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OBSERVABILITY AND CONTROL OF SCHRÖDINGER EQUATIONS*

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Abstract. We propose an exact controllability result for Schrödinger equations in bounded domains under the Bardos–Lebeau–Rauch geometric control condition with an estimate of the control which is explicit with respect to the time of controllability. Also, we prove an explicit in time logarithmic observability estimate for the Schrödinger equation, where no geometrical conditions are supposed on the domain.

Key words. observability, controllability

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1. Introduction. Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$. We consider a nonempty open subset ω of Ω. The question we wish to address is that of controllability for Schrödinger equations with an explicit in time bound of the cost of the following control function. Given a time $\varepsilon > 0$ and considering initial data w_o in some appropriate space X, can we find a control $\vartheta \in L^1(0, \varepsilon; X)$ such that the solution of the system

$$\left\{ \begin{array}{rl} i\partial_t w + \Delta w = \vartheta_{|\omega} & \text{in } \Omega \times]0, \varepsilon[\,, \\ w = 0 & \text{on } \partial\Omega \times]0, \varepsilon[\,, \\ w(\cdot, 0) = w_o & \text{in } \Omega \end{array} \right. \tag{1.1}$$

satisfies $w(\cdot, \varepsilon) \equiv 0$ in Ω, with an estimate of the control ϑ

$$\frac{1}{\sqrt{\varepsilon}} \left\| \vartheta_{|\omega} \right\|_{L^1(0,\varepsilon;X)} \leq \mathcal{C}(\varepsilon) \left\| w_o \right\|_X , \tag{1.2}$$

where $\mathcal{C}$ is an explicit function of ε?

From the work of Lions [Li] on the control for distributed systems, such a result can be obtained with the Hilbert uniqueness method (HUM) by solving the dual observability problem: in the case where $X = L^2(\Omega)$, under which hypothesis the solution of the homogenous Schrödinger equation

$$\left\{ \begin{array}{rl} i\partial_t u + \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_t, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_t, \\ u(\cdot, 0) = u_o & \text{in } \Omega \end{array} \right. \tag{1.3}$$

satisfies

$$\forall \varepsilon > 0, \quad \forall u_o \in L^2(\Omega), \quad \|u_o\|_{L^2(\Omega)} \leq \mathcal{C}(\varepsilon) \|u\|_{L^2(\omega \times]0,\varepsilon[)} . \tag{1.4}$$

The relation (1.4) concerns any initial data but may need suitable geometric conditions on ω. Also, we can establish such an observability estimate which is true for

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any geometric situation but carries information only when $\|u_o\|_{H^s(\Omega)} = O(\|u_o\|_{L^2(\Omega)})$ for some $s \geq 1$. The problem becomes the following. For all $\omega \subset \Omega$, can we find a positive continuous and strictly increasing function $\mathcal{F} : \mathbb{R}_+^* \to \mathbb{R}_+^*$ which satisfies the relation $\lim_{x\to 0}\mathcal{F}(x) = 0$ such that one has the assertion

(1.5)
$$\forall\varepsilon > 0, \quad \forall u_o \in H^s(\Omega) \cap H_0^1(\Omega), \quad \|u_o\|^2_{L^2(\Omega)} \leq \mathcal{D}(\varepsilon)\ \mathcal{F}\left(\frac{\|u\|^2_{L^2(\omega\times]0,\varepsilon[)}}{\|u_o\|^2_{L^2(\Omega)}}\right)\|u_o\|^2_{H^s(\Omega)},$$

where $\mathcal{D}$ is an explicit positive function of ε?

These exact controllability and observability problems were already investigated in [M], [F], and in [LT], [M], [Le], [B], [T] if the control acts on a part of the boundary, but $\mathcal{C}(\varepsilon)$ was not calculated explicitly. The observability problem arises also in the context of parabolic [FI], [F-CZ], [LR1] or hyperbolic [Li], [BLR] systems. In [FI] and [LR1], an exact null controllability result for parabolic problems is established with no restriction on the time of control or on the support of the control function. Also, Fernandez-Cara and Zuazua [F-CZ] proved an explicit in time observability estimate for the heat equation. In [Ru], Russell used a transformation to show that a null controllability result for the heat equation for any time can be obtained from the exact controllability result for the wave equation in some time. Concerning hyperbolic systems, Bardos, Lebeau, and Rauch [BLR] show a link between the propagation of rays of geometric optics and the problem of exact controllability for hyperbolic problems. They give a geometrical control condition which is sufficient and almost necessary to obtain the observability for hyperbolic problems. Without this geometrical control condition, Robbiano [Ro] proved a logarithmic observability estimate for hyperbolic problems (but where ε must be large enough because of the finite speed of propagation) and showed how to use it to obtain an approximate control result with an estimation of the cost of the control.

In this paper, we give simple techniques and results which try to answer the three previous questions in different geometrical situations. Our strategy is to obtain results for the Schrödinger equation from well-known works on observation and controllability for parabolic and hyperbolic problems. Also, we will describe a method to have an exact control result for Schrödinger equations in bounded domain in $\mathbb{R}^n$, $n > 1$, from an observability result for the Schrödinger equation in one space dimension. Even if our approach does not give optimal results, we hope it can be used in other control problems. Let us now state the different results of this paper in the next section. The first result concerns the problem of observability for the Schrödinger equation when no geometrical conditions are required on ω. We give a logarithmic observability estimate. Then we study the case where the Bardos–Lebeau–Rauch geometric control hypothesis [BLR] holds. The second result is about the particular one-dimensional situation. The third result concerns the problem of exact control for the Schrödinger equation in a bounded domain of $\mathbb{R}^n$, $n > 1$.

2. The main results and some remarks. When no geometrical condition is assumed, we propose a logarithmic explicit in time observability estimate.

THEOREM 2.1. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is of class C^∞. Let ω be a nonempty open subset of Ω. Then there exists $C > 0$ such that for all $\varepsilon > 0$, for all initial data $u_o \in H^2(\Omega) \cap H_0^1(\Omega)$, the solution of the homogenous Schrödinger equation* (1.3) *satisfies*

$$\left\|u_o\right\|_{L^2(\Omega)}^2 \leq \frac{C\left(1+1/\varepsilon\right)}{\ln\left(2+\frac{\left\|u_o\right\|_{L^2(\Omega)}^2}{\left\|u\right\|_{L^2(\omega\times]0,\varepsilon[)}^2}\right)}\ \left\|\Delta u_o\right\|_{L^2(\Omega)}^2 . \tag{2.1}$$

The following second result is about the Schrödinger equation in one space dimension. We give an explicit in time observability estimate.

THEOREM 2.2. *Suppose $\Omega\subset\mathbb{R}^n$, $n=1$. Let ω be a nonempty open subset of Ω. Then there exists $C>0$ such that for all $\varepsilon>0$, for all initial data $u_o\in L^2(\Omega)$, the solution of the homogenous Schrödinger equation* (1.3) *satisfies*

$$\left\|u_o\right\|_{L^2(\Omega)}^2 \leq e^{C\left(1+1/\varepsilon^2\right)}\left\|u\right\|_{L^2(\omega\times]0,\varepsilon[)}^2 . \tag{2.2}$$

The last result concerns the problem of exact control for the Schrödinger equation in a bounded domain of $\mathbb{R}^n$, $n>1$. We estimate the size of the control.

THEOREM 2.3. *Suppose $\Omega\subset\mathbb{R}^n$, $n>1$, is of class C^∞, and there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristics of $\partial_t^2-\Delta$. If all generalized bicharacteristic rays meet $\omega\times]0,T_c[$ for some $0<T_c<+\infty$, then for all $\varepsilon>0$, for all initial conditions $w_o\in H_0^1(\Omega)$, there is a control $\vartheta=\vartheta_\varepsilon\in L^1\left(0,\varepsilon;L^2\left(\Omega\right)\right)$ such that the solution $w\in C\left(\left[0,\varepsilon\right];L^2\left(\Omega\right)\right)$ of the Schrödinger problem* (1.1) *satisfies $w\left(\cdot,\varepsilon\right)\equiv 0$ in Ω . Furthermore there exists a constant $C>0$, such that for all $\varepsilon>0$, we have*

$$\left\|\nabla w\left(\cdot,t\right)\right\|_{L^2(\Omega)} \leq \left(\frac{C}{\sqrt{t}}+e^{C\left(1+1/\varepsilon^2\right)}\right)\left\|\nabla w_o\right\|_{L^2(\Omega)} \qquad \forall t>0 \tag{2.3}$$

with an estimate of the control ϑ_ε as follows:

$$\frac{1}{\sqrt{\varepsilon}}\left\|\vartheta_\varepsilon\right\|_{L^1(0,\varepsilon;L^2(\omega))} \leq \left(C+\sqrt{\varepsilon}\right)e^{C\left(1+1/\varepsilon^2\right)}\left\|\nabla w_o\right\|_{L^2(\Omega)} . \tag{2.4}$$

Let us make some comments.

1. Theorem 2.1 expresses a unique continuation property for the Schrödinger equation. Our approach to proving Theorem 2.1 consists of using an explicit in time observability estimate for parabolic equations obtained by Fernandez-Cara and Zuazua [F-CZ]. Next we introduce a Gaussian transformation to return to the solution of the Schrödinger equation. The logarithmic observability estimate (2.1) is equivalent to the following interpolation inequality:

(2.5)

$$\exists C>0,\quad \forall\varepsilon,\delta>0,\quad \left\|u_o\right\|_{L^2(\Omega)}^2 \leq \exp\left(C\left(1+\frac{1}{\varepsilon}\right)\delta\right)\left\|u\right\|_{L^2(\omega\times]0,\varepsilon[)}^2+\frac{1}{\delta}\left\|\Delta u_o\right\|_{L^2(\Omega)}^2 .$$

2. Theorem 2.2 asserts that we have an exact controllability result for the Schrödinger equation in one space dimension, due to the HUM of Lions [Li]. The proof of Theorem 2.2 combines multiplier techniques [M], [F] and interpolation inequalities. The interpolation estimates which are similar to (2.5) allow us to absorb the terms of lower order. The estimate (2.2) of Theorem 2.2 is also true for $n>1$ if we suppose $\omega\subset\Omega\subset\mathbb{R}^n$ to be a neighborhood of $\overline{\Gamma_o}$, where $\Gamma_o=\left\{x\in\partial\Omega\ /\ \left(x-x_o\right)\cdot\nu\left(x\right)>0\right\}$ is either equal to $\partial\Omega$ or is such that the boundary $\partial\Omega\setminus\Gamma_o$ is included in a hyperplan (see [F]), when x_o is a fixed point of $\mathbb{R}^n$, and $\nu\left(x\right)$ is the unit outward normal vector.

