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Note on the cost of the approximate controllability for the heat equation with potential

K.-D. Phung

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Abstract

We prove the approximate controllability for the heat equation with potential with a cost of order $e^{c/\varepsilon}$ when the target is in $H_0^1(\Omega)$ with a precision in $L^2(\Omega)$ norm. Also a quantification estimate of the unique continuation for initial data in $L^2(\Omega)$ of the heat equation with potential is established.

1. Introduction and main results

Throughout this paper, Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with a boundary $\partial\Omega$ of class C^2, ω is a non-empty open subset of Ω and $T > 0$ is a real number. Further, we denote $\|\cdot\|_\infty$ the usual norm in $L^\infty(\Omega \times (0,T))$ and we consider $a = a(x,t)$ a function in $L^\infty(\Omega \times (0,T))$.

In this paper we study the following heat equation with a potential a in $L^\infty(\Omega \times (0,T))$:

$$\begin{cases} \partial_t u - \Delta u + au = f \cdot 1_{|\omega} & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $f \in L^2(\omega \times (0,T))$, $u_0 \in H_0^1(\Omega)$ and $1_{|\omega}$ denotes the characteristic function of the set ω.

It is well known from [6] or [2] that we can act through $f \in L^2(\omega \times (0,T))$ when $u_0 \in L^2(\Omega)$ in order to get the null controllability result $u(\cdot,T) = 0$ and furthermore, the following estimate holds [3]: there exists a constant $c_0 > 0$ such that

E-mail address: kim-dang.phung@cmla.ens-cachan.fr.

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$$\|f\|_{L^2(\omega\times(0,T))} \leqslant \exp\left(c_0\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}\right)\right)\|u_0\|_{L^2(\Omega)}. \tag{1.2}$$

(1.2) is an explicit estimate with respect to both quantities $T > 0$ and $\|a\|_\infty \geqslant 0$, of the control function f and may be viewed as a measure of the cost of the null controllability for the heat equation with potential.

Here we ask whether the following steering property for the heat equation with potential holds when $u_0 = 0$: there exist two constants $D > 1$ and $c = c(T, \|a\|_\infty) > 1$ depending on both quantities $T > 0$ and $\|a\|_\infty \geqslant 0$ such that for all $\varepsilon > 0$, for all $u_d \in H_0^1(\Omega)$, there exists a suitable approximate control function f depending on ε such that

$$\|f\|_{L^2(\omega\times(0,T))} \leqslant c\exp\left(\frac{D\|u_d\|_{H_0^1(\Omega)}}{\varepsilon}\right)\|u_d\|_{L^2(\Omega)} \tag{1.3}$$

and

$$\left\|u(\cdot,T)-u_d\right\|_{L^2(\Omega)} \leqslant \varepsilon. \tag{1.4}$$

Our goal is to measure the cost of the approximate control function f and furthermore to give an explicit estimate with respect to ε, T and $\|a\|_\infty \geqslant 0$.

This problem has received a particular attention from [3] where it is proved that the cost of the approximate controllability for the heat equation with potential is of order $e^{c/\varepsilon}$ if $u_d \in H^2(\Omega)\cap H_0^1(\Omega)$, and of order e^{c/ε^2} if $u_d \in H_0^1(\Omega)$. But when a is a constant or more generally of the form $a(x,t) = a_1(x) + a_2(t)$, where $a_1 \in L^\infty(\Omega)$ and $a_2 \in L^\infty(0,T)$, the order $e^{c/\sqrt{\varepsilon}}$ if $u_d \in H^2(\Omega)\cap H_0^1(\Omega)$ is optimal [3, Theorem 6.2].

Let us make the first observation: by simple changes of variables, we are reduced to check that for all $\varepsilon > 0$, for all $T > 0$, for all $w_d \in H_0^1(\Omega)$ there exists a function ϑ such that

$$\|\vartheta\|_{L^2(\omega\times(0,T))} \leqslant ce^{D/\varepsilon}\|w_d\|_{L^2(\Omega)}, \tag{1.5}$$

and such that the following steering property holds:

$$\left\|w(\cdot,0)-w_d\right\|_{L^2(\Omega)} \leqslant \varepsilon\|w_d\|_{H_0^1(\Omega)}, \tag{1.6}$$

where w is the solution of the following backward heat equation with potential:

$$\begin{cases} \partial_t w + \Delta w - aw = \vartheta\cdot 1_{|\omega} & \text{in } \Omega\times(0,T), \\ w = 0 & \text{on } \partial\Omega\times(0,T), \\ w(\cdot,T) = 0 & \text{in } \Omega. \end{cases} \tag{1.7}$$

