Observability of the Schrödinger Equation

K.-D. Phung

ABSTRACT The goal here is to present two approaches concerning results on observability and control of the Schrödinger equation in a bounded domain. Our results are obtained from different works on the control of the heat equation or of the wave equation. From the theory of exact and approximate controllability, introduced by J.L. Lions [10], we know that observation is equivalent to approximate controllability and stable observation is equivalent to exact controllability.

Our first result is based on a gaussian transform which traduces any estimate of stable observability of the heat equation to an estimate of unstable observability for the Schrödinger equation (see Section 1 below). This work is similar to those done by L. Robbiano [14] for hyperbolic problems on the domain where the geometrical control condition of C. Bardos, G. Lebeau and J. Rauch on the exact controllability of the wave equation [1] is not satisfied.

Our second result is about exact control for the Schrödinger equation (see Section 2) and is inspired by a transform introduced by L. Boutet de Monvel [2] for the study of the propagation of singularities of an analogous solution of the Schrödinger equation. Our strategy is to construct an exact control for the Schrödinger equation from an exact controllability result for the wave equation.

1 Observability results for the Schrödinger equation

Let Ω be an open bounded domain of $\mathbb{R}^n$ with a C^∞ boundary $\partial\Omega$. We consider the Schrödinger equation with the Dirichlet boundary condition:

$$\begin{cases} i\partial_t u + \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_t \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_t \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the solution $u \in C(\mathbb{R}; H_0^1(\Omega))$ if $u_0 \in H_0^1(\Omega)$.

We say that we have stable boundary observability of the heat equation if for all open Γ, non-empty set of $\partial\Omega$, such that $\overline{\Gamma} \subset \partial\Omega$, for all $T > 0$, there exists $C_T > 0$ such that the solution w of the evolution problem

$$\begin{cases} \partial_t w + \Delta w = f & \text{in } \Omega \times]0, T[\\ w = 0 & \text{on } \partial\Omega \times]0, T[\\ w(\cdot, T) \in L^2(\Omega) \end{cases} \tag{1.2}$$

satisfies

$$\int_{\Omega} |w(\cdot,0)|^2 \, dx \leq C_T \left(\int_{\Gamma} \int_0^T |\partial_n w|^2 \, dt dx + \int_{\Omega} \int_0^T |f|^2 \, dt dx \right). \quad (1.3)$$

We propose to establish the following observability estimates:

Theorem 1.1. *If a stable boundary observability of the heat equation is satisfied, then we have a boundary observability estimate in logarithmic type for the Schrödinger equation.*

Corollary 1.2. *For all $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for all initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ of problem (1.1), we have*

$$\int_{\Omega} |\nabla u_0|^2 \, dx \leq \exp\left(C_\varepsilon \frac{\int_{\Omega} |\Delta u_0|^2 \, dx}{\int_{\Omega} |u_0|^2 \, dx} \right) \int_{\Gamma} \int_0^{\varepsilon} |\partial_n u(x,t)|^2 \, dt dx \quad (1.4)$$

Remark 1.3. The estimate (1.4) is equivalent to

$$\int_{\Omega} |u_0|^2 \, dx \leq \frac{C_\varepsilon}{\ln\left(2 + \frac{\int_{\Omega} |\nabla u_0|^2 dx}{\int_{\Gamma} \int_0^{\varepsilon} |\partial_n u(x,t)|^2 dt dx}\right)} \int_{\Omega} |\Delta u_0|^2 \, dx. \quad (1.5)$$

Corollary 1.2 comes from Theorem 1.1 and the work of G. Lebeau and L. Robbiano [11] or of A.V. Fursikov and O.Yu. Imanuvilov [7] on the exact controllability of the heat equation obtained from Carleman inequalities. Note that D. Tataru [15] gives a directly unique continuation estimate for the Schrödinger equation with Dirichlet boundary condition from Carleman's type inequalities. Corollary 1.2 is also valid for internal observability of the Schrödinger equation: Let ω be a non-empty open set included in Ω. For all $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for all initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ of problem (1.1), we have

$$\int_{\Omega} |\nabla u_0|^2 \, dx \leq \exp\left(C_\varepsilon \frac{\int_{\Omega} |\Delta u_0|^2 \, dx}{\int_{\Omega} |u_0|^2 \, dx} \right) \int_{\omega} \int_0^{\varepsilon} |u(x,t)|^2 \, dx dt. \quad (1.6)$$

2 Exact control result for the Schrödinger equation

Let Ω be a bounded open set in $\mathbf{R}^n$, $n > 1$, with a boundary of class C^∞. Let $T > 0$ and $\Theta \in C_c^0(\partial\Omega \times]0,T[;\mathbf{R})$. We say that the function Θ controls Ω exactly for the wave equation with partially null initial data if for all $\Phi_0 \in H_0^1(\Omega)$, there is a boundary control $g \in H^1(\mathbf{R}_t; L^2(\partial\Omega))$ such that the solution of problem

$$\begin{cases} \partial_t^2 \Phi - \Delta \Phi = 0 & \text{in } \Omega \times \mathbf{R}_t \\ \Phi = \Theta g & \text{on } \partial\Omega \times \mathbf{R}_t \\ \Phi(\cdot,0) = \Phi_0, \ \partial_t \Phi(\cdot,0) = 0 & \text{in } \Omega \end{cases} \quad (2.1)$$

satisfies $\Phi \equiv 0$ in $\Omega \times [T, +\infty[$.

We say that the function Θ controls Ω geometrically if any generalized bicharacteristic ray meets the set $\Theta \neq 0$ on a non-diffractive point. (see [4]).

