Any renormalised solution to* (1.3) *satisfies the following weak entropy-entropy dissipation law*

$$
\mathcal{E}(u(s)|u_\infty)\Big|_{s=\tau}^{s=T} + \int_\tau^T \mathcal{D}(u(s))ds \le 0, \quad \forall 0\le \tau < T,
$$

and the conservation laws (1.4), *i.e.*

$$
\int_\Omega q_j\cdot u(x,t)dx = \int_\Omega q_j\cdot u_0(x)dx, \quad \forall j=1,\dots,K,\ \forall t\ge 0.
$$

Consequently, there is $M_0>0$ depending on $\mathcal{E}(u_0|u_\infty)$ such that

$$
\sup_{t\ge 0}\sup_{i=1,\dots,m}\|u_i(t)\|_{L^1(\Omega)} \le M_0. \tag{2.1}
$$

Proof. From the proof of Proposition 2.2, we have

$$
-\sum_{i=1}^m R_i(x,t,u)\log\frac{u_i}{u_{i,\infty}} = \sum_{r=1}^R\int_\Omega k_r(x,t)u_\infty^{y_r}\Psi\left(\frac{u^{y_r}}{u_\infty^{y_r}};\frac{u^{y_r'}}{u_\infty^{y_r'}}\right)dx.
$$

The weak entropy-entropy dissipation law and the conservation laws then follow from [Fis17, Propositions 5 and 6]. To show (2.1) we first note that by choosing $\tau=0$ in the weak entropy-entropy dissipation law, we have in particular

$$
\mathcal{E}(u(t)|u_\infty) \le \mathcal{E}(u_0|u_\infty) \quad \forall t\ge 0.
$$

Using the elementary inequality $u_i \le u_i\log(u_i/u_{i,\infty}) + C$, for a constant C depending only on $u_{i,\infty}$, we get (2.1) immediately. □

The following Csiszár-Kullback-Pinsker type inequality shows that a decay to zero of the relative entropy implies the convergence to equilibrium for solutions in $L^1(\Omega)$-norm.

Lemma 2.4. *There exists a constant $C_{\rm CKP} > 0$ depending on M_0 (see Lemma 2.3), the domain Ω, and the equilibrium u_∞, such that the following inequality holds for any renormalised solution to* (1.3)

$$\mathcal{E}(u|u_\infty) \geq C_{\rm CKP}\sum_{i=1}^m \|u_i - u_{i,\infty}\|_{L^1(\Omega)}^2.$$

Proof. It is straightforward to check that the relative entropy satisfies the additivity

$$\mathcal{E}(u|u_\infty) = \mathcal{E}(u|\,[u]_\Omega) + \mathcal{E}([u]_\Omega\,|u_\infty), \tag{2.2}$$

where

$$\mathcal{E}([u]_\Omega\,|u_\infty) := \sum_{i=1}^m\left([u_i]_\Omega \,\log\left(\frac{[u_i]_\Omega}{u_{i,\infty}}\right) - [u_i]_\Omega + u_{i,\infty}\right). \tag{2.3}$$

By applying Csiszár-Kullback-Pinsker's inequality for bounded domains, see e.g. [FL16, Proposition 4.1],

$$\int_\Omega f\log\frac{f}{[f]_\Omega}dx \geq \frac{1}{2\,[f]_\Omega}\|f - [f]_\Omega\,\|_{L^1(\Omega)}^2$$

and the upper bound $[u_i]_\Omega \leq M_0$ we have

$$\mathcal{E}(u|\,[u]_\Omega) = \sum_{i=1}^m\int_\Omega u_i\log\frac{u_i}{[u_i]_\Omega}dx \geq C_1\sum_{i=1}^m\|u_i - [u_i]_\Omega\,\|_{L^1(\Omega)}^2,$$

where C_1 depends only on Ω, the spatial dimension n and M_0. On the other hand, using the elementary inequality $x\log(x/y) - x + y \geq (\sqrt{x} - \sqrt{y})^2$ and the L^1-bound (2.1), we can estimate

$$\begin{aligned}\mathcal{E}([u]_\Omega\,|u_\infty) \geq \sum_{i=1}^m\left|\sqrt{[u_i]_\Omega} - \sqrt{u_{i,\infty}}\right|^2 &\geq \sum_{i=1}^m\frac{1}{(\sqrt{M_0}+\sqrt{u_{i,\infty}})^2}\left|\,[u_i]_\Omega - u_{i,\infty}\right|^2\\ &\geq C_2\sum_{i=1}^m\|\,[u_i]_\Omega - u_{i,\infty}\|_{L^1(\Omega)}^2.\end{aligned}$$

The proof of Lemma 2.4 is then completed with $C_{\rm CKP} := \min\{C_1, C_2\}/2$. □

A crucial tool for proving Theorem 1.4 is the following functional inequality.

Proposition 2.5. *Assume* (i)–(iv) *in Theorem 1.4. Then there exists a constant $\lambda > 0$ depending on $\Omega, u_\infty, L, y_r, y_r', \beta_j, |\omega_l|$ (see assumption* (**B**)*), such that*

$$\mathcal{D}(u) \geq \lambda\,\mathcal{E}(u|u_\infty)$$

for any non-negative function vector $u : \Omega \to \mathbb{R}_+^m$ satisfying $\sum_{i=1}^m\int_\Omega u_i\log u_i \leq L < +\infty$ and the conservation laws (1.4).

To prove Proposition 2.5, we start with some preliminary results. Recall the notation,

$$U_i = \sqrt{u_i},\; U = (U_1,\ldots,U_m),\quad U_{i,\infty} = \sqrt{u_{i,\infty}},\quad U_\infty = (U_{1,\infty},\ldots,U_{m,\infty}),$$

and for any measurable set A,

$$[f]_A := \frac{1}{|A|}\int_A f(x)dx.$$

Lemma 2.6. *For any renormalised solution to* (1.3)*, the following bounds*

$$\left[U_i^2\right]_\Omega + \left[U_i^2\right]_{\omega_l} + [U_i]_\Omega + [U_i]_{\omega_l} \lesssim 1,$$

hold for all $i = 1,\ldots,m$ and all $l = 1,\ldots,s$.

Proof. The estimates

$$\left[U_i^2\right]_\Omega + \left[U_i^2\right]_{\omega_l} \lesssim 1$$

follow directly from (2.1). By Cauchy-Schwarz inequality

$$[U_i]_\Omega = \int_\Omega U_i(x)dx \lesssim \left(\int_\Omega U_i^2(x)dx\right)^{1/2} \lesssim 1,$$

and the estimate $[U_i]_{\omega_l} \lesssim 1$ can be obtained similarly. □

An immediate estimate is

$$\begin{aligned}
&\sum_{i=1}^m \int_\Omega D_i(x,t)\nabla u_i \cdot \frac{\nabla u_i}{u_i}dx + \sum_{r=1}^R \int_\Omega k_r(x,t)u_\infty^{y_r}\Psi\left(\frac{u^{y_r}}{u_\infty^{y_r}}; \frac{u^{y_r'}}{u_\infty^{y_r'}}\right)dx \\
&= \sum_{i=1}^m \int_\Omega D_i(x,t)\nabla u_i \cdot \frac{\nabla u_i}{u_i}dx + \sum_{l=1}^s \sum_{j=L_{l-1}+1}^{L_l} \int_\Omega k_j(x,t)u_\infty^{y_j}\Psi\left(\frac{u^{y_j}}{u_\infty^{y_j}}; \frac{u^{y_j'}}{u_\infty^{y_j'}}\right)dx \\
&\gtrsim \sum_{i=1}^m \int_\Omega \frac{|\nabla u_i|^2}{u_i}dx + \sum_{l=1}^s \sum_{j=L_{l-1}+1}^{L_l} \int_{\omega_l} u_\infty^{y_j}\Psi\left(\frac{u^{y_j}}{u_\infty^{y_j}}; \frac{u^{y_j'}}{u_\infty^{y_j'}}\right)dx,
\end{aligned} \tag{2.4}$$

where we use the assumptions (**B**), (ii) in Theorem 1.4, and the rewriting

$$D_i(x,t)\nabla u_i \cdot \frac{\nabla u_i}{u_i} = 4\nabla\sqrt{u_i}^\top D_i(x,t)\nabla\sqrt{u_i} \geq 4\underline{D}_i|\nabla\sqrt{u_i}|^2 = \underline{D}_i\frac{|\nabla u_i|^2}{u_i}.$$

Lemma 2.7. *For any renormalised solution to* (1.3) *it holds, for* $l \in \{1,\ldots,s\}$,

$$\begin{aligned}
&\sum_{i=1}^m \int_\Omega \frac{|\nabla u_i|^2}{u_i}dx + \sum_{j=L_{l-1}+1}^{L_l} \int_{\omega_l} u_\infty^{y_j}\Psi\left(\frac{u^{y_j}}{u_\infty^{y_j}}; \frac{u^{y_j'}}{u_\infty^{y_j'}}\right)dx \\
&\qquad\gtrsim \sum_{j=L_{l-1}+1}^{L_l}\left(\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'}\right)^2.
\end{aligned}$$

Here we recall the notation

$$\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} = \prod_{i=1}^m \left[\frac{U_i}{U_{i,\infty}}\right]_{\omega_l}^{y_{j,i}} = \prod_{i=1}^m \frac{[U_i]_{\omega_l}^{y_{j,i}}}{U_{i,\infty}^{y_{j,i}}}.$$

Proof. By Poincaré-Wirtinger inequality and assumption (**B**), we have

$$\int_\Omega \frac{|\nabla u_i|^2}{u_i}dx = 4\int_\Omega |\nabla U_i|^2dx \gtrsim \int_{\omega_l}|\nabla U_i|^2dx \gtrsim \|U_i - [U_i]_{\omega_l}\|_{L^2(\omega_l)}^2.$$