Now let us consider φ the solution of the heat equation without control when the initial data $\varphi_0 \in L^2(\Omega)$:

$$\begin{cases} \partial_t\varphi - \Delta\varphi + a\varphi = 0 & \text{in } \Omega\times(0,T), \\ \varphi = 0 & \text{on } \partial\Omega\times(0,T), \\ \varphi(\cdot,0) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{1.8}$$

Then by classical integrations by parts, we get

$$\int_0^T\int_\omega \vartheta(x,t)\varphi(x,t)\,dx\,dt = -\int_\Omega w(x,0)\varphi(x,0)\,dx$$
$$= \int_\Omega (-w(\cdot,0)+w_d)\varphi_0\,dx - \int_\Omega w_d\varphi_0\,dx. \tag{1.9}$$

Also suppose that the solution w of (1.7) exists with (1.6) and (1.5) then for all $\varepsilon > 0$, for all $w_d \in H_0^1(\Omega)$, we have using Cauchy–Schwarz inequality

$$\int_\Omega w_d\varphi_0\,dx \leqslant \varepsilon\|w_d\|_{H_0^1(\Omega)}\|\varphi_0\|_{L^2(\Omega)} + ce^{D/\varepsilon}\|w_d\|_{L^2(\Omega)}\|\varphi\|_{L^2(\omega\times(0,T))}. \tag{1.10}$$

Consequently, we obtain choosing $w_d = (-\Delta)^{-1}\varphi_0$ that

$$\|\varphi_0\|_{H^{-1}(\Omega)} \leqslant \varepsilon\|\varphi_0\|_{L^2(\Omega)} + ce^{D/\varepsilon}\|\varphi\|_{L^2(\omega\times(0,T))}, \quad \forall\varepsilon > 0, \tag{1.11}$$

or equivalently

$$\|\varphi_0\|_{L^2(\Omega)} \leqslant c\exp\left(D\frac{\|\varphi_0\|_{L^2(\Omega)}}{\|\varphi_0\|_{H^{-1}(\Omega)}}\right)\|\varphi\|_{L^2(\omega\times(0,T))} \quad \text{if } \varphi_0 \neq 0, \tag{1.12}$$

or equivalently

$$\|\varphi_0\|_{H^{-1}(\Omega)} \leqslant \frac{D}{\ln\left(2+\frac{1}{c}\frac{\|\varphi_0\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(\omega\times(0,T))}}\right)}\|\varphi_0\|_{L^2(\Omega)}, \tag{1.13}$$

where the values of the constants $c = c(T, \|a\|_\infty) > 1$ and $D > 1$ may changed from line (1.11) to line (1.13) but not their dependence.

(1.13) is a quantitative estimate for unique continuation for initial data in $L^2(\Omega)$ of the heat equation with potential from an interior observation. This kind of logarithmic estimate already appears in the context of the cost of approximate control and stabilization for hyperbolic equation [5,7]. Here we will follow the strategy in [7] (see also [4]) to prove that (1.13) implies an approximate result with an estimate (1.3) of the cost function.

The first main result of this paper is as follows.

Theorem 1. *There exist two constants $c_1, c_2 > 1$ such that for all $\varepsilon > 0$, for all $T > 0$, for all $a \in L^\infty(\Omega\times(0,T))$, for all $u_d \in H_0^1(\Omega)$, there exists a control function $f_\varepsilon \in L^2(\omega\times(0,T))$ such that*

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant E\exp\left(\frac{D\|\nabla u_d\|_{L^2(\Omega)}}{\varepsilon}\right)\|u_d\|_{L^2(\Omega)},$$
$$\left\|u(\cdot,T) - u_d\right\|_{L^2(\Omega)} \leqslant \varepsilon, \tag{1.14}$$

where $u \in C([0,T]; L^2(\Omega))$ is the unique solution of the heat equation with potential and control function

$$\begin{cases} \partial_t u - \Delta u + au = f_\varepsilon \cdot 1_{|\omega} & \text{in } \Omega\times(0,T),\\ u = 0 & \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.15}$$

and $E > 1$, $D > 1$ are given by

$$\begin{cases} D = D(T, \|a\|_\infty) = c_1\big(Te^{c_1 T\|a\|_\infty^2} + \frac{1}{T}\big), \\ E = E(T, \|a\|_\infty) = \exp(c_2(1 + T\|a\|_\infty(1 + e^{c_2 T\|a\|_\infty^2}) + \|a\|_\infty^{2/3})). \end{cases} \tag{1.16}$$

Of course we will also need to prove the estimate (1.13). Our second main result is as follows.

