Controllability and stabilization of electromagnetic waves

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Titre français: Contrôle et stabilisation d'ondes électromagnétiques

Résumé. Nous étudions la contrôlabilité exacte et la stabilisation des équations de Maxwell par le biais de la propagation des singularités du champ électromagnétique dans un domaine borné. Les résultats présentés s'inspirent des travaux de C. Bardos, J. Rauch et G. Lebeau sur le contrôle géométrique. Notre intéret se portera plus particulièrement sur la stabilisation interne où l'on doit considérer le système de Maxwell avec loi d'Ohm avec une densité de charge non nulle.

Abstract. We consider the exact controllability and stabilization of Maxwell equation by using results on the propagation of singularities of the electromagnetic field. We will assume geometrical control condition and use techniques of the work of C. Bardos, J. Rauch and G. Lebeau on the wave equation. The problem of internal stabilization will be treated with more attention because the condition $divE = 0$ is not preserved by the system of Maxwell with Ohm's law.

AMS: 93B05, 93B07, 35B40, 35Q60

Keywords: Controllability, Stabilization, Maxwell equation

Contents

Introduction

In this article we investigate the control and the stabilization of electromagnetic waves in bounded domain by the use of propagation of singularities.
For such linear problems, the main difficulty consists, as for the scalar wave equation, to get a priori estimates which correspond to the observability ($cf.$ $[Li], [La], [K]$). For the control or stabilization of Maxwell equations, we naturally deal with the orthogonal space of the set of steady states ($cf.$ $[Ba], [BH]$); then we study the propagation of the L^2 microlocal regularity ($cf.$ $[N], [Y]$) and follow the program developed by C.Bardos, G.Lebeau and J.Rauch ($cf.$ $[BLR]$) to get the controllability and the boundary stabilization for the Maxwell equations.
The main novelty appears in the internal stabilization where the condition $divE = 0$ is not preserved by the system of Maxwell with Ohms's law when the conductivity σ acts locally. Moreover, asymptotic in time estimates for the charge density seem not easy to get. Our different results are as follows:

The electromagnetic field of the system of Maxwell with Ohms's law decay uniformly and exponentially when the electric field is damped everywhere.

The same behaviour happens for the electromagnetic field under the geometric condition of the work of C.Bardos, G.Lebeau and J.Rauch ($cf.$ $[BLR]$), if the conductivity σ satisfies the relations: $\sigma(x) \equiv 0\ \forall x \in \Omega\backslash\omega_+$, $\sigma(x) \geq C\ \forall x \in \omega_+$, for some constant $C > 0$.

In the case where the conductivity vanishes on the interface between ω_+ and $\Omega\backslash\omega_+$ in a regular way gives some particular difficulties and may be compared to the Dirichlet stabilization for the wave equation studied by G.Lebeau ($cf.$ $[Le]$). Here, we get a polynomial decay result.

1 Propagation of singularities

Let Ω be a bounded, open, connected region in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We suppose that Ω is occupied by an electromagnetic medium of constant electric permittivity ε and constant magnetic permeability μ. The electromagnetic field (E,H) satisfies Maxwell's equations

$$\left\{\begin{array}{c} \varepsilon\partial_t E - curlH = 0,\ \mu\partial_t H + curlE = 0 \quad \text{in } \Omega\times\mathbb{R} \\ div\,(\varepsilon E) = 0,\ div\,(\mu H) = 0 \quad \text{in } \Omega\times\mathbb{R} \\ E\wedge n = 0,\ H.n = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ E\,(\cdot,0) = E_o,\ H\,(\cdot,0) = H_o \quad \text{in } \Omega\ . \end{array}\right. \tag{1.M}$$

Thus, the electric field E solves the hyperbolic system

$$\left\{\begin{array}{c} \varepsilon\mu\partial_t^2 E - \Delta E = 0 \quad \text{in } \Omega\times\mathbb{R} \\ E\wedge n = 0,\ divE = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ E\,(\cdot,0) = E_o,\ \partial_t E\,(\cdot,0) = \varepsilon^{-1} curlH_o \quad \text{in } \Omega\ , \end{array}\right. \tag{1.E}$$

and, the magnetic field solves the hyperbolic system

$$\left\{\begin{array}{c} \varepsilon\mu\partial_t^2 H - \Delta H = 0 \quad \text{in } \Omega\times\mathbb{R} \\ H.n = 0,\ curlH\wedge n = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ H\,(\cdot,0) = H_o,\ \partial_t H\,(\cdot,0) = -\mu^{-1} curlE_o \quad \text{in } \Omega\ . \end{array}\right. \tag{1.H}$$

Moreover, the fields E and H solution of (1.E) and (1.H) respectively and such that $divE_o = divH_o = 0$ are solution of the Maxwell system (1.M).

We suppose that $\partial\Omega$ is C^∞ and has no contacts of infinite order with its tangents. Let us recall some known results on the singularities of $(U,V)\in\left(\mathcal{D}'\left(\overline{\Omega}\right)\right)^6$ solutions of the following hyperbolic systems

$$\left\{\begin{array}{c} \partial_t^2 U - c_o\Delta U = 0 \quad \text{in } \Omega\times\mathbb{R} \\ U\wedge n = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ divU = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{c} \partial_t^2 V - c_o\Delta V = 0 \quad \text{in } \Omega\times\mathbb{R} \\ V.n = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ curlV\wedge n = 0 \quad \text{on } \partial\Omega\times\mathbb{R}\ , \end{array}\right.$$

where c_o is a positive constant.

Basics of symplectic geometry

Let us begin to define different objects from differential geometry in local coordinates knowing that these definitions may be given in an intrinsic way [MS1].

We denote by $\mathcal{C}$, the characteristic variety of the operator $P = \partial_t^2 - c_o\Delta$,

$$\mathcal{C} = \left\{(x,t,\xi,\tau)\in T^*\left(\overline{\Omega}\times\mathbb{R}\right)\backslash 0\ , -\tau^2 + c_o\left|\xi\right|^2 = 0\right\}\ .$$

We introduce $p\,(x,t,\xi,\tau) = -\tau^2 + c_o\left|\xi\right|^2$, the principal symbol of the operator P, a real-valued function defined on $T^*\,(\Omega\times\mathbb{R})\backslash 0$.

The Hamiltonian vector field associated to p, H_p, on $T^*\,(\Omega\times\mathbb{R})\backslash 0$ is given by

$$H_p = \sum_{j=1}^{3} 2c_o\xi_j\partial_{x_j} - 2\tau\partial_t\ .$$

The integral curves of H_p lying in $p^{-1}(0) = \mathcal{C}$ are the bicharacteristics.

Let $I = [s_0; s_1] \subset \mathbb{R}$ and $\gamma(s) = (x(s), t(s), \xi(s), \tau(s))$ the bicharacteristic associated to p, with $s \in I$. It solves the Hamiltonian system

$$\begin{cases} \frac{d}{ds} x_j = 2c_o \xi_j(s) \\ \frac{d}{ds} t = -2\tau(s) \end{cases} \quad \text{and} \quad \begin{cases} \frac{d}{ds} \xi_j = 0 \\ \frac{d}{ds} \tau = 0 \end{cases}$$

with an initial data such that $\gamma(s_0) \in T^*\left(\overline{\Omega} \times \mathbb{R}\right) \backslash 0$ and $p(\gamma(s)) = 0$ where $s \in I$.

Change of coordinates to the half-space

The study of the singularities of U in a neighborhood of the boundary is reduced to the case of the half-space by a local change of coordinates [MS1].
Suppose that the domain Ω is locally defined by $\Omega = \{(x_1, x_2, x_3) \setminus x_3 - g(x_1, x_2) > 0\}$, the normal vector at the boundary $\partial\Omega$ only depends on the variables (x_1, x_2) and can be written

$$n = \begin{pmatrix} \nu m_1 \\ \nu m_2 \\ \nu \end{pmatrix} \quad \text{where} \quad \nu^2 \left(1 + m_1^2 + m_2^2\right) = 1 \quad \text{and} \quad \begin{cases} m_1 = -\partial_{x_1} g \\ m_2 = -\partial_{x_2} g \ . \end{cases}$$

Let us introduce the following change of coordinates Ψ as follows

$$\Psi : (s, y_1, y_2) \longmapsto (x_1, x_2, x_3) = (y_1, y_2, g(y_1, y_2)) + s(n_1, n_2, n_3) \tag{1.1}$$

where $(n_1, n_2, n_3) = n$ in the variables (y_1, y_2). The Jacobian associated to Ψ is given by

$$Jac_\Psi = \begin{pmatrix} 1 + s\partial_{y_1} n_1 & s\partial_{y_2} n_1 & \nu m_1 \\ s\partial_{y_1} n_2 & 1 + s\partial_{y_2} n_2 & \nu m_2 \\ -m_1 + s\partial_{y_1} n_3 & -m_2 + s\partial_{y_2} n_3 & \nu \end{pmatrix} .$$

Notice that on the boundary, i-e when $s = 0$, we have

$$Jac_\Psi = \begin{pmatrix} 1 & 0 & \nu m_1 \\ 0 & 1 & \nu m_2 \\ -m_1 & -m_2 & \nu \end{pmatrix} , \det(Jac_\Psi) = \frac{1}{\nu} \neq 0, \quad Jac_\Psi^{-1} = \nu^2 \begin{pmatrix} 1 + m_2^2 & -m_1 m_2 & -m_1 \\ -m_1 m_2 & 1 + m_1^2 & -m_2 \\ \frac{m_1}{\nu} & \frac{m_2}{\nu} & \frac{1}{\nu} \end{pmatrix} .$$

Microlocal classification at the boundary of R.Melrose and J.Sjöstrand

The operator P is locally in the form

$$P = \partial_t^2 - c_o \left(\partial_s^2 - H(s, y_1, y_2, \partial_{y_1}, \partial_{y_2})\right)$$

where H has a principal symbol h which is elliptic of second order.
The tangential symbol of P is given by

$$r(0, y_1, y_2, \zeta_1, \zeta_2, \tau) = h(0, y_1, y_2, \zeta_1, \zeta_2) - \frac{1}{c_o} \tau^2 .$$

The classification of the points of $p^{-1}(0) \cap T^*(\partial\Omega \times \mathbb{R}) \backslash 0$ are described in the following way.

- The hyperbolic region $\mathcal{H}$ is defined by $r(0, y_1, y_2, \zeta_1, \zeta_2, \tau) < 0$.
- The glancing region $\mathcal{G}$ is defined by $r(0, y_1, y_2, \zeta_1, \zeta_2, \tau) = 0$.

Moreover, we separate the points in $\mathcal{G}$ following the convexity or the concavity of the boundary. Let us introduce

$$\Sigma_b^0 = \Sigma_b \cap T^*(\Omega \times \mathbb{R}) \backslash 0 \ , \quad \Sigma_b^1 = \mathcal{H}$$
$$\Sigma_b^{2,-} = \mathcal{G} \cap \{\partial_s r(0, y_1, y_2, \zeta_1, \zeta_2, \tau) < 0\} \ , \quad \Sigma_b^{2,+} = \mathcal{G} \cap \{\partial_s r(0, y_1, y_2, \zeta_1, \zeta_2, \tau) > 0\}$$
$$\Sigma_b^{(3)} = \mathcal{G} \cap \{\partial_s r(0, y_1, y_2, \zeta_1, \zeta_2, \tau) = 0\}$$

in order that $\Sigma_b = \Sigma_b^0 \cup \Sigma_b^1 \cup \Sigma_b^{2,-} \cup \Sigma_b^{2,+} \cup \Sigma_b^{(3)}$ where

$$\begin{aligned}\Sigma_b &= \left\{ (x,t,\xi,\tau) \in T^*(\Omega \times \mathbb{R}) \backslash 0 \, ; -\tau^2 + c_o |\xi|^2 = 0 \right\} \\ &\cup \left\{ (y_1, y_2, t, \zeta_1, \zeta_2, \tau) \in T^*\left(\mathbb{R}^2 \times \mathbb{R}\right) \backslash 0 \, ; -\tau^2 + c_o h(0, y_1, y_2, \zeta_1, \zeta_2) \leq 0 \right\} \, .\end{aligned}$$

Definition of the rays

The construction of the generalized bicharacteristic defined on Σ_b is described as follows.

- In Ω, the generalized bicharacteristic propagates like a bicharacteristic following the Hamiltonian system associated to H_p.
- When the bicharacteristic meets the boundary $\partial\Omega$ at a point of $\mathcal{H}$, the generalized bicharacteristic is obtained by reflection following the optic geometric rules.
- When the bicharacteristic meets the boundary $\partial\Omega$ at a point of $\Sigma_b^{2,-}$, the generalized bicharacteristic is extended continuously in Ω.
- When the bicharacteristic meets the boundary $\partial\Omega$ at a point of $\Sigma_b^{2,+} \cup \Sigma_b^{(3)}$, the generalized bicharacteristic is extended continuously on $\partial\Omega$ like a bicharacteristic of $T^*(\partial\Omega \times \mathbb{R}) \backslash 0$ following the Hamiltonian system associated to H_r until it goes out the boundary.

The assumption that $\partial\Omega$ has no contacts of infinite order with its tangents allows to ensure that from any point of Σ_b, the generalized bicharacteristic is uniquely defined ([MS1], [Bu]). The rays are defined as the projection on $\overline{\Omega} \times \mathbb{R}$ of the generalized bicharacteristic. A complete and precise definition is given in [MS1].

Definition of the wave front and of the microlocal Sobolev norms

We define the wave front of U, $WF(U)$, as follows.

Let $\rho = (x_o, t_o, \xi_o, \tau_o) \in T^*(\Omega \times \mathbb{R}) \backslash 0$, $\rho \notin WF(U)$ if there exists $\Phi \in C_0^\infty, \Phi(x_o, t_o) \neq 0$ such that for all N

$$\left| \widehat{\Phi U}(\xi, \tau) \right| \leq \frac{C_N}{(1 + |(\xi, \tau)|)^N}$$

for $(\xi, \tau) \in \Gamma_N(\xi_o, \tau_o)$ where $\Gamma = \Gamma_N(\xi_o, \tau_o)$ is a conical neighborhood of (ξ_o, τ_o). Accordingly, $WF(U)$ consists of the remaining points $(x_o, t_o, \xi_o, \tau_o) \in T^*(\Omega \times \mathbb{R}) \backslash 0$ for which such Φ and Γ do not exist.

Now, we say that $U \in H_\rho^m$ or $U \in H^m$ at ρ, if one of the three equivalent following conditions is satisfied.

_ $U = U_1 + U_2$ where $U_1 \in H^m(\Omega \times \mathbb{R})$ and $\rho \notin WF(U_2)$.
_ There exists $\Phi \in C_0^\infty, \Phi(x_o, t_o) \neq 0$ and Γ a conical neighborhood of (ξ_o, τ_o) such that

$$\int_\Gamma \left(1 + |(\xi, \tau)|^2\right)^m \left| \widehat{\Phi U}(\xi, \tau) \right|^2 d\xi d\tau < \infty \, .$$

_ There exists $\Phi \in C_0^\infty, \Phi = 1$ near (x_o, t_o) et $\Theta \in C_0^\infty, \Theta = 1$ near $\Gamma \ni (\xi_o, \tau_o)$ such that

$$\int_{\mathbb{R}^4} \left(1 + |(\xi, \tau)|^2\right)^m \Theta(\xi, \tau) \left| \widehat{\Phi U}(\xi, \tau) \right|^2 d\xi d\tau < \infty \, .$$

We define the wave front up to the boundary of U, $WF_b(U)$, as a subset of $T^*(\Omega\times\mathbb{R})\backslash 0\cup T^*(\partial\Omega\times\mathbb{R})\backslash 0$ such that

$$\left\{\begin{array}{l}. \ WF_b(U)\cap T^*(\Omega\times\mathbb{R})\backslash 0 = WF(U)\\ . \text{ For } \rho\in T^*(\partial\Omega\times\mathbb{R})\backslash 0\ ,\ \rho\notin WF_b(U) \text{ iff there exists}\\ \qquad \text{a tangential pseudo-differential operator } A\\ \qquad \text{elliptic at } \rho \text{ such that } AU\in C^\infty\left(\overline{\Omega}\times\mathbb{R}\right)\ .\end{array}\right.$$

Also, let U be an extendible distribution in $\Omega\times\mathbb{R}$ such that $\partial_t^2U-c_o\Delta U=0$, then we associate to the spaces H_ρ^m where $m\in\mathbb{R}$, $\rho=(x,t,\xi,\tau)\in\overline{\Omega}\times]T_0;T_1[\times\mathbb{R}^4$ the following semi-norms.