By using the elementary inequality $\Psi(x;y) = x\log(x/y) - x + y \geq (\sqrt{x}-\sqrt{y})^2$ we have

$$\begin{aligned}
&\sum_{i=1}^m \|U_i - [U_i]_{\omega_l}\|_{L^2(\omega_l)}^2 + \sum_{j=L_{l-1}+1}^{L_l} \int_{\omega_l} u_\infty^{y_j}\Psi\left(\frac{u^{y_j}}{u_\infty^{y_j}}; \frac{u^{y_j'}}{u_\infty^{y_j'}}\right)dx \\
&\gtrsim \sum_{i=1}^m \|U_i - [U_i]_{\omega_l}\|_{L^2(\omega_l)}^2 + \sum_{j=L_{l-1}+1}^{L_l} \int_{\omega_l}\left(\frac{U^{y_j}}{U_\infty^{y_j}} - \frac{U^{y_j'}}{U_\infty^{y_j'}}\right)^2dx,
\end{aligned} \tag{2.5}$$

where we used the strict positivity of u_∞. For any $i = L_{l-1}+1,\ldots,L_l$, we use the notation

$$\eta_i(x) := U_i(x) - [U_i]_{\omega_l},\ x \in \omega_l.$$

Fix a constant $\mathfrak{m} > 0$, we consider the domain decomposition

$$\omega_l = \Upsilon_l \cap \Upsilon_l^c$$

where $\Upsilon_l := \{x \in \omega_l : |\eta_i(x)| \le \mathfrak{m} \text{ for all } i \in \{1, \ldots, m\}\}$, and $\Upsilon_l^c = \omega_l \backslash \Upsilon_l$. By Taylor's expansion

$$U_i(x)^{y_{j,i}} = \Big([U_i]_{\omega_l} + \eta_i(x)\Big)^{y_{j,i}} = [U_i]_{\omega_l}^{y_{j,i}} + \widetilde{R}_i \eta_i(x),$$

with

$$\widetilde{R}_i(x) = y_{j,i}(\theta\, [U_i]_{\omega_l} + (1-\theta)\eta_i(x))^{y_{j,i}-1} \text{ for some } \theta = \theta(i,j,l,x) \in (0,1).$$

Thanks to Lemma 2.6 and the definition of Υ_l, it holds that

$$|\widetilde{R}_i(x)| \lesssim_{\mathfrak{m}} 1, \quad \forall x \in \Upsilon_l. \tag{2.6}$$

Therefore, using $|\eta_i(x)| \le \mathfrak{m}$ in Υ_l, (2.6), and the elementary inequality $(x-y)^2 \ge x^2/2 - y^2$, we can estimate

$$\begin{aligned}
&\sum_{j=L_{l-1}+1}^{L_l} \int_{\Upsilon_l} \left(\frac{U^{y_j}}{U_\infty^{y_j}} - \frac{U^{y_j'}}{U_\infty^{y_j'}}\right)^2 dx \\
&= \sum_{j=L_{l-1}+1}^{L_l} \int_{\Upsilon_l} \left(\prod_{i=1}^m \frac{[U_i]_{\omega_l}^{y_{j,i}} + \widetilde{R}_i\eta_i(x)}{U_{i,\infty}^{y_{j,i}}} - \prod_{i=1}^m \frac{[U_i]_{\omega_l}^{y_{j,i}'} + \widetilde{R}_i\eta_i(x)}{U_{i,\infty}^{y_{j,i}'}}\right)^2 dx \\
&\ge \frac{1}{2}|\Upsilon_l| \sum_{j=L_{l-1}+1}^{L_l} \left(\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'}\right)^2 - \mathcal{C}\sum_{i=1}^m \int_{\Upsilon_l} |\eta_i(x)|^2 dx
\end{aligned} \tag{2.7}$$

where $\mathcal{C} = \mathcal{C}(\mathfrak{m})$. On the other hand, on Υ_l^c we know that there exists $i_0 \in \{1, \ldots, m\}$ such that $|\eta_{i_0}(x)| \ge \mathfrak{m}$. Thus, we estimate (using first Cauchy-Schwarz inequality, and then Lemma 2.6)

$$\begin{aligned}
\sum_{i=1}^m \|U_i - [U_i]_{\omega_l}\|_{L^2(\omega_l)}^2 &= \int_{\omega_l} \sum_{i=1}^m |\eta_i(x)|^2 dx \\
&\ge \frac{1}{m} \int_{\Upsilon_l^c} \left(\sum_{i=1}^m |\eta_i(x)|\right)^2 dx \\
&\ge \frac{\mathfrak{m}^2}{m} |\Upsilon_l^c| \gtrsim_{u_\infty, \mathfrak{m}} |\Upsilon_l^c| \sum_{j=L_{l-1}+1}^{L_l} \left(\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'}\right)^2.
\end{aligned} \tag{2.8}$$

From (2.6) and (2.7), we can estimate for any $\delta \in (0,1)$, recalling that $\eta_i(x) := U_i(x) - [U_i]_{\omega_l}$,

$$\begin{aligned}
\text{RHS of (2.5)} &\ge \frac{1}{2}\sum_{i=1}^m \|\eta_i\|_{L^2(\omega_l)}^2 + \frac{1}{2}\sum_{i=1}^m \|\eta_i\|_{L^2(\omega_l)}^2 + \delta \sum_{j=L_{l-1}+1}^{L_l} \int_{\Upsilon_l} \left(\frac{U^{y_j}}{U_\infty^{y_j}} - \frac{U^{y_j'}}{U_\infty^{y_j'}}\right)^2 dx \\
&\gtrsim_{\mathfrak{m}} \frac{1}{2}\sum_{i=1}^m \|\eta_i\|_{L^2(\omega_l)}^2 + \left(|\Upsilon_l^c| + \frac{\delta}{2}|\Upsilon_l|\right) \sum_{j=L_{l-1}+1}^{L_l} \left(\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'}\right)^2 \\
&\quad - \delta\mathcal{C}\sum_{i=1}^m \int_{\Upsilon_l} |\eta_i|^2 dx \\
&\gtrsim_{\mathfrak{m}} |\omega_l| \sum_{j=L_{l-1}+1}^{L_l} \left(\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'}\right)^2,
\end{aligned} \tag{2.9}$$

where we choose δ small enough depending on $\mathcal{C}$, and consequently on $\mathfrak{m}$. It is remarked that the last inequality is *not dependent on* Υ_l. This last estimate and (2.5) allow to conclude the proof of Lemma 2.7. □

Remark 2.8. *Clearly* $\mathfrak{m}$ *can be arbitrary in the proof of Lemma 2.7, and one can choose, for instance,* $\mathfrak{m} = 1$*. We chose to write* $\mathfrak{m}$ *as a constant to leave the room for optimising constants in the desired inequality.*

Lemma 2.9. *For any renormalised solution to* (1.3) *it holds, for* $l \in \{1, \ldots, s\}$,

$$\sum_{j=L_{l-1}+1}^{L_l}\left(\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'}\right)^2 \gtrsim \sum_{j=L_{l-1}+1}^{L_l}\left(\left[\frac{U}{U_\infty}\right]_{\Omega}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\Omega}^{y_j'}\right)^2 - \sum_{i=1}^m \left|[U_i]_\Omega - [U_i]_{\omega_l}\right|^2.$$

Proof. Denote by $\gamma_{i,l} = [U_i]_\Omega - [U_i]_{\omega_l}$. It follows from Lemma 2.6 that

$$|\gamma_{i,l}| \lesssim 1 \quad \forall i = 1, \ldots, m, \quad \forall l = 1, \ldots, s.$$

By Taylor's expansion,

$$\begin{aligned}
\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'} &= \prod_{i=1}^m \frac{([U_i]_\Omega - \gamma_{i,l})^{y_{j,i}}}{U_{i,\infty}^{y_{j,i}}} - \prod_{i=1}^m \frac{([U_i]_\Omega - \gamma_{i,l})^{y_{j,i}'}}{U_{i,\infty}^{y_{j,i}'}}\\
&= \prod_{i=1}^m \frac{[U_i]_\Omega^{y_{j,i}} - \gamma_{i,l}\mathsf{R}([U_i]_\Omega, \gamma_{i,l}, y_j)}{U_{i,\infty}^{y_{j,i}}} - \prod_{i=1}^m \frac{[U_i]_\Omega^{y_{j,i}'} - \gamma_{i,l}\mathsf{R}([U_i]_\Omega, \gamma_{i,l}, y_j')}{U_{i,\infty}^{y_{j,i}'}}\\
&= \left[\frac{U}{U_\infty}\right]_{\Omega}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\Omega}^{y_j'} - \widetilde{\mathsf{R}}([U]_\Omega, \gamma_{i,l}, y_j, y_j', U_\infty)\sum_{i=1}^m |\gamma_{i,l}|,
\end{aligned}$$

where $\mathsf{R}(\cdot)$ denote the rest terms from Taylor expansions and $\tilde{\mathsf{R}}$ is computed from R. Thanks to Lemma 2.6 and the bounds of $\gamma_{i,l}$,

$$|\mathsf{R}([U_i]_\Omega, \gamma_{i,l}, y_j)| + |\mathsf{R}([U_i]_\Omega, \gamma_{i,l}, y_j')| + |\widetilde{\mathsf{R}}([U]_\Omega, \gamma_{i,l}, y_j, y_j', U_\infty)| \lesssim 1.$$

Therefore, by using the elementary inequality $(x+y)^2 \geq \frac{1}{2}x^2 - y^2$, we have

$$\left(\left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\omega_l}^{y_j'}\right)^2 \gtrsim \frac{1}{2}\left(\left[\frac{U}{U_\infty}\right]_{\Omega}^{y_j} - \left[\frac{U}{U_\infty}\right]_{\Omega}^{y_j'}\right)^2 - \sum_{i=1}^m |\gamma_{i,l}|^2.$$