Theorem 2. *There exist two constants $c_1, c_2 > 1$ such that for all $T > 0$, for all $a \in L^\infty(\Omega \times (0,T))$, for all initial data $u_0 \in L^2(\Omega)$ such that $u_0 \neq 0$, the solution u of the homogeneous heat equation with potential*

$$\begin{cases} \partial_t u - \Delta u + au = 0 \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{1.17}$$

satisfies

$$\|u_0\|_{H^{-1}(\Omega)} \leqslant \frac{D}{\ln\big(2 + \frac{1}{E}\frac{\|u_0\|_{L^2(\Omega)}}{\|u\|_{L^2(\omega\times(0,T))}}\big)}\|u_0\|_{L^2(\Omega)}, \tag{1.18}$$

where $E > 1$, $D > 1$ are given by

$$\begin{cases} D = c_1\big(Te^{c_1 T\|a\|_\infty^2} + \frac{1}{T}\big), \\ E = \exp(c_2(1 + T\|a\|_\infty(1 + e^{c_2 T\|a\|_\infty^2}) + \|a\|_\infty^{2/3})). \end{cases}$$

Remarks. (1) Note that $E(T, 0)$ is a constant not depending on $T > 0$.

(2) An application of Theorem 1 to get a space of exact controllable target data may be possible following [7] based on properties of Riesz basis (see [8]).

The plan of the paper is as follows. In Section 2 we prove Theorem 1 as an application of Theorem 2. Section 3 contains the proof of Theorem 2. Finally in the last section some comments are added.

2. Proof of Theorem 1

Theorem 1 is easily deduced from the following result.

Theorem 3. *There exist two constants $c_1, c_2 > 1$ such that for all $\varepsilon > 0$, for all $T > 0$, for all $a \in L^\infty(\Omega \times (0,T))$, for all $w_d \in H_0^1(\Omega)$, there exists a control function $\vartheta_\varepsilon \in L^2(\omega \times (0,T))$ such that*

$$\begin{aligned} &\|\vartheta_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant \exp\left(\frac{D}{\varepsilon}\right)\|w_d\|_{L^2(\Omega)}, \\ &\big\|w(\cdot,0) - w_d\big\|_{L^2(\Omega)} \leqslant \varepsilon\|w_d\|_{H_0^1(\Omega)}, \end{aligned} \tag{2.1}$$

where $w \in C([0,T]; L^2(\Omega))$ is the unique solution of the backward heat equation with potential and control function

$$\begin{cases} -\partial_t w - \Delta w + aw = E\vartheta_\varepsilon \cdot 1_{|\omega} & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T), \\ w(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \tag{2.2}$$

and $E > 1$, $D > 1$ are given by

$$\begin{cases} D = c_1\big(Te^{c_1 T\|a\|_\infty^2} + \frac{1}{T}\big), \\ E = \exp(c_2(1 + T\|a\|_\infty(1 + e^{c_2 T\|a\|_\infty^2}) + \|a\|_\infty^{2/3})). \end{cases}$$

Proof. We will use the logarithmic estimate (1.18) from Theorem 2.

Let us introduce the operator

$$C : \vartheta \in L^2\big(\omega \times (0,T)\big) \to w(\cdot, 0) \in L^2(\Omega), \tag{2.3}$$

where w is the solution of

$$\begin{cases} \partial_t w + \Delta w - aw = -E\vartheta \cdot 1_{|\omega} & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T), \\ w(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

It is well known that if $\vartheta \in L^2(\omega \times (0,T))$, then $w \in C([0,T]; H_0^1(\Omega))$ and in particular $w(\cdot, 0) \in H_0^1(\Omega) \subset L^2(\Omega)$ with compact injection. Thus, the operator C is linear, continuous and compact from $L^2(\omega \times (0,T))$ to $L^2(\Omega)$. We define $F = \operatorname{Im} C$ the space of exact controllability initial data with the following norm:

$$\|w_0\|_F = \inf\big\{\|\vartheta\|_{L^2(\omega\times(0,T))} \mid C\vartheta = w_0\big\}. \tag{2.5}$$