We propose to establish the following exact control result:

Theorem 2.1. *If the function $\Theta : (x,t) \mapsto \Xi(x)\theta(t)$ controls Ω exactly for the wave equation with partially null initial data, then for all $\varepsilon > 0$, for all initial data $w_o \in H_0^1(\Omega)$, there exists a control $\vartheta_\varepsilon \in L^2(\partial\Omega \times]0,\varepsilon[)$ such that the solution of problem*

$$\begin{cases} i\partial_t w + \Delta w = 0 & \text{in } \Omega \times]0,\varepsilon[\\ w = \Xi\vartheta_\varepsilon & \text{on } \partial\Omega \times]0,\varepsilon[\\ w(\cdot,0) = w_o & \text{in } \Omega \end{cases} \tag{2.2}$$

satisfies $w \equiv 0$ in $\Omega \times \{t \geq \varepsilon\}$.

Corollary 2.2. *We suppose there is no infinite order of contact between the boundary $\partial\Omega \times]0,T[$ and the bicharacteristics of $\partial_t^2 - \Delta$. If the function $\Theta : (x,t) \mapsto \Xi(x)\theta(t)$ controls Ω geometrically, then for all $\varepsilon > 0$, for all initial data $w_o \in H_0^1(\Omega)$, there exists a control $\vartheta_\varepsilon \in L^2(\partial\Omega \times]0,\varepsilon[)$ such that the solution of problem*

$$\begin{cases} i\partial_t w + \Delta w = 0 & \text{in } \Omega \times]0,\varepsilon[\\ w = \Xi\vartheta_\varepsilon & \text{on } \partial\Omega \times]0,\varepsilon[\\ w(\cdot,0) = w_o & \text{in } \Omega \end{cases} \tag{2.3}$$

satisfies $w \equiv 0$ in $\Omega \times \{t \geq \varepsilon\}$. Furthermore, we have an estimate of the control ϑ_ε, as follows

$$\|\vartheta_\varepsilon\|_{L^2(\partial\Omega \times]0,\varepsilon[)} \leq \sqrt{\varepsilon} C_T (1 + \varepsilon\beta_\varepsilon) \|\nabla w_o\|_{L^2(\Omega)}. \tag{2.4}$$

Remark 2.3. Corollary 2.2 comes from Theorem 2.1 and the work of C. Bardos, G. Lebeau and J. Rauch [1] or of N. Burq and P. Gérard [4] on the exact controllability of the wave equation from a microlocal analysis. The constant β_ε is given by an observability estimate in the one dimensional case. Our result is not optimal in norm in the sense that it is sufficient to choose initial data $w_o \in H^{-1}(\Omega)$ to have an exact control result for the Schrödinger equation, with hypothesis of the multiplier method [12] [13] [5]. Also, G. Lebeau [9] has proved the exact controllability for the Schrödinger equation with the geometrical control condition of the wave equation [1] and an analytic boundary. Furthermore, there exist open sets which do not satisfy the geometrical control condition and in which it is possible to control exactly with regular initial data [3]. Here, our goal is to use knowledge of the exact controllability for the wave equation to obtain an exact control result for the Schrödinger equation.

3 Proof of the unstable observability results for the Schrödinger equation

3.1 The parabolic problem

The proof of Theorem 1.1 comes from the work of Lebeau and Robbiano [11] or of Fursikov and Imanuvilov [7] on the exact controllability for the heat equation from Carleman inequalities. We recall the result in [11] to be complete:

Let Ω be a Riemanian compact manifold with a boundary $\partial\Omega$ of class C^∞, and let Δ be the laplacian on Ω. For all $0 < a < b < T$, there exists a continuous operator $S_\Gamma : L^2(\Omega) \to C_0^\infty(\Gamma\times]a,b[)$ when $\partial\Omega \neq \emptyset$ (resp. $S_\omega : L^2(\Omega) \to C_0^\infty(\omega\times]a,b[)$ with eventually $\partial\Omega = \emptyset$) such that for all $v_0 \in L^2(\Omega)$, the solution of the heat equation

$$\begin{cases} \partial_t v - \Delta v = 0 \text{ (resp. } = S_\omega(v_0)) & \text{in } \Omega\times]0,T[\\ v = S_\Gamma(v_0) \text{ (resp. } = 0) & \text{on } \partial\Omega\times]0,T[\\ v(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \tag{3.1}$$

satisfies $v(\cdot,T) \equiv 0$.

We have the following estimates:

Lemma 3.1. *Let w be the solution of the following evolution problem:*

$$\begin{cases} \partial_t w + \Delta w = f & \text{in } \Omega\times]0,T[\\ w = 0 & \text{on } \partial\Omega\times]0,T[\\ w(\cdot,T) \in L^2(\Omega). \end{cases}$$

Then,

$$\exists C_T > 0 \quad \int_\Omega |w(\cdot,0)|^2\, dx \le C_T \left(\int_\Gamma \int_0^T |\partial_n w|^2\, dt dx + \int_\Omega \int_0^T |f|^2\, dt dx \right) \tag{3.2}$$

$$\exists C_T > 0 \quad \int_\Omega |w(\cdot,0)|^2\, dx \le C_T \left(\int_\omega \int_0^T |w|^2\, dt dx + \int_\Omega \int_0^T |f|^2\, dt dx \right). \tag{3.3}$$

Furthermore if $w(\cdot,T) \in H^2 \cap H_0^1(\Omega)$, then

$$\exists C_T > 0 \quad \int_\Omega |w(\cdot,0)|^2\, dx \le C_T \left(\int_\omega \int_0^T |\Delta w|^2\, dt dx + \int_\Omega \int_0^T |f|^2\, dt dx \right). \tag{3.4}$$