By summing for $j = L_{l-1}+1, \ldots, L_l$, we can conclude the proof of Lemma 2.9. □

Lemma 2.10. *For any renormalised solution to* (1.3) *it holds*

$$\sum_{i=1}^m \int_\Omega \frac{|\nabla u_i|^2}{u_i}dx \gtrsim \sum_{l=1}^s\sum_{i=1}^m \left|[U_i]_\Omega - [U_i]_{\omega_l}\right|^2.$$

Proof. For any $i \in \{1, \ldots m\}$ and any $l \in \{1, \ldots, s\}$, we use Poincaré-Wirtinger inequality to estimate

$$\int_\Omega \frac{|\nabla u_i|^2}{u_i}dx = 4\|\nabla U_i\|_{L^2(\Omega)}^2 \gtrsim \|U_i - [U_i]_\Omega\|_{L^2(\Omega)}^2 \geq \|U_i - [U_i]_\Omega\|_{L^2(\omega_l)}^2$$

and

$$\int_\Omega \frac{|\nabla u_i|^2}{u_i}dx = 4\|\nabla U_i\|_{L^2(\Omega)}^2 \gtrsim \|\nabla U_i\|_{L^2(\omega_l)}^2 \gtrsim \|U_i - [U_i]_{\omega_l}\|_{L^2(\omega_l)}^2.$$

Therefore,

$$\int_\Omega \frac{|\nabla u_i|^2}{u_i}dx \gtrsim \int_{\omega_l}\left(|U_i - [U_i]_\Omega|^2 + \left|U_i - [U_i]_{\omega_l}\right|^2\right)dx \gtrsim |\omega_l|\left|[U_i]_\Omega - [U_i]_{\omega_l}\right|^2$$

which concludes the proof of Lemma 2.10. □

Lemma 2.11. *For any renormalised solution to* (1.3) *it holds*

$$\mathcal{D}(u) \gtrsim \sum_{r=1}^{R}\left(\left[\frac{U}{U_\infty}\right]_\Omega^{y_r} - \left[\frac{U}{U_\infty}\right]_\Omega^{y_r'}\right)^2.$$

Proof. Let $\theta \in (0,1)$ to be chosen later. It follows from (2.4), Lemmas 2.7 and 2.9 that

$$\theta\mathcal{D}(u) \gtrsim \theta\sum_{r=1}^{R}\left(\left[\frac{U}{U_\infty}\right]_\Omega^{y_r} - \left[\frac{U}{U_\infty}\right]_\Omega^{y_r'}\right)^2 - \theta\sum_{l=1}^{s}\sum_{i=1}^{m}\left|[U_i]_\Omega - [U_i]_{\omega_l}\right|^2.$$

On the other hand, by Lemma 2.10,

$$(1-\theta)\mathcal{D}(u) \gtrsim (1-\theta)\sum_{l=1}^{s}\sum_{i=1}^{m}\left|[U_i]_\Omega - [U_i]_{\omega_l}\right|^2.$$

Thus, by choosing θ small enough we get the desired estimate in Lemma 2.11. □

Proof of Proposition 2.5. Proposition 2.5 now follows immediately from Lemma 2.11 and [FT18, Lemmas 2.7 and 2.8]. For the convenience of the reader, we nevertheless provide the proof. First, from (2.4) and Lemma 2.11, we have

$$\mathcal{D}(u) \gtrsim \sum_{i=1}^{m}\int_\Omega \frac{|\nabla u_i|^2}{u_i}dx + \sum_{r=1}^{R}\left(\left[\frac{U}{U_\infty}\right]_\Omega^{y_r} - \left[\frac{U}{U_\infty}\right]_\Omega^{y_r'}\right)^2. \tag{2.10}$$

Thanks to the Logarithmic Sobolev inequality in bounded Lipschitz domains, see e.g. [DF14], we have

$$\mathcal{D}(u) \gtrsim \sum_{i=1}^{m}\int_\Omega u_i\log\frac{u_i}{[u_i]_\Omega}dx = \mathcal{E}(u|\,[u]_\Omega), \tag{2.11}$$

with $\mathcal{E}(u|\,[u]_\Omega)$ appearing in (2.2). From Poincaré-Wirtinger's inequality

$$\int_\Omega \frac{|\nabla u_i|^2}{u_i}dx = 4\int_\Omega |\nabla U_i|^2 dx \gtrsim_\Omega \|U_i - [U_i]_\Omega\|_{L^2(\Omega)}^2 =: \|\mu_i\|_{L^2(\Omega)}^2, \tag{2.12}$$

where we denote by $\mu_i(x) := U_i(x) - [U_i]_\Omega$ for $x \in \Omega$, $i = 1,\ldots,m$. Thanks to Lemma 2.6,

$$\|\mu_i\|_{L^2(\Omega)} \lesssim 1. \tag{2.13}$$

From $\|\mu_i\|_{L^2(\Omega)}^2 = \left[U_i^2\right]_\Omega - [U_i]_\Omega^2$,

$$\frac{[U_i]_\Omega}{U_{i,\infty}} = \frac{1}{U_{i,\infty}}\left(\sqrt{\left[U_i^2\right]_\Omega} - \frac{\|\mu_i\|_{L^2(\Omega)}^2}{\sqrt{\left[U_i^2\right]_\Omega} + [U_i]_\Omega}\right) = \sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}} - \mathfrak{R}(U_i)\|\mu_i\|_{L^2(\Omega)},$$

with

$$\mathfrak{R}(U_i) := \frac{\|\mu_i\|_{L^2(\Omega)}}{U_{i,\infty}\left(\sqrt{\left[U_i^2\right]_\Omega} + [U_i]_\Omega\right)} \geq 0$$

satisfying

$$\mathfrak{R}(U_i)^2 = \frac{\|\mu_i\|_{L^2(\Omega)}^2}{U_{i,\infty}^2\left(\sqrt{\left[U_i^2\right]_\Omega} + [U_i]_\Omega\right)^2} = \frac{\sqrt{\left[U_i^2\right]_\Omega} - [U_i]_\Omega}{U_{i,\infty}^2\left(\sqrt{\left[U_i^2\right]_\Omega} + [U_i]_\Omega\right)} \leq \frac{1}{U_{i,\infty}^2}. \tag{2.14}$$

Therefore, we can estimate using Taylor's expansion and the bounds (2.13), (2.14)

$$\begin{aligned}
&\sum_{r=1}^{R}\left(\left[\frac{U}{U_\infty}\right]_\Omega^{y_r} - \left[\frac{U}{U_\infty}\right]_\Omega^{y_r'}\right)^2 \\
&= \sum_{r=1}^{R}\left[\prod_{i=1}^{m}\left(\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}} - \mathfrak{R}(U_i)\|\mu_i\|_{L^2(\Omega)}\right)^{y_{r,i}} - \prod_{i=1}^{m}\left(\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}} - \mathfrak{R}(U_i)\|\mu_i\|_{L^2(\Omega)}\right)^{y_{r,i}'}\right]^2 \qquad (2.15)\\
&\gtrsim_{u_\infty} \frac{1}{2}\sum_{r=1}^{R}\left[\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r} - \sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r'}\right]^2 - \sum_{i=1}^{m}\|\mu_i\|_{L^2(\Omega)}^2.
\end{aligned}$$

Let $\theta \in (0,1)$ be a parameter, to be chosen later. It follows from (2.10), (2.12) and (2.15) that

$$\begin{aligned}
\mathcal{D}(u) &\gtrsim \sum_{i=1}^{m}\int_\Omega \frac{|\nabla u_i|^2}{u_i}dx + \theta\sum_{r=1}^{R}\left(\left[\frac{U}{U_\infty}\right]_\Omega^{y_r} - \left[\frac{U}{U_\infty}\right]_\Omega^{y_r'}\right)^2 \\
&\gtrsim \sum_{i=1}^{m}\|\mu_i\|_{L^2(\Omega)}^2 + \frac{\theta}{2}\sum_{r=1}^{R}\left[\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r} - \sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r'}\right]^2 - \theta\sum_{i=1}^{m}\|\mu_i\|_{L^2(\Omega)}^2 \qquad (2.16)\\
&\gtrsim_\theta \sum_{r=1}^{R}\left[\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r} - \sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r'}\right]^2,
\end{aligned}$$

by choosing θ small enough. Applying then [FT18, Inequality (11)], we have the following finite dimensional inequality

$$\sum_{r=1}^{R}\left[\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r} - \sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}}^{y_r'}\right]^2 \gtrsim \sum_{i=1}^{m}\left(\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}} - 1\right)^2. \qquad (2.17)$$

On the other hand, using the elementary inequality $z\log z - z + 1 \le (z-1)^2$, we estimate

$$\begin{aligned}
\mathcal{E}([u]_\Omega\,|u_\infty) &= \sum_{i=1}^{m}u_{i,\infty}\left(\frac{[u_i]_\Omega}{u_{i,\infty}}\log\frac{[u_i]_\Omega}{u_{i,\infty}} - \frac{[u_i]_\Omega}{u_{i,\infty}} + 1\right) \\
&\le \sum_{i=1}^{m}u_{i,\infty}\left(\frac{[u_i]_\Omega}{u_{i,\infty}} - 1\right)^2 \\
&= \sum_{i=1}^{m}u_{i,\infty}\left(\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}} + 1\right)^2\left(\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}} - 1\right)^2 \qquad (2.18)\\
&\lesssim \sum_{i=1}^{m}\left(\sqrt{\frac{[u_i]_\Omega}{u_{i,\infty}}} - 1\right)^2.
\end{aligned}$$

Combining (2.16), (2.17) and (2.18) yields

$$\mathcal{D}(u) \gtrsim \mathcal{E}([u]_\Omega\,|u_\infty).$$

From this, (2.11) and (2.2), we get the proof of Proposition 2.5.