We will need to construct the dual operator of C. Let $u_0 \in L^2(\Omega)$, we consider the unique solution $u \in C([0,T]; L^2(\Omega))$ of

$$\begin{cases} \partial_t u - \Delta u + au = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.6}$$

and we know from Theorem 2 that

$$\|u_0\|_{H^{-1}(\Omega)} \leqslant \frac{D}{\ln\big(2 + \frac{\|u_0\|_{L^2(\Omega)}}{\|Eu\|_{L^2(\omega\times(0,T))}}\big)} \|u_0\|_{L^2(\Omega)}. \tag{2.7}$$

Let us introduce the operator

$$K : u_0 \in L^2(\Omega) \to Eu_{|\omega} \in L^2\big(\omega \times (0,T)\big), \tag{2.8}$$

where u is the solution of (2.6). The operator K is linear, continuous, compact from $L^2(\Omega)$ to $L^2(\omega \times (0,T))$. Remark that the operator K is the adjoint of C and we denote $C^* = K$. Indeed, by multiplying (2.6) by w the solution of (2.4) where $\vartheta \in L^2(\omega \times (0,T))$ and by applying the Green formula, we obtain the following duality relation:

$$\int_0^T \int_\omega \vartheta(x,t) Eu(x,t)\,dx\,dt = \int_\Omega w(x,0)u(x,0)\,dx,$$

i.e., for all $\vartheta \in L^2(\omega \times (0,T))$, for all $u_0 \in L^2(\Omega)$, we have

$$\int_0^T \int_\omega \vartheta K(u_0)\,dx\,dt = \int_\Omega u_0 C(\vartheta)\,dx. \tag{2.9}$$

We define $F' = (\operatorname{Im} C)'$ with the norm

$$\|u_0\|_{F'} = \big\|K(u_0)\big\|_{L^2(\omega\times]0,T[)} = \left(\int_0^T \int_\omega \big|Eu(x,t)\big|^2\,dx\,dt\right)^{1/2}. \tag{2.10}$$

Note that (2.9) and Holmgren theorem imply that $F = \operatorname{Im} C$ is dense in $L^2(\Omega)$.

From the duality relation (2.9), we have by choosing $\vartheta = K(u_0) \in L^2(\omega \times (0,T))$ that

$$\big\|K(u_0)\big\|^2_{L^2(\omega\times(0,T))} = \int_\Omega u_0 C\big(K(u_0)\big)\,dx. \tag{2.11}$$

We use the notation $(\cdot,\cdot)$ to describe the scalar product on $L^2(\Omega)$.

Let $B = C \circ K$. Then (2.11) becomes: for all $u_0 \in L^2(\Omega)$, $\|K(u_0)\|^2_{L^2(\omega\times(0,T))} = (B(u_0), u_0)$.

The operator B is non-negative, compact from $L^2(\Omega)$ to $L^2(\Omega)$ and self-adjoint on $L^2(\Omega)$. We deduce that it has a discrete spectrum and we associate in $L^2(\Omega)$ the Hilbert basis with eigenfunctions ξ_n of B and eigenvalues μ_n, where $\mu_n > 0$, is non-increasing and tends to zero.

Consequently, for every element $g \in L^2(\Omega)$ we have the Fourier expansion $g = \sum_{n>0} (g,\xi_n)\xi_n$, $\|g\|^2_{L^2(\Omega)} = \sum_{n>0} |(g,\xi_n)|^2 < +\infty$ and also

$$\big\|K(g)\big\|^2_{L^2(\omega\times(0,T))} = (Bg, g) = \sum_{n>0} \mu_n \big|(g,\xi_n)\big|^2. \tag{2.12}$$

Note that from (2.10), (2.12) and by duality, we deduce that

$$\|g\|^2_F = \sum_{n>0} \frac{1}{\mu_n} \big|(g,\xi_n)\big|^2.$$

Let us introduce the sets $S_n = \{m > 0 \mid \alpha_{n+1} < \mu_m \leqslant \alpha_n\}$, where $\alpha_n = e^{\mu_1 + e} e^{-e^n}$ for all $n > 0$. Then each function $g \in L^2(\Omega)$ can be represented in the form $g = \sum_{n>0} g_n$, where

$$g_n = \sum_{m\in S_n} (g,\xi_m)\xi_m. \tag{2.13}$$

Also, we have $\|g\|^2_{L^2(\Omega)} = \sum_{n>0} \|g_n\|^2_{L^2(\Omega)}$, where $\|g_n\|^2_{L^2(\Omega)} = \sum_{m\in S_n} |(g,\xi_m)|^2$.