The author is indebted to Professor Zuazua, who called his attention to the papers [MZ], [I] and who pointed out that Theorem 2.2 can also be obtained from [MZ, Thm. 3.4], [I, Thm. 1], and a Fourier analysis.

3. Lebeau [Le] has proved the exact boundary controllability for the Schrödinger equation with an analytic boundary under the geometrical control condition of the work of Bardos, Lebeau, and Rauch on exact controllability for the wave equation [BLR]. Moreover, Burq [B] has proved the existence of open subsets of $\partial\Omega$ which do not geometrically control Ω for the wave equation in which it is possible to construct an exact boundary control for the Schrödinger equation if initial data are more regular than those with finite energy. Here the result of Theorem 2.3 is not optimal in norm in the sense that it should be enough to choose the initial condition in $w_o \in L^2(\Omega)$ to obtain a result of exact controllability for the Schrödinger equation with a control in $L^2(\omega \times]0,\varepsilon[)$ when it satisfies suitable geometric conditions (see the previous comment or the multiplier techniques [M], [F]). Also, Theorem 2.3 only implies the following observability estimate under the geometric control condition:

$$\forall \varepsilon > 0, \quad \forall u_o \in L^2(\Omega), \quad \|u_o\|_{H^{-1}(\Omega)} \leq \mathcal{C}(\varepsilon) \sup_{[0,\varepsilon]} \|u\|_{L^2(\omega)} . \tag{2.6}$$

4. Here our construction of the control given by Theorem 2.3 provides more precise information on the cost of the control. We propose a proof based on the theorem of Bardos, Lebeau, and Rauch [BLR] on exact controllability for hyperbolic equations, and a transformation inspired from the work of Boutet de Monvel [BdM] on the propagation of singularities in Schrödinger-type equations (see also [KS]). We will also use the estimate (2.2) of Theorem 2.2 in one space dimension to establish an explicit estimate on the size of the control function for the problem of exact controllability for the Schrödinger equation in $\Omega \subset \mathbb{R}^n$, $n > 1$, when the control region ω controls geometrically Ω. Furthermore, our method described (in section 5) below also applies to the case of Schrödinger equations with nonconstant principal part [HL]. We have the following control result.

Theorem 2.4. *Let $\Delta_{\mathcal{A}} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + a_o(x)$, where the coefficients of $\Delta_{\mathcal{A}}$ are real, smooth, and satisfy the following conditions: $a_{ij}(x) = a_{ji}(x)$ and $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2$. Suppose $\Omega \subset \mathbb{R}^n$, $n > 1$, of class C^∞, and there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristics of $\partial_t^2 - \Delta_{\mathcal{A}}$. If all generalized bicharacteristic rays of $\partial_t^2 - \Delta_{\mathcal{A}}$ meet $\omega \times]0, T_c[$ for some $0 < T_c < +\infty$, then for all $\varepsilon > 0$, for all initial conditions $w_o \in H_0^1(\Omega)$, there is a control $\vartheta_\varepsilon \in L^1(0,\varepsilon; L^2(\Omega))$ such that the solution $w \in C([0,\varepsilon]; L^2(\Omega))$ of the Schrödinger problem*

$$\begin{cases} i\partial_t w + \Delta_{\mathcal{A}} w = \vartheta_{\varepsilon|\omega} & \text{in } \Omega \times]0,\varepsilon[\,, \\ w = 0 & \text{on } \partial\Omega \times]0,\varepsilon[\,, \\ w(\cdot,0) = w_o & \text{in } \Omega \end{cases} \tag{2.7}$$

satisfies $w(\cdot,\varepsilon) \equiv 0$ in Ω . Furthermore, there exists a constant $C > 0$, such that for all $\varepsilon > 0$, the estimates (2.3)–(2.4) *hold.*

5. The main goal of this paper is to point out that an observability result for the heat equation gives a logarithmic observability estimate for the Schrödinger equation but also that an exact control result for the wave equation gives the exact controllability for the Schrödinger equation. Moreover, we show that an exact control result for the Schrödinger equation in one space dimension implies the exact controllability

for the Schrödinger equation in bounded domain $\Omega \subset \mathbb{R}^n$, $n > 1$, when a geometrical condition is assumed. The observability results for the Schrödinger equation are established under two different kinds of geometry: either when no geometrical hypothesis is assumed (Theorem 2.1) or when the Bardos–Lebeau–Rauch geometric control condition is satisfied (Theorem 2.2 ($n = 1$) and Theorem 2.3 ($n > 1$); see (2.6) in comment 3). Also, these techniques allow us to have explicit estimates with respect to the time of controllability.

The paper is organized in the following way. The proofs of Theorems 2.1 and 2.2 (see comment 2) rest on interpolation estimates which are established in section 3. These interpolation estimates can be seen as low frequency estimates. The proof of Theorem 2.1 is then easily described. In section 4, we prove Theorem 2.2. Section 5 is devoted to the construction of the control stated in Theorem 2.3.

3. Low frequency estimates. In this section, we first state interpolation inequalities in Theorem 3.1 below, which are the key results to prove Theorems 2.1 and 2.2 (see comment 2). Next, we recall some results on the observability for parabolic problems obtained by Fernandez-Cara and Zuazua [F-CZ]. Finally, we prove Theorem 3.1.

3.1. Interpolation inequalities and the proof of Theorem 2.1. We have the following interpolation inequalities.

THEOREM 3.1. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is of class C^∞. Let ω be a nonempty open set included in Ω. Then there exist $C > 0$, $\varepsilon_o > 0$, $\mu_o > 0$, such that for all $\varepsilon \leq \varepsilon_o$, for all $\mu \geq \mu_o$, for all initial data $u_o \in H^2(\Omega) \cap H_0^1(\Omega)$, the solution of the homogenous Schrödinger equation* (1.3) *satisfies*

$$\int_\Omega |u_o(x)|^2 \, dx \leq \exp\left(\frac{C\mu}{\varepsilon}\right) \int_\omega \int_0^\varepsilon |u(x,t)|^2 \, dtdx + \frac{\varepsilon^3}{\mu} \int_\Omega |\Delta u_o(x)|^2 \, dx. \tag{3.1}$$

Furthermore, there exist $C > 0$, $\varepsilon_o > 0$, $\mu_o > 0$, such that for all $\varepsilon \leq \varepsilon_o$, for all $\mu \geq \mu_o$, for all initial data $u_o \in H^2(\Omega) \cap H_0^1(\Omega)$, the solution of the homogenous Schrödinger equation (1.3) *satisfies*

$$\int_\Omega |u_o(x)|^2 \, dx \leq \exp\left(\frac{C\mu}{\varepsilon}\right) \int_\omega \int_0^\varepsilon |\Delta u(x,t)|^2 \, dtdx + \frac{\varepsilon^3}{\mu} \int_\Omega |\Delta u_o(x)|^2 \, dx. \tag{3.2}$$

Let us assume that the interpolation inequality (3.1) holds in order to prove Theorem 2.1. The proof of Theorem 3.1 will be given at the end of section 3.

Proof of Theorem 2.1. By taking μ such that $\mu = C_0 \frac{\int_\Omega |\Delta u_0|^2 dx}{\int_\Omega |u_0|^2 dx} \geq \mu_o$, the estimate (3.1) becomes

$$\int_\Omega |u_0|^2 \, dx \leq \exp\left(\frac{CC_0}{\varepsilon} \frac{\int_\Omega |\Delta u_0|^2 \, dx}{\int_\Omega |u_0|^2 \, dx}\right) \int_\omega \int_0^\varepsilon |u|^2 \, dtdx + \frac{\varepsilon^3}{C_0} \int_\Omega |u_0|^2 \, dx. \tag{3.3}$$

So, $\exists C > 0, \ \exists \varepsilon_o > 0,$ for all $\varepsilon \leq \varepsilon_o$,

$$\int_\Omega |u_0|^2 \, dx \leq \exp\left(\frac{C}{\varepsilon} \frac{\int_\Omega |\Delta u_0|^2 \, dx}{\int_\Omega |u_0|^2 \, dx}\right) \int_\omega \int_0^\varepsilon |u|^2 \, dtdx.$$

And also, $\exists C > 0,$ for all $\varepsilon > 0$,

$$\int_\Omega |u_0|^2 \, dx \leq \exp\left(C\left(1 + \frac{1}{\varepsilon}\right) \frac{\int_\Omega |\Delta u_0|^2 \, dx}{\int_\Omega |u_0|^2 \, dx}\right) \int_\omega \int_0^\varepsilon |u|^2 \, dtdx. \tag{3.4}$$

The estimate (2.1) of Theorem 2.1 is equivalent to (3.4) by using properties of the logarithmic and exponential functions. That concludes the proof of Theorem 2.1. Let us remark here that we obtain (2.5) from (3.4) by studying the case where either $\|\Delta u_o\| \leq \delta \|u_o\|$ or $\|\Delta u_o\| > \delta \|u_o\|$ (see comment 1). □

It will be useful to recall some explicit in time observability results for parabolic problems before proving Theorem 3.1.