- When $x\in\Omega$, $\|U\|^2_{H_\rho^m}=\int_{\mathbb{R}^4}\left(1+|(\xi,\tau)|^2\right)^m\Theta(\xi,\tau)\left|\widehat{\Phi U}(\xi,\tau)\right|^2 d\xi d\tau$ where Φ is localized around a neighborhood of (x,t) and $\Theta=\Theta(\xi,\tau)$ is a localized function around a conical neighborhood of (ξ,τ).

- When $x\in\partial\Omega$, $\|U\|^2_{H_\rho^m}=\int_0^1 ds\int_{\mathbb{R}^3}\left(1+|(\zeta',\tau)|^2\right)^m\Xi^2(\zeta',\tau)\left|\widehat{\Phi W}'(\zeta',\tau)\right|^2 d\zeta' d\tau$ where by a change of coordinates we are in the half-space, $\Phi U(x,t)=\Phi U(\Psi(s,y'),t)=\Phi W(s,y',t)$, $\widehat{\ }'$ denotes the Fourier transform on the variables (y_1,y_2,t) tangential of $(s,y',t)=(s,y_1,y_2,t)\in\mathbb{R}^4$ and $\Xi=\Xi(\zeta',\tau)$ is a localized function around a conical neighborhood of $(\zeta_1,\zeta_2,\tau)=(\zeta',\tau)$.

Now, some formulas of K.Yamamoto [Y] are deduced from the change of coordinates (1.1):

Let $E(s,y_1,y_2,t)=U(\Psi(s,y_1,y_2),t)=U(x_1,x_2,x_3,t)$, we have

$$U\wedge n=\nu\left(\begin{array}{c}E_2-m_2E_3\\ -E_1+m_1E_3\\ m_2E_1-m_1E_2\end{array}\right)\ , \tag{1.2}$$

$$divU=\left(\begin{array}{c}\nu m_1\\ \nu m_2\\ \nu\end{array}\right)\partial_sE+\nu^2\left(\begin{array}{c}1+m_2^2\\ -m_1m_2\\ -m_1\end{array}\right)\partial_{y_1}E+\nu^2\left(\begin{array}{c}-m_1m_2\\ 1+m_1^2\\ -m_2\end{array}\right)\partial_{y_2}E\ . \tag{1.3}$$

Thus, if $U\wedge n=0$ and $divU=0$, then E becomes $\quad E=\left(\begin{array}{c}m_1\\ m_2\\ 1\end{array}\right)E_3$ and

$$\left(\begin{array}{c}m_1\\ m_2\\ 1\end{array}\right)\partial_sE+\nu\left[\left(\begin{array}{c}1+m_2^2\\ -m_1m_2\\ -m_1\end{array}\right)\partial_{y_1}\left(\begin{array}{c}m_1\\ m_2\\ 1\end{array}\right)+\left(\begin{array}{c}-m_1m_2\\ 1+m_1^2\\ -m_2\end{array}\right)\partial_{y_2}\left(\begin{array}{c}m_1\\ m_2\\ 1\end{array}\right)\right]E_3=0=\frac{1}{\nu}divU. \tag{1.4}$$

Finally, if $U\wedge n=0$ and $divU=0$, then from (1.2) and (1.4),

$$\left\{\begin{array}{l}E\wedge n=0,\quad E.n=\frac{1}{\nu}E_3\\ \partial_sE.n+\ell\nu^2E.n=0\quad \text{where } \ell \text{ is a real-valued } C^\infty \text{ function}\ .\end{array}\right. \tag{1.5}$$

The boundary conditions $U\wedge n=0$ and $divU=0$ will allow to apply the proof of the theorem on the propagation of singularities due to R.Melrose and J.Sjöstrand [MS1].

Let $B(s,y_1,y_2,t)=V(\Psi(s,y_1,y_2),t)=V(x_1,x_2,x_3,t)$, we have

$$V.n=\nu\left(m_1B_1+m_2B_2+B_3\right)\ , \tag{1.6}$$

thus,

$$\partial_{y_j}\left(V.\frac{n}{\nu}\right)=\left(\begin{array}{ccc}m_1 & m_2 & 1\end{array}\right)\partial_{y_j}B+\partial_{y_y}m_1B_1+\partial_{y_j}m_2B_2\ . \tag{1.7}$$

On another hand,

$$\begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & -m_2 \\ 0 & 0 & 1 \end{pmatrix} rotV\wedge n \; = \begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & -m_2 \\ -m_1 & -m_2 & m_1^2+m_2^2 \end{pmatrix}\partial_s B$$
$$+\nu\begin{pmatrix} -m_1 & -m_2 & -1 \\ 0 & 0 & 0 \\ m_1^2 & m_1m_2 & m_1 \end{pmatrix}\partial_{y_1}B+\nu\begin{pmatrix} 0 & 0 & 0 \\ -m_1 & -m_2 & -1 \\ m_1m_2 & m_2^2 & m_2 \end{pmatrix}\partial_{y_2}B\;. \tag{1.8}$$

This last equality (1.8) is deduced by writing the curl operator as follows.

$$curlV=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\partial_{x_1}V+\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\partial_{x_2}V+\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\partial_{x_3}V\;.$$

After a change of variables, we next multiply the both side by the matrix

$$\begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & -m_2 \\ 0 & 0 & 1 \end{pmatrix}\;.$$

Finally, if $V.n=0$ and $rotV\wedge n=0$, then from (1.6) and (1.8),

$$\begin{cases} B_3=-m_1B_1-m_2B_2 \\ \begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & -m_2 \end{pmatrix}\partial_s B=-\begin{pmatrix} \partial_1 m_1 & \partial_1 m_2 \\ \partial_2 m_1 & \partial_2 m_2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \end{cases} \tag{1.9}$$

that is

$$\begin{cases} B.n=0 \\ \partial_s B\wedge n+\mathcal{A}\left(B\wedge n\right)=0 \quad \text{where } \mathcal{A} \text{ is a real-valued } C^\infty \text{ matrix.} \end{cases} \tag{1.10}$$

The boundary conditions $V.n=0$ and $curlV\wedge n=0$ will allow to apply the proof of the theorem on the propagation of singularities due to R.Melrose and J.Sjöstrand [MS1].

Let $H\left(s,y_1,y_2,t\right)=V\left(\Psi\left(s,y_1,y_2\right),t\right)=V\left(x_1,x_2,x_3,t\right)$, we have that
if $divV=0$ and $curlV\wedge n=0$, then from (1.3) and (1.8),

$$\frac{1}{\nu}\begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & -m_2 \\ m_1 & m_2 & 1 \end{pmatrix}\partial_s H=\begin{pmatrix} m_1 & m_2 & 1 \\ 0 & 0 & 0 \\ -1-m_2^2 & m_1m_2 & m_1 \end{pmatrix}\partial_{y_1}H+\begin{pmatrix} 0 & 0 & 0 \\ m_1 & m_2 & 1 \\ m_1m_2 & -1-m_1^2 & m_2 \end{pmatrix}\partial_{y_2}H.$$

Remark that the matrix in front of $\partial_s H=\partial_n V$ is invertible and

$$\begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & -m_2 \\ m_1 & m_2 & 1 \end{pmatrix}^{-1}=\begin{pmatrix} 1-\nu^2m_1^2 & -\nu m_1m_2 & \nu^2m_1 \\ -\nu m_1m_2 & 1-\nu^2m_2^2 & \nu^2m_2 \\ -\nu^2m_1 & -\nu^2m_2 & \nu^2 \end{pmatrix}\;.$$

Consequently,

$$\partial_s H-\nu\begin{pmatrix} 0 & m_2 & 1 \\ -m_2 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\partial_{y_1}H-\nu\begin{pmatrix} 0 & -m_1 & 0 \\ m_1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\partial_{y_2}H=0\;. \tag{1.11}$$

The boundary conditions $divV=0$ and $curlV\wedge n=0$ will allow to apply the boundary condition $\partial_n V+\mathcal{L}V=0$ where $\mathcal{L}$ is a differential matrix operator of order 1.

We recall the theorem on propagation of singularities due to R.Melrose and J.Sjöstrand [MS1].

Let U be an extendible distribution in $\Omega \times \mathbb{R}$ such that $PU = f$, with the Dirichet or Neumann boundary condition, where $f \in C^{\infty}(\Omega \times \mathbb{R})$. Let $\rho \in \Sigma_b$ and γ be the generalized bicharacteristic starting from ρ. Then $WF_b(U)$ is contained in Σ_b, and if $\rho \in WF_b(U)$, then $\gamma(s) \in WF_b(U)$ for all s.

We need the following result.

Theorem 1.1 Let (U, V) be a solution of

$$\left\{ \begin{array}{c} \partial_t^2 U - c_o \Delta U = f \quad \text{in } \Omega \times I \\ U \wedge n = 0 \quad \text{on } \partial\Omega \times I \\ divU = 0 \quad \text{on } \partial\Omega \times I \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{c} \partial_t^2 V - c_o \Delta V = f \quad \text{in } \Omega \times I \\ V.n = 0 \quad \text{on } \partial\Omega \times I \\ curlV \wedge n = 0 \quad \text{on } \partial\Omega \times I \end{array} \right.$$

where $I = [s_0; s_1] \subset \mathbb{R}$, $f \in \left(H^{-1}(\Omega \times I)\right)^3$. If $\rho \in \Sigma_b$ and $U \in L^2_\rho$ (resp. $V \in L^2_\rho$), then $U \in L^2_{\rho'}$ (resp. $V \in L^2_{\rho'}$) at any point $\rho' \in \gamma(I)$ of the generalized bicharacteristic γ crossing $\rho = \gamma(s_0)$.

Moreover, we have the following estimates $\quad \exists c, d > 0$

$$\|U\|_{L^2_{\rho'}} \leq c \|U\|_{L^2_\rho} + d \left(\|U\|_{H^{-1/2}(\Omega \times I)} + \|f\|_{H^{-1}(\Omega \times I)} \right) , \tag{1.12}$$

$$\|V\|_{L^2_{\rho'}} \leq c \|V\|_{L^2_\rho} + d \left(\|V\|_{H^{-1/2}(\Omega \times I)} + \|f\|_{H^{-1}(\Omega \times I)} \right) . \tag{1.13}$$

Proof.- The proof of theorem 1.1 comes from the work of R.Melrose and J.Sjöstrand [MS1] and the fact that the system of Maxwell is a well-posed problem [N]. The estimates (1.12) and (1.13) will be then deduced from the closed graph theorem. Here, we only establish the proof for the solution U because the proof for the solution V follows the same strategy.

We begin with the particular case where $f \equiv 0$.

In Ω, i-e $\rho \in \Sigma_b^0$, we apply the theorem of propagation of singularities due to L.Hörmander [T1]. The reflection and the propagation of the H^m regularity for hyperbolic systems with the perfect conductor boundary condition through a convex regular boundary, i-e when $\rho \in \Sigma_b^1 \cup \Sigma_b^{2,-}$ have been done by M.Taylor ([T2], [T3]).
Following the proof given by R.Melrose et J.Sjöstrand in [MS1], we obtain an analysis of the wave front up to the boundary of each component of U at a point in the glancing region:

Locally, we are reduced to the half-space $\{s > 0\}$. Let $E(s, y_1, y_2, t) = U(\Psi(s, y_1, y_2), t) = U(x_1, x_2, x_3, t)$, the boundary conditions $U \wedge n = 0$ and $divU = 0$ on the boundary imply that

$$\left\{ \begin{array}{c} P(E_i) = \left(-\partial_s^2 + R(s, y, D_{y,t})\right)(E_i) = 0 \quad \text{in } \{s > 0\} \quad \text{where } i = \{1, 2, 3\} \\ E = \frac{n}{\nu} E_3 \quad \text{on } \{s = 0\} \\ \frac{n}{\nu}.\partial_s E + \ell E_3 = 0 \quad \text{on } \{s = 0\} \end{array} \right.$$

where ℓ is a real-valued smooth function and R is a second order differential operator with smooth coefficients and its principal symbol is a real-valued function for any fixed s.

We begin to compute $\sum_{i=1}^{3} \int_0^\infty \int_{\mathbb{R}^3} \left(QE_i \overline{PE_i}\right) dsdy$ where Q is a tangential pseudo-differential operator with compact support and such that $Q'_s = 0$ on $\{s = 0\}$. We obtain then

$$\sum_{i=1}^{3} \int_0^\infty \int \left([P, Q] + (R^* - R) Q\right) E_i \overline{E_i} = \int \left(Q\partial_s E.\overline{E}\right)_{s=0} - \int \left(QE.\overline{\partial_s E}\right)_{s=0} \tag{1.14}$$

but,

$$\begin{array}{ll}\int\left(Q\partial_s E.\overline{E}\right)_{s=0} & =\int\left(Q\partial_s E.\left(\overline{\frac{n}{\nu}E_3}\right)\right)_{s=0}\\ & =\int\left(Q\left(-\ell\nu E_3\right)\left(\overline{\frac{1}{\nu}E_3}\right)\right)_{s=0}+\sum_{i=1}^{3}\int\left(\left[n_i,Q\right]\left(\partial_s E_i\right)\overline{\frac{1}{\nu}E_3}\right)_{s=0}\\ \int\left(QE.\overline{\partial_s E}\right)_{s=0} & =\int\left(Q\left(\frac{n}{\nu}E_3\right).\overline{\partial_s E}\right)_{s=0}\\ & =\int\left(Q\left(\frac{1}{\nu}E_3\right)\left(\overline{-\ell\nu E_3}\right)\right)_{s=0}+\sum_{i=1}^{3}\int\left(\left[Q,n_i\right]\left(\frac{1}{\nu}E_3\right)\overline{\partial_s E_i}\right)_{s=0}\ .\end{array} \tag{1.15}$$

On another hand, by integration by parts,

$$\begin{array}{ll}\int\left(\left[n_i,Q\right]\left(\partial_s E_i\right)\overline{\frac{1}{\nu}E_3}+\left[n_i,Q\right]\left(\frac{1}{\nu}E_3\right)\overline{\partial_s E_i}\right)_{s=0} & =\int_0^\infty\int\left(\partial_s^2\left(\left[n_i,Q\right]E_i\right)\overline{\frac{1}{\nu}E_3}-\left[n_i,Q\right]E_i\overline{\partial_s^2\left(\frac{1}{\nu}E_3\right)}\right)\\ & \quad+\int_0^\infty\int\left[n_i,Q\right]\left(\frac{1}{\nu}E_3\right)\overline{\partial_s^2 E_i}-\partial_s^2\left(\left[n_i,Q\right]\left(\frac{1}{\nu}E_3\right)\right)\overline{E_i}\\ & \quad+\int\left(\left[n_i,Q\right]E_i\overline{\partial_s\left(\frac{1}{\nu}E_3\right)}+\left[n_i,Q\right]\partial_s\left(\frac{1}{\nu}E_3\right)\overline{E_i}\right)_{s=0}\ .\end{array} \tag{1.16}$$