Further, $\|g\|^2_{F'} = \sum_{n>0} \|K(g_n)\|^2_{L^2(\omega\times(0,T))}$, where

$$\big\|K(g_n)\big\|^2_{L^2(\omega\times(0,T))} = \sum_{m\in S_n} \mu_m \big|(g,\xi_m)\big|^2. \tag{2.14}$$

From (2.14), we have, if $g_n \neq 0$, that

$$\frac{1}{\alpha_n} \leqslant \frac{\|g_n\|^2_{L^2(\Omega)}}{\|K(g_n)\|^2_{L^2(\omega\times(0,T))}} < \frac{1}{\alpha_{n+1}}. \tag{2.15}$$

From (2.7) and (2.15), we obtain, if $g_n \neq 0$, that

$$\|g_n\|_{H^{-1}(\Omega)} \leqslant \frac{D}{[\ln(2+1/\sqrt{\alpha_n})]}\|g_n\|_{L^2(\Omega)}. \tag{2.16}$$

Let us introduce, with g_n given by (2.13),

$$I = \left\{ g = \sum_{n>0} g_n \;\Big|\; \|g\|_I^2 = \sum_{n>0} \alpha_{n+1}\|g_n\|^2_{L^2(\Omega)} < +\infty \right\}. \tag{2.17}$$

Then the dual space of I is defined by

$$I' = \left\{ g = \sum_{n>0} g_n \;\Big|\; \|g\|_{I'}^2 = \sum_{n>0} \frac{1}{\alpha_{n+1}}\|g_n\|^2_{L^2(\Omega)} < +\infty \right\}. \tag{2.18}$$

From (2.15) and more exactly $F' \subset I$, we have that

$$I' \subset F. \tag{2.19}$$

Now, let $N > 0$. Let $z \in H_0^1(\Omega)$. Then we can write, in $L^2(\Omega)$,

$$z = \sum_{n\leqslant N} z_n + \sum_{n>N} z_n \quad \text{with } z_n = \sum_{m\in S_n} (z,\xi_m)\xi_m. \tag{2.20}$$

The properties of the sequence $(\alpha_n)_{n>0}$ and (2.19), (2.16) imply that there exists a constant $c_3 > 0$ such that the following relations hold:

$$\begin{aligned}
\left(\frac{1}{c_3}\left\|\sum_{n\leqslant N} z_n\right\|_F\right)^2 &\leqslant \left\|\sum_{n\leqslant N} z_n\right\|^2_{I'} \\
&= \sum_{n>0}\frac{1}{\alpha_{n+1}}\sum_{m\in S_n}\left|\left(\sum_{p\leqslant N} z_p,\xi_m\right)\right|^2 \\
&= \sum_{n>0}\frac{1}{\alpha_{n+1}}\sum_{m\in S_n}\left|\sum_{p\leqslant N}\sum_{q\in S_p}(z,\xi_q)(\xi_q,\xi_m)\right|^2 \\
&= \sum_{n\leqslant N}\frac{1}{\alpha_{n+1}}\sum_{m\in S_n}\left|\sum_{q\in S_1\cup\cdots\cup S_N}(z,\xi_q)(\xi_q,\xi_m)\right|^2 \\
&\leqslant \frac{1}{\alpha_{N+1}}\sum_{m\in S_1\cup\cdots\cup S_N}\left|\sum_{q\in S_1\cup\cdots\cup S_N}(z,\xi_q)(\xi_q,\xi_m)\right|^2 \\
&\leqslant \frac{1}{\alpha_{N+1}}\sum_{m\in S_1\cup\cdots\cup S_N}\left|(z,\xi_m)\right|^2 \leqslant \frac{1}{\alpha_{N+1}}\|z\|^2_{L^2(\Omega)},
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\Big\|\sum_{n\leqslant N} z_n - z\Big\|^2_{L^2(\Omega)} &= \sum_{n>N}\|z_n\|^2_{L^2(\Omega)} = \sum_{n>N}\big|(z,z_n)\big| \\
&\leqslant \sum_{n>N}\|z\|_{H_0^1(\Omega)}\|z_n\|_{H^{-1}(\Omega)} \\
&\leqslant \|z\|_{H_0^1(\Omega)}\sum_{n>N}\frac{D}{[\ln(2+1/\sqrt{\alpha_n}\,)]}\|z_n\|_{L^2(\Omega)} \\
&\leqslant \|z\|_{H_0^1(\Omega)}\sqrt{\sum_{n>N}\frac{D^2}{[\ln(2+1/\sqrt{\alpha_n}\,)]^2}}\sqrt{\sum_{n>N}\|z_n\|^2_{L^2(\Omega)}} \\
&\leqslant D\sqrt{\sum_{n>N}\frac{1}{[\ln(2+1/\sqrt{\alpha_n}\,)]^2}}\|z\|_{H_0^1(\Omega)}\Big\|\sum_{n\leqslant N} z_n - z\Big\|_{L^2(\Omega)} \\
&\leqslant c_3\frac{D}{\ln(2+1/\sqrt{\alpha_{N+1}}\,)}\|z\|_{H_0^1(\Omega)}\Big\|\sum_{n\leqslant N} z_n - z\Big\|_{L^2(\Omega)}. \qquad (2.22)
\end{aligned}$$