3.2. Observability for the parabolic problem. We recall the result in [F-CZ] in the particular case of a null potential.

Theorem [F-CZ]. Let Ω be a connected bounded domain in $\mathbb{R}^n$, with smooth boundary. Let v be the solution of the following adjoint parabolic equation:

$$\left\{ \begin{array}{rl} \partial_t v + \Delta v = 0 & \text{in } \Omega\times]0,T[, \\ v = 0 & \text{on } \partial\Omega\times]0,T[. \end{array} \right. \tag{3.5}$$

Then there is $C > 0$, such that for all $T > 0$,

$$\int_\Omega |v(x,0)|^2 \, dx \leq \exp\left(C \left(1 + \frac{1}{T} \right) \right) \int_\omega \int_0^T |v|^2 \, dt dx. \tag{3.6}$$

Remark 3.2. Theorem [F-CZ] is obtained from the works of Fursikov and Imanuvilov [FI] on Carleman estimates for adjoint parabolic equations. Another approach, based on the work of Lebeau and Robbiano [LR1] on the exact controllability of the heat equation on a Riemannian compact manifold with boundary, and Dirichlet boundary conditions, in both cases of interior or boundary controls, gives us the estimates (3.6) but not explicitly in time. Nevertheless, a logarthmic boundary observability estimate for the Schrödinger equation is presented with that approach in [P].

We deduce from Theorem [F-CZ] the following corollary.

Corollary 3.3. *Let W be the solution of the following adjoint parabolic problem:*

$$\left\{ \begin{array}{rl} \partial_t W + \Delta W = f & \text{in } \Omega\times]0,T[, \\ W = 0 & \text{on } \partial\Omega\times]0,T[, \\ W(\cdot,T) \in L^2(\Omega). & \end{array} \right. \tag{3.7}$$

Then

$$\exists C_T > 0, \qquad \int_\Omega |W(x,0)|^2 \, dx \leq C_T \left(\int_\omega \int_0^T |W|^2 \, dt dx + \int_\Omega \int_0^T |f|^2 \, dt dx \right). \tag{3.8}$$

If, moreover, $W(\cdot,T) \in H^2 \cap H_0^1(\Omega)$, then

$$\exists C_T > 0, \qquad \int_\Omega |W(x,0)|^2 \, dx \leq C_T \left(\int_\omega \int_0^T |\Delta W|^2 \, dt dx + \int_\Omega \int_0^T |f|^2 \, dt dx \right). \tag{3.9}$$

Here, the constant C_T of the estimates (3.8), (3.9) *is of the order of*

$$C_T = \exp\left(C \left(1 + \frac{1}{T} \right) \right), \tag{3.10}$$

where $C > 0$ is a constant independent of $T > 0$.

Proof of Corollary 3.3. It is easy to see that (3.8) holds from (3.6) with a classical energy method. Let us prove (3.9). We consider $z(x,t) = W(x,t) - a(x,t)$, where

$$\begin{cases} \partial_t a + \Delta a = f & \text{in } \Omega\times]0,T[, \\ a = 0 & \text{on } \partial\Omega\times]0,T[, \\ a(\cdot,T) = 0 & \text{in } \Omega. \end{cases} \tag{3.11}$$

As $\partial_t z$ is a solution of (3.5), the regularity of $W(\cdot,T)$ and (3.6) allow us to obtain the estimate

$$\int_\Omega |\Delta z(x,0)|^2 \, dx \leq \exp\left(C\left(1+\frac{1}{T}\right)\right) \int_\omega \int_0^T |\partial_t z|^2 \, dt dx. \tag{3.12}$$

Now we give equalities on the solution a by a classical energy method:

$$\begin{aligned} \frac{1}{2}\int_\Omega |a(x,0)|^2 \, dx + \int_0^T \int_\Omega |\nabla a|^2 \, dx dt &= -\int_0^T \int_\Omega f a dx dt, \\ \frac{1}{2}\int_\Omega |\nabla a(x,0)|^2 \, dx + \int_0^T \int_\Omega |\partial_t a|^2 \, dx dt &= \int_0^T \int_\Omega f \partial_t a dx dt. \end{aligned} \tag{3.13}$$

By Cauchy–Schwarz and Poincaré inequalities and from (3.13) we have

$$\int_\Omega |a(x,0)|^2 \, dx + \int_0^T \int_\Omega |\partial_t a|^2 \, dx dt \leq c \int_0^T \int_\Omega |f|^2 \, dx dt. \tag{3.14}$$

We obtain from (3.12) and (3.14)

(3.15)

$$\begin{aligned} \int_\Omega |W(x,0)|^2 \, dx &\leq 2\int_\Omega |z(x,0)|^2 \, dx + 2\int_\Omega |a(x,0)|^2 \, dx \\ &\leq c\int_\Omega |\Delta z(x,0)|^2 \, dx + 2\int_\Omega |a(x,0)|^2 \, dx \\ &\leq \exp\left(C\left(1+\frac{1}{T}\right)\right) \int_\omega \int_0^T |\partial_t z|^2 \, dt dx + c\int_0^T \int_\Omega |f|^2 \, dx dt \\ &\leq \exp\left(C\left(1+\frac{1}{T}\right)\right) \left(\int_\omega \int_0^T |\partial_t W|^2 \, dt dx + c\int_0^T \int_\Omega |f|^2 \, dx dt\right) \\ &\quad + c\int_0^T \int_\Omega |f|^2 \, dx dt \\ &\leq \exp\left(C\left(1+\frac{1}{T}\right)\right) \left(\int_\omega \int_0^T |\Delta W|^2 \, dt dx + c\int_0^T \int_\Omega |f|^2 \, dx dt\right) \\ &\quad + c\int_0^T \int_\Omega |f|^2 \, dx dt. \end{aligned}$$

That concludes the proof of (3.9) and Corollary 3.3. □

3.3. Proof of Theorem 3.1. We begin to prove (3.1) as follows.

Let $F(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau} e^{-\tau^2} d\tau$; then $F(z) = \frac{\sqrt{\pi}}{2\pi} e^{\frac{1}{4}\left(|\mathrm{Im}\, z|^2 - |\mathrm{Re}\, z|^2\right)} e^{-\frac{i}{2}(\mathrm{Im}\, z \,\mathrm{Re}\, z)}$. Also, with $\lambda > 0$, let us consider

$$F_\lambda(z) = \lambda F(\lambda z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau} e^{-\left(\frac{\tau}{\lambda}\right)^2} d\tau.$$

We have

$$|F_\lambda(z)| = \frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}\left(|\mathrm{Im}\, z|^2 - |\mathrm{Re}\, z|^2\right)}. \tag{3.16}$$

Let $s, \ell_0 \in \mathbb{R}$, and

$$W_{\ell_0,\lambda}(s,x) = \int_{\mathbb{R}} F_\lambda(\ell_0 + is - \ell)\Phi(\ell)u(x,\ell)d\ell, \tag{3.17}$$

where $\Phi \in C_0^\infty(\mathbb{R})$. The Gaussian transformation (3.17) is inspired from the Fourier–Bros–Iagolnitzer transformation in [LR2]. We remark that $\partial_s F_\lambda(\ell_0 + is - \ell) = -i\partial_\ell F_\lambda(\ell_0 + is - \ell)$, so

$$\begin{aligned}\partial_s W_{\ell_0,\lambda}(s,x) &= \int_{\mathbb{R}} -i\partial_\ell F_\lambda(\ell_0 + is - \ell)\Phi(\ell)u(x,\ell)d\ell \\ &= \int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\left\{\frac{d}{d\ell}\Phi(\ell)u(x,\ell) + \Phi(\ell)\frac{\partial}{\partial\ell}u(x,\ell)\right\}d\ell.\end{aligned}$$

As $u : (x,t) \longmapsto u(x,t)$ is the solution of (1.3), $W_{\ell_0,\lambda}$ satisfies

$$\left\{\begin{array}{c}\partial_s W_{\ell_0,\lambda}(s,x) + \Delta W_{\ell_0,\lambda}(s,x) = \int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\Phi'(\ell)u(x,\ell)d\ell, \\ W_{\ell_0,\lambda}(s,x) = 0 \quad \forall x \in \partial\Omega, \\ W_{\ell_0,\lambda}(0,x) = \left(F_\lambda * \Phi u(x,\cdot)\right)(\ell_0) \quad \forall x \in \Omega.\end{array}\right. \tag{3.18}$$

We define $\Phi \in C_0^\infty(\mathbb{R})$ such that the following holds. Let $L > 0$, and we choose $\Phi \in C_0^\infty(]0,L[)$, $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on $\left[\frac{L}{4};\frac{3L}{4}\right]$ and such that $|\Phi'| \leq \frac{8}{L}$. We take $K = \left[0;\frac{L}{4}\right] \cup \left[\frac{3L}{4};L\right]$ and $K_0 = \left[\frac{3L}{8};\frac{5L}{8}\right]$. So, $\mathrm{mes}K_0 = \frac{L}{4}$, $\mathrm{mes}K = \frac{L}{2}$, $\mathrm{supp}(\Phi') = K$, and $\mathrm{dist}(K;K_o) = \frac{L}{8}$. We will choose $\ell_0 \in K_0$.