Another approach, based on the work of Fursikov and Imanuvilov [7], gives the following uniform time estimates [6]:

Let Ω be a connected bounded open set included in $\mathbb{R}^n$ with boundary $\partial\Omega \subset C^\infty$. Let w be the solution of the following evolution problem:

$$\begin{cases} \partial_t w + \Delta w = 0 & \text{in } \Omega\times]0,T[\\ w = 0 & \text{on } \partial\Omega\times]0,T[. \end{cases} \tag{3.5}$$

Then, there is $C > 0$, such that for all $T > 0$, if $w(\cdot,T) \in L^2(\Omega)$, we have

$$\int_\Omega |w(\cdot,0)|^2\, dx \leq \exp\left(C\left(1+\frac{1}{T}\right)\right) \int_\omega \int_0^T |w|^2\, dt dx \tag{3.6}$$

and, there is $C > 0$, such that for all $T > 0$, if $w(\cdot,T) \in H^2 \cap H_0^1(\Omega)$, we have

$$\int_\Omega |w(\cdot,0)|^2\, dx \leq \exp\left(C\left(1+\frac{1}{T}\right)\right) \left(\int_\omega \int_0^T |\Delta w|^2\, dt dx\right). \tag{3.7}$$

Thus the constant C_ε of estimate (1.6) could be written explicitly in ε.

3.2 Proof of Theorem 1.1

Let $F(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau} e^{-\tau^2} d\tau$; then $|F(z)| = \frac{\sqrt{\pi}}{2\pi} e^{\frac{1}{4}(|\mathrm{Im}\, z|^2 - |\mathrm{Re}\, z|^2)}$. Also, let $\lambda > 0$, and

$$F_\lambda(z) = \lambda F(\lambda z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau} e^{-\left(\frac{\tau}{\lambda}\right)^2} d\tau.$$

We have

$$|F_\lambda(z)| = \frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}(|\mathrm{Im}\, z|^2 - |\mathrm{Re}\, z|^2)}. \tag{3.8}$$

Let $s, \ell_0 \in \mathbb{R}$ and

$$W_{\ell_0,\lambda}(s,x) = \int_{\mathbb{R}} F_\lambda(\ell_0 + is - \ell)\Phi(\ell)u(x,\ell)d\ell$$

where $\Phi \in C_0^\infty(\mathbb{R})$. We remark that $\partial_s F_\lambda(\ell_0 + is - \ell) = -i\partial_\ell F_\lambda(\ell_0 + is - \ell)$ and thus

$$\begin{aligned} \partial_s W_{\ell_0,\lambda}(s,x) &= \int_{\mathbb{R}} -i\partial_\ell F_\lambda(\ell_0 + is - \ell)\Phi(\ell)u(x,\ell)d\ell \\ &= \int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\left\{\frac{d}{d\ell}\Phi(\ell)u(x,\ell) + \Phi(\ell)\frac{\partial}{\partial\ell}u(x,\ell)\right\} d\ell. \end{aligned}$$

As u is the solution of (1.1), $W_{\ell_0,\lambda}$ satisfies

$$\begin{cases} \partial_s W_{\ell_0,\lambda}(s,x) + \Delta W_{\ell_0,\lambda}(s,x) = \int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\Phi'(\ell)u(x,\ell)d\ell \\ W_{\ell_0,\lambda}(s,x) = 0 \quad \forall x \in \partial\Omega \\ W_{\ell_0,\lambda}(0,x) = (F_\lambda * \Phi u(x,\cdot))(\ell_0) \quad \forall x \in \Omega. \end{cases} \tag{3.9}$$

We define $\Phi \in C_0^\infty(\mathbb{R})$. Let $L > 0$. We choose $\Phi \in C_0^\infty(]0, L[)$, $0 \le \Phi \le 1$, $\Phi \equiv 1$ on $\left[\frac{L}{4}; \frac{3L}{4}\right]$ such that $|\Phi'| \le \frac{4}{L}$, $|\Phi''| \le \frac{4\times 8}{L^2}$. We take $K = [0; \frac{L}{4}] \cup [\frac{3L}{4}; L]$ and $K_0 = \left[\frac{3L}{8}; \frac{5L}{8}\right]$. So, $\mathrm{mes} K_0 = \frac{L}{4}$, $\mathrm{mes} K = \frac{L}{2}$, $\mathrm{supp}(\Phi') = K$ and $\mathrm{dist}(K; K_o) = \frac{L}{8}$. We will choose $\ell_0 \in K_0$.

By application of (3.2), $W_{\ell_0,\lambda}$ satisfies the following estimate:

$$\int_\Omega |(F_\lambda * \Phi u(x,\cdot))(\ell_0)|^2\, dx \le C_T \int_\Gamma \int_0^T |\partial_n W_{\ell_0,\lambda}(s,x)|^2\, dsdx$$
$$+ C_T \int_\Omega \int_0^T \left| \int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\Phi'(\ell)u(x,\ell)d\ell \right|^2 dsdx \qquad (3.10)$$