□

We are now ready to prove the first main result.

Proof of Theorem 1.4. Thanks to Lemma 2.3, any renormalised solution satisfies for $0 \le \tau < T$

$$\mathcal{E}(u(s)|u_\infty)\Big|_{s=\tau}^{s=T} + \int_\tau^T \mathcal{D}(u(s))ds \le 0. \tag{2.19}$$

Moreover, still thanks to Lemma 2.3, all renormalised solutions satisfy the conservation laws (1.4). Note that (2.19) also implies

$$\sum_{i=1}^m \int_\Omega u_i(x,t)\log u_i(x,t)dx \le L$$

for some L depending on $\mathcal{E}(u_0|u_\infty)$ and u_∞. Therefore, we can apply Proposition 2.5 to get

$$\mathcal{D}(u(s)) \ge \lambda\mathcal{E}(u(s)|u_\infty), \quad \forall s \ge 0.$$

Inserting this into (2.19) gives for all $0 \le \tau < T$

$$\mathcal{E}(u(s)|u_\infty)\Big|_{s=\tau}^{s=T} + \lambda\int_\tau^T \mathcal{E}(u(s)|u_\infty)ds \le 0.$$

Defining

$$\varphi(\tau) = \int_\tau^T \mathcal{E}(u(s)|u_\infty)ds,$$

we see that

$$\varphi'(\tau) = -\mathcal{E}(u(\tau)|u_\infty) \le -\mathcal{E}(u(T)|u_\infty) - \lambda\int_\tau^T \mathcal{E}(u(s)|u_\infty)ds = -\mathcal{E}(u(T)|u_\infty) - \lambda\varphi(\tau).$$

Gronwall's lemma implies then that

$$e^{\lambda T}\varphi(T) + \mathcal{E}(u(T)|u_\infty)\frac{e^{\lambda T} - e^{\lambda\tau}}{\lambda} \le e^{\lambda\tau}\varphi(\tau).$$

Using $\varphi(T) = 0$ and $\lambda\varphi(\tau) \le \mathcal{E}(u(\tau)|u_\infty) - \mathcal{E}(u(T)|u_\infty)$ leads to

$$\mathcal{E}(u(T)|u_\infty)(e^{\lambda T} - e^{\lambda\tau}) \le e^{\lambda\tau}(\mathcal{E}(u(\tau)|u_\infty) - \mathcal{E}(u(T)|u_\infty)),$$

and thus to

$$\mathcal{E}(u(T)|u_\infty) \le e^{-\lambda(T-\tau)}\mathcal{E}(u(\tau)|u_\infty) \quad \forall 0 \le \tau < T.$$

Setting $\tau = 0$ entails the global exponential decay of the relative entropy, which finally yields the decay of the solution towards equilibrium, thanks to Lemma 2.4. $\square$

3. Reactions in measurable sets - Proof of Theorem 1.5

3.1. Preliminary estimates.

Proposition 3.1. *Assume the assumptions in Theorem 1.5. Then, for any bounded and non-negative initial data $u_0 \in L^\infty_+(\Omega)^3$, there exists a unique global non-negative weak solution to* (1.8), *which is bounded uniformly in time, i.e.*

$$\sup_{t\ge 0}\sup_{i=1,2,3}\|u_i(t)\|_{L^\infty(\Omega)} < +\infty. \tag{3.1}$$

Proof. Denote by $f_j(x,t,u)$ the nonlinearity in the equation for u_j in (1.8). It is easy to check that these nonlinearities are locally Lipschitz continuous in u, uniformly in (x,t), quasi-positive, i.e.

$$f_j(x,t,u) \ge 0 \quad \text{ for all } (x,t,u) \in \Omega\times\mathbb{R}_+\times\mathbb{R}^3_+ \text{ with } u_j = 0,$$

and satisfy the following (weighted) conservation of mass condition

$$4f_1(x,t,u) + 2f_2(x,t,u) + f_3(x,t,u) = 0, \quad \forall (x,t,u) \in \Omega\times\mathbb{R}_+\times\mathbb{R}^3_+.$$

Therefore, we can apply [FMTY21, Theorem 1.1] to obtain the global existence of a unique weak solution, together with the uniform-in-time boundedness in $L^\infty(\Omega)$-norm. $\square$

In the following, we prove certain estimates linked to the entropy and entropy dissipation of (1.8). Thanks to the $L^\infty(\Omega)$ bound (3.1) in Proposition 3.1, we denote

$$C_0 := \sup_{t\geq 0}(\|u_1(t)\|_{L^\infty(\Omega)} + \|u_2(t)\|_{L^\infty(\Omega)} + \|u_3(t)\|_{L^\infty(\Omega)}). \tag{3.2}$$

As in the previous section, we consider the relative entropy

$$\mathcal{E}(u|u_\infty) = \sum_{j=1}^{3}\int_\Omega \left(u_j\,(\cdot,t)\log\frac{u_j\,(\cdot,t)}{u_{j,\infty}} - u_j\,(\cdot,t) + u_{j,\infty}\right)dx, \tag{3.3}$$

where the equilibrium u_∞ is defined as in (1.10), and the corresponding entropy dissipation

$$\mathcal{D}(u) := \sum_{j=1}^{3}\int_\Omega d_j\frac{|\nabla u_j|^2}{u_j}dx + \int_\Omega k_1\left(u_2^2 - u_1\right)\log\frac{u_2^2}{u_1}dx + \int_\Omega k_2\left(u_3^2 - u_2\right)\log\frac{u_3^2}{u_2}dx. \tag{3.4}$$

Lemma 3.2. *It holds*

$$\sum_{j=1}^{3}\|u_j - u_{j,\infty}\|^2_{L^1(\Omega)} \lesssim_{u_\infty, C_0} \mathcal{E}(u|u_\infty) \lesssim_{u_\infty} \sum_{j=1}^{3}\int_\Omega |u_j - u_{j,\infty}|^2 dx.$$

Proof. The first estimate is a special case of the Csiszár-Kullback-Pinsker inequality in Lemma 2.4, and the second one follows directly form the elementary inequality

$$x\log\frac{x}{y} - x + y \leq \frac{1}{y}|x-y|^2.$$

□

Lemma 3.3. *For solutions to* (1.8)*, it holds, for* $j = 1, 2, 3$*,*

$$\left\|\sqrt{u_j} - \left[\sqrt{u_j}\right]_\Omega\right\|^2_{L^2(\Omega)} \lesssim_\Omega \int_\Omega \frac{|\nabla u_j|^2}{u_j}dx, \tag{3.5}$$

and

$$\left\|u_j - [u_j]_\Omega\right\|^2_{L^2(\Omega)} \lesssim_{\Omega, C_0} \int_\Omega \frac{|\nabla u_j|^2}{u_j}dx, \tag{3.6}$$

with C_0 *defined in* (3.2).

Proof. The estimate (3.5) is a consequence of Poincaré-Wirtinger's inequality and the fact that $\frac{|\nabla u_j|^2}{u_j} = 4|\nabla\sqrt{u_j}|^2$. For (3.6), we estimate

$$\int_\Omega |u_j - [u_j]_\Omega|^2 dx = \int_\Omega \left|\sqrt{u_j} + \sqrt{[u_j]_\Omega}\right|^2 \left|\sqrt{u_j} - \left[\sqrt{u_j}\right]_\Omega\right|^2 dx \lesssim_{C_0} \left\|\sqrt{u_j} - \left[\sqrt{u_j}\right]_\Omega\right\|^2_{L^2(\Omega)},$$

hence (3.6) follows from (3.5). □

Lemma 3.4. *It holds*

$$\mathcal{D}(u) \gtrsim \sum_{j=1}^{3}\int_\Omega d_j\frac{|\nabla u_j|^2}{u_j}dx + \int_{\omega_1}\left|u_2 - \sqrt{u_1}\right|^2 dx + \int_{\omega_2}\left|u_3 - \sqrt{u_2}\right|^2 dx.$$

Proof. The proof is straightforward thanks to the inequality $(x-y)\log(x/y) \geq (\sqrt{x} - \sqrt{y})^2$, and assumptions (1.11). □

From the equations defining the equilibrium and the conservation of total mass, that is

$$\begin{cases} u_{2,\infty}^2 - u_{1,\infty} = 0, \\ u_{3,\infty}^2 - u_{2,\infty} = 0, \\ 4u_{1,\infty} + 2u_{2,\infty} + u_{3,\infty} = \int_\Omega (4u_1 + 2u_2 + u_3)\, dx, \end{cases} \tag{3.7}$$

we have

$$\begin{aligned} 4u_{1,\infty} + 2\sqrt{u_{1,\infty}} + \sqrt[4]{u_{1,\infty}} &= 4u_{2,\infty}^2 + 2u_{2,\infty} + \sqrt{u_{2,\infty}} \\ &= 4u_{3,\infty}^4 + 2u_{3,\infty}^2 + u_{3,\infty} \\ &= \int_\Omega (4u_1 + 2u_2 + u_3) dx. \end{aligned} \tag{3.8}$$

Lemma 3.5. *We have the following pointwise estimates for all* $(x,t) \in \Omega \times \mathbb{R}_+$

$$|u_1 - u_{1,\infty}|^2 \lesssim_{C_0, u_\infty} |u_2 - \sqrt{u_1}|^2 + |u_2 - u_{2,\infty}|^2,$$

$$|u_3 - u_{3,\infty}|^2 \lesssim_{u_\infty} |u_3 - \sqrt{u_2}|^2 + |u_2 - u_{2,\infty}|^2,$$

and

$$|u_2 - u_{2,\infty}|^2 \lesssim_{C_0} \sum_{j=1}^3 |u_j - [u_j]_\Omega|^2 + |u_2 - \sqrt{u_1}|^2 + |u_3 - \sqrt{u_2}|^2. \tag{3.9}$$