As an application of (3.8), $W_{\ell_0,\lambda}$ satisfies the following estimate:

(3.19)

$$\begin{aligned}\int_\Omega \left|\left(F_\lambda * \Phi u(x,\cdot)\right)(\ell_0)\right|^2 dx &\leq C_T \int_\omega \int_0^T \left|W_{\ell_0,\lambda}(s,x)\right|^2 dsdx \\ &\quad + C_T \int_\Omega \int_0^T \left|\int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\Phi'(\ell)u(x,\ell)d\ell\right|^2 dsdx.\end{aligned}$$

On the other hand, from (3.16)

$$\begin{aligned}\int_\omega \int_0^T \left|W_{\ell_0,\lambda}(s,x)\right|^2 dsdx &= \int_\omega \int_0^T \left|\int_{\mathbb{R}} F_\lambda(\ell_0 + is - \ell)\Phi(\ell)u(x,\ell)d\ell\right|^2 dsdx \\ &\leq \int_0^T \int_\omega \left|\int_{\mathbb{R}} \frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}\left(s^2 - |\ell_0 - \ell|^2\right)}\Phi(\ell)\left|u(x,\ell)\right| d\ell\right|^2 dxds \\ &\leq \frac{\lambda^2}{4\pi}\left(\int_0^T e^{\frac{\lambda^2}{2}s^2}ds\right)\left|\sup \Phi\right|^2 \int_\omega \left|\int_0^L \left|u(x,\ell)\right| d\ell\right|^2 dx \\ &\leq \frac{\lambda^2}{4\pi}e^{\frac{\lambda^2}{2}T^2}T \;\left|\sup \Phi\right|^2\; L \int_\omega \int_0^L \left|u(x,\ell)\right|^2 d\ell dx\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \int_0^T \left| \int_{\mathbb{R}} iF_\lambda(\ell_0 + is - \ell)\Phi'(\ell)u(x,\ell)d\ell \right|^2 dsdx \\
&\leq \int_0^T \int_\Omega \left| \int_{\mathbb{R}} \frac{\sqrt{\pi}}{2\pi} \lambda e^{\frac{\lambda^2}{4}\left(s^2 - |\ell_0 - \ell|^2\right)} |\Phi'(\ell)| \, |u(x,\ell)| \, d\ell \right|^2 dxds \\
&\leq \frac{\lambda^2}{4\pi} e^{\frac{\lambda^2}{2}T^2} T \int_\Omega \left(\int_K e^{-\frac{\lambda^2}{2}|\ell_0 - \ell|^2} |\Phi'(\ell)|^2 |u(x,\ell)|^2 d\ell \right) \operatorname{mes}(K) \, dx \\
&\leq \frac{\lambda^2}{4\pi} e^{\frac{\lambda^2}{2}T^2} T \; e^{-\frac{\lambda^2}{2}\operatorname{dist}(K,K_0)^2} \; \sup |\Phi'(\ell)|^2 \operatorname{mes}(K) \int_\Omega \int_K |u(x,\ell)|^2 d\ell dx \\
&\leq \frac{\lambda^2}{4\pi} e^{\frac{\lambda^2}{2}T^2} T \; e^{-\frac{\lambda^2}{2}\operatorname{dist}(K,K_0)^2} \; \sup |\Phi'(\ell)|^2 \operatorname{mes}(K)^2 \int_\Omega |u_o|^2 dx \\
&\leq \frac{\lambda^2 T}{4\pi} \exp\left[\frac{\lambda^2}{2} \left(T^2 - \left(\frac{L}{8}\right)^2 \right) \right] \; \frac{8^2}{L^2} \frac{L^2}{4} \int_\Omega |u_o|^2 dx \\
&\leq \frac{4\lambda^2 T}{\pi} \exp\left[\frac{\lambda^2}{2} \left(T^2 - \left(\frac{L}{8}\right)^2 \right) \right] \int_\Omega |u_o|^2 dx.
\end{aligned}$$

So, the inequality (3.19) becomes

$$\begin{aligned}
\int_\Omega |(F_\lambda * \Phi u(x,\cdot))(\ell_0)|^2 dx &\leq C_T \frac{\lambda^2 TL}{4\pi} \exp\left(\frac{\lambda^2}{2} T^2 \right) \int_\omega \int_0^L |u(x,\ell)|^2 d\ell dx \\
&+ C_T \frac{4\lambda^2 T}{\pi} \exp\left[\frac{\lambda^2}{2} \left(T^2 - \left(\frac{L}{8}\right)^2 \right) \right] \int_\Omega |u_o|^2 dx. \qquad (3.20)
\end{aligned}$$

With the Parseval relation, we have

$$\begin{aligned}
&\int_{\mathbb{R}} |\Phi(\ell_0)u(x,\ell_0) - (F_\lambda * \Phi u(x,\cdot))(\ell_0)|^2 d\ell_0 \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \widehat{\Phi(\ell_0)u(x,\ell_0)}(\tau) \right|^2 \left(1 - e^{-\left(\frac{\tau}{\lambda}\right)^2} \right)^2 d\tau \\
&\leq \frac{1}{\pi\lambda^2} \int_{\mathbb{R}} \left| \tau \widehat{\Phi(\ell_0)u(x,\ell_0)}(\tau) \right|^2 d\tau \\
&\leq \frac{2}{\lambda^2} \int_{\mathbb{R}} |\Phi'(\ell_0)u(x,\ell_0) + \Phi(\ell_0)\partial_{\ell_0} u(x,\ell_0)|^2 d\ell_0 \\
&\leq \frac{4}{\lambda^2} \left[\frac{8^2}{L^2} \int_K |u(x,\ell_0)|^2 d\ell_0 + \int_0^L |\partial_{\ell_0} u(x,\ell_0)|^2 d\ell_0 \right].
\end{aligned}$$

By integrating on Ω, we obtain

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}} |\Phi(\ell_0)u(x,\ell_0) - (F_\lambda * \Phi u(x,\cdot))(\ell_0)|^2 d\ell_0 dx \qquad (3.21) \\
&\leq \frac{4}{\lambda^2} \left[\frac{c^2}{L^2} \int_K \int_\Omega |u(x,\ell_0)|^2 d\ell_0 dx + \int_0^L \int_\Omega |\partial_{\ell_0} u(x,\ell_0)|^2 d\ell_0 dx \right] \\
&\leq \frac{4}{\lambda^2} \left[\frac{8^2}{L^2} \frac{L}{2} \int_\Omega |u_o|^2 dx + L \int_\Omega |\Delta u_o|^2 dx \right].
\end{aligned}$$

So, with (3.20) and (3.21),

$$\begin{aligned}\operatorname{mes}(K_0)\int_\Omega |u_o|^2\,dx &= \int_{K_0}\int_\Omega |\Phi(\ell_0)u(x,\ell_0)|^2\,dx d\ell_0\\ &\le \operatorname{mes}(K_0)\,C_T\frac{\lambda^2 TL}{4\pi}\exp\left(\frac{\lambda^2}{2}T^2\right)\int_\omega\int_0^L |u(x,\ell)|^2\,d\ell dx\\ &+\operatorname{mes}(K_0)\,C_T\frac{4\lambda^2 T}{\pi}\exp\left[\frac{\lambda^2}{2}\left(T^2-\left(\frac{L}{8}\right)^2\right)\right]\int_\Omega |u_o|^2\,dx\\ &+\frac{4}{\lambda^2}\left[\frac{8^2}{L^2}\frac{L}{2}\int_\Omega |u_o|^2\,dx + L\int_\Omega |\Delta u_o|^2\,dx\right].\end{aligned}$$

Finally,

$$\begin{aligned}\int_\Omega |u_o|^2\,dx &\le C_T\frac{\lambda^2 TL}{4\pi}\exp\left(\frac{\lambda^2}{2}T^2\right)\int_\omega\int_0^L |u(x,\ell)|^2\,d\ell dx\\ &+C_T\frac{4\lambda^2 T}{\pi}\exp\left[\frac{\lambda^2}{2}\left(T^2-\left(\frac{L}{8}\right)^2\right)\right]\int_\Omega |u_o|^2\,dx\\ &+\frac{1}{\lambda^2}\frac{4^2}{L}\left[\frac{8^2}{2L}\int_\Omega |u_o|^2\,dx + L\int_\Omega |\Delta u_o|^2\,dx\right].\end{aligned}$$

Let us consider $A>0$ real such that $\left(1-A^2\right)<0$. By choosing $L=8AT$, it becomes

$$\begin{aligned}\int_\Omega |u_o|^2\,dx \le{}& \frac{2A}{\pi}C_T\lambda^2T^2\exp\left(\frac{\lambda^2T^2}{2}\right)\int_\omega\int_0^{8AT}|u(x,\ell)|^2\,d\ell dx + 16\frac{1}{\lambda^2}\int_\Omega |\Delta u_o|^2\,dx\\ &+\left[\frac{8}{A^2}\frac{1}{\lambda^2T^2}+\frac{4}{\pi}C_T\lambda^2T\exp\left(-\frac{A^2-1}{2}\lambda^2T^2\right)\right]\int_\Omega |u_o|^2\,dx.\end{aligned}$$

With the relation (3.10), we have the following uniform in time interpolation estimate:

$$\begin{aligned}\int_\Omega |u_o|^2\,dx &\le Ae^Ce^{C/T}\lambda^2T^2\exp\left(\frac{\lambda^2T^2}{2}\right)\int_\omega\int_0^{8AT}|u(x,\ell)|^2\,d\ell dx\\ &+\left[\frac{C}{A^2}\frac{1}{\lambda^2T^2}+e^Ce^{C/T}\lambda^2T\exp\left(-\frac{A^2-1}{2}\lambda^2T^2\right)\right]\int_\Omega |u_o|^2\,dx\\ &+\frac{16}{\lambda^2}\int_\Omega |\Delta u_o|^2\,dx.\end{aligned}$$

We introduce $\lambda^2=\frac{\mu}{T^3}$. Let $\alpha>0$ be real such that $\left(2\alpha+1-A^2\right)<0$; hence,

$$\begin{aligned}\int_\Omega |u_o|^2\,dx &\le Ae^Ce^{C/T}\frac{\mu}{T}\exp\left(\frac{\mu}{2T}\right)\int_\omega\int_0^{8AT}|u(x,\ell)|^2\,d\ell dx\\ &+\left[\frac{C}{A^2}\frac{T}{\mu}+e^Ce^{C/T}\frac{\mu}{T^2}\exp\left(-\frac{A^2-1}{2}\frac{\mu}{T}\right)\right]\int_\Omega |u_o|^2\,dx\\ &+16\frac{T^3}{\mu}\int_\Omega |\Delta u_o|^2\,dx\\ &\le Ae^Ce^{C/T}\frac{\mu}{T}\exp\left(\frac{\mu}{2T}\right)\int_\omega\int_0^{8AT}|u(x,\ell)|^2\,d\ell dx\end{aligned}$$

$$+\left[\frac{C}{A^2}\frac{T}{\mu}+e^{C}e^{C/T}\frac{1}{\mu}\frac{4}{\alpha^2}\exp\left(-\frac{A^2-1-2\alpha}{2}\frac{\mu}{T}\right)\right]\int_\Omega |u_o|^2\,dx$$
$$+16\frac{T^3}{\mu}\int_\Omega |\Delta u_o|^2\,dx.$$