Furthermore, from (3.8)

$$\begin{aligned}
&\int_\Gamma \int_0^T |\partial_n W_{\ell_0,\lambda}(s,x)|^2\, dsdx \\
&\quad = \int_\Gamma \int_0^T \left| \int_{\mathbb{R}} F_\lambda(\ell_0 + is - \ell)\Phi(\ell)\partial_n u(x,\ell)d\ell \right|^2 dsdx \\
&\quad \le \int_0^T \int_\Gamma \left| \int_{\mathbb{R}} \frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}(s^2 - |\ell_0-\ell|^2)}\Phi(\ell)\, |\partial_n u(x,\ell)|\, d\ell \right|^2 dxds \\
&\quad \le \frac{\lambda^2}{4\pi}\left(\int_0^T e^{\frac{\lambda^2}{2}s^2} ds \right) |\sup \Phi|^2 \int_\Gamma \left| \int_0^L |\partial_n u(x,\ell)|\, d\ell \right|^2 dx \\
&\quad \le \frac{\lambda^2}{4\pi} e^{\frac{\lambda^2}{2}T^2} T\ |\sup \Phi|^2\ L \int_\Gamma \int_0^L |\partial_n u(x,\ell)|^2\, d\ell dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \int_0^T \left| \int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\Phi'(\ell)u(x,\ell)d\ell \right|^2 dsdx \\
&\le \int_0^T \int_\Omega \left| \int_{\mathbb{R}} \frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}(s^2 - |\ell_0-\ell|^2)}\, |\Phi'(\ell)|\, |u(x,\ell)|\, d\ell \right|^2 dxds \\
&\le \frac{\lambda^2}{4\pi} e^{\frac{\lambda^2}{2}T^2} T \int_\Omega \left(\int_K e^{-\frac{\lambda^2}{2}|\ell_0-\ell|^2} |\Phi'(\ell)|^2 |u(x,\ell)|^2 d\ell \right) \mathrm{mes}\,(K)\, dx \\
&\le \frac{\lambda^2}{4\pi} e^{\frac{\lambda^2}{2}T^2} T\, e^{-\frac{\lambda^2}{2}\mathrm{dist}(K,K_0)^2} \sup |\Phi'(\ell)|^2 \mathrm{mes}(K) \int_\Omega \int_K |u(x,\ell)|^2 d\ell dx \\
&\le \frac{\lambda^2}{4\pi} e^{\frac{\lambda^2}{2}T^2} T\, e^{-\frac{\lambda^2}{2}\mathrm{dist}(K,K_0)^2} \sup |\Phi'(\ell)|^2\, \mathrm{mes}\,(K)^2 \int_\Omega |u_0|^2\, dx \\
&\le \frac{\lambda^2 T}{4\pi} \exp\left[\frac{\lambda^2}{2}\left(T^2 - \left(\frac{L}{8}\right)^2 \right) \right] \frac{4^2}{L^2}\frac{L^2}{4} \int_\Omega |u_0|^2\, dx \\
&\le \frac{\lambda^2 T}{\pi} \exp\left[\frac{\lambda^2}{2}\left(T^2 - \left(\frac{L}{8}\right)^2 \right) \right] \int_\Omega |u_0|^2\, dx.
\end{aligned}$$

Thus, inequality (3.10) becomes:

$$\int_{\Omega} |(F_{\lambda} * \Phi u(x,\cdot))(\ell_0)|^2 \, dx$$
$$\leq C_T \frac{\lambda^2 TL}{4\pi} \exp\left(\frac{\lambda^2}{2} T^2\right) \int_{\Gamma}\int_0^L |\partial_n u(x,\ell)|^2 \, d\ell dx$$
$$+ C_T \frac{\lambda^2 T}{\pi} \exp\left[\frac{\lambda^2}{2}\left(T^2 - \left(\frac{L}{8}\right)^2\right)\right] \int_{\Omega} |u_0|^2 \, dx \quad (3.11)$$

By the Parseval relation, we have:

$$\int_{\mathbf{R}} |\Phi(\ell_0) u(x,\ell_0) - (F_{\lambda} * \Phi u(x,\cdot))(\ell_0)|^2 \, d\ell_0$$
$$= \frac{1}{2\pi} \int_{\mathbf{R}} \left|\widehat{\Phi(\ell_0) u(x,\ell_0)}(\tau)\right|^2 \left(1 - e^{-\left(\frac{\tau}{\lambda}\right)^2}\right)^2 d\tau$$
$$\leq \frac{1}{\pi\lambda^2} \int_{\mathbf{R}} \left|\tau \widehat{\Phi(\ell_0) u(x,\ell_0)}(\tau)\right|^2 d\tau$$
$$\leq \frac{2}{\lambda^2} \int_{\mathbf{R}} |\Phi'(\ell_0) u(x,\ell_0) + \Phi(\ell_0) \partial_{\ell_0} u(x,\ell_0)|^2 \, d\ell_0$$
$$\leq \frac{4}{\lambda^2} \left[\left(\frac{4}{L}\right)^2 \int_K |u(x,\ell_0)|^2 \, d\ell_0 + \int_0^L |\partial_{\ell_0} u(x,\ell_0)|^2 \, d\ell_0\right].$$

By integrating on Ω, we obtain

$$\int_{\Omega}\int_{\mathbf{R}} |\Phi(\ell_0) u(x,\ell_0) - (F_{\lambda} * \Phi u(x,\cdot))(\ell_0)|^2 \, d\ell_0 dx$$
$$\leq \frac{4}{\lambda^2} \left[\left(\frac{4}{L}\right)^2 \int_K \int_{\Omega} |u(x,\ell_0)|^2 \, d\ell_0 dx + \int_0^L \int_{\Omega} |\partial_{\ell_0} u(x,\ell_0)|^2 \, d\ell_0 dx\right]$$
$$\leq \frac{4}{\lambda^2} \left[\left(\frac{4}{L}\right)^2 \frac{L}{2} \int_{\Omega} |u_0|^2 \, dx + L \int_{\Omega} |\Delta u_0|^2 \, dx\right]. \quad (3.12)$$