Proof. By using (3.2) and elementary computations, we have

$$\begin{aligned} |u_1 - u_{1,\infty}|^2 = |u_1 - u_{2,\infty}^2|^2 &\lesssim |u_1 - u_2^2|^2 + |u_2^2 - u_{2,\infty}^2|^2 \\ &\lesssim_{C_0, u_\infty} |u_2 - \sqrt{u_1}|^2 + |u_2 - u_{2,\infty}|^2, \end{aligned}$$

and similarly,

$$\begin{aligned} |u_3 - u_{3,\infty}|^2 = |u_3 - \sqrt{u_{2,\infty}}|^2 &\lesssim |u_3 - \sqrt{u_2}|^2 + |\sqrt{u_2} - \sqrt{u_{2,\infty}}|^2 \\ &\lesssim_{u_\infty} |u_3 - \sqrt{u_2}|^2 + |u_2 - u_{2,\infty}|^2. \end{aligned}$$

Defining $f(z) := 4z^2 + 2z + \sqrt{z}$, $z > 0$ we see that

$$|f(w) - f(z)| = |w - z| \left| 4(w+z) + 2 + \frac{1}{\sqrt{w} + \sqrt{z}} \right| \geq 2|w - z|.$$

Therefore,

$$\begin{aligned} |u_2 - u_{2,\infty}|^2 &\leq |f(u_2) - f(u_{2,\infty})|^2 \\ &= \left| 4u_2^2 + 2u_2 + \sqrt{u_2} - \int_\Omega (4u_1 + 2u_2 + u_3) dx \right|^2 \quad \text{(using (3.8))} \\ &\lesssim |u_2^2 - u_1|^2 + |u_1 - [u_1]_\Omega|^2 + |u_2 - [u_2]_\Omega|^2 + |\sqrt{u_2} - u_3|^2 + |u_3 - [u_3]_\Omega|^2 \\ &\lesssim_{C_0} \sum_{j=1}^3 |u_j - [u_j]_\Omega|^2 + |u_2 - \sqrt{u_1}|^2 + |u_3 - \sqrt{u_2}|^2. \end{aligned}$$

□

3.2. Proof of Theorem 1.5.

Lemma 3.6. *Under the assumptions of Theorem 1.5, it holds*

$$\mathcal{E}(u|u_\infty) \lesssim \frac{1}{|\omega_1|}\sum_{j=1}^{3}\|u_j - [u_j]_\Omega\|_{L^2(\Omega)}^2 + \frac{1}{|\omega_1|}\int_{\omega_1}\left(|u_2-\sqrt{u_1}|^2 + |u_3 - \sqrt{u_2}|^2\right)dx.$$

Proof. First, we have

$$\int_\Omega |u_1 - u_{1,\infty}|^2 dx \lesssim \|u_1 - [u_1]_\Omega\|_{L^2(\Omega)}^2 + |\,[u_1]_\Omega - u_{1,\infty}|^2.$$

Then, it holds

$$\begin{aligned}|\,[u_1]_\Omega - u_{1,\infty}|^2 &= \frac{1}{|\omega_1|}\int_{\omega_1}|\,[u_1]_\Omega - u_{1,\infty}|^2dx\\ &\lesssim \frac{1}{|\omega_1|}\int_{\omega_1}|u_1 - [u_1]_\Omega|^2dx + \frac{1}{|\omega_1|}\int_{\omega_1}|u_1 - u_{1,\infty}|^2dx\\ &\lesssim \frac{1}{|\omega_1|}\|u_1 - [u_1]_\Omega\|_{L^2(\Omega)}^2 + \frac{1}{|\omega_1|}\int_{\omega_1}\left(|u_2 - \sqrt{u_1}|^2 + |u_2 - u_{2,\infty}|^2\right)dx,\end{aligned}$$

where we used Lemma 3.5 at the last estimate. Therefore, we have

$$\int_\Omega |u_1 - u_{1,\infty}|^2dx \lesssim \frac{1}{|\omega_1|}\left(\|u_1 - [u_1]_\Omega\|_{L^2(\Omega)}^2 + \int_\Omega |u_2 - u_{2,\infty}|^2dx + \int_{\omega_1}|u_2 - \sqrt{u_1}|^2dx\right). \tag{3.10}$$

Similarly,

$$\int_\Omega |u_3 - u_{3,\infty}|^2dx \lesssim \frac{1}{|\omega_1|}\left(\|u_3 - [u_3]_\Omega\|_{L^2(\Omega)}^2 + \int_\Omega |u_2 - u_{2,\infty}|^2dx + \int_{\omega_1}|u_3 - \sqrt{u_2}|^2dx\right). \tag{3.11}$$

Integrating both sides of (3.9) over ω_1 yields

$$\begin{aligned}\int_{\omega_1}|u_2 - u_{2,\infty}|^2dx &\lesssim \sum_{j=1}^{3}\int_{\omega_1}|u_j - [u_j]_\Omega|^2dx + \int_{\omega_1}\left(|u_2 - \sqrt{u_1}|^2 + |u_3 - \sqrt{u_2}|^2\right)dx\\ &\lesssim \sum_{j=1}^{3}\|u_j - [u_j]_\Omega\|_{L^2(\Omega)}^2 + \int_{\omega_1}\left(|u_2 - \sqrt{u_1}|^2 + |u_3 - \sqrt{u_2}|^2\right)dx.\end{aligned} \tag{3.12}$$

Thus

$$\begin{aligned}\int_\Omega |u_2 - u_{2,\infty}|^2dx &\lesssim \|u_2 - [u_2]_\Omega\|_{L^2(\Omega)}^2 + \frac{1}{|\omega_1|}\int_{\omega_1}|\,[u_2]_\Omega - u_{2,\infty}|^2dx\\ &\lesssim \|u_2 - [u_2]_\Omega\|_{L^2(\Omega)}^2 + \frac{1}{|\omega_1|}\int_{\omega_1}|\,[u_2]_\Omega - u_2|^2dx + \frac{1}{|\omega_1|}\int_{\omega_1}|u_2 - u_{2,\infty}|^2dx\\ &\lesssim \frac{1}{|\omega_1|}\sum_{j=1}^{3}\|u_j - [u_j]_\Omega\|_{L^2(\Omega)}^2 + \frac{1}{|\omega_1|}\int_{\omega_1}\left(|u_2 - \sqrt{u_1}|^2 + |u_3 - \sqrt{u_2}|^2\right)dx,\end{aligned} \tag{3.13}$$

where (3.12) was used at the last step. Lemma 3.6 then follows directly from Lemma 3.2 and (3.10)–(3.13). □

It is clear that we only have to control the term $\int_{\omega_1}|u_3 - \sqrt{u_2}|^2dx$ since the reaction $\mathsf{S}_2 \leftrightarrows 2\mathsf{S}_3$ happens in ω_2 and not necessarily in ω_1. This is done in the following lemma.

Lemma 3.7. *Under the assumptions of Theorem 1.5, it holds*

$$\frac{1}{|\omega_1|}\int_{\omega_1}|u_3 - \sqrt{u_2}|^2dx \lesssim_{C_0} \left(\frac{1}{|\omega_1|} + \frac{1}{|\omega_2|}\right)\mathcal{D}(u).$$

Proof. Using triangular inequality, we estimate

$$\begin{aligned}
&\frac{1}{|\omega_1|}\int_{\omega_1}|u_3-\sqrt{u_2}|^2\,dx\\
&\lesssim \frac{1}{|\omega_1|}\int_{\omega_1}\bigg(|u_3-[u_3]_\Omega|^2+\big|[u_3]_\Omega-[u_3]_{\omega_2}\big|^2+\big|[u_3]_{\omega_2}-[\sqrt{u_2}]_{\omega_2}\big|^2\\
&\qquad +\big|[\sqrt{u_2}]_{\omega_2}-[\sqrt{u_2}]_\Omega\big|^2+\big|[\sqrt{u_2}]_\Omega-\sqrt{u_2}\big|^2\bigg)dx\\
&\lesssim \frac{1}{|\omega_1|}\int_\Omega\left(|u_3-[u_3]_\Omega|^2+|\sqrt{u_2}-[\sqrt{u_2}]_\Omega|^2\right)dx\\
&\qquad +\underbrace{\left(\big|[u_3]_\Omega-[u_3]_{\omega_2}\big|^2+\big|[\sqrt{u_2}]_\Omega-[\sqrt{u_2}]_{\omega_2}\big|^2\right)}_{(I)}+\underbrace{\big|[u_3]_{\omega_2}-[\sqrt{u_2}]_{\omega_2}\big|^2}_{(II)}.
\end{aligned} \tag{3.14}$$

For (I), we use Cauchy-Schwarz inequality to get

$$\big|[u_3]_\Omega-[u_3]_{\omega_2}\big|^2=\left|\frac{1}{|\omega_2|}\int_{\omega_2}([u_3]_\Omega-u_3)dx\right|^2\le\frac{1}{|\omega_2|}\int_{\omega_2}|\,[u_3]_\Omega-u_3|^2dx\le\frac{1}{|\omega_2|}\int_\Omega|\,[u_3]_\Omega-u_3|^2dx,$$

and similarly

$$\big|[\sqrt{u_2}]_\Omega-[\sqrt{u_2}]_{\omega_2}\big|^2=\left|\frac{1}{|\omega_2|}\int_{\omega_2}(\sqrt{u_2}-[\sqrt{u_2}]_\Omega)dx\right|^2\le\frac{1}{|\omega_2|}\int_\Omega\big|\sqrt{u_2}-[\sqrt{u_2}]_\Omega\big|^2\,dx.$$