We conclude the proof of Theorem 3 by choosing $N > 0$ such that

$$\frac{D}{\varepsilon} \approx \ln\left(2+\frac{1}{\sqrt{\alpha_{N+1}}}\right)$$

in order to have, with $z = w_d$,

$$\begin{aligned}
&\|C\vartheta - w_d\|_{L^2(\Omega)} \leqslant c_3\varepsilon\|w_d\|_{H_0^1(\Omega)}, \\
&\|\vartheta\|_{L^2(\omega\times(0,T))} \leqslant c_3\exp\left(\frac{D}{\varepsilon}\right)\|w_d\|_{L^2(\Omega)}. \qquad \Box \qquad (2.23)
\end{aligned}$$

3. Proof of Theorem 2

We proceed in three steps.

Step 1. We apply the observability estimate [3, Theorem 1.2] with a simple change of variable in time.

Then there exists a constant $c_0 > 0$ such that for any solution u of (1.17), for any $L > 0$, one has

$$\big\|u(\cdot,L)\big\|^2_{L^2(\Omega)} \leqslant \exp\left(c_0\left(1+\frac{1}{L}+L\|a\|_\infty+\|a\|_\infty^{2/3}\right)\right)\int_0^L\int_\omega\big|u(x,t)\big|^2\,dx\,dt. \qquad (3.1)$$

Step 2. We look for an estimate of the form

$$\big\|u(\cdot,0)\big\|^2_{H^{-1}(\Omega)} \leqslant \text{constant}\big\|u(\cdot,L)\big\|^2_{L^2(\Omega)}. \qquad (3.2)$$

This is a backward estimate for the heat equation. To do this, we apply the ideas in [1]. Let us consider for almost all $t \in [0, L]$ such that $u(x, t) \neq 0$ the following quantity:

$$\Psi(t) = \frac{\|u(\cdot, t)\|^2_{L^2(\Omega)}}{\|u(\cdot, t)\|^2_{H^{-1}(\Omega)}}. \tag{3.3}$$

We first prove that

$$\frac{d}{dt}\Psi(t) \leqslant \frac{1}{2}\frac{\|au(\cdot, t)\|^2_{H^{-1}(\Omega)}}{\|u(\cdot, t)\|^2_{H^{-1}(\Omega)}} \leqslant c_4\|a\|^2_\infty \Psi(t), \tag{3.4}$$

where $c_4 > 0$ is a constant only depending on the geometry and may change of value in the lines below.

Indeed we have the two following energy equalities: posing $f = -au$,

$$\frac{1}{2}\frac{d}{dt}\big\|u(\cdot, t)\big\|^2_{L^2(\Omega)} + \big\|u(\cdot, t)\big\|^2_{H^1_0(\Omega)} = \big(f(\cdot, t), u(\cdot, t)\big), \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\big\|u(\cdot, t)\big\|^2_{H^{-1}(\Omega)} + \big\|u(\cdot, t)\big\|^2_{L^2(\Omega)} = \big(f(\cdot, t), (-\Delta)^{-1}u(\cdot, t)\big). \tag{3.6}$$

Then from (3.5), (3.6) and (3.3), we get

$$\begin{aligned}\frac{d}{dt}\Psi(t) = \frac{2}{\|u\|^4_{H^{-1}(\Omega)}}\Big(&\big[-\|u\|^2_{H^1_0(\Omega)} + (f, u)\big]\|u\|^2_{H^{-1}(\Omega)} \\ &+ \big[\|u\|^4_{L^2(\Omega)} - \|u\|^2_{L^2(\Omega)}\big(f, (-\Delta)^{-1}u\big)\big]\Big).\end{aligned} \tag{3.7}$$