So, from (3.11) and (3.12)

$$\operatorname{mes}(K_0) \int_{\Omega} |u_0|^2 \, dx = \int_{K_0}\int_{\Omega} |\Phi(\ell_0) u(x,\ell_0)|^2 \, dx d\ell_0$$
$$\leq \operatorname{mes}(K_0)\, C_T \frac{\lambda^2 TL}{4\pi} \exp\left(\frac{\lambda^2}{2} T^2\right) \int_{\Gamma}\int_0^L |\partial_n u(x,\ell)|^2 \, d\ell dx$$
$$+ \operatorname{mes}(K_0)\, C_T \frac{\lambda^2 T}{\pi} \exp\left[\frac{\lambda^2}{2}\left(T^2 - \left(\frac{L}{8}\right)^2\right)\right] \int_{\Omega} |u_0|^2 \, dx$$
$$+ \frac{4}{\lambda^2} \left[\left(\frac{4}{L}\right)^2 \frac{L}{2} \int_{\Omega} |u_0|^2 \, dx + L \int_{\Omega} |\Delta u_0|^2 \, dx\right].$$

Finally

$$\int_\Omega |u_0|^2\,dx \le C_T \frac{\lambda^2 TL}{4\pi} \exp\left(\frac{\lambda^2}{2}T^2\right) \int_\Gamma \int_0^L |\partial_n u(x,\ell)|^2\,d\ell dx + C_T \frac{\lambda^2 T}{\pi} \exp\left[\frac{\lambda^2}{2}\left(T^2 - \left(\frac{L}{8}\right)^2\right)\right] \int_\Omega |u_0|^2 dx + \frac{1}{\lambda^2}\frac{4^2}{L}\left[\frac{8}{L}\int_\Omega |u_0|^2 dx + L\int_\Omega |\Delta u_0|^2 dx\right].$$

By choosing $L = 8AT$, with $(1 - A^2) < 0$ and $T \le 1$, this becomes

$$\begin{aligned}\int_\Omega |u_0|^2\,dx &\le \frac{2A}{\pi} C_T \lambda^2 T^2 \exp\left(\frac{\lambda^2 T^2}{2}\right) \int_\Gamma \int_0^{8AT} |\partial_n u(x,\ell)|^2\,d\ell dx \\ &\quad + 16\frac{1}{\lambda^2}\int_\Omega |\Delta u_0|^2\,dx \\ &\quad + \left[\frac{2}{A^2}\frac{1}{\lambda^2 T^2} + \frac{1}{\pi} C_T \lambda^2 T \exp\left(-\frac{A^2-1}{2}\lambda^2 T^2\right)\right] \int_\Omega |u_0|^2\,dx \\ &\le \frac{2A}{\pi} C_T \lambda^2 T^2 \exp\left(\frac{\lambda^2 T^2}{2}\right) \int_\Gamma \int_0^{8AT} |\partial_n u(x,\ell)|^2\,d\ell dx \\ &\quad + \left[\frac{2}{A^2}\frac{1}{\lambda^2 T^2} + \frac{C_T}{\pi (A^2-1)^2}\frac{1}{\lambda^2 T^3}\right] \int_\Omega |u_0|^2 dx + 16\frac{1}{\lambda^2}\int_\Omega |\Delta u_0|^2 dx.\end{aligned}$$

We write

$$\int_\Omega |u_0|^2 dx \le C \exp(\lambda^2 T^2) \int_\Gamma \int_0^{8AT} |\partial_n u(x,\ell)|^2 d\ell dx + \frac{C}{\lambda^2 T^3}\int_\Omega |\Delta u_0|^2 dx. \tag{3.13}$$

Now, we take

$$\frac{1}{\lambda^2} = \frac{T^3}{C}\frac{1}{2}\frac{\int_\Omega |u_0|^2\,dx}{\int_\Omega |\Delta u_0|^2\,dx}.$$

So, estimate (3.13) becomes

$$\int_\Omega |u_0|^2\,dx \le C \exp\left(\frac{C}{T}\frac{\int_\Omega |\Delta u_0|^2\,dx}{\int_\Omega |u_0|^2\,dx}\right) \int_\Gamma \int_0^{8AT} |\partial_n u(x,\ell)|^2\,d\ell dx.$$

We conclude by choosing $T = \frac{\varepsilon}{8A} \le 1$, with $\varepsilon > 0$. The estimate (1.4) of Theorem 1.1 is obtained by interpolation.

4 Proof of the exact control result for the Schrödinger equation

4.1 The Schrödinger equation in one space dimension

We give two results on the Schrödinger equation in one dimension.

Proposition 4.1. *If $n = 1$, then for all $\omega \subset \Omega =]A, B[$ non-empty open set included in $\mathbb{R}$ and a neighbourhood of the point $x = B$, for all $\varepsilon > 0$, there is $\beta_\varepsilon > 0$ such that for all $u_o \in L^2(\Omega)$ initial data of problem (1.1), we have*

$$\|u_o\|^2_{L^2(\Omega)} \leq \beta_\varepsilon \int_0^\varepsilon \int_\omega |u|^2 \, dxdt. \tag{4.1}$$

Proposition 4.2. *There exists a triplet (f, u, F) such that*

$$\begin{cases} i\partial_t F + \partial_s^2 F = f.1_{|]3T/2,2T[} - \partial_s u \otimes \delta(s + 2T) \\ \quad +\partial_s u \otimes \delta(s - 2T) \ \ in \ ]0, \varepsilon[\times \mathbb{R}_s \\ F(0, \cdot) = \delta(\cdot) \\ F(\varepsilon, \cdot) = 0 \quad in \ [-T, T]. \end{cases} \tag{4.2}$$

Proof of the Proposition 4.1. Comes from the multiplier method [12], [13], [5]. The constant β_ε could be written explicitly in ε, from (3.6).