Concerning (II), we estimate

$$(II)=\left|\frac{1}{|\omega_2|}\int_{\omega_2}(u_3-\sqrt{u_2})dx\right|^2\le\frac{1}{|\omega_2|}\int_{\omega_2}|u_3-\sqrt{u_2}|^2dx.$$

Therefore, it follows from (3.14) that

$$\begin{aligned}
&\frac{1}{|\omega_1|}\int_{\omega_1}|u_3-\sqrt{u_2}|^2\,dx\\
&\lesssim\left(\frac{1}{|\omega_1|}+\frac{1}{|\omega_2|}\right)\int_\Omega\left(|u_3-[u_3]_\Omega|^2+|\sqrt{u_2}-[\sqrt{u_2}]_\Omega|^2\right)dx+\frac{1}{|\omega_2|}\int_{\omega_2}|u_3-\sqrt{u_2}|^2\,dx\\
&\lesssim\left(\frac{1}{|\omega_1|}+\frac{1}{|\omega_2|}\right)\left(\int_\Omega\left(|u_3-[u_3]_\Omega|^2+|\sqrt{u_2}-[\sqrt{u_2}]_\Omega|^2\right)dx+\int_{\omega_2}|u_3-\sqrt{u_2}|^2\,dx\right)\\
&\lesssim_{\Omega,C_0}\left(\frac{1}{|\omega_1|}+\frac{1}{|\omega_2|}\right)\mathcal{D}(u),
\end{aligned}$$

where we used Lemmas 3.3 and 3.4 at the last step. □

Lemma 3.8. *Under the assumptions in Theorem 1.5, it holds*

$$\mathcal{E}(u|u_\infty)\lesssim\left(\frac{1}{|\omega_1|}+\frac{1}{|\omega_2|}\right)\mathcal{D}(u).$$

Proof. The proof of this lemma follows immediately from Lemmas 3.3, 3.4, 3.6 and 3.7. □

Proof of Theorem 1.5 The global existence and boundedness of a unique weak non-negative solution is given in Proposition 3.1. It can be easily checked that the solution fulfills the weak entropy-entropy dissipation relation

$$\mathcal{E}(u(t)|u_\infty)+\int_s^t\mathcal{D}(u(\tau))d\tau\le\mathcal{E}(u(\tau)|u_\infty)\quad\forall t\ge\tau\ge0.$$

Now using the entropy-entropy dissipation inequality in Lemma 3.8 and a Gronwall's inequality (as in the proof of Theorem 1.4), we get the exponentially fast decay of the relative entropy

$$\mathcal{E}(u(t)|u_\infty) \le \mathcal{E}(u_0|u_\infty)e^{-\lambda t}$$

where $\lambda^{-1} \sim |\omega_1|^{-1} + |\omega_2|^{-1}$. The convergence in $L^1(\Omega)$-norm follows directly from Lemma 3.2. □

4. Degenerate diffusion and reactions - Proof of Theorem 1.7

Due to the degeneracy of d_3, the global existence of solution to (1.8) is not obtained as quickly as previously. Nevertheless, under assumption (1.16), we can obtain the following global existence and boundedness of solutions.

Proposition 4.1. *We work under the assumptions of Theorem 1.7. Then for any non-negative, bounded initial data $u_0 \in L^\infty_+(\Omega)^3$, there exists a unique global non-negative weak solution to* (1.8)*, which is bounded uniformly in time, i.e.*

$$\sup_{t\ge 0}\sup_{i=1,2,3}\|u_i(t)\|_{L^\infty(\Omega)} < +\infty.$$

Remark 4.2. *In the proof of Proposition 4.1, we need to know that $\nabla d_3 \in L^q(\Omega)$ for some $q > n$, in order to deal with the degeneracy* (1.16)*. If d_3 is merely continuous on Ω and its zero-set $\{x \in \Omega : d_3(x) = 0\}$ is a finite union of $(n-1)$-dimensional smooth manifolds, then we can show the global existence and boundedness of solutions without imposing any conditions on the gradient of d_3. The idea is that strong compactness will hold (for the third component of the solution to an approximated problem) outside of an ε-neighbourhood of the considered manifolds, and that convergence a.e. of a subsequence on the whole domain can be obtained thanks to a diagonal extraction. We leave the details to the interested reader.*

Proof. We regularize the system (1.8) as follows: for any $\varepsilon \in (0,1)$, we define

$$d_{\varepsilon 1}(x,t) := d_1(x,t), \quad d_{\varepsilon 2}(x,t) := d_2(x,t), \quad d_{\varepsilon 3}(x) := d_3(x) + \varepsilon, \tag{4.1}$$

$$f_{\varepsilon i}(x,t,u) := f_i(x,t,u)\left(1 + \varepsilon\sum_{j=1}^{3}|f_j(x,t,u)|\right)^{-1}, \quad i = 1,2,3. \tag{4.2}$$

Consider now the approximating system for $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2}, u_{\varepsilon 3})$, that is

$$\begin{cases} \partial_t u_{\varepsilon i} - \nabla\cdot(d_{\varepsilon i}\nabla u_{\varepsilon i}) = f_{\varepsilon i}(x,t,u_\varepsilon), & \text{in } \Omega\times\mathbb{R}_+,\\ d_{\varepsilon i}\nabla u_{\varepsilon i}\cdot\nu = 0, & \text{on } \partial\Omega\times\mathbb{R}_+,\\ u_{\varepsilon i}(x,0) = u_{i,0}, & \text{in } \Omega. \end{cases} \tag{4.3}$$

By applying [FMTY21, Theorem 1.1], (4.3) has a unique global weak solution, which is also bounded uniformly in time. It is remarked, however, that this bound depends on ε and could in principle tend to ∞ as $\varepsilon \to 0$. In the following, we show therefore some uniform-in-time bounds for $u_{\varepsilon i}$, which are independent of ε. In order to do that, we consider for $p \in \mathbb{N}$ the energy functional

$$\mathcal{H}_p[u_\varepsilon] := \int_\Omega \left(\frac{4}{p+1}(u_{\varepsilon 1})^{p+1} + \frac{2}{2p+1}(u_{\varepsilon 2})^{2p+1} + \frac{1}{4p+1}(u_{\varepsilon 3})^{4p+1}\right)dx.$$

Differentiating $\mathcal{H}_p[u_\varepsilon]$ in t, using the system leads to

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_p[u_\varepsilon](t) = &- 4p\int_\Omega d_{\varepsilon 1}(u_{\varepsilon 1})^{p-1}|\nabla u_{\varepsilon 1}|^2dx - 4p\int_\Omega d_{\varepsilon 2}(u_{\varepsilon 2})^{2p-1}|\nabla u_{\varepsilon 2}|^2dx\\ &- 4p\int_\Omega d_{\varepsilon 3}(u_{\varepsilon 3})^{4p-1}|\nabla u_{\varepsilon 3}|^2dx\end{aligned}$$

$$
\begin{aligned}
&- 4\int_\Omega k_1(u_{\varepsilon 2}^2 - u_{\varepsilon 1})(u_{\varepsilon 2}^{2p} - u_{\varepsilon 1}^p)\left(1+\varepsilon\sum_{j=1}^3|f_j(x,t,u_\varepsilon)|\right)^{-1}dx\\
&- 2\int_\Omega k_2(u_{\varepsilon 3}^2 - u_{\varepsilon 2})(u_{\varepsilon 3}^{4p} - u_{\varepsilon 2}^{2p})\left(1+\varepsilon\sum_{j=1}^3|f_j(x,t,u_\varepsilon)|\right)^{-1}dx \le 0.
\end{aligned}
$$

Therefore

$$
\mathcal{H}_p[u_\varepsilon](t) \le \mathcal{H}_p[u_\varepsilon](0), \quad \forall t \ge 0,
$$

which entails

$$
\begin{aligned}
&\frac{4}{p+1}\|u_{\varepsilon 1}(t)\|_{L^{p+1}(\Omega)}^{p+1} + \frac{2}{2p+1}\|u_{\varepsilon 2}(t)\|_{L^{2p+1}(\Omega)}^{2p+1} + \frac{1}{4p+1}\|u_{\varepsilon 3}(t)\|_{L^{4p+1}(\Omega)}^{4p+1}\\
&\quad \le \frac{4}{p+1}\|u_{1,0}\|_{L^{p+1}(\Omega)}^{p+1} + \frac{2}{2p+1}\|u_{2,0}\|_{L^{2p+1}(\Omega)}^{2p+1} + \frac{1}{4p+1}\|u_{3,0}\|_{L^{4p+1}(\Omega)}^{4p+1}.
\end{aligned}
$$

We can take the root of order $p+1$ of both sides, then let $p\to\infty$, and obtain that

$$
\|u_{\varepsilon 1}(t)\|_{L^\infty(\Omega)} + \|u_{\varepsilon 2}(t)\|_{L^\infty(\Omega)}^2 + \|u_{\varepsilon 3}(t)\|_{L^\infty(\Omega)}^4 \le C\left(\|u_{1,0}\|_{L^\infty(\Omega)} + \|u_{2,0}\|_{L^\infty(\Omega)}^2 + \|u_{3,0}\|_{L^\infty(\Omega)}^4\right), \tag{4.4}
$$

for a constant C *independent of* $\varepsilon \in (0,1)$. Thanks to this and (4.2), we have

$$
\sup_{\varepsilon\in(0,1)}\sup_{i=1,2,3}\|f_i(\cdot,\cdot,u_\varepsilon)\|_{L^\infty(0,T;L^\infty(\Omega))} < +\infty. \tag{4.5}
$$

Since $d_{\varepsilon 1}$ and $d_{\varepsilon 2}$ are in fact independent of ε, see (4.1), the classical Aubin-Lions lemma gives the strong convergence of $u_{\varepsilon 1}$ and $u_{\varepsilon 2}$, up to some subsequence, i.e.