Let $\widetilde{R}=\frac{1}{\nu}R\nu$ and $\widetilde{P}=-\partial_s^2+\widetilde{R}$ then

$$\begin{array}{l}\sum_{i=1}^{3}\int\left(\left[n_i,Q\right]\left(\partial_s E_i\right)\overline{\frac{1}{\nu}E_3}+\left[n_i,Q\right]\left(\frac{1}{\nu}E_3\right)\overline{\partial_s E_i}\right)_{s=0}\\ \quad=\ \sum_{i=1}^{3}\int_0^\infty\int\left(-\left(-\partial_s^2+\widetilde{R}^*\right)\left(\left[n_i,Q\right]E_i\right)\overline{\frac{1}{\nu}E_3}\right)\\ \qquad+\sum_{i=1}^{3}\int_0^\infty\int\left(-\partial_s^2+R^*\right)\left[n_i,Q\right]\left(\frac{1}{\nu}E_3\right)\overline{E_i}\\ \qquad+\sum_{i=1}^{3}\int\left(\left[n_i,Q\right]\left(n_i\left(\frac{1}{\nu}E_3\right)\right)\overline{\partial_s\left(\frac{1}{\nu}E_3\right)}+\left[n_i,Q\right]\partial_s\left(\frac{1}{\nu}E_3\right)\overline{\left(n_i\left(\frac{1}{\nu}E_3\right)\right)}\right)_{s=0}\\ \quad=\ \sum_{i=1}^{3}\int_0^\infty\int\left(-\left(\left[P,\left[n_i,Q\right]\right]+\left(\widetilde{R}^*-R\right)\left[n_i,Q\right]\right)E_i\overline{\frac{1}{\nu}E_3}\right)\\ \qquad+\sum_{i=1}^{3}\int_0^\infty\int\left(\left[\widetilde{P},\left[n_i,Q\right]\right]+\left(R^*-\widetilde{R}\right)\left[n_i,Q\right]\right)\left(\frac{1}{\nu}E_3\right)\overline{E_i}\\ \qquad+\sum_{i=1}^{3}\int_0^\infty\int\left(-\partial_s^2+\widetilde{R}^*\right)\left(n_iQn_i\right)\left(\frac{1}{\nu}E_3\right)\overline{\frac{1}{\nu}E_3}\\ \qquad+\int_0^\infty\int-\left(-\partial_s^2+\widetilde{R}^*\right)Q\left(\frac{1}{\nu}E_3\right)\overline{\frac{1}{\nu}E_3}\ .\end{array} \tag{1.17}$$

Consequently, from (1.15), (1.16) and (1.17), the equality (1.14) becomes

$$\begin{array}{ll}\sum_{i=1}^{3}\int_0^\infty\int\left(\left[P,Q\right]+\left(R^*-R\right)Q\right)E_i\overline{E_i} & =-\int\left(\left[Q,\ell\nu^2\right]\left(\frac{1}{\nu}E_3\right)\overline{\frac{1}{\nu}E_3}\right)_{s=0}\\ & \quad+\int_0^\infty\int\left(-\left(\left[P,\left[n,Q\right]\right]+\left(\widetilde{R}^*-R\right)\left[n,Q\right]\right)E\overline{\frac{1}{\nu}E_3}\right)\\ & \quad+\int_0^\infty\int\left(\left[\widetilde{P},\left[n,Q\right]\right]+\left(R^*-\widetilde{R}\right)\left[n,Q\right]\right)\left(\frac{1}{\nu}E_3\right)\overline{E}\\ & \quad+\int_0^\infty\int\left(\left[\widetilde{P},\left[n,Q\right]n\right]+\left(\widetilde{R}^*-\widetilde{R}\right)\left[n,Q\right]n\right)\left(\frac{1}{\nu}E_3\right)\overline{\frac{1}{\nu}E_3}\ .\end{array} \tag{1.18}$$

Remark that the principal symbols of the operators
$i\left(\widetilde{R}^*-R\right)\left[n_j,Q\right]$, $i\left(R^*-\widetilde{R}\right)\left[n_j,Q\right]$, $i\left(\widetilde{R}^*-\widetilde{R}\right)\left[n_j,Q\right]n_j$,$\frac{1}{i}\left[\widetilde{R},\left[n_j,Q\right]\right]$ and $\frac{1}{i}\left[R,\left[n_j,Q\right]\right]$ are homogeneous of order m, if Q has a homogeneous principal symbol of order m.

Following the energy method, we need to construct Q satisfying the following properties:

$$\frac{1}{i}\left(\left[P,Q\right]+\left(R^*-R\right)Q\right)=A^*A+B$$

$$\left|\int_0^\infty\int B\left(E\right).E\right|<\infty$$

$\left|\int\left(\left[Q,\ell\nu^2\right]\left(\frac{1}{\nu}E_3\right)\overline{\frac{1}{\nu}E_3}\right)_{s=0}\right|<\infty$

$\left|\int_0^\infty\int\left(-\left(\left[P,\left[n,Q\right]\right]+\left(\widetilde{R}^*-R\right)\left[n,Q\right]\right)E\overline{\frac{1}{\nu}E_3}\right)\right|<\infty$

$\left|\int_0^\infty\int\left(\left[\widetilde{P},\left[n,Q\right]\right]+\left(R^*-\widetilde{R}\right)\left[n,Q\right]\right)\left(\frac{1}{\nu}E_3\right)\overline{E}\right|<\infty$

$\left|\int_0^\infty\int\left(\left[\widetilde{P},\left[n,Q\right]n\right]+\left(\widetilde{R}^*-\widetilde{R}\right)\left[n,Q\right]n\right)\left(\frac{1}{\nu}E_3\right)\overline{\frac{1}{\nu}E_3}\right|<\infty$

where A is a tangential pseudo-differential operator, elliptic at ρ' and B is a pseudo-differential operator.

The assertion $\|A(E)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^3)}<\infty$ will give the conclusion.

Now, we denote $(x,y)=(x,(y_1,y_2,y_3))$ the variable $(s,(y_1,y_2,t))$ and let (ξ,η) be the dual variable of (x,y). Let

$r(x,y,\eta)$	, the principal symbol of R, of order 2
$k(x,y,\eta)$	, the principal symbol of $i(R^*-R)$, homogeneous of order 1
$p(x,y,\xi,\eta)=\xi^2+r$	, the principal symbol of P, homogeneous of order 2
$q(x,y,\eta)$	, the principal symbol of Q, tangential
$\sigma([P,Q])=iH_pq$	, the principal symbol of $[P,Q]$
$-(H_p+k)q$	, the principal symbol of $\frac{1}{i}([P,Q]+(R^*-R)Q)$.

We recall that

$$\begin{aligned} H_r &= \sum_{i=1}^{3}\left(\frac{\partial r}{\partial\eta_i}\frac{\partial}{\partial y_i}-\frac{\partial r}{\partial y_i}\frac{\partial}{\partial\eta_i}\right) \quad = \nabla_\eta r.\nabla_y-\nabla_y r.\nabla_\eta \\ H_p &= 2\xi\frac{\partial}{\partial x}-\frac{\partial r}{\partial x}\frac{\partial}{\partial\xi}+H_r\ . \end{aligned}$$

Let $\mathcal{G}$ be the glancing surface given by

$$\mathcal{G}=\left\{(y,\eta)\in\mathbb{R}^3\times\mathbb{R}^3\backslash0\ \backslash\ r_0(y,\eta)\equiv r(0,y,\eta)=0\right\}$$

and let $(y_0,\eta_0)\in\mathcal{G}$. Let $H_{r_0}=H_{r(0,y,\eta)}=H_r\,|_{x=0}$.
Let Γ be a conical neighborhood of (y_0,η_0). We introduce $L\subset\Gamma$ a conical hypersurface of (y_0,η_0) and transversal to H_{r_0}.
We introduce for small τ,$\varepsilon>0$,

$$L^+(\varepsilon,\tau)=\left\{e^{tH_{r_0}}(y,\eta)\in\Gamma\ \middle\backslash\ \begin{array}{c}(y,\eta)\in L\ ,\\ \left|\left(y,\frac{\eta}{|\eta|}\right)-\left(y_0,\frac{\eta_0}{|\eta_0|}\right)\right|\le\varepsilon^2\ ,\\ 0\le t\le\frac{\tau}{|\eta|}\end{array}\right\}$$

$$L^-(\varepsilon,\tau)=\left\{e^{tH_{r_0}}(y,\eta)\in\Gamma\ \middle\backslash\ \begin{array}{c}(y,\eta)\in L\ ,\\ \left|\left(y,\frac{\eta}{|\eta|}\right)-\left(y_0,\frac{\eta_0}{|\eta_0|}\right)\right|\le\varepsilon^2\ ,\\ -\frac{\tau}{|\eta|}\le t\le0\end{array}\right\}$$

$$F^\pm(\varepsilon,\tau)=\left\{(x,y,\eta)\ \middle\backslash\ \begin{array}{c}0\le x\le\varepsilon^2\ ,\\ (y,\eta)\in L^\pm(\varepsilon,\tau)\end{array}\right\}\ .$$

We will choose τ_0,$\varepsilon_0>0$ small enough and $C_1>0$ large in order that

$$\left.\begin{array}{c}0<\varepsilon\le\varepsilon_0,\ 0<\tau\le\tau_0,\\ (x,y,\eta)\in F^\pm(\varepsilon,\tau)\end{array}\right\}\implies|r(x,y,\eta)|\le\left(\tfrac{1}{2}C_1|\eta|\varepsilon\right)^2\ . \tag{1.19}$$

We will also need the following conical closed sets

$$V^\pm(\varepsilon,\tau)=\left\{(x,y,\xi,\eta)\ \middle\backslash (y,\eta)\in L^\pm(\varepsilon,\tau)\ ,\left[\text{or}\ \begin{array}{l}0\le x\le\frac{1}{2}\varepsilon^2\\ \frac{1}{2}\varepsilon^2\le x\le\varepsilon^2,\ \ |\xi|\le C_1\varepsilon|\eta|\end{array}\right]\right\}$$

$$W^{\pm}(\varepsilon,\tau)=\left\{(x,y,\xi,\eta)\Big\backslash(y,\eta)\in L^{\pm}(\varepsilon,\tau)\ ,\left[\text{or}\ \begin{array}{l}0\le x\le\frac{1}{2}\varepsilon^2\\ \frac{1}{2}\varepsilon^2\le x\le\varepsilon^2,\ \ |\xi|\le 2C_1\varepsilon|\eta|\end{array}\right]\right\}$$

$$V(\varepsilon)=V^{+}(\varepsilon,\tau_0)\cup V^{-}(\varepsilon,\tau_0)$$

$$W(\varepsilon)=W^{+}(\varepsilon,\tau_0)\cup W^{-}(\varepsilon,\tau_0)\ .$$

We begin to construct $q_\varepsilon(x,y,\eta)\in C_0^\infty([0,1[\times\Gamma)$:

In the neighborhood of $[0,1[\times\Gamma$, we introduce new coordinates $(x,s,t)\in[0,1[\times\mathbb{R}^5\times\mathbb{R}$, in order that $(y_0,\eta_0)=(0,0)$, $L=\{t=0\}$ and $H_{r_0}=\frac{\partial}{\partial t}$. Also,

$$\begin{array}{ll}H_r&=\frac{\partial}{\partial t}+O(x)\frac{\partial}{\partial s}+O(x)\frac{\partial}{\partial t}\\ H_p&=2\xi\frac{\partial}{\partial x}-\frac{\partial r}{\partial x}\frac{\partial}{\partial\xi}+H_r\ .\end{array}$$

Let χ,$\beta\in C^\infty(\mathbb{R})$ be positive, well-chosen functions and $f\in C^\infty(\mathbb{R})$, null on $\left]-\infty,\frac{1}{2}\right[$, decreasing and convex on $\left[\frac{1}{2},+\infty\right[$ ([MS1](p.601)). We introduce

$$q_\varepsilon(x,s,t)=\beta\left(\frac{t}{\varepsilon^2}\right)\chi\left(\frac{t}{\delta\varepsilon}+\frac{s^2}{\varepsilon^4}+f\left(\frac{x}{\varepsilon^2}\right)\right)$$

in order that

q_ε does not depend on x for $0\le x<\frac{1}{2}\varepsilon^2$,

$\text{supp}q_\varepsilon=\left\{(x,s,t)\setminus-\varepsilon^2\le t,\ \frac{t}{\delta\varepsilon}+\frac{s^2}{\varepsilon^4}+f\left(\frac{x}{\varepsilon^2}\right)\le 1\right\}\subset\left\{x\le\varepsilon^2,\ \ |s|\le\varepsilon^2,\ \ -\varepsilon^2\le t\le\delta\varepsilon\right\}$,

$q_\varepsilon\ge 0$, $q_{\varepsilon'}>0$ in $\text{supp}q_\varepsilon$ if $0<\varepsilon<\varepsilon'\le\varepsilon_0$, $q_\varepsilon\left(0,e^{tH_{r_0}}(y_0,\eta_0)\right)\ne 0$ for $0\le t<\delta\varepsilon$.

It remains to compute $(H_p+k)\,q=2\xi\frac{\partial q}{\partial x}-\frac{\partial r}{\partial x}\frac{\partial q}{\partial\xi}+H_rq+kq$

$$\begin{array}{ll}-(H_p+k)\,q_\varepsilon&=-\beta(H_p+k)\,\chi-\chi H_p\beta\\ &=g_\varepsilon-\chi H_p\beta\\ &=g_\varepsilon-\chi\left(\frac{\partial}{\partial t}+O(x)\frac{\partial}{\partial t}\right)\beta\\ &=g_\varepsilon-\chi\left(\frac{1}{\varepsilon^2}+O(x)\frac{1}{\varepsilon^2}\right)\beta'\\ &=g_\varepsilon+h_\varepsilon\ .\end{array}\tag{1.20}$$

When $0\le x\le\frac{1}{2}\varepsilon^2$,

$$\begin{array}{ll}-(H_p+k)\,\chi&=-\left(\frac{\partial}{\partial t}+O(x)\frac{\partial}{\partial s}+O(x)\frac{\partial}{\partial t}+k\right)\chi\\ &=-\left(\frac{\partial}{\partial t}+O(x)\frac{\partial}{\partial s}+O(x)\frac{\partial}{\partial t}+O(1)\right)\chi\\ &=-\left(\frac{1}{\delta\varepsilon}+O\left(\varepsilon^2\right)\frac{|s|}{\varepsilon^4}+O\left(\varepsilon^2\right)\frac{1}{\delta\varepsilon}\right)\chi'+O(1)\,\chi\\ &=-\left(\frac{1}{\delta\varepsilon}+O(1)\right)\chi'+O(1)\,\chi\\ &=-\left(\frac{1}{\delta\varepsilon}+O(1)\right)\chi'\quad\text{if }\chi=O(\chi')\\ &=-O\left(\frac{1}{\delta\varepsilon}\right)\chi'\quad\text{if }0<\varepsilon\le\varepsilon_{0,}\text{ where }\varepsilon_0,\delta>0\text{ is small.}\end{array}\tag{1.21}$$

When $\frac{1}{2}\varepsilon^2\le x\le\varepsilon^2$ and $|\xi|\le C_1\varepsilon|\eta|$,

$$\begin{array}{ll}-(H_p+k)\,\chi&=-\left(2\xi\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+O(x)\frac{\partial}{\partial s}+O(x)\frac{\partial}{\partial t}+\kappa\right)\chi\\ &=-\left(\xi O\left(\frac{1}{\varepsilon^2}\right)+\frac{1}{\delta\varepsilon}+O\left(\varepsilon^2\right)\frac{|s|}{\varepsilon^4}+O\left(\varepsilon^2\right)\frac{1}{\delta\varepsilon}\right)\chi'+O(1)\,\chi\\ &=-\left(O\left(\frac{1}{\varepsilon}\right)+\frac{1}{\delta\varepsilon}+O(1)\right)\chi'+O(1)\,\chi\\ &=-\left(O\left(\frac{1}{\varepsilon}\right)+\frac{1}{\delta\varepsilon}+O(1)\right)\chi'\quad\text{if }\chi=O(\chi')\\ &=-O\left(\frac{1}{\delta\varepsilon}\right)\chi'\quad\text{if }0<\varepsilon\le\varepsilon_{0,}\text{ where }\varepsilon_0,\delta>0\text{ is small.}\end{array}\tag{1.22}$$

The functions $g_\varepsilon(x,y,\xi,\eta)\,,h_\varepsilon(x,y,\xi,\eta)\in C^\infty\left([0,1[\times[-1,1]^3\times\mathbb{R}_\xi\times\mathbb{R}^3\backslash 0\right)$ defined by (1.20), satisfy, from (1.21) and (1.22), the following properties:

supp$g_\varepsilon \cup$supp$h_\varepsilon \subset$suppq_ε, and $g_\varepsilon, h_\varepsilon$ do not depend on ξ for $0 \leq x \leq \frac{1}{2}\varepsilon^2$.