Proof of the Proposition 4.2. From the HUM method and Proposition 4.1, for all data $u_o \in L^2(]-2T, 2T[)$, there exists a control $f \in L^2(]0, \varepsilon[\times]3T/2, 2T[)$ such that the solution $u : (t, s) \longmapsto u(t, s) \in C([0, \varepsilon]; L^2(]-2T, 2T[))$ satisfies

$$\begin{cases} i\partial_t u + \partial_s^2 u = f.1_{|]3T/2,2T[} \quad \text{in }]0, \varepsilon[\times]-2T, 2T[\\ u(\cdot, -2T) = 0, \ u(\cdot, 2T) = 0 \quad \text{on }]0, \varepsilon[\\ u(0, \cdot) = 0 \quad \text{in }]-2T, 2T[\\ u(\varepsilon, \cdot) = u_o \quad \text{in }]-2T, 2T[\end{cases} \tag{4.3}$$

and

$$\|f\|_{L^2(]0,\varepsilon[\times]3T/2,2T[)} \leq \beta_\varepsilon \|u_o\|_{L^2(]-2T,2T[)}. \tag{4.4}$$

In particular, we take $u_o(\varepsilon, s) = -\chi(s) \frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi\varepsilon}} e^{i\frac{s^2}{4\varepsilon}}$ where $s \in]-2T, 2T[$, $\chi \in C_0^\infty(]-2T, 2T[)$, $0 \leq \chi \leq 1$, $\chi_{|[-T,T]} = 1$. Thus,

$$\begin{aligned} \|u\|_{L^2(]0,\varepsilon[\times]-2T,2T[)} &\leq \varepsilon \|u\|_{L^\infty(0,\varepsilon;L^2(]-2T,2T[))} \\ &\leq 2\varepsilon \|f\|_{L^1(0,\varepsilon,L^2(]3T/2,2T[))} \leq 2\varepsilon\sqrt{\varepsilon}\beta_\varepsilon \end{aligned} \tag{4.5}$$

Let

$$H(t, s) = \left| \begin{array}{ll} u(t, s) & \text{in } [0, \varepsilon] \times [-2T, 2T] \\ 0 & \text{in } [0, \varepsilon] \times (]-\infty, -2T[\cup]2T, +\infty[) \end{array} \right. \tag{4.6}$$

where u is the solution of (4.3). Thus,

$$\begin{cases} i\partial_t H + \partial_s^2 H = f.1_{]]3T/2,2T[} - \partial_s u \otimes \delta(s+2T) \\ \quad +\partial_s u \otimes \delta(s-2T) \quad \text{in }]0,\varepsilon[\times \mathbb{R}_s \\ H(0,\cdot) = 0 \quad \text{in } \mathbb{R}_s \\ H(\varepsilon,s) = -\chi(s) \frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi\varepsilon}} e^{i\frac{s^2}{4\varepsilon}} \end{cases} \tag{4.7}$$

and

$$\|H\|_{L^2(]0,\varepsilon[\times]-2T,2T[)} \leq \varepsilon \|H\|_{L^\infty(0,\varepsilon;L^2(]-2T,2T[))} \leq \varepsilon\sqrt{\varepsilon}\beta_\varepsilon. \tag{4.8}$$

Let $E(t,s)$ be the fundamental solution of the Schrödinger equation in one dimension:

$$E(t,s) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi t}} e^{i\frac{s^2}{4t}}. \tag{4.9}$$

The solution $E \in C^\infty(\{t>0\} \times \mathbb{R}_s) \cap C([0,+\infty[;H^{-1/2-\epsilon}(\mathbb{R}_s))$ satisfies

$$\begin{cases} i\partial_t E + \partial_s^2 E = 0 \quad \text{in } \{t>0\} \times \mathbb{R}_s \\ E(0,\cdot) = \delta(\cdot) \in H^{-1/2-\epsilon}(\mathbb{R}_s). \end{cases} \tag{4.10}$$

We finally choose $F(t,s) = E(t,s) + H(t,s)$, which is the solution of (4.2).

4.2 The hyperbolic problem

We give a result for the exact control of the wave equation.

Proposition 4.3. *If the function* $\Theta : (x,t) \mapsto \Xi(x)\theta(t) \in C_c^0(\partial\Omega \times]0,T[;\mathbb{R})$ *controls exactly* Ω *for the wave equation with partially null initial data then, for all initial data* $\Phi_o \in H_0^1(\Omega)$*, there exists a control* $\varrho \in H^1(]-T,T[;L^2(\partial\Omega))$ *such that the solution* $y \in C(\mathbb{R};H^1(\Omega)) \cap C^1(\mathbb{R};L^2(\Omega))$ *satisfies*

$$\begin{cases} \partial_t^2 y - \Delta y = 0 \quad \text{in } \Omega \times \mathbb{R} \\ y = \Xi\varrho.1_{|\partial\Omega\times]-T,T[} \quad \text{on } \partial\Omega \times \mathbb{R} \\ y(\cdot,0) = \Phi_o,\ \partial_t y(\cdot,0) = 0 \quad \text{in } \Omega \\ y \equiv 0 \quad \text{in } \Omega \times (]-\infty,-T] \cup [T,+\infty[). \end{cases} \tag{4.11}$$