Also,

$$\begin{aligned}&\|u\|^4_{L^2(\Omega)} - \|u\|^2_{L^2(\Omega)}\big(f, (-\Delta)^{-1}u\big) \\ &\quad = \left[\left(\Delta u + \frac{f}{2}, (-\Delta)^{-1}u\right)\right]^2 - \left|\left(\frac{f}{2}, (-\Delta)^{-1}u\right)\right|^2 \\ &\quad \leqslant \left\|\Delta u + \frac{f}{2}\right\|^2_{H^{-1}(\Omega)}\big\|(-\Delta)^{-1}u\big\|^2_{H^1_0(\Omega)} - \left|\left(\frac{f}{2}, (-\Delta)^{-1}u\right)\right|^2 \\ &\quad \leqslant \left(\|u\|^2_{H^1_0(\Omega)} + \left\|\frac{f}{2}\right\|^2_{H^{-1}(\Omega)} - (f, u)\right)\|u\|^2_{H^{-1}(\Omega)} - \left|\left(\frac{f}{2}, (-\Delta)^{-1}u\right)\right|^2.\end{aligned} \tag{3.8}$$

Finally, (3.7) and (3.8) imply

$$\frac{d}{dt}\Psi(t) \leqslant \frac{2}{\|u\|^4_{H^{-1}(\Omega)}}\left(\left\|\frac{f}{2}\right\|^2_{H^{-1}(\Omega)}\|u\|^2_{H^{-1}(\Omega)} - \left|\left(\frac{f}{2}, (-\Delta)^{-1}u\right)\right|^2\right). \tag{3.9}$$

Replacing $f = -au$, we get the desired inequalities (3.4).

Consequently, from (3.4) we obtain for all $t \in (0, L)$,

$$\Psi(t) \leqslant e^{c_4 L\|a\|^2_\infty}\Psi(0). \tag{3.10}$$

Secondly, remark that from (3.6) and (3.10), we have

$$\begin{aligned}
0 &= \frac{1}{2}\frac{d}{dt}\left\|u(\cdot,t)\right\|^2_{H^{-1}(\Omega)} + \Psi(t)\left\|u(\cdot,t)\right\|^2_{H^{-1}(\Omega)} + \left(a(\cdot,t)u(\cdot,t), (-\Delta)^{-1}u(\cdot,t)\right) \\
&\leqslant \frac{1}{2}\frac{d}{dt}\left\|u(\cdot,t)\right\|^2_{H^{-1}(\Omega)} + \Psi(t)\left\|u(\cdot,t)\right\|^2_{H^{-1}(\Omega)} + c_4\|a\|_\infty\sqrt{\Psi(t)}\left\|u(\cdot,t)\right\|^2_{H^{-1}(\Omega)} \\
&\leqslant \frac{1}{2}\frac{d}{dt}\left\|u(\cdot,t)\right\|^2_{H^{-1}(\Omega)} + e^{c_4L\|a\|^2_\infty}\left(\Psi(0) + c_4\|a\|_\infty\sqrt{\Psi(0)}\right)\|u(\cdot,t)\|^2_{H^{-1}(\Omega)}.
\end{aligned} \tag{3.11}$$

Consequently, integrating (3.11) on $(0, L)$, we get the desired estimate

$$\left\|u(\cdot,0)\right\|^2_{H^{-1}(\Omega)} \leqslant \exp\left(2e^{c_4L\|a\|^2_\infty}\left(\Psi(0) + c_4\|a\|_\infty\sqrt{\Psi(0)}\right)L\right)\left\|u(\cdot,L)\right\|^2_{H^{-1}(\Omega)}. \tag{3.12}$$

Step 3. We choose an adequate $L > 0$ to conclude the proof of Theorem 2. Let us denote $\lambda_1 > 0$ a constant such that

$$\lambda_1 \leqslant \Psi(0). \tag{3.13}$$

Now, we choose

$$L = T\sqrt{\frac{\lambda_1}{\Psi(0)}} \leqslant T, \tag{3.14}$$

then we obtain from (3.1) and (3.12) that the solution u of (1.17) satisfies if $u(x,t) \neq 0$ for almost all $t \in [0, T]$,

$$\left\|u(\cdot,0)\right\|^2_{H^{-1}(\Omega)} \leqslant E_1 \exp\left(D_1\sqrt{\Psi(0)}\right)\int_0^T\int_\omega \left|u(x,t)\right|^2 dx\,dt, \tag{3.15}$$

where E_1 and D_1 are two constants such that

$$\begin{cases} D_1 = 2e^{c_4T\|a\|^2_\infty}T\sqrt{\lambda_1} + \frac{c_0}{T\sqrt{\lambda_1}}, \\ E_1 = c_4\exp(c_4\sqrt{\lambda_1}T\|a\|_\infty e^{c_4T\|a\|^2_\infty} + c_0(1 + T\|a\|_\infty + \|a\|^{2/3}_\infty)). \end{cases} \tag{3.16}$$