By choosing χ decreasing, we have : in $W(\varepsilon)$, $g_\varepsilon \geq 0$ and $g_\varepsilon > 0$ if $q_\varepsilon \neq 0$.

By choosing adequately β and χ, we have : $\partial^\alpha g_\varepsilon = O\left(g_\varepsilon^{1 \backslash \rho}\right)$ uniformly in $W(\varepsilon)$.

An adequate choice of β' implies that : supp$h_\varepsilon \subset [0,1[\times L^-\left(\varepsilon, \varepsilon^2\right) \times \mathbb{R}_\xi$.

We keep going with the construction of $q_\varepsilon^{m,\lambda}(x,y,\eta) \in C^\infty\left([0,1[\times [-1,1]^3 \times \mathbb{R}^3 \backslash 0\right)$:

Let $m \in \mathbb{R}$, $1 \leq \lambda \leq \infty$, we generate the symbols (homogeneous of degree m if $\lambda = \infty$) $q_\varepsilon^{m,\lambda}(x,y,\eta)$, $g_\varepsilon^{m,\lambda}(x,y,\xi,\eta)$, $h_\varepsilon^{m,\lambda}(x,y,\xi,\eta)$ from $q_\varepsilon(x,y,\eta)$, $g_\varepsilon(x,y,\xi,\eta)$, $h_\varepsilon(x,y,\xi,\eta)$ ([MS1](p.603)). Also, the following properties will be satisfied:

$q_\varepsilon^{m,\infty}$ is homogeneous of degree m, and $q_\varepsilon^{m,\lambda}$ does not depend on x for $0 \leq x < \frac{1}{2}\varepsilon^2$.

supp$q_\varepsilon^{m,\lambda} \subset F^+(\varepsilon, \delta\varepsilon) \cup F^-\left(\varepsilon, \varepsilon^2\right)$, $q_\varepsilon^{m,\lambda}\left(0, e^{tH_{r_0}}(y_0,\eta_0)\right) \neq 0$ for $0 \leq t < \delta\varepsilon$.

$q_\varepsilon^{m,\lambda} \geq 0$, and $q_{\varepsilon'}^{m,\lambda} > 0$ in suppq_ε if $0 < \varepsilon < \varepsilon' \leq \varepsilon_0$.

$-(H_p + k)\, q_\varepsilon^{m,\lambda} = g_\varepsilon^{m+1,\lambda} + h_\varepsilon^{m+1,\lambda}$.

supp$g_\varepsilon^{m,\lambda} \cup$supp$h_\varepsilon^{m,\lambda} \subset$supp$q_\varepsilon^{m,\lambda}$, et $g_\varepsilon^{m,\lambda}, h_\varepsilon^{m,\lambda}$ do not depend on ξ for $0 \leq x \leq \frac{1}{2}\varepsilon^2$.

supp$h_\varepsilon^{m,\lambda} \subset [0,1[\times L^-\left(\varepsilon, \varepsilon^2\right) \times \mathbb{R}_\xi$.

In $W(\varepsilon)$, $g_\varepsilon^{m,\lambda} \geq 0$ and $g_\varepsilon^{m,\lambda} > 0$ if $q_\varepsilon \neq 0$.

$\partial^\alpha g_\varepsilon^{m,\lambda} = O\left(\left(g_\varepsilon^{m,\lambda}\right)^{1 \backslash \rho}\right)$ uniformly in $W(\varepsilon)$ where $\rho > 1$.

Let $a_\varepsilon^{m,\lambda} = \sqrt{g_\varepsilon^{2m,\lambda}}\,|_{W(\varepsilon)} \in C^\infty(W(\varepsilon))$ (because $\partial^\alpha g_\varepsilon^{m,\lambda} = O\left(\left(g_\varepsilon^{m,\lambda}\right)^{1 \backslash \rho}\right)$ uniformly in $W(\varepsilon)$ where $\rho > 1$ and $g_\varepsilon^{s,\lambda} \geq 0$). Then, $a_\varepsilon^{m,\lambda}$ does not depend on ξ for $0 \leq x \leq \frac{1}{2}\varepsilon^2$, and $a_\varepsilon^{m,\infty} \geq 0$ is homogeneous of degree m.

Let $\sigma_2(x,y,\xi,\eta) \in C^\infty\left([0,1[\times [-1,1]^3 \times \mathbb{R}^4 \backslash 0\right)$ be a well-chosen function ([MS1](p.604)) depending on ε and let $\sigma_1(x) \in C^\infty(\mathbb{R})$ be equal to 1 on $\left[0, \frac{1}{4}\varepsilon^2\right]$, null on $\left[\frac{1}{2}\varepsilon^2, +\infty\right[$, and such that

$$\sigma_1^2 + \sigma_2^2 = 1 \quad \text{near } V(\varepsilon) \subset W(\varepsilon) \ .$$

We introduce

$a_{\varepsilon,1}^{m,\lambda}(x,y,\eta) = \sigma_1(x)\, a_\varepsilon^{m,\lambda}(x,y,\xi,\eta)$ independent on ξ,

$a_{\varepsilon,2}^{m,\lambda}(x,y,\xi,\eta) = \sigma_2(x,y,\xi,\eta)\, a_\varepsilon^{m,\lambda}(x,y,\xi,\eta) \in C^\infty\left([0,1[\times [-1,1]^3 \times \mathbb{R}^4 \backslash 0\right)$,

$h_{\varepsilon,1}^{m,\lambda}(x,y,\eta) = \sigma_1^2(x)\, h_\varepsilon^{m,\lambda}(x,y,\xi,\eta)$ independent on ξ,

$$h_{\varepsilon,2}^{m,\lambda}(x,y,\xi,\eta)=\sigma_2^2(x,y,\xi,\eta)\,h_{\varepsilon}^{m,\lambda}(x,y,\xi,\eta)\in C^\infty\left([0,1[\times[-1,1]^3\times\mathbb{R}^4\backslash 0\right).$$

Then, we generate the pseudo-differential operators $Q_{\varepsilon}^{m,\lambda}$, $A_{\varepsilon,1}^{m,\lambda}$, $A_{\varepsilon,2}^{m,\lambda}$, $B_{\varepsilon,1}^{m,\lambda}$, $B_{\varepsilon,2}^{m,\lambda}$, respectively from the symbols $q_{\varepsilon}^{m,\lambda}$, $a_{\varepsilon,1}^{m,\lambda}$, $a_{\varepsilon,2}^{m,\lambda}$, $h_{\varepsilon,1}^{m,\lambda}$, $h_{\varepsilon,2}^{m,\lambda}$ in an adequate way ([MS1]($p.604$)).

From the relation $-(H_p+k)\,q_{\varepsilon}^{m,\lambda}=g_{\varepsilon}^{m+1,\lambda}+h_{\varepsilon}^{m+1,\lambda}$, we have that
$N_{\varepsilon}^{m+1,\lambda}=\frac{1}{i}\left(\left[P,Q_{\varepsilon}^{m,\lambda}\right]+(R^*-R)\,Q_{\varepsilon}^{m,\lambda}\right)$ is a pseudo-differential operator of order $(m+1)$,

$$\sigma_1^2(x)\,N_{\varepsilon}^{m+1,\lambda}=\left(A_{\varepsilon,1}^{(m+1)/2,\lambda}\right)^*\left(A_{\varepsilon,1}^{(m+1)/2,\lambda}\right)+B_{\varepsilon,1}^{m+1,\lambda}+C_{\varepsilon,1}^{m,\lambda}$$

$$\sigma_2^2(x,y,D_x,D_y)\,N_{\varepsilon}^{m+1,\lambda}=\left(A_{\varepsilon,2}^{(m+1)/2,\lambda}\right)^*\left(A_{\varepsilon,2}^{(m+1)/2,\lambda}\right)+B_{\varepsilon,2}^{m+1,\lambda}+C_{\varepsilon,2}^{m,\lambda}$$

where $C_{\varepsilon,1}^{s,\lambda}$ and $C_{\varepsilon,2}^{s,\lambda}$ are pseudo-differential operators of order m.

Thus

$$\begin{aligned}N_{\varepsilon}^{m+1,\lambda}&=\left(N_{\varepsilon}^{m+1,\lambda}\right)_{|V(\varepsilon)}+\left(N_{\varepsilon}^{m+1,\lambda}\right)_{|\left(suppq_{\varepsilon}^{m,\lambda}\times\mathbb{R}_\xi\right)\backslash V(\varepsilon)}\\&=\left(\sigma_1^2(x)+\sigma_2^2(x,y,D_x,D_y)\right)\left(N_{\varepsilon}^{m+1,\lambda}\right)_{|V(\varepsilon)}+\left(N_{\varepsilon}^{m+1,\lambda}\right)_{|\left(suppq_{\varepsilon}^{m,\lambda}\times\mathbb{R}_\xi\right)\backslash V(\varepsilon)}\\&=\left(\sigma_1^2(x)+\sigma_2^2(x,y,D_x,D_y)\right)\left(N_{\varepsilon}^{m+1,\lambda}\right)_{|V(\varepsilon)}+B_{\varepsilon,3}^{m+1,\lambda}\\&=\sum_{j=1,2}\left(A_{\varepsilon,j}^{(m+1)/2,\lambda}\right)^*\left(A_{\varepsilon,j}^{(m+1)/2,\lambda}\right)+\sum_{j=1,2,3}B_{\varepsilon,j}^{m+1,\lambda}+\sum_{j=1,2}C_{\varepsilon,j}^{m,\lambda}\end{aligned}$$

where $B_{\varepsilon,3}^{m+1,\lambda}=\left(N_{\varepsilon}^{m+1,\lambda}\right)_{|\left(suppq_{\varepsilon}^{m,\lambda}\times\mathbb{R}_\xi\right)\backslash V(\varepsilon)}$ is a pseudo-differential operator of order $(m+1)$ such that $B_{\varepsilon,3}^{m+1,\lambda}(E_i)\in C_0^\infty\left([0,1[\times[-1,1]^3\right)$ because $WF(E_i)\subset p^{-1}(0)$ and $\left(\left(suppq_{\varepsilon}^{m,\lambda}\times\mathbb{R}_\xi\right)\backslash V(\varepsilon)\right)\cap p^{-1}(0)=\emptyset$ from the estimate (1.19) on $r(x,y,\eta)$ in $F^{\pm}(\varepsilon,\tau)$. On another hand, under the assumption

$$\left\{(x,y,\xi,\eta)\in WF_b(E_i)\backslash 0\leq x<\varepsilon_1^2,\quad (y,\eta)\in L^-\left(\varepsilon_1,\varepsilon_1^2\right)\right\}=\emptyset\quad\text{for }0<\varepsilon_1\leq\varepsilon_0$$

we have by construction $\left|\int_0^\infty\int B_{\varepsilon,j}^{m+1,\lambda}(E_i)\,E_i\right|<\infty$ for $j=1,2$ and this uniformly with respect to λ. In order to bound the term $\left|\int_0^\infty\int C_{\varepsilon,j}^{m,\lambda}(E_i)\,E_i\right|$ and the second member of the inequality (1.17), we choose adequately the support of $\sigma_2(x,y,\xi,\eta)$ and we use an iterative process on the Sobolev regularity order m ([MS1]).

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In the case with the solution $B(s,y_1,y_2,t)=V(\Psi(s,y_1,y_2),t)=V(x_1,x_2,x_3,t)$, we get the following equality

$$\begin{aligned}\sum_{i=1}^{3}\int_0^\infty\int\left([P,Q]+(R^*-R)\,Q\right)B_i\overline{B_i}&=\sum_{i=1}^{2}\int\left([\partial_i(m_1+m_2),Q]\,B_i\overline{B_i}\right)_{s=0}\\&-\sum_{i=1}^{2}\int\left(\partial_i(m_1+m_2)\,[Q,n_i]\left(\tfrac{1}{\nu}B_3\right)\overline{B_i}\right)_{s=0}\\&-\sum_{i=1}^{2}\int\left([Q,n_i]\left(\partial_i(m_1+m_2)\,B_i\right)\overline{\tfrac{1}{\nu}B_3}\right)_{s=0}\\&+\int_0^\infty\int\left(-\left([P,[n,Q]]+\left(\widetilde{R}^*-R\right)[n,Q]\right)B\overline{\tfrac{1}{\nu}B_3}\right)\\&+\int_0^\infty\int\left(\left[\widetilde{P},[n,Q]\right]+\left(R^*-\widetilde{R}\right)[n,Q]\right)\left(\tfrac{1}{\nu}B_3\right)\overline{B}\\&+\int_0^\infty\int\left(\left[\widetilde{P},[n,Q]\,n\right]+\left(\widetilde{R}^*-\widetilde{R}\right)[n,Q]\,n\right)\left(\tfrac{1}{\nu}B_3\right)\overline{\tfrac{1}{\nu}B_3}.\end{aligned}\tag{1.23}$$

Using the construction of the rays done by R.Melrose and J.Sjöstrand ([MS2],(p.153)), we establish that $WF_b(U) = \bigcup_{i=1}^{3} WF_b(U_i) \subset \Sigma_b$ and, if $\rho \in WF_b(U)$, then $\gamma(s) \in WF_b(U)$ for all s where γ is the generalized bicharacteristic starting from ρ.

Recall that such result of propagation of singularities has been proved by K.Yamamoto [Y] by working with a matrix system of differential equation of first order.

Now, we deduce the propagation of the L^2_ρ regularity:

Let $\rho \in T^*(\partial\Omega \times I)\backslash 0$ such that $U \in L^2_\rho$. There exists a tangential pseudo-differential operator A , of degree, with support near ρ and microlocally equivalent to the identity in a neighborhood of ρ such that $AU \in \left(L^2(\Omega \times I)\right)^3$.

Let W be a solution of

$$\begin{cases} \left(\partial_t^2 - c_o\Delta\right)(W - AU) = 0 & \text{in } \Omega \times I \\ (W - AU) \wedge n = 0 & \text{on } \partial\Omega \times I \\ div\,(W - AU) = 0 & \text{on } \partial\Omega \times I \end{cases}$$

such that $W \in \left(L^2(\Omega \times I)\right)^3$.