Futhermore

$$\|\varrho\|^2_{L^2(\Gamma\times]-T,T[)} + \|\partial_t\varrho\|^2_{L^2(\Gamma\times]-T,T[)} \leq C_T \|\nabla\Phi_o\|^2_{L^2(\Omega)}. \tag{4.12}$$

Proof of Proposition 4.3.- We extend the solution $\Phi(x,t)$ of (2.1) by symmetry:

$$y(x,t) = \left| \begin{array}{ll} \Phi(x,t) & \text{in } \Omega \times [0,T] \\ \Phi(x,-t) & \text{in } \Omega \times [-T,0[. \end{array} \right. \tag{4.13}$$

And from the HUM method, we have

$$\|\varrho\|^2_{L^2(\partial\Omega\times]-T,T[)} + \|\partial_t\varrho\|^2_{L^2(\partial\Omega\times]-T,T[)} \leq C_T \|\nabla\Phi_o\|^2_{L^2(\Omega)}. \tag{4.14}$$

4.3 Proof of Theorem 2.1

Let $0 < t < \varepsilon$. We define $w(x,t)$ such that

$$w(x,t) = \int_{\mathbb{R}} F(t,\ell)\, y(x,\ell)\, d\ell \tag{4.15}$$

where $y : (x,\ell) \mapsto y(x,\ell)$ and $F : (t,\ell) \mapsto F(t,\ell)$ are solutions of the problems

$$\begin{cases} \partial_\ell^2 y - \Delta y = 0 \quad \text{in } \Omega \times \{-T < \ell < T\} \\ y(x,\ell)_{|\partial\Omega \times]-T,T[} = \Xi(x)\, \varrho(x,\ell) .1_{|\partial\Omega\times]-T,T[} \\ y(\cdot,0) = w_o \in H_0^1(\Omega),\ \partial_\ell y(\cdot,0) = 0 \quad \text{in } \Omega \\ y \equiv 0 \quad \text{in } \Omega \times (]-\infty,-T] \cup [T,+\infty[) \end{cases} \tag{4.16}$$

and

$$\begin{cases} i\partial_t F + \partial_\ell^2 F = 0 \quad \text{in }]0,\varepsilon[\times [-T,T] \\ F(0,\cdot) = \delta(\cdot) \quad \text{in } \mathbb{R}_\ell \\ F(\varepsilon,\cdot) = 0 \quad \text{in } [-T,T]. \end{cases} \tag{4.17}$$

The existence of y is given by the Proposition 4.3 with the hypothesis of exact controllability for the wave equation with partially initial data. The existence of F is given by the Proposition 4.2 where the support of the second member of (4.2) does not meet $]0,\varepsilon[\times]-T,T[$.

We calculate $(i\partial_t + \Delta)\, w(x,t)$:

$$\begin{aligned} i\partial_t w(x,t) + \Delta w(x,t) &= \textstyle\int_{\mathbb{R}} i\partial_t F(t,\ell)\, y(x,\ell)\, d\ell + \int_{\mathbb{R}} F(t,\ell)\, \Delta y(x,\ell)\, d\ell \\ &= \textstyle\int_{\mathbb{R}} -\partial_\ell^2 F(t,\ell)\, y(x,\ell)\, d\ell + \int_{\mathbb{R}} F(t,\ell)\, \partial_\ell^2 y(x,\ell)\, d\ell \\ &= 0. \end{aligned} \tag{4.18}$$

Conclusion

$$\begin{cases} i\partial_t w + \Delta w = 0 \quad \text{in } \Omega \times]0,\varepsilon[\\ w(x,t) = \Xi(x)\, \vartheta_\varepsilon(x,t) \quad \text{on } \partial\Omega \times]0,\varepsilon[\\ w(\cdot,0) = w_o \quad \text{in } \Omega \\ w(\cdot,\varepsilon) = 0 \quad \text{in } \Omega \end{cases} \tag{4.19}$$

with an estimate of the control $\vartheta_\varepsilon(x,t)$ on $\partial\Omega \times]0,\varepsilon[$, given by

$$\begin{aligned} \vartheta_\varepsilon(x,t) &= \textstyle\int_{-T}^{T} -F(t,\ell)\, \varrho(x,\ell)\, d\ell \\ &= \textstyle\int_{-T}^{T} -(E+H)(t,\ell)\, \varrho(x,\ell)\, d\ell \\ &= \vartheta_{\varepsilon,1}(x,t) + \vartheta_{\varepsilon,2}(x,t) \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \|\vartheta_{\varepsilon,2}\|_{L^2(\partial\Omega\times]0,\varepsilon[)}^2 &= \textstyle\int_0^\varepsilon \int_{\partial\Omega} \left| \int_{-T}^{T} H(t,\ell)\, \varrho(x,\ell)\, d\ell \right|^2 dx dt \\ &\le \textstyle\int_0^\varepsilon \|H(t,\cdot)\|_{L^2(]-T,T[)}^2\, dt \int_{\partial\Omega} \|\varrho(x,\cdot)\|_{L^2(]-T,T[)}^2\, dx \\ &\le \left(\varepsilon\sqrt{\varepsilon}\beta_\varepsilon\right)^2 C_T\, \|\nabla w_o\|_{L^2(\Omega)}^2 \end{aligned} \tag{4.21}$$