We take $\mu>\frac{2C}{A^2-1-2\alpha}$, so

$$\int_\Omega |u_o|^2\,dx \le Ae^{C}\frac{\mu}{T}\exp\left(\frac{A^2\mu}{2T}\right)\int_\omega\int_0^{8AT}|u(x,\ell)|^2\,d\ell dx$$
$$+\left[\frac{C}{A^2}\frac{T}{\mu}+e^{C}\frac{4}{\alpha^2}\frac{1}{\mu}\right]\int_\Omega |u_o|^2\,dx+16\frac{T^3}{\mu}\int_\Omega|\Delta u_o|^2\,dx,$$

$$\int_\Omega |u_o|^2\,dx \le \frac{e^C}{A}\exp\left(\frac{A^2\mu}{2T}\right)\int_\omega\int_0^{8AT}|u(x,\ell)|^2\,d\ell dx \tag{3.22}$$
$$+\left[\frac{CT}{A^2}+\frac{4e^C}{\alpha^2}\right]\frac{1}{\mu}\int_\Omega |u_o|^2\,dx+16\frac{T^3}{\mu}\int_\Omega|\Delta u_o|^2\,dx,$$

and with $\mu_o=\max(\frac{2C}{A^2-1-2\alpha};2(\frac{C}{A^2}+\frac{4e^C}{\alpha^2}))$

$$\forall T\le 1,\quad \forall \mu>\mu_o,\qquad \frac{1}{2}\int_\Omega |u_o|^2\,dx \tag{3.23}$$
$$\le \frac{e^C}{A}\exp\left(\frac{A^2\mu}{2T}\right)\int_\omega\int_0^{8AT}|u(x,\ell)|^2\,d\ell dx+16\frac{T^3}{\mu}\int_\Omega|\Delta u_o|^2\,dx.$$

Therefore, (3.1) is proved by choosing $T=\frac{\varepsilon}{8A}\le 1$.

The proof of (3.2) follows the same approach by using (3.9).

As application of (3.9), $W_{\ell_0,\lambda}$ satisfies the following estimate:

$$\int_\Omega |(F_\lambda*\Phi u(x,\cdot))(\ell_0)|^2\,dx\le C_T\int_\omega\int_0^T|\Delta W_{\ell_0,\lambda}(s,x)|^2\,dsdx \tag{3.24}$$
$$+C_T\int_\omega\int_0^T\left|\int_{\mathbb{R}} iF_\lambda(\ell_0+is-\ell)\Phi'(\ell)u(x,\ell)d\ell\right|^2 dsdx.$$

On the other hand, from (3.16)

$$\int_\omega\int_0^T|\Delta W_{\ell_0,\lambda}(s,x)|^2\,dsdx\le\frac{\lambda^2}{4\pi}e^{\frac{\lambda^2}{2}T^2}T\ \ |\sup\Phi|^2\ \ L\int_\omega\int_0^L|\Delta u(x,\ell)|^2\,d\ell dx.$$

Consequently, the inequality (3.24) becomes

$$\int_\Omega |(F_\lambda*\Phi u(x,\cdot))(\ell_0)|^2\,dx\le C_T\frac{\lambda^2 TL}{4\pi}\exp\left(\frac{\lambda^2}{2}T^2\right)\int_\omega\int_0^L|\Delta u(x,\ell)|^2\,d\ell dx$$
$$+C_T\frac{4\lambda^2T}{\pi}\exp\left[\frac{\lambda^2}{2}\left(T^2-\left(\frac{L}{8}\right)^2\right)\right]\int_\Omega|u_o|^2\,dx. \tag{3.25}$$

Now, due to (3.25) and (3.21), for $(1-A^2)<0$ and $T\le 1$, we have

$$\int_\Omega|u_o|^2\,dx\le Ae^Ce^{C/T}\lambda^2T^2\exp\left(\frac{\lambda^2T^2}{2}\right)\int_\omega\int_0^{8AT}|\Delta u(x,\ell)|^2\,d\ell dx+\frac{16}{\lambda^2}\int_\Omega|\Delta u_o|^2\,dx$$

$$+\left[\frac{C}{A^2}\frac{1}{\lambda^2 T^2}+e^C e^{C/T}\lambda^2 T\exp\left(-\frac{A^2-1}{2}\lambda^2 T^2\right)\right]\int_\Omega |u_o|^2\,dx.$$

We introduce $\lambda^2=\frac{\mu}{T^3}$ and $0<\alpha<\frac{1}{2}\left(A^2-1\right)$, so that

$$\begin{aligned}\int_\Omega |u_o|^2\,dx &\leq Ae^{C/T}\frac{\mu}{T}\exp\left(\frac{\mu}{2T}\right)\int_\omega\int_0^{8AT}|\Delta u(x,\ell)|^2\,d\ell dx+16\frac{T^3}{\mu}\int_\Omega|\Delta u_o|^2\,dx\\ &+\left[\frac{C}{A^2}T+e^C e^{C/T}\left(\frac{\mu}{T}\right)^2\exp\left(-\frac{A^2-1}{2}\frac{\mu}{T}\right)\right]\frac{1}{\mu}\int_\Omega|u_o|^2\,dx\\ &\leq Ae^{C/T}\frac{\mu}{T}\exp\left(\frac{\mu}{2T}\right)\int_\omega\int_0^{8AT}|\Delta u(x,\ell)|^2\,d\ell dx+16\frac{T^3}{\mu}\int_\Omega|\Delta u_o|^2\,dx\\ &+\left[\frac{C}{A^2}T+e^C e^{C/T}\frac{4}{\alpha^2}\exp\left(-\frac{A^2-1-2\alpha}{2}\frac{\mu}{T}\right)\right]\frac{1}{\mu}\int_\Omega|u_o|^2\,dx.\end{aligned}$$

We choose μ large enough such that

$$\int_\Omega |u_o|^2\,dx \leq Ae^{C/T}\frac{\mu}{T}\exp\left(\frac{A\mu}{2T}\right)\int_\omega\int_0^{8AT}|\Delta u(x,\ell)|^2\,d\ell dx+C\frac{T^3}{\mu}\int_\Omega|\Delta u_o|^2\,dx.$$

Consequently, we have the following assertion: $\exists C>0$, $\exists\mu_o>0$, for all $T\leq 1$, for all $\mu>\mu_o$,

$$\int_\Omega |u_o|^2\,dx\leq C\exp\left(\frac{A\mu}{T}\right)\int_\omega\int_0^{8AT}|\Delta u(x,\ell)|^2\,d\ell dx+C\frac{T^3}{\mu}\int_\Omega|\Delta u_o|^2\,dx. \tag{3.26}$$

That concludes the proof of Theorem 3.1. ∎

4. Proof of Theorem 2.2. We now prove Theorem 2.2 by using (3.2) when $n=1$ and multiplier techniques. Let A, B, β, ε be four reals such that $A<B$, $0<2\beta<B-A$, and $\varepsilon>0$. Let $\varphi:(t,s)\in\,]0,\varepsilon[\,\times]A,B[\longmapsto\varphi(t,s)$ be the solution of the Schrödinger equation in one space dimension:

$$\begin{cases} i\partial_t\varphi+\partial_s^2\varphi=0 & \text{in }]0,\varepsilon[\,\times]A,B[,\\ \varphi(\cdot,A)=\varphi(\cdot,B)=0 & \text{on }]0,\varepsilon[\,,\\ \varphi(0,\cdot)=\varphi_o & \text{in }]A,B[. \end{cases} \tag{4.1}$$

We will prove the following stable observability estimate: $\exists C>0,$ for all $\varepsilon>0,$

$$\int_A^B\left|\partial_s^2\varphi_o\right|^2 ds\leq e^{C\left(1+1/\varepsilon^2\right)}\int_{B-2\beta}^{B-\beta}\int_0^\varepsilon\left|\partial_s^2\varphi\right|^2 dtds. \tag{4.2}$$

Indeed, let $q\in C^2([A,B])$ be a real function

$$\begin{aligned} iq\frac{d}{dt}\left(\partial_s\varphi\partial_s^2\overline{\varphi}\right) &= q\left(i\frac{d}{dt}\partial_s\varphi\partial_s^2\overline{\varphi}+i\partial_s\varphi\frac{d}{dt}\partial_s^2\overline{\varphi}\right)\\ &= q\left(-\partial_s^3\varphi\partial_s^2\overline{\varphi}+\partial_s\varphi\partial_s^4\overline{\varphi}\right)\\ &= q\left(-\partial_s^3\varphi\partial_s^2\overline{\varphi}+\partial_s\left(\partial_s\varphi\partial_s^3\overline{\varphi}\right)-\partial_s^2\varphi\partial_s^3\overline{\varphi}\right)\\ &= -q\partial_s\left(\partial_s^2\varphi\partial_s^2\overline{\varphi}\right)+q\partial_s\left(\partial_s\varphi\partial_s^3\overline{\varphi}\right). \end{aligned}$$