We introduce $U = U_1 + U_2$ where $U_1 = W - AU$. Then we have $U_1 \in \left(L^2(\Omega \times I)\right)^3$ and U_2 solves

$$\begin{cases} \left(\partial_t^2 - c_o\Delta\right)U_2 = 0 & \text{in } \Omega \times I \\ U_2 \wedge n = 0 & \text{on } \partial\Omega \times I \\ divU_2 = 0 & \text{on } \partial\Omega \times I \end{cases}$$

with $\rho \notin WF_b(U_2)$. Consequently, we have $\rho' \notin WF_b(U_2)$ and $U \in L^2_\rho$ at any point $\rho' \in \gamma(I)$ of the generalized bicharacteristic crossing $\rho = \gamma(s_0)$.

The estimate of theorem 1.1 is deduced from the two following systems

$$\begin{cases} \left(\partial_t^2 - c_o\Delta\right)U_1 = 0 & \text{in } \Omega \times I \\ U_1 \wedge n = 0 & \text{on } \partial\Omega \times I \\ divU_1 = 0 & \text{on } \partial\Omega \times I \end{cases} \quad \text{and} \quad \begin{cases} \left(\partial_t^2 - c_o\Delta\right)U_2 = f & \text{in } \Omega \times I \\ divU_2 = 0,\ U_2 \wedge n = 0 & \text{on } \partial\Omega \times I \\ U_2(\cdot, s_0) = \partial_t U_2(\cdot, s_0) = 0 & \text{on } \Omega\ . \end{cases}$$

This ends the proof of theorem 1.1.

Now, we know that the propagation of the L^2 regularity follows the generalized bicharacyeristics in Σ_b. We complete this result by showing that the singularities are concentrated on Σ_b, from a theorem of elliptic regularity:

Theorem 1.2 Let (U, V) be a solution of

$$\begin{cases} \partial_t^2 U - c_o\Delta U = f & \text{in } \Omega \times I \\ U \wedge n = 0 & \text{on } \partial\Omega \times I \\ divU = 0 & \text{on } \partial\Omega \times I \end{cases} \quad \text{and} \quad \begin{cases} \partial_t^2 V - c_o\Delta V = f & \text{in } \Omega \times I \\ V.n = 0 & \text{on } \partial\Omega \times I \\ curlV \wedge n = 0 & \text{on } \partial\Omega \times I \end{cases}$$

where $I = [s_0; s_1] \subset \mathbb{R}$, $f \in \left(L^2(\Omega \times I)\right)^3$. Then we have $(U, V) \in H^1_\rho$ at any point $\rho \notin \Sigma_b$.

Moreover, we have the following estimates: $\forall \rho \notin \Sigma_b \quad \exists c > 0$

$$\|U\|_{H^1_\rho} \le c\left(\|U\|_{L^2(\Omega\times I)} + \|f\|_{L^2(\Omega\times I)}\right) \tag{1.24}$$

$$\|V\|_{H^1_\rho} \le c\left(\|V\|_{L^2(\Omega\times I)} + \|f\|_{L^2(\Omega\times I)}\right)\ . \tag{1.25}$$

Proof of theorem 1.2.- The proof of theorem 1.2 follows the techniques to solve elliptic problems. We give the proof for the solution U.

Let $\rho \in \left\{(x,t,\xi,\tau) \in T^*(\omega \times I)\backslash 0\,; -\tau^2 + c_o|\xi|^2 \neq 0\right\}$ where ω is an open set strictly included in Ω. We choose $\Phi \in C_0^\infty(\omega \times I)$ and Θ a homogeneous function of degree 0, localized in a conical neighborhood of (ξ,τ). By homogeneity of the symbol $p(x,t,\xi,\tau) = -\tau^2 + c_o|\xi|^2$, we have

$$\Theta p(x,t,\xi,\tau) \neq 0 \quad \Longrightarrow \exists c, R > 0 \quad \forall |\xi,\tau| > R \quad \Theta^2 |p(x,t,\xi,\tau)| \geq c\Theta^2 |\xi,\tau|^2 \ .$$

We conclude that $\exists c > 0$

$$\begin{aligned} \int_{\mathbb{R}^4} \Theta^2 \left|\widehat{\Phi U}\right|^2 \left(1 + |\xi,\tau|^2\right) d\xi d\tau &\leq c \left\|\Theta\widehat{\Phi U}\right\|^2_{L^2(\mathbb{R}^4)} + \int_{|\xi,\tau|>R} \Theta^2 \left|\widehat{\Phi U}\right|^2 \left(1 + |\xi,\tau|^2\right) d\xi d\tau \\ &\leq c\left(\|U\|^2_{L^2_\rho} + \|\Phi P(U)\|^2_{H^{-1}(\mathbb{R}^4)}\right) \ . \end{aligned}$$

In a neighborhood of the boundary, a change of coordinates reduces to the case of the half-space. The principal symbol of P becomes

$$p(s,y_1,y_2,t,\varsigma,\zeta_1,\zeta_2,\tau) = c_o\varsigma^2 + c_o h(s,y_1,y_2,\zeta_1,\zeta_2) - \tau^2 \ .$$

We will take the notations given previously for the change of coordinates Ψ (1.1). Let $E(s,y_1,y_2,t) = U(\Psi(s,y_1,y_2),t) = U(x_1,x_2,x_3,t)$, we recall that the boundary conditions $U \wedge n = 0$ and $divU = 0$ on the boundary imply (1.5). We write E in the form $E = n \wedge F + Dn$ where $F = (E \wedge n)$ and $D = (E.n)$.

Let $\rho \in \left\{(s,y_1,y_2,t,\varsigma,\zeta_1,\zeta_2,\tau) \in T^*\left(\mathbb{R}^+ \times \mathbb{R}^3\right)\backslash 0\,; -\tau^2 + c_o h(0,y_1,y_2,\zeta_1,\zeta_2) > 0\right\}$ and Ξ be a homogeneous function of degree 0, localized in a conical neighborhood of $(\zeta_1,\zeta_2,\tau) = (\zeta',\tau)$. We introduce the pseudo-differential operators R and L with principal symbol $-\tau^2 + c_o h(0,y_1,y_2,\zeta_1,\zeta_2)$ and $\Xi(\zeta_1,\zeta_2,\tau)$ respectively. By Garding inequality, we have that $\exists c > 0$

$$\begin{aligned} &\tfrac{1}{c}\int_{\mathbb{R}^3} \left(1 + |(\zeta',\tau)|^2\right) \Xi^2(\zeta',\tau) \left|\widehat{\Phi D}'(\zeta',\tau)\right|^2 d\zeta' d\tau \\ &\quad \leq c\|D\|^2_{L^2_{loc}(\mathbb{R}^3)} + ((L\Phi R)D, L\Phi D)_{L^2(\mathbb{R}^3)} \ . \end{aligned} \tag{1.26}$$

Integrating (1.26) over the variable s, we get

$$\begin{aligned} &\int_0^1 ds \int_{\mathbb{R}^3} \left(1 + |\zeta',\tau|^2\right) \Xi^2 \left|\widehat{\Phi D}'\right|^2 d\zeta' d\tau + \int_0^1 \left(-\partial_s^2 (L\Phi D), L\Phi D\right)_{L^2(\mathbb{R}^3)} ds \\ &\quad \leq c\|E\|^2_{L^2\left([0,1[;L^2_{loc}(\mathbb{R}^3)\right)} + c\int_0^1 (L\Phi[R,n]E, L\Phi D)_{L^2(\mathbb{R}^3)} ds + \|f\|^2_{L^2(\Omega\times I)} \end{aligned}$$

where $L\Phi[R,n]$ is a pseudo-differential operator of order 1.

By integration by parts, we have

$$\begin{aligned} \int_0^1 \left(-\partial_s^2(L\Phi D), L\Phi D\right)_{L^2(\mathbb{R}^3)} ds &= \tfrac{1}{(2\pi)^3}\int_0^1 \int_{\mathbb{R}^3} \Xi^2 \left|\partial_s \widehat{\Phi D}'\right|^2 d\zeta' d\tau ds \\ &\quad + \left(\partial_s L\Phi D_{|s=0}, L\Phi D_{|s=0}\right)_{L^2(\mathbb{R}^3)} \ . \end{aligned}$$

On another hand, the trace theorem and our boundary condition imply that $\exists c > 0 \quad \exists \alpha \in]0,1[$

$$-\left(L\Phi\partial_s D_{|s=0}, L\Phi D_{|s=0}\right)_{L^2(\mathbb{R}^3)} \leq c\|L\Phi D\|^\alpha_{H^1([0,1[\times\mathbb{R}^3)} \|E\|^{1-\alpha}_{L^2\left([0,1[;L^2_{loc}(\mathbb{R}^3)\right)} \ .$$

Finally, $\exists c > 0$

$$\begin{aligned} &\int_0^1 ds \int_{\mathbb{R}^3} \left(1 + |\zeta',\tau|^2\right) \Xi^2 \left|\widehat{\Phi D}'\right|^2 d\zeta' d\tau + \int_0^1 \int_{\mathbb{R}^3} \Xi^2 \left|\partial_s \widehat{\Phi D}'\right|^2 d\zeta' d\tau ds \\ &\quad \leq c\|E\|^2_{L^2\left([0,1[;L^2_{loc}(\mathbb{R}^3)\right)} + c\|f\|^2_{L^2(\Omega\times I)} \end{aligned} \tag{1.27}$$

and also

$$\begin{array}{l}\int_0^1 ds \int_{\mathbb{R}^3} \left(1+|\zeta',\tau|^2\right) \Xi^2 \left|\widehat{\Phi F}'\right|^2 d\zeta' d\tau + \int_0^1 \int_{\mathbb{R}^3} \Xi^2 \left|\partial_s \widehat{\Phi F}'\right|^2 d\zeta' d\tau ds \\ \quad \leq c\,\|E\|^2_{L^2\left([0,1[;L^2_{loc}(\mathbb{R}^3)\right)} + c\,\|f\|^2_{L^2(\Omega\times I)} \ .\end{array} \tag{1.28}$$

Consequently, (1.27) and (1.28) imply

$$\begin{array}{l}\int_0^1 ds \int_{\mathbb{R}^3} \left(1+|\zeta',\tau|^2\right) \Xi^2 \left|\widehat{\Phi E}'\right|^2 d\zeta' d\tau + \int_0^1 \int_{\mathbb{R}^3} \Xi^2 \left|\partial_s \widehat{\Phi E}'\right|^2 d\zeta' d\tau ds \\ \quad \leq c\,\|U\|^2_{L^2(\Omega\times I)} + c\,\|f\|^2_{L^2(\Omega\times I)} \ .\end{array}$$

This concludes the proof for the solution $U(x_1,x_2,x_3,t)$. The proof for the solution $V(x_1,x_2,x_3,t) = V(\Psi(s,y_1,y_2),t) = B(s,y_1,y_2,t)$ follows the same strategy with the boundary condition (1.9).

From theorem 1.2 and by interpolation, we have the following result.

Theorem 1.3 Let (U,V) be a solution of

$$\left\{\begin{array}{c}\partial_t^2 U - c_o \Delta U = 0 \quad \text{in } \Omega\times I \\ U\wedge n = 0 \quad \text{on } \partial\Omega\times I \\ divU = 0 \quad \text{on } \partial\Omega\times I\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{c}\partial_t^2 V - c_o \Delta V = 0 \quad \text{in } \Omega\times I \\ V.n = 0 \quad \text{on } \partial\Omega\times I \\ curlV\wedge n = 0 \quad \text{on } \partial\Omega\times I\end{array}\right.$$

where $I = [s_0; s_1] \subset \mathbb{R}$. Then, we have the following estimates $\quad \forall\rho\notin\Sigma_b \quad \exists c>0$

$$\|U\|_{L^2_\rho} \leq c\,\|U\|^{1/2}_{L^2(\Omega\times I)}\,\|U\|^{1/2}_{H^{-1}(\Omega\times I)} \tag{1.29}$$

$$\|V\|_{L^2_\rho} \leq c\,\|V\|^{1/2}_{L^2(\Omega\times I)}\,\|V\|^{1/2}_{H^{-1}(\Omega\times I)} \ . \tag{1.30}$$

1.1 Applications : Observation for waves

Here, we establish observability from geometrical conditions.

Definition 1.1 Let ω be an open set of $\mathbb{R}^3$ and $T_c > 0$. We say that ω geometrically controls Ω if there exists a compact set $\widetilde{\omega}$ in ω such that any ray meets $\left(\widetilde{\omega}\cap\overline{\Omega}\right)\times\,]0;T_c[$.

We need the following observability result.

Theorem 1.4 Let ω be an open set of $\mathbb{R}^3$ and $T_c>0$ such that ω geometrically controls Ω. Let U be a solution of

$$\left\{\begin{array}{c}\partial_t^2 U - c_o \Delta U = f \quad \text{in } \Omega\times\,]0,T[\\ U\wedge n = 0 \quad \text{on } \partial\Omega\times\,]0,T[\\ divU = 0 \quad \text{on } \partial\Omega\times\,]0,T[\end{array}\right.$$

where $T > T_c$, $f\in\left(H^{-1}\left(\Omega\times\,]0,T[\right)\right)^3$. Then, we have $\quad \exists c>0 \quad \exists\widehat{\omega}\subset\subset\omega$

$$\|U\|_{L^2(\Omega\times]0,T[)} \leq c\left(\|U\|_{L^2((\widehat{\omega}\cap\Omega)\times]0,T[)} + \|f\|_{H^{-1}(\Omega\times]0,T[)}\right) \ . \tag{1.31}$$

Proof of theorem 1.4.- The proof of theorem 1.4 comes from theorem 1.1 and 1.3 in the following way.