$$
u_{\varepsilon i} \to u_i \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;H^1(\Omega))
$$

for some $u_1, u_2 \in L^2(0,T;H^1(\Omega))$. At the same time, (4.4) implies that

$$
u_{\varepsilon 3} \rightharpoonup u_3 \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{4.6}
$$

We will now show the strong convergence of $u_{\varepsilon 3}$. By multiplying by $u_{\varepsilon 3}$ the third equation, and by integrating on $\Omega\times(0,T)$, we get

$$
\int_0^T\int_\Omega d_{\varepsilon 3}\,|\nabla u_{\varepsilon 3}|^2 dxdt \le \frac{1}{2}\|u_{3,0}\|_{L^2(\Omega)}^2 + \int_0^T\int_\Omega f_{\varepsilon 3}(x,t,u_\varepsilon)u_{\varepsilon 3}dxdt. \tag{4.7}
$$

Thanks to (4.1), (4.4) and (4.5), we see that

$$
\sup_{\varepsilon\in(0,1)}\int_0^T\int_\Omega d_{\varepsilon 3}|\nabla u_{\varepsilon 3}|^2 dx < +\infty. \tag{4.8}
$$

Thanks to this estimate, the assumption on the gradient of d_3, and the elementary computation

$$
|\nabla(d_3 u_{\varepsilon 3})| \le |\nabla d_3|u_{\varepsilon 3} + \sqrt{d_3}\cdot\sqrt{d_3}|\nabla u_{\varepsilon 3}|,
$$

it follows that

$$
\sup_{\varepsilon\in(0,1)}\|\nabla(d_3 u_{\varepsilon 3})\|_{L^2(0,T;L^2(\Omega))} < +\infty. \tag{4.9}
$$

For $\varphi \in L^2(0,T;H^1(\Omega))$, we have

$$\begin{aligned}
\left|\int_0^T \langle \partial_t(d_3u_{\varepsilon 3}), \varphi\rangle dt\right| &\le \left|\int_0^T\int_\Omega d_{\varepsilon 3}\nabla u_{\varepsilon 3}\cdot\nabla(d_3\varphi)dxdt\right| + \int_0^T\int_\Omega |f_3(x,t,u_\varepsilon)||d_3\varphi|dxdt\\
&\le \left|\int_0^T\int_\Omega \sqrt{d_{\varepsilon 3}}\left(\sqrt{d_{\varepsilon 3}}\nabla u_{\varepsilon 3}\right)\,\varphi\,\nabla d_3\,dxdt\right|\\
&\quad + \left|\int_0^T\int_\Omega \sqrt{d_{\varepsilon 3}}\left(\sqrt{d_{\varepsilon 3}}\nabla u_{\varepsilon 3}\right)\,d_3\,\nabla\varphi\,dxdt\right|\\
&\quad + \|d_3f_3(\cdot,\cdot,u_\varepsilon)\|_{L^\infty(0,T;L^\infty(\Omega))}\sqrt{T|\Omega|}\|\varphi\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}\tag{4.10}$$

The second term and the first term on the right hand side of (4.10) are estimated as

$$\left|\int_0^T\int_\Omega \sqrt{d_{\varepsilon 3}}\left(\sqrt{d_{\varepsilon 3}}\nabla u_{\varepsilon 3}\right)d_3\nabla\varphi\,dxdt\right| \le \|\sqrt{d_{\varepsilon 3}}d_3\|_{L^\infty(\Omega)}\|\sqrt{d_{\varepsilon 3}}\nabla u_{\varepsilon 3}\|_{L^2(0,T;L^2(\Omega))}\|\nabla\varphi\|_{L^2(0,T;L^2(\Omega))}$$

and

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \sqrt{d_{\varepsilon 3}}\left(\sqrt{d_{\varepsilon 3}}\nabla u_{\varepsilon 3}\right)\,\varphi\,\nabla d_3\,dxdt\right|\\
&\le C\|\sqrt{d_{\varepsilon 3}}\|_{L^\infty(\Omega)}\|\sqrt{d_{\varepsilon 3}}\nabla u_{\varepsilon 3}\|_{L^2(0,T;L^2(\Omega))}\|\nabla d_3\|_{L^q(\Omega)}\|\varphi\|_{L^2(0,T;L^{2_n^*}(\Omega))}
\end{aligned}$$

thanks to $d_3 \in W^{1,q}(\Omega)$, $q > n$, and $H^1(\Omega) \subset L^{2_n^*}(\Omega)$, where $2_n^* = +\infty$ for $n = 1$, $2_n^* < \infty$ arbitrary for $n = 2$, and $2_n^* = 2n/(n-2)$ for $n \ge 3$. Thus

$$\{\partial_t(d_3u_{\varepsilon 3})\}_{\varepsilon\in(0,1)} \text{ is bounded in } L^2(0,T;(H^1(\Omega))'). \tag{4.11}$$

From (4.9) and (4.11), we can use Aubin-Lions lemma to get, up to a subsequence

$$d_3u_{\varepsilon 3} \to \xi \quad \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

Since $|\{x\in\Omega : d_3(x)\}| = 0$, we have $d_3(x) > 0$ a.e. in Ω. Therefore, it follows that

$$u_{\varepsilon 3} \to \frac{\xi}{d_3} \quad \text{a.e. in } \Omega\times(0,T).$$

From this and (4.4), we have $u_{\varepsilon 3} \to \dfrac{\xi}{d_3}$ strongly in $L^2(\Omega\times(0,T))$, which in combination with (4.6) yields $u_3 = \dfrac{\xi}{d_3}$. Moreover, interpolating with the bounds in (4.4) gives

$$u_{\varepsilon 3} \to u_3 \quad \text{strongly in } L^p(0,T;L^p(\Omega)) \quad \forall p\in[1,\infty).$$

From (4.8) and boundedness of d_3,

$$d_{\varepsilon 3}\nabla u_{\varepsilon 3} \rightharpoonup \chi \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{4.12}$$

For any smooth vector field $\psi \in C_c^\infty((0,T)\times\Omega)^n$, using $\nabla d_{\varepsilon 3} = \nabla d_3$,

$$\begin{aligned}
&\int_0^T\int_\Omega (d_{\varepsilon 3}\nabla u_{\varepsilon 3} - [\nabla(d_3u_3) - u_3\nabla d_3])\cdot\psi dxdt\\
&= \int_0^T\int_\Omega [(d_{\varepsilon 3}\nabla u_{\varepsilon 3} + u_3\nabla d_3)\cdot\psi + d_3u_3\nabla\cdot\psi]\,dxdt\\
&= -\int_0^T\int_\Omega (u_{\varepsilon 3}-u_3)(\nabla d_3\cdot\psi + d_{\varepsilon 3}\nabla\cdot\psi)dxdt - \int_0^T\int_\Omega u_3(d_{\varepsilon 3}-d_3)\nabla\cdot\psi dxdt\\
&\longrightarrow 0 \quad \text{ as } \varepsilon\to 0.
\end{aligned}$$

This means that $d_{\varepsilon 3}\nabla u_{\varepsilon 3}$ converges to $\nabla(d_3u_3) - u_3\nabla d_3$ in the sense of distributions. Together with (4.12), we finally obtain

$$d_{\varepsilon 3}\nabla u_{\varepsilon 3} \rightharpoonup \nabla(d_3u_3) - u_3\nabla d_3 \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

Now we can pass to the limit in the weak formulation of the approximating system

$$\int_0^T \langle \partial_t u_{\varepsilon i}, \varphi\rangle dt + \int_0^T\int_\Omega d_{\varepsilon i}\nabla u_{\varepsilon i}\cdot\nabla\varphi dxdt = \int_0^T\int_\Omega f_{\varepsilon i}(u_\varepsilon)\varphi dxdt, \qquad \varphi\in L^2(0,T;H^1(\Omega)),$$

to conclude that $u = (u_1,u_2,u_3)$ is a weak solution to (1.8) on $(0,T)$ for $T>0$ arbitrary. Moreover, this solution is bounded uniformly in time thanks to (4.4), and consequently is unique due to the local Lipschitz continuity of the nonlinearities. □

To show the convergence to equilibrium, we use some estimates which are similar to those used in the proof of Theorem 1.5. We present them below:

Lemma 4.3. *Under the assumptions in Theorem 1.7, we have the following estimates for solutions to* (1.8)

$$\begin{aligned}
&\sum_{j=1}^3 \|u_j - u_{j,\infty}\|_{L^1(\Omega)}^2 \lesssim_{u_\infty, C_0} \mathcal{E}(u|u_\infty) \lesssim_{u_\infty} \sum_{j=1}^3\int_\Omega |u_j - u_{j,\infty}|^2dx,\\
&\left\|\sqrt{u_j} - \left[\sqrt{u_j}\right]_\Omega\right\|_{L^2(\Omega)}^2 \lesssim_\Omega \int_\Omega d_j\frac{|\nabla u_j|^2}{u_j}dx, \quad j=1,2,\\
&\left\|u_j - [u_j]_\Omega\right\|_{L^2(\Omega)}^2 \lesssim_{\Omega,C_0} \int_\Omega d_j\frac{|\nabla u_j|^2}{u_j}dx, \quad j=1,2,\\
&\mathcal{D}(u) \gtrsim \sum_{j=1}^3\int_\Omega d_j\frac{|\nabla u_j|^2}{u_j}dx + \int_{\omega_1}\left|u_2 - \sqrt{u_1}\right|^2dx + \int_{\omega_2}\left|u_3-\sqrt{u_2}\right|^2dx,\\
&|u_1 - u_{1,\infty}|^2 \lesssim_{C_0,u_\infty} |u_2 - \sqrt{u_1}|^2 + |u_2 - u_{2,\infty}|^2,\\
&|u_3 - u_{3,\infty}|^2 \lesssim_{u_\infty} |u_3 - \sqrt{u_2}|^2 + |u_2 - u_{2,\infty}|^2,\\
&|u_2 - u_{2,\infty}|^2 \lesssim_{C_0} \sum_{j=1}^3\left|u_j - [u_j]_\Omega\right|^2 + |u_2 - \sqrt{u_1}|^2 + |u_3 - \sqrt{u_2}|^2,
\end{aligned}$$

where

$$C_0 := \sup_{t\geq 0}\sum_{i=1}^3 \|u_i(t)\|_{L^\infty(\Omega)}.$$

Proof. The proofs of these estimates are the same as the proofs of Lemmas 3.2, 3.3, 3.4, 3.5, since we do not need a positive lower bound for d_3 in these proofs. □

Comparing to the proof of Theorem 1.5, we need some estimates to compensate the lack of diffusion of S_3 in some part of Ω. This is done in the following lemma.