So

$$
\begin{aligned}
\int_0^\varepsilon \int_A^B iq\frac{d}{dt}\left(\partial_s\varphi\partial_s^2\overline{\varphi}\right)dsdt &= -\int_0^\varepsilon \int_A^B q\partial_s\left|\partial_s^2\varphi\right|^2 + \int_0^\varepsilon \int_A^B q\partial_s\left(\partial_s\varphi\partial_s^3\overline{\varphi}\right) \\
&= -\int_0^\varepsilon \left[q\left|\partial_s^2\varphi\right|^2\right]_A^B + \int_0^\varepsilon \int_A^B q'\left|\partial_s^2\varphi\right|^2 \\
&+ \int_0^\varepsilon \left[q\partial_s\varphi\partial_s^3\overline{\varphi}\right]_A^B - \int_0^\varepsilon \int_A^B q'\left(\partial_s\varphi\partial_s^3\overline{\varphi}\right) \\
&= \int_0^\varepsilon \int_A^B q'\left|\partial_s^2\varphi\right|^2 + \int_0^\varepsilon \left[q\partial_s\varphi\partial_s^3\overline{\varphi}\right]_A^B \\
&- \int_0^\varepsilon \left[q'\partial_s\varphi\partial_s^2\overline{\varphi}\right]_A^B + \int_0^\varepsilon \int_A^B \left(q''\partial_s\varphi + q'\partial_s^2\varphi\right)\partial_s^2\overline{\varphi} \\
&= \int_0^\varepsilon \left[q\partial_s\varphi\partial_s^3\overline{\varphi}\right]_A^B + 2\int_0^\varepsilon \int_A^B q'\left|\partial_s^2\varphi\right|^2 + \int_0^\varepsilon \int_A^B q''\partial_s\varphi\partial_s^2\overline{\varphi}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\int_A^B \left[iq\partial_s\varphi\partial_s^2\overline{\varphi}\right]_0^\varepsilon ds &= \int_0^\varepsilon \left[q\partial_s\varphi\partial_s^3\overline{\varphi}\right]_A^B dt \\
&+2\int_0^\varepsilon \int_A^B q'\left|\partial_s^2\varphi\right|^2 dsdt + \int_0^\varepsilon \int_A^B q''\partial_s\varphi\partial_s^2\overline{\varphi}dsdt.
\end{aligned}
\tag{4.3}
$$

Consequently, by taking the real part of (4.3), we obtain

$$
\begin{aligned}
&2\int_0^\varepsilon \int_A^B q'\left|\partial_s^2\varphi\right|^2 dsdt + \operatorname{Re}\int_0^\varepsilon \left[q\partial_s\varphi\partial_s^3\overline{\varphi}\right]_A^B dt \\
&= -\operatorname{Im}\int_A^B \left[q\partial_s\varphi\partial_s^2\overline{\varphi}\right]_0^\varepsilon ds - \operatorname{Re}\int_0^\varepsilon \int_A^B q''\partial_s\varphi\partial_s^2\overline{\varphi}dsdt.
\end{aligned}
$$

By choosing $q(s) = s - A$, we have

$$
\begin{aligned}
2\int_0^\varepsilon \int_A^B \left|\partial_s^2\varphi\right|^2 dsdt &= -(B-A)\operatorname{Re}\int_0^\varepsilon \partial_s\varphi(t,B)\,\partial_s^3\overline{\varphi}(t,B)\,dt \\
&-\operatorname{Im}\int_A^B \left[(s-A)\,\partial_s\varphi\partial_s^2\overline{\varphi}\right]_0^\varepsilon ds.
\end{aligned}
\tag{4.4}
$$

By choosing $q(s) = \chi(s)$ with $\operatorname{supp}\chi \subset [B-2\beta, B]$ and $\chi(B) \neq 0$, we have

$$
\begin{aligned}
-\chi(B)\operatorname{Re}\int_0^\varepsilon \partial_s\varphi(t,B)\,\partial_s^3\overline{\varphi}(t,B)\,dt &= 2\int_0^\varepsilon \int_A^B \chi'(s)\left|\partial_s^2\varphi\right|^2 dsdt \\
&+\operatorname{Im}\int_A^B \left[\chi(s)\,\partial_s\varphi\partial_s^2\overline{\varphi}\right]_0^\varepsilon ds + \operatorname{Re}\int_0^\varepsilon \int_A^B \chi''(s)\,\partial_s\varphi\partial_s^2\overline{\varphi}dsdt.
\end{aligned}
\tag{4.5}
$$

Due to (4.4) and (4.5), if, moreover, $\operatorname{supp}\chi' \subset [B-2\beta, B-\beta]$, we obtain the following assertion:

$\exists c>0, \quad \forall \varepsilon>0,$

$$\int_0^\varepsilon \int_A^B \left|\partial_s^2 \varphi\right|^2 ds dt \leq c\left(\int_0^\varepsilon \int_{B-2\beta}^{B-\beta}\left|\partial_s^2 \varphi\right|^2 ds dt+(1+\varepsilon)\left\|\partial_s \varphi_o\right\|\left\|\partial_s^2 \varphi_o\right\|_{L^2(]A, B[)}\right).$$

Hence

$$\int_A^B\left|\partial_s^2 \varphi_o\right|^2 ds \leq \frac{c}{\varepsilon} \int_{B-2\beta}^{B-\beta} \int_0^\varepsilon\left|\partial_s^2 \varphi\right|^2 ds dt+c\left(1+\frac{1}{\varepsilon}\right)\left\|\partial_s \varphi_o\right\|_{L^2(]A, B[)}\left\|\partial_s^2 \varphi_o\right\|_{L^2(]A, B[)}.$$

Finally,

$$\exists c>0, \quad \forall \varepsilon \leq 1, \qquad \int_A^B\left|\partial_s^2 \varphi_o\right|^2 ds \leq \frac{c}{\varepsilon} \int_{B-2\beta}^{B-\beta} \int_0^\varepsilon\left|\partial_s^2 \varphi\right|^2 dt ds+\frac{c}{\varepsilon^2}\left\|\partial_s \varphi_o\right\|_{L^2(]A, B[)}^2.$$

By interpolation, we have

(4.6)

$$\exists c>0, \quad \forall \varepsilon \leq 1, \qquad \int_A^B\left|\partial_s^2 \varphi_o\right|^2 ds \leq \frac{c}{\varepsilon} \int_{B-2\beta}^{B-\beta} \int_0^\varepsilon\left|\partial_s^2 \varphi\right|^2 dt ds+\frac{c}{\varepsilon^4}\left\|\varphi_o\right\|_{L^2(]A, B[)}^2.$$

But the interpolation inequality (3.2) of Theorem 3.1 in the one-dimensional case implies that

$\exists C>0, \exists \varepsilon_o>0, \exists \mu_o>0, \forall \varepsilon \leq \varepsilon_o, \forall \mu \geq \mu_o,$

$$\text{(4.7)} \qquad \int_A^B\left|\varphi_o\right|^2 ds \leq \exp\left(C \frac{\mu}{\varepsilon}\right) \int_{B-2\beta}^{B-\beta} \int_0^\varepsilon\left|\partial_s^2 \varphi\right|^2 dt ds+\frac{\varepsilon^3}{\mu} \int_A^B\left|\partial_s^2 \varphi_o\right|^2 ds.$$

Let D_o be real such that $D_o=\max(2; \frac{\min(1;\varepsilon_o)\mu_o}{c})$. By choosing $\mu=D_o \frac{c}{\varepsilon} \geq \mu_o$, we conclude the proof of (4.2) from (4.7) and (4.6). And, in a standard way [Li], [F], we also have $\exists C>0$, for all $\varepsilon>0$,

$$\text{(4.8)} \qquad \int_A^B\left|\varphi_o\right|^2 ds \leq e^{C\left(1+1/\varepsilon^2\right)} \int_{B-2\beta}^{B-\beta} \int_0^\varepsilon|\varphi|^2 dt ds.$$

That concludes the proof of Theorem 2.2. □

Remark 4.1. To prove (2.2), we used the multiplier methods and we absorbed the terms of lower order with the interpolation inequality (3.2) for $n=1$. The same method can be used for $n>1$ (see comment 2). Indeed, from the equality (1.21) of the work of Fabre [F, Lemma 1.9] on the exact internal controllability of the Schrödinger equation, we choose $\theta=g^2(t)$, where $g \in C_0^\infty(]0, \varepsilon[)$, $g=1$ in $]\varepsilon/3; 2\varepsilon/3[$, and $0 \leq g \leq 1$, to obtain with standard bootstrap arguments the following assertion:

$$\text{(4.9)} \quad \exists c>0, \quad \forall \varepsilon \leq 1, \qquad \int_\Omega\left|\Delta u_o\right|^2 dx dt \leq \frac{c}{\varepsilon} \int_0^\varepsilon \int_\omega|\Delta u|^2 dx dt+\frac{c}{\varepsilon^2}\left\|u_o\right\|_{H^1(\Omega)}^2.$$

Under the hypothesis of Lemma 1.9 of the work of Fabre [F, p. 350],we have (2.2) for $n>1$ by applying the interpolation inequality (3.2) of Theorem 3.1.

5. Proof of Theorem 2.3. This section is devoted to proving the exact control result for Schrödinger equations with an explicit in time estimate of the control. We proceed in three steps.

5.1. Step 1. The Schrödinger equation on $\mathbb{R}$. In this section, we prove the existence of the following solution of the Schrödinger equation on $\mathbb{R}$.

PROPOSITION 5.1. *Let $T > 0$ be real, and let δ be the Dirac measure. There exists a distribution $f = f(t,s)$ defined on $]0,\varepsilon[\times\mathbb{R}_s$ such that $f_t : s \in \mathbb{R}_s \longmapsto f(t,s)$ has a support included in $J = (]-\infty,-2T[\cup]2T,+\infty[)$ and the solution $F : (t,s) \in [0,\varepsilon]\times\mathbb{R}_s \longmapsto F(t,s)$ of the Schrödinger equation*

$$\left\{\begin{array}{ll} i\partial_t F + \partial_s^2 F = f_{|J} & \text{in }]0,\varepsilon[\times\mathbb{R}_s, \\ F(0,\cdot) = \delta(\cdot) & \text{in } \mathbb{R}_s \end{array}\right. \tag{5.1}$$

satisfies $F(\varepsilon,\cdot) \equiv 0$ in $[-T,T]$. Moreover, if $H = F - E$, where E is the fundamental solution of the Schrödinger equation in one space dimension, then

$$\exists C > 0, \quad \forall\varepsilon > 0, \qquad \|H\|_{L^\infty(0,\varepsilon;L^2(]-T,T[))} \leq e^{C\left(1+1/\varepsilon^2\right)}. \tag{5.2}$$

Remark 5.2. The result of Proposition 5.1 simply says that the Schrödinger equation on the whole line can be controlled to zero with a control concentrated in the exterior of the ball. This is also true in several dimensions. The proof of this can be easily obtained from the result on a bounded domain by a cut-off argument (see also [Z]). In our case, we obtain an explicit estimate with respect to the time ε of controllability. Here T does not denote time, and let us adopt the variables $(t,s) \in [0,\varepsilon]\times\mathbb{R}_s$ when we consider the one-dimensional case.