We write U as follows $U = W + V$ where W and V are solutions of the hyperbolic systems

$$\left\{\begin{array}{c}\left(\partial_t^2 - c_o\Delta\right) W = 0 \quad \text{in } \Omega \times]0,T[\\ W \wedge n = 0 \quad \text{on } \partial\Omega \times]0,T[\\ divW = 0 \quad \text{on } \partial\Omega \times]0,T[\end{array}\right. \tag{1.32}$$

$$\left\{\begin{array}{c}\left(\partial_t^2 - c_o\Delta\right) V = f \quad \text{in } \Omega \times]0,T[\\ divV = 0,\ V \wedge n = 0 \quad \text{on } \partial\Omega \times]0,T[\\ V(\cdot,0) = \partial_t V(\cdot,0) \quad \text{in } \Omega\ .\end{array}\right.$$

The solution V satisfies

$$\exists c > 0 \qquad \|V\|_{L^2(\Omega\times]0,T[)} \leq c\,\|f\|_{H^{-1}(\Omega\times]0,T[)}\ . \tag{1.33}$$

Let $\epsilon = \frac{1}{2}(T - T_c) > 0$. We will show that $\exists c, d > 0$

$$\|W\|_{L^2(\Omega\times]T-\epsilon,T[)} \leq c\,\|W\|_{L^2((\widehat{\omega}\cap\Omega)\times]0,T[)} + d\,\|W\|_{H^{-1}(\Omega\times]0,T[)}\ . \tag{1.34}$$

As $\overline{\Omega}$ is a compact set, there exists a finite family $\{\mathcal{O}_i\}_{i=0,..,m}$ of bounded open set recovering $\overline{\Omega}$ such that $\bigcup_{i=1}^{m} \mathcal{O}_i \supset \partial\Omega$ and for any $i = 1,..,m$ there exists a diffeomorphism Ψ_i from $\mathcal{Q} =]-1,1[^3$ to $\mathcal{O}_i$ such that

$$\Psi_i^{-1}(\mathcal{O}_i \cap \Omega) = \{(s, y_1, y_2) \in \mathcal{Q} \backslash s > 0\}\,, \quad \Psi_i^{-1}(\mathcal{O}_i \cap \partial\Omega) = \{(s, y_1, y_2) \in \mathcal{Q} \backslash s = 0\}\ .$$

Let $\{\alpha_i\}_{i=0,..,m}$ be a partition of unity by $\left\{\mathcal{O}_i \cap \overline{\Omega}\right\}_{i=0,..,m}$ of $\overline{\Omega}$.

$$\alpha_0 \in C_0^\infty(\Omega)\,, \quad \alpha_i \in C_0^\infty\left(\mathbb{R}^3\right), \quad \text{supp}(\alpha_i) \subset \mathcal{O}_i \cap \overline{\Omega}\ .$$

Let $\chi \in C_0^\infty(]T_c, T + \epsilon[)$, $\chi = 1$ in $[T - \epsilon, T]$. For all $(x,t) \in \Omega \times]T - \epsilon, T[$, $W(x,t)$ can be written as follows

$$W = \chi\alpha_0 W + \sum_{i=1}^{m} \chi\alpha_i W = \Phi_0 W + \sum_{i=1}^{m} \Phi_i W\ .$$

Thus, we obtain

$$\|W\|^2_{L^2(\Omega\times]T-\epsilon,T[)} \leq c\left(\int_{\mathbb{R}^4}\left|\widehat{\Phi_0 W}\right|^2 d\xi' d\tau + \sum_{i=1}^{m}\int_0^1 ds \int_{\mathbb{R}^3}\left|\widehat{\Phi_i W \circ \Psi_i}'\right|^2 d\zeta' d\tau\right)\ .$$

Now, we decompose $T^*\left(\overline{\Omega} \times]T - \epsilon, T[\right) \backslash 0$ with respect to Σ_b as follows

$$\|W\|^2_{L^2(\Omega\times]T-\epsilon,T[)} \leq c\left(\sum_{j=0}^{m}\|W\|^2_{L^2_{\rho_j}} + \sum_{j=0}^{m}\|W\|^2_{L^2_{\rho'_j}}\right) \tag{1.35}$$

where $\rho_j = \left\{(x_j, t, \xi, \tau) \in \Sigma_b \cap T^*\left(\overline{\Omega} \times]T - \epsilon, T[\right) \backslash 0\,; x_j \in \mathcal{O}_j\right\}$
and $\rho'_j = \left\{(x_j, t, \xi, \tau) \in \left(T^*\left(\overline{\Omega} \times]T - \epsilon, T[\right) \backslash 0\right) \backslash \Sigma_b\,; x_j \in \mathcal{O}_j\right\}$.

In a neighborhood of $p(x, t, \xi, \tau) \neq 0$, we apply the elliptic regularity theorem 1.3. In a neighborhood of Σ_b, we apply the theorem 1.1 of propagation of singularities with the geometric control condition. It implies that : $\exists c, d > 0 \quad \exists\left(\widetilde{\omega} \cap \overline{\Omega}\right) \subset \widehat{\omega} \subset\subset \omega$

$$\|W\|_{L^2(\Omega\times]T-\epsilon,T[)} \leq c\,\|\Phi W\|_{L^2(\Omega\times]0,T[)} + d\,\|W\|_{H^{-1}(\Omega\times]0,T[)} \tag{1.36}$$

where $\Phi \in C_0^\infty\left(\widehat{\omega} \times]0,T[\right)$, $\Phi = 1$ in $\left(\widetilde{\omega} \cap \overline{\Omega}\right) \times]\epsilon, T-\epsilon[$.

By conservation of the energy of W, we finally get $\quad \exists c, d > 0$

$$\|W\|_{L^2(\Omega\times]0,T[)} \leq c\,\|\Phi W\|_{L^2(\Omega\times]0,T[)} + d\,\|W\|_{H^{-1}(\Omega\times]0,T[)} \ .$$

By a contradiction argument, we show that $\quad \|W\|_{H^{-1}(\Omega\times]0,T[)} \leq c\,\|W\|_{L^2((\widehat{\omega}\cap\Omega)\times]0,T[)}$.

Indeed, if there exists a sequence (W_n) such that

$$\|W_n\|_{H^{-1}(\Omega\times]0,T[)} = 1 \text{ and } \lim_{n\to+\infty} \|W_n\|_{L^2((\widehat{\omega}\cap\Omega)\times]0,T[)} = 0$$

then, by compact injection of $L^2\left(\Omega \times]0,T[\right)$ in $H^{-1}\left(\Omega \times]0,T[\right)$, there exists $W \in \left(L^2\left(\Omega \times]0,T[\right)\right)^3$ solution of the hyperbolic system (1.32) such that

$$\|W\|_{H^{-1}(\Omega\times]0,T[)} = 1 \text{ and } W = 0 \quad \text{in } \left(\widehat{\omega} \cap \Omega\right) \times]0,T[\ . \tag{1.37}$$

Let us introduce $\mathcal{N}$ the space of solutions of the hyperbolic system (1.32) such that (1.37) holds. This is a nonempty closed set of $L^2\left(\Omega \times]0,T[\right)$ included in $H^1\left(\Omega \times]0,T[\right)$ because we have $\|\partial_t W\|_{L^2(\Omega\times]0,T[)} \leq d\,\|\partial_t W\|_{H^{-1}(\Omega\times]0,T[)}$, and W solves the elliptic system

$$\begin{cases} \Delta W \in \left(H^1(\Omega)\right)' \\ divW = 0,\ W \wedge n = 0 \quad \text{on } \partial\Omega \ . \end{cases}$$

By compact injection from $H^1\left(\Omega \times]0,T[\right)$ in $L^2\left(\Omega \times]0,T[\right)$, we deduce that $\mathcal{N}$ is a space of finite dimension.

As ∂_t is a linear application from $\mathcal{N}$ to $\mathcal{N}$, there exist λ and $W \in \mathcal{N}$ such that $\partial_t W = \lambda W$. We conclude that W solves

$$\begin{cases} \lambda^2 W - c_o \Delta W = 0 \quad \text{in } \Omega \\ W = 0 \quad \text{in } \left(\widehat{\omega} \cap \Omega\right) \ . \end{cases}$$

Consequently, $W \equiv 0$, which contradicts $\|W\| = 1$.

Conclusion

$$\exists c > 0 \qquad \|W\|_{L^2(\Omega\times]0,T[)} \leq c\,\|\Phi W\|_{L^2((\widehat{\omega}\cap\Omega)\times]0,T[)} \tag{1.38}$$

This ends the proof of theorem 1.4.

The study of the boundary observability is done with the following definitions.

Definition 1.2 We say that a point $\rho \in T^*\left(\partial\Omega \times \mathbb{R}\right)\backslash 0$ is non-diffractive if $\rho \in \mathcal{H}$ or, if $\rho \in \mathcal{G}$ and if $\widetilde{\rho} \in \mathcal{C}$ is the unique point such that $\rho = \widetilde{\rho}$ (in the change of coordinates), the bicharacteristic associated to H_p such that $\gamma(0) = \widetilde{\rho}$ satisfies for all $\epsilon > 0$ there is $-\epsilon < s < \epsilon$ such $\gamma(s) \notin \left(T^*\left(\mathbb{R}^4\right)\backslash 0\right)_{|\overline{\Omega}\times\mathbb{R}}$

Definition 1.3 Let Γ be an open subset of $\partial\Omega$ and $T_c > 0$. We say that Γ geometrically controls Ω if there exists a compact set $\widetilde{\Gamma}$ in Γ such that any ray meets $\widetilde{\Gamma} \times]0;T_c[$ in a non-diffractive point.

We need the following observability result.

Theorem 1.5 Let Γ be an open subset of $\partial\Omega$ and $T_c > 0$ such that Γ geometrically controls Ω. Let $T > T_c$ and V be a solution of

$$\begin{cases} \partial_t^2 V - c_o \Delta V = 0 \quad \text{in } \Omega \times]0,T[\\ V.n = 0, \quad curl V \wedge n = 0 \quad \text{on } (\partial\Omega \setminus \Gamma) \times]0,T[\end{cases},$$

such that $V_{|\partial\Omega} \in \left(L^2(\Gamma \times]0,T[)\right)^3$ and $\partial_n V \in \left(H^{-1}(\Gamma \times]0,T[)\right)^3$. Then we have $\exists c > 0$

$$\|V\|_{L^2(\Omega\times]0,T[)} \leq c\left(\|\partial_n V\|_{H^{-1}(\Gamma\times]0,T[)} + \|V\|_{L^2(\Gamma\times]0,T[)}\right) . \tag{1.39}$$

Proof of theorem 1.5.- The proof of theorem 1.5 comes from a lifting lemma [BLR] which is recall here.

Lemma 1.1 Let U be an extendible distribution in $\Omega \times \mathbb{R}$ solution of $P(U) = 0$ and $\rho \in T^*(\partial\Omega \times \mathbb{R}) \setminus 0$ is a non-diffractive point. If $U_{|\partial\Omega\times\mathbb{R}} \in L^2_\rho$ and $\partial_n U_{|\partial\Omega\times\mathbb{R}} \in H^{-1}_\rho$, then $U \in L^2_\rho$.

Using same strategy than in the proof of theorem 1.4, the geometric condition of theorem 1.5 implies that $\exists c, d > 0$

$$\|V\|_{L^2(\Omega\times]0,T[)} \leq c\|V\|_{L^2_\rho} + d\|V\|_{H^{-1}(\Omega\times]0,T[)}$$

where $\rho \in T^*(\Gamma \times]\epsilon, T-\epsilon[) \setminus 0$ is a non-diffractive point. From Lemma 1.1, we deduce that $\exists c, d > 0$

$$\|V\|_{L^2(\Omega\times]0,T[)} \leq c\left(\|\partial_n V\|_{H^{-1}(\Gamma\times]0,T[)} + \|V\|_{L^2(\Gamma\times]0,T[)}\right) + d\|V\|_{H^{-1}(\Omega\times]0,T[)} .$$

By a contradiction argument, we show that $\|V\|_{H^{-1}(\Omega\times]0,T[)} \leq c\left(\|\partial_n V\|_{H^{-1}(\Gamma\times]0,T[)} + \|V\|_{L^2(\Gamma\times]0,T[)}\right)$. Indeed, if there exists a sequence (V_n) such that

$$\|V_n\|_{H^{-1}(\Omega\times]0,T[)} = 1 \text{ and } \lim_{n\to+\infty} \|V_n\|_{L^2(\Gamma\times]0,T[)} = \lim_{n\to+\infty} \|\partial_n V_n\|_{H^{-1}(\Gamma\times]0,T[)} = 0$$

then, by compact injection of $L^2(\Omega \times]0,T[)$ in $H^{-1}(\Omega \times]0,T[)$, there exists $V \in \left(L^2(\Omega \times]0,T[)\right)^3$ solution of the hyperbolic system $\left(\partial_t^2 - c_o\Delta\right) V = 0$ in $\Omega \times]0,T[$ such that

$$\|V\|_{H^{-1}(\Omega\times]0,T[)} = 1 \text{ and } V = \partial_n V = 0 \quad \text{on } \Gamma \times]0,T[.$$

From Holmgren theorem, we conclude that $V \equiv 0$. This contradicts the fact that $\|V\| = 1$.

Conclusion

$$\exists c > 0 \qquad \|V\|_{L^2(\Omega\times]0,T[)} \leq c\left(\|\partial_n V\|_{H^{-1}(\Gamma\times]0,T[)} + \|V\|_{L^2(\Gamma\times]0,T[)}\right) .$$

This ends the proof of theorem 1.5.

2 Functional framework for the Maxwell equations

In this section, we define the functional spaces naturally used for electromagnetism ([Ba], [C], [DL]). We will make the following geometrical assumptions ([DL]$(5, p.252)$).

- Let Γ_0,..,Γ_m be the connected components of $\partial\Omega$.

- Let Σ_1,..,Σ_N be the cuttings of Ω so that the domain $\Omega \backslash \cup \Sigma_j$ is simply connected.

We begin to recall the cohomology spaces $\mathbb{H}_1(\Omega)$ and $\mathbb{H}_2(\Omega)$.

$$\mathbb{H}_1(\Omega) = \left\{ g \in \left(L^2(\Omega)\right)^3 \setminus \; divg = 0,\; g.n_{|\partial\Omega} = 0,\; curlg = 0 \right\}$$

$$\begin{array}{rl} \mathbb{H}_2(\Omega) & = \left\{ f \in \left(L^2(\Omega)\right)^3 \setminus \; divf = 0,\; curlf = 0,\; f \wedge n_{|\partial\Omega} = 0 \right\} \\ & = \left\{ f = \nabla\varphi \setminus \; \varphi \in H^1(\Omega),\; \Delta\varphi = 0,\; \varphi_{|\Gamma_i} = constant \text{ for } i = 0 \text{ to } m \right\} \; . \end{array}$$

Thus, we get the following particular cases.

If Ω is simply connected, then $\mathbb{H}_1(\Omega) \equiv \{0\}$.
If $\partial\Omega$ has only one connected component, then $\mathbb{H}_2(\Omega) \equiv \{0\}$.

Let us introduce $\mathcal{M}_E$ (resp. $\mathcal{M}_H$) the orthogonal space to $\mathbb{H}_2(\Omega)$ (resp. $\mathbb{H}_1(\Omega)$) for the $\left(L^2(\Omega)\right)^3$ norm. Moreover, we set $\mathcal{S}_E = \mathcal{M}_E \times \left(L^2(\Omega)\right)^3$ and $\mathcal{S}_H = \left(L^2(\Omega)\right)^3 \times \mathcal{M}_H$. Then, we have $\mathcal{S}_E = \mathcal{S}_H = \left(L^2(\Omega)\right)^6$, if the hypothesis of the Poincaré lemma hold (i-e Ω is simply connected and $\partial\Omega$ has only one connected component).

2.1 Existence and uniqueness

We recall the main well-posedness result.

Let $\mathcal{A}_o$ be an unbounded operator on the Hilbert space $\mathcal{H}_o = \left(L^2(\Omega)\right)^6$ with domain $D(\mathcal{A}_o)$, defined as follows.

$$\begin{array}{l} \|(f,g)\|^2_{\mathcal{H}_o} = \varepsilon \|f\|^2_{L^2(\Omega)} + \mu \|g\|^2_{L^2(\Omega)} \\ \mathcal{A}_o = \begin{pmatrix} 0 & -\varepsilon^{-1} curl \\ \mu^{-1} curl & 0 \end{pmatrix} \\ D(\mathcal{A}_o) = \left\{ (f,g) \in \mathcal{H}_o \setminus \; (curlf, curlg) \in \mathcal{H}_o,\; f \wedge n_{|\partial\Omega} = 0 \right\} \; . \end{array}$$

We recall that $-\mathcal{A}_o$ is the infinitesimal generator of an unitary group in $\mathcal{H}_o$ of class $\mathcal{C}^o$ ([DL]$(8, p.519)$ or [C]$(p.255)$). By application of Hille-Yosida theorem, we have the following well-posedness result.

$$\forall (E_o, H_o) \in D(\mathcal{A}_o) \quad \exists! \, (E,H) \in C^0(\mathbb{R}, D(\mathcal{A}_o)) \cap C^1(\mathbb{R}, D(\mathcal{H}_o)) \quad \text{solution of the system}$$

$$\left\{ \begin{array}{c} \varepsilon\partial_t E - curlH = 0,\; \mu\partial_t H + curlE = 0 \quad \text{in } \Omega \times \mathbb{R} \\ E \wedge n = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \\ E(\cdot,0) = E_o,\; H(\cdot,0) = H_o \quad \text{in } \Omega \; . \end{array} \right.$$

On another hand, let

$$\mathcal{V} = \left\{ f \in \left(L^2(\Omega)\right)^3 \setminus \; divf = 0 \right\} \times \left\{ g \in \left(L^2(\Omega)\right)^3 \setminus \; divg = 0, \; g.n_{|\partial\Omega} = 0 \right\}$$

$$\mathcal{W} = \Big\{ (f,g) \in \left(L^2(\Omega)\right)^3 \times \left(L^2(\Omega)\right)^3 \setminus \; curlf \in \left(L^2(\Omega)\right)^3, \; f \wedge n = 0, \; divf = 0, \; divg = 0, \; g.n_{|\partial\Omega} = 0, \; curlg \in \left(L^2(\Omega)\right)^3 \Big\} \; .$$

Noticing that the space $\mathcal{V}$ is stable by the semi-group, we get the following well-posedness result.

$$\forall (E_o, H_o) \in \mathcal{W} \quad \exists! \, (E,H) \in C^0(\mathbb{R}, \mathcal{W}) \cap C^1(\mathbb{R}, \mathcal{V}) \quad \text{solution of the system}$$

$$\left\{ \begin{array}{c} \varepsilon \partial_t E - curlH = 0, \; \mu \partial_t H + curlE = 0 \quad \text{in } \Omega \times \mathbb{R} \\ div(\varepsilon E) = 0, \; div(\mu H) = 0 \quad \text{in } \Omega \times \mathbb{R} \\ E \wedge n = 0, \; H.n = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \\ E(\cdot, 0) = E_o, \; H(\cdot, 0) = H_o \quad \text{in } \Omega \; . \end{array} \right. \tag{2.1}$$

Moreover, posing $\mathcal{E}(t) = \frac{1}{2} \int_\Omega \left(\varepsilon |E|^2 + \mu |H|^2 \right)$ the energy of the system (2.1), we have

$$\forall (E_o, H_o) \in \mathcal{W} \qquad \frac{d}{dt} \mathcal{E}(t) = 0 \; .$$

This last equality imply that the energy is a constant function of time, $\mathcal{E}(t) = \mathcal{E}(0) \quad \forall t \in \mathbb{R}$.