and

$$\|\vartheta_{\epsilon,1}\|^2_{L^2(\partial\Omega\times]0,\epsilon[)} = \int_0^\epsilon \int_{\partial\Omega} \left|\int_{-T}^T E(t,\ell)\,\varrho(x,\ell)\,d\ell\right|^2 dxdt$$
$$= \int_0^\epsilon \int_{\partial\Omega} \left|\int_{\mathbb{R}} \frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi t}} e^{i\frac{\ell^2}{4t}}\varrho(x,\ell)\,d\ell\right|^2 dxdt \tag{4.22}$$

and

$$\begin{aligned}
&\int_{\partial\Omega}\left|\int_{\mathbb{R}} \frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi t}} e^{i\frac{\ell^2}{4t}}\varrho(x,\ell)\,d\ell\right|^2 dx \leq c\int_{\partial\Omega}\left|\int_{\mathbb{R}} e^{i\xi^2 t}\widehat{\varrho(x,\ell)}(\xi)\,d\xi\right|^2 dx\\
&\leq c\int_{\partial\Omega}\left|\int_{|\xi|<M}\left|\widehat{\varrho(x,\ell)}(\xi)\right|d\xi\right|^2 dx\\
&\qquad + c\int_{\partial\Omega}\left|\int_{|\xi|\geq M} e^{i\xi^2 t}\widehat{\varrho(x,\ell)}(\xi)\,d\xi\right|^2 dx\\
&\leq cM\int_{\partial\Omega}\int_{|\xi|<M}\left|\widehat{\varrho(x,\ell)}(\xi)\right|^2 d\xi dx\\
&\qquad + c\int_{\partial\Omega}\left|\int_{|\xi|\geq M}\frac{|\xi|}{M+|\xi|}\left|\widehat{\varrho(x,\ell)}(\xi)\right|d\xi\right|^2 dx\\
&\leq cM\int_{\partial\Omega}\int_{\mathbb{R}}\left|\widehat{\varrho(x,\ell)}(\xi)\right|^2 d\xi dx + c\int_{\partial\Omega}\left|\int_{|\xi|\geq M}\frac{1}{M+|\xi|}\left|\xi\widehat{\varrho(x,\ell)}(\xi)\right|d\xi\right|^2 dx\\
&\leq cM\int_{\partial\Omega}\int_{\mathbb{R}}|\varrho(x,\ell)|^2 d\ell dx\\
&\qquad + c\int_{\partial\Omega}\left(\int_{|\xi|\geq M}\left|\frac{1}{M^2+|\xi|^2}\right|\right)\left(\int_{|\xi|\geq M}\left|\xi\widehat{\varrho(x,\ell)}(\xi)\right|^2 d\xi\right)dx\\
&\leq cM\|\varrho\|^2_{L^2(\partial\Omega\times\mathbb{R})} + cM^{-1}\int_{\partial\Omega}\int_{\mathbb{R}}\left|\partial_\ell\widehat{\varrho(x,\ell)}(\xi)\right|^2 d\xi dx\\
&\leq C_T\|\nabla w_o\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{4.23}$$

Remark 4.4. The proofs of Theorems 1.1 and Theorem 2.1 are still true if we change the laplacien operator by an elliptic, autoadjoint, regular in espace operator. We complete the result of control in [8].

References

[1] C. Bardos, G. Lebeau, and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30**(5) (1992), 1024–1065.

[2] L. Boutet de Monvel, *Propagation des singularités des solutions d'équations analogues à l'équation de Schrödinger*, Lecture Notes in Mathematics, **459**, 1975.

[3] N. Burq, Contrôle de l'équation des plaques en présence d'obstacles strictement convexes, *Mémoires S.M.F.*, 55, nouvelle série, 1993.

[4] N. Burq and P. Gérard, condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, *C. R. Acad. Sci. Paris*, t. 325, série 1, 1997, 749–752.

[5] C. Fabre, Résultats de contrôlabilité exacte interne pour l'équation de Schrödinger et leurs limites asymptotiques: application à certaines équations de plaques vibrantes, *Asymptotic Analysis*, **5** (1992), 343-379.

[6] E. Fernandez-Cara and E. Zuazua, Conférence à Cortona, 1999.

[7] A.V. Fursikov and O.Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series, no. 34.

[8] M.A. Horn and W. Littman, *Boundary Control of a Schrödinger Equation with Nonconstant Principal Part*, Lecture Notes in Pure and Applied Mathematics, no. 174.

[9] G. Lebeau, Contrôle de l'équation de Schrödinger, *J. Math. Pures et Applic.*, **71** (1992), 267-291.

[10] J.-L. Lions, Contrôlabilité exacte, stabilisation et perturbation des systèmes distribués, 1, *Coll. RMA*, Masson, Paris, 1998.

[11] G. Lebeau and L. Robbiano, Contrôle exacte de l'équation de la chaleur, *Comm. Part. Diff. Eq.*, **20** (1995), 335-356.

[12] I. Lasiecka and R.Triggiani, Optimal regularity, exact controllability and uniform stabilisation of Schrödinger equations with Dirichlet control, *Differential and Integral Equations*, **5** (1992), 521-535.

[13] E. Machtyngier, Exact controllability for the Schrödinger equation, *SIAM J. Control Optm.*, **32**(1) (1994), 24-34.

[14] L. Robbiano, Fonction de coût et contrôle des solutions des équations hyperboliques, *Asymptotic Analysis*, **10** (1995), 95-115.

[15] D. Tataru, Carleman estimates and unique continuation for solutions to boundary value problems, *J. Math. Pures et Applic.*, **75** (1996), 367-408.

Kim dang Phung
17 rue Leonard Mafrand
92320 Chatillon, France
Kim-Dang.Phung@cmla.ens-cachan.fr