Multiplying (3.15) by $\Psi(0)$, we get

$$\left\|u(\cdot,0)\right\|^2_{L^2(\Omega)} \leqslant E_1\Psi(0)\exp\left(D_1\sqrt{\Psi(0)}\right)\|u\|^2_{L^2(\omega\times(0,T))}. \tag{3.17}$$

So, if $u_0 \neq 0$, we have

$$\begin{aligned}
\|u_0\|_{L^2(\Omega)} &\leqslant \sqrt{E_1}\sqrt{\Psi(0)}\exp\left(\frac{1}{2}D_1\sqrt{\Psi(0)}\right)\|u\|_{L^2(\omega\times(0,T))} \\
&\leqslant \frac{2\sqrt{E_1}}{D_1}\exp\left(D_1\sqrt{\Psi(0)}\right)\|u\|_{L^2(\omega\times(0,T))},
\end{aligned} \tag{3.18}$$

and also clearly for all $c_1 > 1$,

$$\frac{D_1}{2\sqrt{E_1}}\frac{\|u_0\|_{L^2(\Omega)}}{\|u\|_{L^2(\omega\times(0,T))}} \leqslant \exp\left(c_1D_1\sqrt{\Psi(0)}\right). \tag{3.19}$$

We choose $c_1 > 1$ large enough to get

$$2 + \frac{D_1}{4\sqrt{E_1}} \frac{\|u_0\|_{L^2(\Omega)}}{\|u\|_{L^2(\omega\times(0,T))}} \leqslant \exp\bigl(c_1 D_1 \sqrt{\Psi(0)}\bigr), \tag{3.20}$$

and finally we have

$$\|u_0\|_{H^{-1}(\Omega)} \leqslant \frac{c_1 D_1}{\ln\bigl(2 + \frac{D_1}{4\sqrt{E_1}} \frac{\|u_0\|_{L^2(\Omega)}}{\|u\|_{L^2(\omega\times(0,T))}}\bigr)} \|u_0\|_{L^2(\Omega)}. \tag{3.21}$$

That completes the proof of Theorem 2. □

4. Further comments

A quantitative estimate of unique continuation for initial data in $H_0^1(\Omega)$ of the heat equation with potential, solution u of (1.17), may be realized by using properties of the quantity

$$\Psi(t) = \frac{\|\nabla u(\cdot,t)\|^2_{L^2(\Omega)}}{\|u(\cdot,t)\|^2_{L^2(\Omega)}}, \tag{4.1}$$

which are well described in [1].

We find that there exist two constants $c_1, c_2 > 1$ such that for all $T > 0$, for all $a \in L^\infty(\Omega \times (0,T))$, for all initial data $u_0 \in H_0^1(\Omega)$ such that $u_0 \neq 0$, the solution u of (1.17) satisfies

$$\|u_0\|_{L^2(\Omega)} \leqslant \frac{D_1}{\ln\bigl(2 + \frac{1}{E_1} \frac{\|\nabla u_0\|_{L^2(\Omega)}}{\|u\|_{L^2(\omega\times(0,T))}}\bigr)} \|\nabla u_0\|_{L^2(\Omega)}, \tag{4.2}$$

where E_1 and D_1 are two constants such that

$$\begin{cases} D_1 = c_1\bigl(1 + \frac{1}{T}\bigr), \\ E_1 = \exp(c_2(1 + T^2\|a\|^2_\infty + T\|a\|_\infty + \|a\|^{2/3}_\infty)). \end{cases} \tag{4.3}$$

This improves the finite dimensional observability estimate of [3, Theorem 1.5] in the sense that if $u_0 \neq 0$,

$$\|u_0\|_{H_0^1(\Omega)} \leqslant \biggl[\exp\bigl(c_2\bigl(1 + T^2\|a\|^2_\infty + T\|a\|_\infty + \|a\|^{2/3}_\infty\bigr)\bigr) \times \exp\biggl(c_1\biggl(1 + \frac{1}{T}\biggr)\frac{\|u_0\|_{H_0^1(\Omega)}}{\|u_0\|_{L^2(\Omega)}}\biggr)\biggr] \|u\|_{L^2(\omega\times(0,T))}. \tag{4.4}$$

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