We remark that the spaces $\mathcal{S}_E$ and $\mathcal{S}_H$ are stable for the system (2.1). Indeed, it is enough to multiply the equation $\varepsilon \partial_t E - curlH = 0$ (resp. $\mu \partial_t H + curlE = 0$) by $h_2 \in \mathbb{H}_2(\Omega)$ (resp. $h_1 \in \mathbb{H}_1(\Omega)$) and then to integrate by parts over Ω to check it. Consequently, we obtain the following well-posedness results.

$$\forall (E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_E \quad \exists! \, (E,H) \in C^0(\mathbb{R}, \mathcal{W} \cap \mathcal{S}_E) \cap C^1(\mathbb{R}, \mathcal{V} \cap \mathcal{S}_E) \quad \text{solution of the system (2.1).}$$

$$\forall (E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_H \quad \exists! \, (E,H) \in C^0(\mathbb{R}, \mathcal{W} \cap \mathcal{S}_H) \cap C^1(\mathbb{R}, \mathcal{V} \cap \mathcal{S}_H) \quad \text{solution of the system (2.1).}$$

2.2 Orthogonal decompositions

In the framework of electromagnetic problems, it is natural to work with the scalar potential and the vector potential associated to the electromagnetic field.

We need the following decomposition results for the electromagnetic field.

Lemma 2.1

$$\forall (E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_E \quad \exists! \, (h_1, A) \in C^1(\mathbb{R}, \mathbb{H}_1(\Omega)) \times C^2\left(\mathbb{R}, \left(H^1(\Omega)\right)^3\right) \quad \text{such that}$$

$$\left\{ \begin{array}{l} \quad (E,H) \text{ solution of the system (2.1)} \\ \varepsilon E = curlA \\ H = \partial_t A + h_1 \\ divA = 0, \; A.n_{|\partial\Omega} = 0, \; \int_{\Sigma_i} A.n d\Gamma = 0 \quad \text{for } i = 0 \text{ to } N \; . \end{array} \right.$$

Moreover, we have the following relations

$$\|H\|^2_{L^2(\Omega)} = \|\partial_t A\|^2_{L^2(\Omega)} + \|h_1\|^2_{L^2(\Omega)} \; , \tag{2.2}$$

$$\exists c>0 \qquad \|A\|^2_{L^2(\Omega)} \leq c\,\|curlA\|^2_{L^2(\Omega)} \ . \tag{2.3}$$

Lemma 2.2
$\forall\,(E_o,H_o)\in\mathcal{W}\cap\mathcal{S}_H \quad \exists!\,(h_2,A)\in C^1(\mathbb{R},\mathbb{H}_2(\Omega))\times C^2\left(\mathbb{R},\left(H^1(\Omega)\right)^3\right)$ such that

$$\begin{cases} \quad (E,H) \text{ solution of the system (2.1)} \\ E=-\partial_t A+h_2 \\ \mu H = curlA \\ divA=0,\ A\wedge n_{|\partial\Omega}=0,\ \int_{\Gamma_i} A.nd\Gamma=0 \quad \text{for } i=0 \text{ to } m\ . \end{cases}$$

Moreover, we have the following relations

$$\|E\|^2_{L^2(\Omega)} = \|\partial_t A\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Omega)}\ , \tag{2.4}$$

$$\exists c>0 \qquad \|A\|^2_{L^2(\Omega)} \leq c\,\|curlA\|^2_{L^2(\Omega)} \ . \tag{2.5}$$

Proof of lemma 2.1.- The electric field $E\in\left(L^2(\Omega)\right)^3$ can be written in an unique way as follows

$$\varepsilon E=\nabla p+h_2+curlA, \text{ where } p\in H^1_0(\Omega)\,,\ h_2\in\mathbb{H}_2(\Omega)\,,\ A\in\left(H^1(\Omega)\right)^3$$

such that $divA=0,\ A.n_{|\partial\Omega}=0,\ \int_{\Sigma_i} A.nd\Gamma=0 \quad i=0$ to N ([C](p.55)).

But $divE=0$, and consequently $\Delta p=0$, which implies that $p=0$. Moreover, $E\in\mathcal{M}_E$, and thus $h_2=0$.

Going back to the system (2.1), we get

$$\begin{cases} curl\,(\partial_t A-H)=0 \\ div\,(\partial_t A-H)=0 \\ (\partial_t A-H)\,.n_{|\partial\Omega}=0\ . \end{cases}$$

This implies the desired decomposition for the electromagnetic field, with $h_1=\partial_t A-H$. Moreover, A is orthogonal to h_1 ([DL]](5, p.256)) and we get the relation (2.2).

The inequality (2.3) can be proved by a contradiction argument. Indeed, if it is false, then there exists a sequence (A_n) of bounded function in $\left(H^1(\Omega)\right)^3$ such that

$$\|A_n\|_{L^2(\Omega\times]0,T[)}=1 \text{ and } \lim_{n\to+\infty}\|curlA_n\|_{L^2(\Omega)}=0,\ divA_n=0,\ A_n.n_{|\partial\Omega}=0\ .$$

By compact injection of $H^1(\Omega)$ in $L^2(\Omega)$, there is $A\in\mathbb{H}_1(\Omega)$ such that $\|A\|_{L^2(\Omega)}=1$ and $\int_\Omega Ah_1=0 \quad \forall h_1\in\mathbb{H}_1(\Omega)$, which is impossible.

This completes the proof of lemma 2.1.

Proof of lemma 2.2.- The magnetic field $H\in\left(L^2(\Omega)\right)^3$ can be written in an unique way as follows

$$\mu H=\nabla q+h_1+curlA, \text{ where } q\in H^1(\Omega)\,,\ h_1\in\mathbb{H}_1(\Omega)\,,\ A\in\left(H^1(\Omega)\right)^3$$

such that $divA=0,\ A\wedge n_{|\partial\Omega}=0,\ \int_{\Gamma_i} A.nd\Gamma=0 \quad i=0$ to m ([DL](5, p.261)).

But $divH=0$ and $H.n_{|\partial\Omega}=0$, and consequently $\Delta q=0$ and $\partial_n q=0$, which implies that q is a constant function. Moreover, $H\in\mathcal{M}_H$, and thus $h_1=0$.

Going back to system (2.1), we have

$$\left\{ \begin{array}{l} curl\,(\partial_t A + E) = 0 \\ div\,(\partial_t A + E) = 0 \\ (\partial_t A + E) \wedge n_{|\partial\Omega} = 0 \ . \end{array} \right.$$

This implies the desired decomposition result for the electromagnetic field, with $h_2 = \partial_t A + E$. By the second definition of $\mathbb{H}_2(\Omega)$, A is orthogonal to h_2, $\int_\Omega Ah_2 = \int_\Omega A.\nabla\varphi = -\int_\Omega \varphi div A + \int_{\partial\Omega} \varphi A.n = 0$,

and it implies the relations (2.4) and (2.5).

This completes the proof of lemma 2.1.

3 Boundary and internal controllability for the Maxwell equations

In this section, we complete the boundary controllability results of J.Lagnese [La] and of V.Komornik [K]. Also, we recall that the work of C.Bardos, J.Rauch and G.Lebeau for the observability and controllability of the scalar wave equation [BLR] has been extended to the Maxwell equations by O.Nalin [N].

3.1 Presentation of the boundary controllability problem

Let Ω be a bounded open connected region in $\mathbb{R}^3$, with a smooth boundary $\partial\Omega$ (C^∞ having no contacts of infinite order with its tangents). The domain Ω is occupied by an electromagnetic medium of constant electric permittivity ε and constant magnetic permeability μ. Let E and H denote the electric field and the magnetic field respectively. The Maxwell equations are described by

$$\left\{ \begin{array}{c} \varepsilon\partial_t E - curl H = 0,\ \mu\partial_t H + curl E = 0 \quad \text{in } \Omega \times [0, +\infty[\\ div\,(\varepsilon E) = 0,\ div\,(\mu H) = 0 \quad \text{in } \Omega \times [0, +\infty[\\ E\,(\cdot, 0) = E_o,\ H\,(\cdot, 0) = H_o \quad \text{in } \Omega \ . \end{array} \right.$$

It is assumed that the electromagnetic field is driven by an externally applied density of current J flowing tangentially in a part Γ of $\partial\Omega$. We get the following boundary condition

$$H \wedge n = J.1_{|\Gamma} \quad \text{on } \partial\Omega \times]0, +\infty[$$

where n denotes the outward normal vector to $\partial\Omega$. Also, let

$$\begin{array}{l} L^2_n\,(\partial\Omega) = \left\{ X \in \left(L^2\,(\partial\Omega)\right)^3 \setminus\ X.n_{|\partial\Omega} = 0 \right\} \\ \mathcal{V}^* = \left\{ f \in \left(L^2\,(\Omega)\right)^3 \setminus\ div f = 0,\ f.n_{|\partial\Omega} = 0 \right\} \times \left\{ g \in \left(L^2\,(\partial\Omega)\right)^3 \setminus\ div g = 0 \right\} \ . \end{array}$$

Our goal is to find the control function J in order that at time $T > 0$, the electromagnetic field returns to the equilibrium state i-e $E\,(\cdot, T) = H\,(\cdot, T) = 0$ in Ω. Such control will be constructed by the HUM method of J.L.Lions [Li], which need an observability estimate [La].

3.2 Statement of the results

We show the following control results.

Theorem 3.1 Suppose that Γ geometrically controls Ω. Then for any initial data $(E_o, H_o) \in \mathcal{V}^* \cap \left(\left(L^2(\Omega)\right)^3 \times \mathcal{M}_E\right)$, there exists a control $J \in L^2\left(0,T; L^2_n(\partial\Omega)\right)$, such that the solution of the system

$$\begin{cases} \varepsilon\partial_t E - curl H = 0,\ \mu\partial_t H + curl E = 0 \quad \text{in } \Omega\times[0,T[\\ div E = 0,\ div H = 0 \quad \text{in } \Omega\times[0,T[\\ E(\cdot,0) = E_o,\ H(\cdot,0) = H_o \quad \text{in } \Omega \\ H\wedge n = J.1_{|\Gamma} \quad \text{on } \partial\Omega\times]0,T[\end{cases}$$

satisfies $(E,H)(\cdot,t) \equiv 0$ for $t \geq T$.

The proof of theorem 2.1 comes from the HUM method of J.L.Lions and an observability estimate given below.

Theorem 3.2 Suppose that Γ geometrically controls Ω. Then there exists $c > 0$ such that for any initial data $(E_o, H_o) \in \mathcal{V}\cap\mathcal{S}_E$ of the system

$$\begin{cases} \varepsilon\partial_t E - curl H = 0,\ \mu\partial_t H + curl E = 0 \quad \text{in } \Omega\times]0,T[\\ div E = 0,\ div H = 0 \quad \text{in } \Omega\times]0,T[\\ E(\cdot,0) = E_o,\ H(\cdot,0) = H_o \quad \text{in } \Omega \\ E\wedge n = 0,\ H.n = 0 \quad \text{on } \partial\Omega\times]0,T[\end{cases}$$

we have the observability estimate

$$\int_0^T\int_\Omega |(E,H)|^2\, dxdt \leq c\int_0^T\int_\Gamma |H|^2\, d\sigma dt\ . \tag{3.1}$$

Proof .- The proof of theorem 3.2 comes from theorem 1.5. We divide the proof into two steps.
Step 1 :
We link E and H, by integration by parts and using the decomposition of the electromagnetic field done in lemma 2.1 $(\varepsilon E, \mu H) = (curl A, \mu\partial_t A + h_1)$ where the potential vector A is orthogonal to $h_1 \in \mathbb{H}_1(\Omega)$ and satisfies the inequalities (2.5) and

$$\|\mu H\|^2_{L^2(\Omega)} = \|\mu\partial_t A\|^2_{L^2(\Omega)} + \|h_1\|^2_{L^2(\Omega)}\ .$$

Let $\Phi \in C_0^\infty(]0,T[)$, we have from the Green formula, the following relations

$$\begin{array}{ll} \int_0^T\int_\Omega \varepsilon|\Phi E|^2 & = \frac{1}{\varepsilon}\int_0^T\int_\Omega \Phi^2 A curl curl A + \int_0^T\int_{\partial\Omega}\Phi^2(E\wedge n)A \\ & = -\varepsilon\mu\int_0^T\int_\Omega \Phi^2 A\partial_t H \\ & = \int_0^T\int_\Omega \varepsilon\mu 2\Phi\partial_t\Phi AH + \int_0^T\int_\Omega \varepsilon\mu\Phi^2\partial_t AH\ . \end{array}$$

By Cauchy-Schwarz inequality, we conclude that

$$\int_0^T\int_\Omega |\Phi E|^2 + \int_0^T\int_\Omega |\Phi H|^2 \leq c\int_0^T\int_\Omega |H|^2\ . \tag{3.2}$$

Choosing Φ such that $\Phi = 1$ in $[T/3, 2T/3]$, we get from the conservation of the energy,

$$\varepsilon\int_0^T\int_\Omega |E|^2 + \mu\int_0^T\int_\Omega |H|^2 = 3\left(\frac{T}{3}\mathcal{E}\left(\frac{2T}{3}\right)\right) \leq c\int_0^T\int_\Omega |H|^2\ . \tag{3.3}$$

Conclusion.- We see that it is sufficient to study the observability problem only for the magnetic field.