Proof of Proposition 5.1. Using the HUM of Lions [Li] and estimate (2.2) of Theorem 2.2, we know that for all data $v_\varepsilon \in L^2(]-3T,4T[$, there exists a control $h \in L^2(]0,\varepsilon[\times]3T,4T[)$ such that the solution $v : (t,s) \longmapsto v(t,s) \in C([0,\varepsilon];L^2(]-3T,4T[))$ satisfies

$$\left\{\begin{array}{l} i\partial_t v + \partial_s^2 v = h_{|]3T,4T[} \quad \text{in }]0,\varepsilon[\times]-3T,4T[\,, \\ v(\cdot,-3T) = 0,\ v(\cdot,4T) = 0 \quad \text{on }]0,\varepsilon[\,, \\ \quad v(0,\cdot) = 0 \quad \text{in }]-3T,4T[\,, \\ \quad v(\varepsilon,\cdot) = v_\varepsilon \quad \text{in }]-3T,4T[\end{array}\right. \tag{5.3}$$

and

$$\frac{1}{\sqrt{\varepsilon}}\|h\|_{L^1(0,\varepsilon;L^2(]3T,4T[))} \leq \|h\|_{L^2(]0,\varepsilon[\times]3T,4T[)} \leq e^{C\left(1+1/\varepsilon^2\right)}\|v_\varepsilon\|_{L^2(]-3T,4T[)}\,. \tag{5.4}$$

In particular, we take $v_\varepsilon(\varepsilon,s) = -\chi(s)\frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi\varepsilon}}e^{i\frac{s^2}{4\varepsilon}}$, where $s \in]-3T,4T[$, $\chi \in C_0^\infty(]-3T,4T[)$, $0 \leq \chi \leq 1$, $\chi_{|[-T,T]} = 1$. So

$$\|v\|_{L^\infty(0,\varepsilon;L^2(]-3T,4T[))} \leq 2\|h\|_{L^1(0,\varepsilon;L^2(]3T,4T[))} \leq Ce^{C\left(1+1/\varepsilon^2\right)}. \tag{5.5}$$

Let us consider

$$H(t,s) = \left|\begin{array}{ll} v(t,s) & \text{in } [0,\varepsilon]\times[-3T,4T]\,, \\ 0 & \text{in } [0,\varepsilon]\times(]-\infty,-3T[\cup]4T,+\infty[)\,, \end{array}\right. \tag{5.6}$$

where v is the solution of (5.3). So

(5.7)
$$\left\{\begin{array}{l} i\partial_t H + \partial_s^2 H = h_{|]3T,4T[} - \partial_s v\otimes\delta(s-4T) + \partial_s v\otimes\delta(s+3T) \quad \text{in }]0,\varepsilon[\times\mathbb{R}_s, \\ \quad H(0,\cdot) = 0 \quad \text{in } \mathbb{R}_s, \\ \quad H(\varepsilon,s) = -\chi(s)\frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi\varepsilon}}e^{i\frac{s^2}{4\varepsilon}} \end{array}\right.$$

and

$$\exists C>0,\quad \forall \varepsilon>0,\qquad \|H\|_{L^\infty(0,\varepsilon;L^2(]-3T,4T[))}\leq e^{C\left(1+1/\varepsilon^2\right)}. \tag{5.8}$$

Let E be the fundamental solution of the Schrödinger equation on the whole line

$$E(t,s)=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi t}}e^{i\frac{s^2}{4t}}. \tag{5.9}$$

The solution $E\in C^\infty(\{t>0\}\times\mathbb{R}_s)\cap C\left([0,+\infty[\,;H^{-1/2-\epsilon}(\mathbb{R}_s)\right)$ satisfies

$$\begin{cases} i\partial_t E+\partial_s^2 E=0 \quad \text{in } \{t>0\}\times\mathbb{R}_s, \\ E(0,\cdot)=\delta(\cdot)\in H^{-1/2-\epsilon}(\mathbb{R}_s). \end{cases} \tag{5.10}$$

We finally consider $f=h_{|]3T,4T[}-\partial_s v\otimes\delta(s-4T)+\partial_s v\otimes\delta(s+3T)$ such that the solution $F=E+H$ satisfies

$$\begin{cases} i\partial_t F+\partial_s^2 F=f_{|J} \quad \text{in }]0,\varepsilon[\times\mathbb{R}_s, \\ i\partial_t F+\partial_s^2 F=0 \quad \text{in }]0,\varepsilon[\times[-T,T], \\ F(0,\cdot)=\delta(\cdot) \quad \text{in } \mathbb{R}_s, \\ F(\varepsilon,\cdot)\equiv 0 \quad \text{in } [-T,T]. \end{cases} \tag{5.11}$$

That concludes the proof of Proposition 5.1. □

5.2. Step 2. Controllability for the hyperbolic problem. The following exact controllability result holds.

LEMMA 5.3. *Suppose $\Omega\subset\mathbb{R}^n$, $n\geq 1$, is of class C^∞, and there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristics of $\partial_t^2-\Delta$. If all generalized bicharacteristic rays meet $\omega\times]0,T_c[$ for some $0<T_c<+\infty$, then for all $T>T_c$, for all initial condition $w_o\in H_0^1(\Omega)$, there exists a control $g\in L^2(\Omega\times]-T,T[)$ such that the solution $y\in C\left(\mathbb{R}_t;H_0^1(\Omega)\right)\cap C^1\left(\mathbb{R}_t;L^2(\Omega)\right)$ satisfies*

$$\begin{cases} \partial_t^2 y-\Delta y=g_{|\omega\times]-T,T[} \quad \text{in } \Omega\times\mathbb{R}_t, \\ y=0 \quad \text{on } \partial\Omega\times\mathbb{R}_t, \\ y(\cdot,0)=w_o,\ \partial_t y(\cdot,0)=0 \quad \text{in } \Omega, \\ y\equiv 0 \quad \text{in } \Omega\times(]-\infty,-T]\cup[T,+\infty[) \end{cases} \tag{5.12}$$

and

$$\|g\|_{L^2(\omega\times]-T,T[)}\leq C_{\omega,T}\|\nabla w_o\|_{L^2(\Omega)}. \tag{5.13}$$

Remark 5.4. The result of Lemma 5.3 holds by a simple reflection argument as a consequence of the theorem of Bardos, Lebeau, and Rauch [BLR] on the exact controllability for hyperbolic equations which are obtained with microlocal techniques and propagation of singularities of the solution of hyperbolic systems. We recall their result to be complete.

Theorem [BLR]. Suppose $\Omega\subset\mathbb{R}^n$, $n\geq 1$, is of class C^∞, and there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristics of $\partial_t^2-\Delta$. If all generalized bicharacteristic rays meet $\omega\times]0,T_c[$ for some $0<T_c<+\infty$, then for all $T>T_c$, for all $\theta\in C_0^\infty(]0,T[)$, for all initial conditions $(w_o,w_1)\in H_0^1(\Omega)\times L^2(\Omega)$, there exists a control $\varrho\in L^2(\Omega\times\mathbb{R}_t)$ such that the solution $\Psi\in C\left(\mathbb{R}_t;H_0^1(\Omega)\right)\cap C^1\left(\mathbb{R}_t;L^2(\Omega)\right)$ satisfies

$$\begin{cases} \partial_t^2\Psi-\Delta\Psi=\theta\varrho_{|\omega} \quad \text{in } \Omega\times\mathbb{R}_t, \\ \Psi=0 \quad \text{on } \partial\Omega\times\mathbb{R}_t, \\ \Psi(\cdot,0)=w_o,\ \partial_t\Psi(\cdot,0)=w_1 \quad \text{in } \Omega, \\ \Psi(\cdot,T)=\partial_t\Psi(\cdot,T)=0 \quad \text{in } \Omega \end{cases}$$

and

$$\|\theta\varrho\|_{L^2(\omega\times]0,T[)} \leq C_{\omega,T} \|(\nabla w_o, w_1)\|_{L^2(\Omega)} .$$

Proof of Lemma 5.3. We choose $w_1 = 0$ and extend Ψ in a symmetric way by taking

$$y(x,t) = \left| \begin{array}{ll} \Psi(x,t) & \text{in } \Omega \times [0,T], \\ \Psi(x,-t) & \text{in } \Omega \times [-T,0[. \end{array} \right.$$

The control g will be given by

(5.14)

$$g(x,t)_{|\omega\times]-T,T[} = \theta(t)\,\varrho(x,t)_{|\omega\times]0,T[} + \theta(-t)\,\varrho(x,-t)_{|\omega\times]-T,0[} \quad \text{in } \Omega \times]-T,T[,$$

where $\theta \in C_0^\infty(]0,T[)$ so that $g(\cdot,-T) = g(\cdot,0) = g(\cdot,T) = 0$. □

5.3. Step 3. Construction of the control. Now we are able to construct and estimate the control of Theorem 2.3 as follows.