Lemma 4.4. *Under the assumptions of Theorem 1.7, it holds for solutions to* (1.8)*,*

$$\int_\Omega \left|u_3 - [u_3]_\Omega\right|^2dx \lesssim \mathcal{D}(u).$$

Proof. Because ω_2 is open with Lipschitz boundary, the Poincaré-Wirtinger inequality holds for ω_2. The assumption $d_3\in W^{1,q}(\Omega)$ implies $d_3\in C(\overline{\Omega})$ thanks to Sobolev embedding. From the assumption $\{x\in\overline{\Omega}: d_3(x) = 0\}\subset\omega_2$, there is B an open set of class C^1 such that $\{x\in\overline{\Omega}: d_3(x) = 0\}\subset B\Subset\omega_2$ with the properties:

- the Poincaré-Wirtinger inequality in $B^c = \Omega \backslash B$ holds;
- $d_3(x) \geq \varrho$, $\forall x \in B^c$ for some $\varrho > 0$;
- $\omega_2^c \subset B^c$;
- $\omega_2 \backslash B \subset \omega_2$ and $\omega_2 \backslash B \subset B^c$.

Next, using triangle inequalities, we estimate

$$\begin{aligned}\int_\Omega |u_3 - [u_3]_\Omega|^2 dx &\lesssim \int_{\omega_2^c} |u_3 - [u_3]_\Omega|^2 dx + \int_{\omega_2} |u_3 - [u_3]_\Omega|^2 dx \\ &\lesssim \int_{\omega_2^c} |u_3 - [u_3]_{B^c}|^2 dx + |[u_3]_{B^c} - [u_3]_\Omega|^2 \\ &\quad + \int_{\omega_2} |u_3 - \sqrt{u_2}|^2 dx + \int_{\omega_2} \left|\sqrt{u_2} - [\sqrt{u_2}]_{\omega_2}\right|^2 dx \\ &\quad + \left|[\sqrt{u_2}]_{\omega_2} - [u_3]_{\omega_2}\right|^2 + \left|[u_3]_{\omega_2} - [u_3]_\Omega\right|^2. \end{aligned} \tag{4.13}$$

We treat $|[u_3]_{B^c} - [u_3]_\Omega|^2 + \left|[u_3]_{\omega_2} - [u_3]_\Omega\right|^2$ with the aim to drop the term $[u_3]_\Omega$. Since $|\omega_2| + |\omega_2^c| = |\Omega| = 1$ and $[u_3]_\Omega = |\omega_2|\,[u_3]_{\omega_2} + |\omega_2^c|\,[u_3]_{\omega_2^c}$, it holds

$$\left|[u_3]_{\omega_2} - [u_3]_\Omega\right|^2 = |\omega_2^c|^2 \left|[u_3]_{\omega_2} - [u_3]_{\omega_2^c}\right|^2.$$

Therefore, by triangle inequality, we have

$$\begin{aligned}|[u_3]_{B^c} - [u_3]_\Omega|^2 + \left|[u_3]_{\omega_2} - [u_3]_\Omega\right|^2 &\lesssim \left|[u_3]_{B^c} - [u_3]_{\omega_2}\right|^2 + \left|[u_3]_{\omega_2} - [u_3]_\Omega\right|^2 \\ &\lesssim \left|[u_3]_{B^c} - [u_3]_{\omega_2}\right|^2 + \left|[u_3]_{\omega_2} - [u_3]_{\omega_2^c}\right|^2 \\ &\lesssim \left|[u_3]_{B^c} - [u_3]_{\omega_2}\right|^2 + \left|[u_3]_{B^c} - [u_3]_{\omega_2^c}\right|^2. \end{aligned} \tag{4.14}$$

We bound $\left|[u_3]_{B^c} - [u_3]_{\omega_2^c}\right|^2$ thanks to Cauchy-Schwarz inequality as follows

$$\left|[u_3]_{B^c} - [u_3]_{\omega_2^c}\right|^2 = \left|\frac{1}{|\omega_2^c|}\int_{\omega_2^c} ([u_3]_{B^c} - u_3)\, dx\right|^2 \leq \frac{1}{|\omega_2^c|}\int_{\omega_2^c} |[u_3]_{B^c} - u_3|^2 dx. \tag{4.15}$$

Now, we estimate $\left|[u_3]_{B^c} - [u_3]_{\omega_2}\right|^2$. Let $\vartheta := \omega_2 \backslash B$. Since $\vartheta \subset \omega_2$ and $\vartheta \subset B^c$, it holds

$$\begin{aligned}\left|[u_3]_{B^c} - [u_3]_{\omega_2}\right|^2 &= \frac{1}{|\vartheta|}\int_\vartheta \left|[u_3]_{B^c} - u_3 + u_3 - [u_3]_{\omega_2}\right|^2 dx \qquad (4.16) \\ &\lesssim \int_\vartheta |u_3 - [u_3]_{B^c}|^2 dx \\ &\quad + \int_\vartheta |u_3 - \sqrt{u_2}|^2 dx + \int_\vartheta \left|\sqrt{u_2} - [\sqrt{u_2}]_{\omega_2}\right|^2 dx + \left|[\sqrt{u_2}]_{\omega_2} - [u_3]_{\omega_2}\right|^2 \\ &\lesssim \int_{B^c} |u_3 - [u_3]_{B^c}|^2 dx \\ &\quad + \int_{\omega_2} |u_3 - \sqrt{u_2}|^2 dx + \int_{\omega_2} \left|\sqrt{u_2} - [\sqrt{u_2}]_{\omega_2}\right|^2 dx + \left|[\sqrt{u_2}]_{\omega_2} - [u_3]_{\omega_2}\right|^2. \end{aligned}$$

Combining the estimates (4.14), (4.15) and (4.16) with (4.13), one can conclude that

$$\begin{aligned}\int_\Omega |u_3 - [u_3]_\Omega|^2 dx &\lesssim \int_{\omega_2^c} |u_3 - [u_3]_{B^c}|^2 dx + \int_{B^c} |u_3 - [u_3]_{B^c}|^2 dx \\ &\quad + \int_{\omega_2} |u_3 - \sqrt{u_2}|^2 dx + \int_{\omega_2} \left|\sqrt{u_2} - [\sqrt{u_2}]_{\omega_2}\right|^2 dx \end{aligned}$$

$$+\left|[\sqrt{u_2}]_{\omega_2}-[u_3]_{\omega_2}\right|^2$$
$$=: (I)+(II)+(III)+(IV)+(V)\,.$$

Due to $\omega_2^c \subset B^c$, Poincaré-Wirtinger inequality on B^c and $d_3 \geq \varrho > 0$ on B^c,

$$(I)+(II) \lesssim \int_{B^c} |u_3-[u_3]_{B^c}|^2\,dx \lesssim \int_{B^c} |\nabla u_3|^2\,dx \lesssim \int_\Omega d_3\frac{|\nabla u_3|^2}{u_3}dx \leq \mathcal{D}(u)\,.$$

By Poincaré-Wirtinger inequality on ω_2, we have

$$(IV) \lesssim \int_{\omega_2} \frac{|\nabla u_2|^2}{u_2}dx \lesssim \int_\Omega d_2\frac{|\nabla u_2|^2}{u_2}dx \leq \mathcal{D}(u)\,.$$

Besides, thanks to Cauchy-Schwarz inequality and Lemma 4.3,

$$(III)+(V) \lesssim \int_{\omega_2} |u_3-\sqrt{u_2}|^2 \lesssim \mathcal{D}(u)\,.$$

This concludes the proof of Lemma 4.4. □

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Thanks to Lemma 4.4, we can use the same arguments in the proof of Theorem 1.5 to obtain the entropy-entropy dissipation inequality

$$\mathcal{E}(u|u_\infty) \lesssim \mathcal{D}(u).$$

Indeed, the same arguments as in Lemma 3.6 gives

$$\mathcal{E}(u|u_\infty) \lesssim \sum_{j=1}^{3} \|u_j-[u_j]_\Omega\|_{L^2(\Omega)}^2 + \int_{\omega_1} \left(|u_2-\sqrt{u_1}|^2+|u_3-\sqrt{u_2}|^2\right)dx.$$

It follows from Lemma 4.3 that

$$\sum_{j=1}^{2} \|u_j-[u_j]_\Omega\|_{L^2(\Omega)}^2 + \int_{\omega_1} |u_2-\sqrt{u_1}|^2dx \lesssim \mathcal{D}(u).$$

Lemma 4.4 gives

$$\|u_3-[u_3]_\Omega\|_{L^2(\Omega)}^2 \lesssim \mathcal{D}(u).$$

With this estimate at hand, we can finally estimate

$$\int_{\omega_1} |u_3-\sqrt{u_2}|^2dx \lesssim \mathcal{D}(u)$$

by using the same arguments as in Lemma 3.7.

The convergence to equilibrium then follows in the standard way, as in Proof of Theorem 1.5, so we omit the details. □

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