We define $w : (x,t) \in \Omega \times [0,\varepsilon] \longmapsto w(x,t)$ such that

$$w(x,t) = \int_{\mathbb{R}} F(t,\ell)\, y(x,\ell)\, d\ell, \tag{5.15}$$

where $F : (t,\ell) \in [0,\varepsilon] \times \mathbb{R}_\ell \longmapsto F(t,\ell)$ is obtained from Proposition 5.1:

$$\left\{ \begin{array}{l} i\partial_t F + \partial_\ell^2 F = f_{|J} \quad \text{in }]0,\varepsilon[\times \mathbb{R}_\ell, \\ i\partial_t F + \partial_\ell^2 F = 0 \quad \text{in }]0,\varepsilon[\times [-T,T], \\ \qquad F(0,\cdot) = \delta(\cdot) \quad \text{in } \mathbb{R}_\ell, \\ \qquad F(\varepsilon,\cdot) \equiv 0 \quad \text{in } [-T,T], \end{array} \right. \tag{5.16}$$

and $y : (x,\ell) \in \Omega \times \mathbb{R}_\ell \longmapsto y(x,\ell)$ given by Lemma 5.3 satisfies

$$\left\{ \begin{array}{l} \partial_\ell^2 y - \Delta y = g_{|\omega\times]-T,T[} \quad \text{in } \Omega \times \mathbb{R}_\ell, \\ \qquad y = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_\ell, \\ \quad y(\cdot,0) = w_o \in H_0^1(\Omega),\ \partial_\ell y(\cdot,0) = 0 \quad \text{in } \Omega, \\ \qquad y \equiv 0 \quad \text{in } \Omega \times (]-\infty,-T] \cup [T,+\infty[) . \end{array} \right. \tag{5.17}$$

Let us calculate $i\partial_t w(x,t)$:

$$\begin{aligned} i\partial_t w(x,t) &= \int_{\mathbb{R}} i\partial_t F(t,\ell)\, y(x,\ell)\, d\ell \\ &= \int_{\mathbb{R}} \left[-\partial_\ell^2 F(t,\ell) + f_{|J}\right] y(x,\ell)\, d\ell \\ &= \int_{\mathbb{R}} -F(t,\ell)\, \partial_\ell^2 y(x,\ell)\, d\ell + \int_{-T}^{T} \left[f_{|J}\right] y(x,\ell)\, d\ell \\ &= \int_{\mathbb{R}} F(t,\ell) \left[-\Delta y(x,\ell) - g(x,\ell)_{|\omega\times]-T,T[}\right] d\ell. \end{aligned}$$

Remark 5.5. The key point is that the solution $y : (x,\ell) \longmapsto y(x,\ell)$, where $\ell \in \mathbb{R}$, is identically null for ℓ out of the domain $]-T,T[$. Next, we need that the solution $F : (t,\ell) \longmapsto F(t,\ell)$ is defined for $\ell \in \mathbb{R}$. Moreover, F must satisfy $F(0,\cdot) = \delta(\cdot)$

in $\mathbb{R}$, $F(\varepsilon,\cdot)\equiv 0$ in $[-T,T]$, and also the homogenous Schrödinger equation in the domain $]0,\varepsilon[\times[-T,T]$. Out of the domain $[-T,T]$, F is solution of the Schrödinger equation with a second member f, but we do not see it in the integrations by parts because y is null on the support of f.

Our conclusion is

$$\left\{\begin{array}{rl} i\partial_t w+\Delta w=\vartheta_{\varepsilon|\omega} & \text{in } \Omega\times\,]0,\varepsilon[\,, \\ w=0 & \text{on } \partial\Omega\times\,]0,\varepsilon[\,, \\ w(\cdot,0)=w_o & \text{in } \Omega, \\ w(\cdot,\varepsilon)=0 & \text{in } \Omega \end{array}\right. \tag{5.18}$$

with an estimate of the control ϑ_ε in $\omega\times\,]0,\varepsilon[$, given by

$$\begin{aligned} \vartheta_\varepsilon(x,t) &= \int_{-T}^{T} -F(t,\ell)\,g(x,\ell)\,d\ell \\ &= \int_{-T}^{T} -(E+H)(t,\ell)\,g(x,\ell)\,d\ell \\ &= \vartheta_{\varepsilon,1}(x,t)+\vartheta_{\varepsilon,2}(x,t)\,, \end{aligned} \tag{5.19}$$

where, from Proposition 5.1, (5.2), and (5.13), we have

$$\begin{aligned} \|\vartheta_{\varepsilon,1}(\cdot,t)\|_{L^2(\omega)} &= \left(\int_\omega \left|\int_{-T}^{T} E(t,\ell)\,g(x,\ell)\,d\ell\right|^2 dx\right)^{1/2} \\ &\leq c\left(\int_\omega \left|\frac{1}{\sqrt{t}}\,\|g\|_{L^2(]-T,T[)}\right|^2 dx\right)^{1/2} \\ &\leq \frac{1}{\sqrt{t}} C_{\omega,T}\,\|\nabla w_o\|_{L^2(\Omega)}, \end{aligned} \tag{5.20}$$

$$\begin{aligned} \|\vartheta_{\varepsilon,2}(\cdot,t)\|_{L^2(\omega)} &= \left(\int_\omega \left|\int_{-T}^{T} H(t,\ell)\,g(x,\ell)\,d\ell\right|^2 dx\right)^{1/2} \\ &\leq \left(\|H(t,\cdot)\|^2_{L^2(]-T,T[)}\int_\omega \|g(x,\cdot)\|^2_{L^2(]-T,T[)}\,dx\right)^{1/2} \\ &\leq e^{C\left(1+1/\varepsilon^2\right)}\;C_{\omega,T}\,\|\nabla w_o\|_{L^2(\Omega).} \end{aligned} \tag{5.21}$$

We conclude with an estimate of $(\int_\Omega|\nabla w(x,t)|^2dx)^{1/2}=(\int_\Omega|\int_{\mathbb{R}}(E+H)(t,\ell)\nabla y(x,\ell)d\ell|^2dx)^{1/2}$:

(5.22)

$$\begin{aligned} \left(\int_\Omega\left|\int_{\mathbb{R}} E(t,\ell)\,\nabla y(x,\ell)\,d\ell\right|^2 dx\right)^{1/2} &= \left(\int_\Omega\left|\int_{-T}^{T} E(t,\ell)\,\nabla y(x,\ell)\,d\ell\right|^2 dx\right)^{1/2} \\ &\leq c\left(\int_\Omega\left|\frac{1}{\sqrt{t}}\,\|\nabla y(x,\cdot)\|_{L^2(]-T,T[)}\right|^2 dx\right)^{1/2} \\ &\leq \frac{1}{\sqrt{t}} C_{\omega,T}\,\|\nabla w_o\|_{L^2(\Omega)}\,, \end{aligned}$$

(5.23)

$$\begin{aligned}\left(\int_\Omega \left|\int_{\mathbb{R}} H(t,\ell)\,\nabla y(x,\ell)\,d\ell\right|^2 dx\right)^{1/2} &= \left(\int_\Omega \left|\int_{-T}^{T} H(t,\ell)\,\nabla y(x,\ell)\,d\ell\right|^2 dx\right)^{1/2} \\ &\leq \left(\|H(t,\cdot)\|^2_{L^2(]-T,T[)} \int_\Omega \|\nabla y(x,\cdot)\|^2_{L^2(]-T,T[)}\,dx\right)^{1/2} \\ &\leq e^{C\left(1+1/\varepsilon^2\right)}\, C_{\omega,T}\, \|\nabla w_o\|_{L^2(\Omega)}\,.\end{aligned}$$

Remark 5.6. If we choose $w(x,t) = \int_{\mathbb{R}} F(t,\ell)\,y(x,\ell)\,d\ell$ with F given by (5.16) and y given by (5.17) where the operator Δ is replaced by $\Delta_{\mathcal{A}}$, then w is solution of (2.7).

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REFERENCES

[BdM] L. Boutet de Monvel, *Propagation des singularités des solutions d'équations analogues à l'équation de Schrödinger*, Lecture Notes in Math. 459, Springer, New York, 1975.

[B] N. Burq, *Contrôle de l'équation des plaques en présence d'obstacles strictement convexes*, Mém. Soc. Math. France (N.S.) 55, Marseille, 1993.

[BLR] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024–1065.

[F] C. Fabre, *Résultats de contrôlabilité exacte interne pour l'équation de Schrödinger et leurs limites asymptotiques: Application à certaines équations de plaques vibrantes*, Asymptot. Anal., 5 (1992), pp. 343–379.

[F-CZ] E. Fernandez-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: The linear case*, Adv. Differential Equations, 5 (2000), pp. 465–514.

[FI] A.V. Fursikov and O. Yu Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Ser. 34, Seoul National University, Seoul, Korea, 1996.

[HL] M.A. Horn and W. Littman, *Boundary control of a Schrödinger equation with nonconstant principal part*, in Control of Partial Differential Equations and Applications (Laredo, 1994), Lecture Notes in Pure and Appl. Math. 174, Dekker, New York, 1996, pp. 101–106.

[I] A.E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Z., 41 (1936), pp. 367–369.

[KS] L. Kapitanski and Y. Safarov, *Dispersive Smoothing for Schrödinger Equations*, preprint, 1995.

[Le] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. (9), 71 (1992), pp. 267–291.

[Li] J.-L. Lions, *Contrôlabilité exacte, stabilisation et perturbation des systèmes distribués*, Rech. Math. Appl. 1, Masson, Paris, 1998.

[LR1] G. Lebeau and L. Robbiano, *Contrôle exacte de l'équation de la chaleur*, Comm. Partial Differential Equations, 20 (1995), pp. 335–356.

[LR2] G. Lebeau and L. Robbiano, *Stabilisation de l'équation des ondes par le bord*, Duke Math. J., 86 (1997), pp. 465–491.

[LT] I. Lasiecka and R. Triggiani, *Optimal regularity, exact controllability and uniform stabilisation of Schrödinger equations with Dirichlet control*, Differential Integral Equations, 5 (1992), pp. 521–535.

[M] E. Machtyngier, *Exact controllability for the Schrödinger equation*, SIAM J. Control Optim., 32 (1994), pp. 24–34.

[MZ] S. Micu and E. Zuazua, *Boundary controllability of a linear hybrid system arising in the control of noise*, SIAM J. Control Optim., 35 (1997), pp. 1614–1637.

[P] K.-D. Phung, *Observabilité de l'équation de Schrödinger*, Prépublication LAPT/UMR 99-11, l'Université de Provence, Provence, France, 1999.

[Ro] L. Robbiano, *Fonction de coût et contrôle des solutions des équations hyperboliques*, Asymptot. Anal., 10 (1995), pp. 95–115.

[Ru] D. Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, Stud. Appl. Math., 52 (1973), pp. 189–212.

[T] D. Tataru, *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pures Appl. (9), 75 (1996), pp. 367–408.

[Z] E. Zuazua, *Exponential decay for the semilinear wave equation with localized damping in unbounded domains*, J. Math. Pures Appl. (9), 70 (1991), pp. 513–529.