Step 2 :
We go back to the wave equation

$$\left\{ \begin{array}{c} \varepsilon\mu\partial_t^2 H - \Delta H = 0 \quad \text{in } \Omega \times]0,T[\\ H.n = 0, \quad curlH \wedge n = 0 \quad \text{on } \partial\Omega \times]0,T[\end{array} \right. .$$

From (1.39), we have $\exists c > 0$

$$\|H\|_{L^2(\Omega\times]0,T[)} \leq c\left(\|\partial_n H\|_{H^{-1}(\Gamma\times]0,T[)} + \|H\|_{L^2(\Gamma\times]0,T[)}\right) . \tag{3.4}$$

The boundary conditions $divH = 0$ and $curlH \wedge n = 0$ on $\partial\Omega \times]0,T[$ allow to get the system (1.11) i-e $\partial_n H + \mathcal{L}H = 0$ where $\mathcal{L}$ is a matrix differential operator of order 1.

This completes the proof of theorem 3.2.

Theorem 3.2 gives an uniqueness result. The result of propagation of the L^2 regularity of theorem 1.5 (which says that the propagation of singularities for the magnetic field is similar that the one for a scalar wave equation) and the condition (1.11) show that if $H = 0$ on a part of the boundary then $H \equiv 0$, with more precisely the estimate [N]

$$\|H\|_{L^2(\Omega\times]0,T[)} \leq c\,\|H\|_{L^2(\Gamma\times]0,T[)}$$

under the geometric control condition of the work of C.Bardos, G.Lebeau and J.Rauch [BLR]. It remains to prove that if $H \equiv 0$, then $E \equiv 0$. To this ends, it is useful to restrict to the initial electric fields E_o which are orthogonal to $\mathbb{H}_2(\Omega)$.

In the particular case when $\partial\Omega$ has only one connected component and under the geometric control condition, then for any $(E_o, H_o) \in \mathcal{V}$, $\|H\|_{L^2(\Gamma\times]0,T[)} = 0$ implies that $(E_o, H_o) \equiv 0$ with the observability estimate (3.1). Such result was obtained by O.Nalin [N] in the H^1 functional space framework: if $\partial\Omega$ has only one connected component and under the geometric control condition, then for any $(E_o, H_o) \in \mathcal{W}$, $\|\partial_t H\|_{L^2(\Gamma\times]0,T[)} = 0$ implies that $(E_o, H_o) \equiv 0$ with the corresponding H^1 observability estimate.

3.3 Presentation of the internal controllability problem

Let $\omega \subset \Omega$ where Ω is a bounded open connected region in $\mathbb{R}^3$, with a smooth boundary $\partial\Omega$ (C^∞ having no contacts of infinite order with its tangents). The domain Ω is occupied by an electromagnetic medium of constant electric permittivity ε and constant magnetic permeability μ. Let E and H denote the electric field and the magnetic field respectively. The Maxwell equations are described by

$$\left\{ \begin{array}{c} \varepsilon\partial_t E - curlH = J.1_{|\omega\times]0,T[}\,, \ \mu\partial_t H + curlE = 0 \quad \text{in } \Omega \times [0,+\infty[\\ div\,(\mu H) = 0 \quad \text{in } \Omega \times [0,+\infty[\\ E \wedge n = 0, \ H.n = 0 \quad \text{on } \partial\Omega \times [0,+\infty[\\ E\,(\cdot,0) = E_o, \ H\,(\cdot,0) = H_o \quad \text{in } \Omega \ . \end{array} \right. \tag{3.5}$$

Such system (3.5) is well-posed. Also, let

$$L^2_{div0}\,(\Omega) = \left\{ X \in \left(L^2\,(\partial\Omega)\right)^3 \setminus \ divX = 0 \right\} .$$

Our goal is to find the control function J in order that at time $T > 0$, the electromagnetic field returns to the equilibrium state i-e $E\,(\cdot,T) = H\,(\cdot,T) = 0$ in Ω. Such control will be constructed by the HUM method of J.L.Lions [Li], which need an observability estimate.

3.4 Statement of the results

We show the following control results.

Theorem 3.3 Let $\widetilde{\omega}$ be an open set of $\mathbb{R}^3$ such that $\widetilde{\omega}$ geometrically controls Ω. Suppose that $\omega = (\widetilde{\omega} \cap \Omega)$. Then, for any initial data $(E_o, H_o) \in \mathcal{V} \cap \mathcal{S}_H$, there exists a control $J \in L^2\left(0,T; L^2_{div0}(\Omega)\right)$, such that the solution of the system (3.5) satisfies $(E,H)(\cdot,t) \equiv 0 \quad \text{for } t \geq T$.

The proof of theorem 3.3 comes from the HUM method of J.L.Lions and an observability estimate given below.

Theorem 3.4 Let $\widetilde{\omega}$ be an open set of $\mathbb{R}^3$ such that $\widetilde{\omega}$ geometrically controls Ω. Suppose that $\omega = (\widetilde{\omega} \cap \Omega)$. Then there exists $c > 0$ such that for any initial data $(E_o, H_o) \in \mathcal{V} \cap \mathcal{S}_H$ of the system

$$\left\{ \begin{array}{c} \varepsilon\partial_t E - curl H = 0,\ \mu\partial_t H + curl E = 0 \quad \text{in } \Omega \times [0,T[\\ div E = 0,\ div H = 0 \quad \text{in } \Omega \times [0,T[\\ E \wedge n = 0,\ H.n = 0 \quad \text{on } \partial\Omega \times]0,T[\\ E(\cdot,0) = E_o,\ H(\cdot,0) = H_o \quad \text{in } \Omega\ , \end{array} \right.$$

we have the observability estimate

$$\int_0^T \int_\Omega |(E,H)|^2 \, dxdt \leq c \int_0^T \int_\omega |E|^2 \, dxdt\ . \tag{3.6}$$

Proof .- The proof of theorem 3.4 comes from theorem 1.4. We divide the proof into two steps.
Step 1 :
We link E and H, by integration by parts and using the decomposition of the electromagnetic field done in lemma 2.2. $(\varepsilon E, \mu H) = (\varepsilon\partial_t A + h_2, curl A)$ where the potential vector A is orthogonal to $h_2 \in \mathbb{H}_2(\Omega)$ and satisfies the inequalities (2.3) and

$$\|\varepsilon H\|^2_{L^2(\Omega)} = \|\varepsilon\partial_t A\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Omega)}\ .$$

Let $\Phi \in C_0^\infty(]0,T[)$, we have from the Green formula, the following relations

$$\begin{array}{ll} \int_0^T \int_\Omega \mu |\Phi H|^2 & = \frac{1}{\mu} \int_0^T \int_\Omega \Phi^2 A curl curl A \\ & = \varepsilon\mu \int_0^T \int_\Omega \Phi^2 A \partial_t E \\ & = -\int_0^T \int_\Omega \varepsilon\mu 2\Phi\partial_t\Phi A E - \int_0^T \int_\Omega \varepsilon\mu\Phi^2 \partial_t A E\ . \end{array}$$

From Cauchy-Schwarz inequality, we conclude that

$$\int_0^T \int_\Omega |\Phi E|^2 + \int_0^T \int_\Omega |\Phi H|^2 \leq c \int_0^T \int_\Omega |E|^2\ . \tag{3.7}$$

Choosing Φ such that $\Phi = 1$ in $[T/3, 2T/3]$, we get by conservation of the energy,

$$\varepsilon \int_0^T \int_\Omega |E|^2 + \mu \int_0^T \int_\Omega |H|^2 = 3\left(\frac{T}{3}\mathcal{E}\left(\frac{2T}{3}\right)\right) \leq c \int_0^T \int_\Omega |E|^2\ . \tag{3.8}$$

Conclusion.- We see that it is sufficient to study the observability problem only for the electric field.

Step 2 :
We go back to the wave equation

$$\left\{ \begin{array}{c} \varepsilon\mu\partial_t^2 E - \Delta E = 0 \quad \text{in } \Omega \times]0,T[\\ E \wedge n = 0, \quad div E = 0 \quad \text{on } \partial\Omega \times]0,T[\ . \end{array} \right.$$

From (1.31), we have the following observability estimate $\quad \exists c>0$

$$\|E\|_{L^2(\Omega\times]0,T[)} \leq c\,\|E\|_{L^2(\omega\times]0,T[)} \ .$$

This completes the proof of theorem 3.4.

4 Boundary stabilization for the Maxwell equations

In this section, we complete the study of the asymptotic behaviour in time of the electromagnetic field with an absorbing boundary condition done by H.Barucq et B.Hanouzet ([Ba], [BH]). The problem of stabilization for the Maxwell equation with the Silver-Müller's absorbing boundary condition has also been studied by V.Komornik [K] in the framework of particular geometry using the multipliers method.

4.1 Presentation of the boundary stabilization problem

Let Ω be a bounded open connected region in $\mathbb{R}^3$, with a smooth boundary $\partial\Omega$ (C^∞ having no contacts of infinite order with its tangents). The domain Ω is occupied by an electromagnetic medium of constant electric permittivity ε and constant magnetic permeability μ. We consider that $\partial\Omega=\Gamma_o\cup\Gamma$ and $\Gamma_o\cap\Gamma=\emptyset$. Let $z=\sqrt{\frac{\mu}{\varepsilon}}$. Let E and H denote the electric field and the magnetic field respectively. The Maxwell equations are described by

$$\left\{\begin{array}{c}\varepsilon\partial_t E - curl H=0,\ \mu\partial_t H+curl E=0 \quad \text{in } \Omega\times[0,+\infty[\\ div\,(\varepsilon E)=0,\ div\,(\mu H)=0 \quad \text{in } \Omega\times[0,+\infty[\\ E\,(\cdot,0)=E_o,\ H\,(\cdot,0)=H_o \quad \text{in } \Omega\\ E\wedge n=0,\ H.n=0 \quad \text{on } \Gamma_o\times[0,+\infty[\\ (E\wedge n)\wedge n+z\,(H\wedge n)=0 \quad \text{on } \Gamma\times[0,+\infty[\ .\end{array}\right. \tag{4.1}$$

We recall the main results of H.Barucq and B.Hanouzet ([Ba], [BH]).

The system (4.1) is well-posed. Let

$$\begin{array}{r}\mathcal{V}=\left\{f\in\left(L^2\left(\Omega\right)\right)^3 \setminus\ div f=0\right\}\times\left\{g\in\left(L^2\left(\Omega\right)\right)^3\setminus\ div g=0,\ g.n_{|\Gamma_o}=0\right\}\\ \mathcal{W}=\left\{(f,g)\in\left(H^1\left(\Omega\right)\right)^6\setminus\ div f=0,\ f\wedge n_{|\Gamma_o}=0,\ div g=0,\ g.n_{|\Gamma_o}=0,\right.\\ \left.,(f\wedge n)\wedge n+z\,(g\wedge n)_{|\Gamma}=0\right\}\end{array}$$

$$\forall\,(E_o,H_o)\in\mathcal{W}\quad \exists!\,(E,H)\in C^0\left(]0,+\infty[\,,\mathcal{W}\right)\cap C^1\left(]0,+\infty[\,,\mathcal{V}\right)\quad \text{solution of the system (4.1).}$$

Such result is obtained by Hille-Yosida theorem applied to the unbounded operator $\mathcal{A}$ on the Hilbert space $\mathcal{V}$,

$$\mathcal{A}=\left(\begin{array}{cc}0 & -\varepsilon^{-1}curl\\ \mu^{-1}curl & 0\end{array}\right)\ .$$

It follows that $\mathcal{W}$ coincides with $D(\mathcal{A})$ the domain of $\mathcal{A}$ and

$$D\left(\mathcal{A}^2\right) = \Big\{(f,g) \in \left(H^2(\Omega)\right)^6 \setminus \ divf = divg = 0,\ f \wedge n_{|\Gamma_o} = g.n_{|\Gamma_o} = 0, \\ ,(f \wedge n) \wedge n + z\,(g \wedge n)_{|\Gamma} = 0\Big\} \ .$$

Moreover, posing $\mathcal{E}(t) = \frac{1}{2}\int_\Omega \left(\varepsilon\,|E|^2 + \mu\,|H|^2\right)$ the energy of the system (4.1), we have

$$\forall\,(E_o,H_o) \in \mathcal{W} \qquad -\frac{d}{dt}\mathcal{E}(t) = \frac{1}{z}\int_\Gamma |E \wedge n|^2 = z\int_\Gamma |H \wedge n|^2 \ .$$

This last relation implies that the energy is a decreasing function of time.

On another hand, we know that $\forall\,(E_o,H_o) \in \mathcal{W} \cap \mathcal{M}_\Gamma \quad \lim_{t\to+\infty}(E,H) = (0,0)$ in $\mathcal{V}$, where $\mathcal{M}_\Gamma$ is the orthogonal space to the steady states for the $\left(L^2(\Omega)\right)^6$ norm. We recall that $\mathcal{W} \cap \mathcal{M}_\Gamma$ is stable for the system (4.1) ([Ba], [BH]).

Posing $\mathcal{E}'$ the decreasing functional $\mathcal{E}'(t) = \frac{1}{2}\int_\Omega \left(\varepsilon\,|\partial_t E|^2 + \mu\,|\partial_t H|^2\right)$, we have

$$\forall\,(E_o,H_o) \in D\left(\mathcal{A}^2\right) \qquad -\frac{d}{dt}\mathcal{E}'(t) = z\int_\Gamma |\partial_t H \wedge n|^2 \ .$$

Remark that the Silver-Müller's boundary condition $(E \wedge n) \wedge n + z\,(H \wedge n) = 0$ can be written $(E \wedge n) - +z\,(H \wedge n) \wedge n = 0$ and we obtain $(curlH \wedge n) + z\,(\partial_t H \wedge n) \wedge n = 0$.

4.2 Dissipation under the geometric control condition

We show the following control results.

Theorem 4.1 Suppose that Γ geometrically controls Ω. Then there exist $c > 0$ and $\beta > 0$ such that

$$\forall\,(E_o,H_o) \in \mathcal{V} \cap \mathcal{M}_\Gamma \text{ initial data for the system (4.1)} \qquad \forall t \geq 0 \quad \mathcal{E}(t) \leq ce^{-\beta t}\ \mathcal{E}(0).$$

The proof of theorem 4.1 comes from theorem 4.2 below. Remark that the geometric control condition of theorem 4.1 is almost a necessary condition by constructing solutions with localized energy ([BLR], [Ph1], [R]).

Theorem 4.2 Suppose that Γ geometrically controls Ω. Then there exist $c > 0$ and $\beta > 0$ such that

$$\forall\,(E_o,H_o) \in \mathcal{W} \cap \mathcal{M}_\Gamma \text{ initial data for system (4.1)} \qquad \forall t \geq 0 \quad (\mathcal{E} + \mathcal{E}')(t) \leq ce^{-\beta t}\ (\mathcal{E} + \mathcal{E}')(0).$$

Proof of theorem 4.2.- The proof of theorem 4.2 comes from the following lemma.

Lemma 4.1 Under the assumptions of theorem 4.2, if (E,H) is a solution of the system (4.1) with initial data in $\mathcal{W} \cap \mathcal{M}_\Gamma$, then we have

$$\exists c > 0 \quad \int_0^T \int_\Omega \left(|(E,H)|^2 + |(\partial_t E, \partial_t H)|^2\right) \leq c\int_0^T \int_\Gamma |\partial_t H \wedge n|^2 \ . \tag{4.2}$$

Proof of lemma 4.1.- We divide the proof into five steps.

Step 1 :

We link $\partial_t E$ and $\partial_t H$, by integration by parts.

Let $\Phi \in C_0^\infty(]0,T[)$, we get using the Green formula, the following relations

$$\begin{aligned}\int_0^T \int_\Omega \varepsilon \left|\Phi \partial_t E\right|^2 &= \frac{1}{\varepsilon} \int_0^T \int_\Omega \Phi^2 H curlcurl H + \frac{1}{\varepsilon} \int_0^T \int_\Gamma \Phi^2 \left(rot H \wedge n\right) H \\ &= -\mu \int_0^T \int_\Omega \Phi^2 H \partial_t^2 H + \int_0^T \int_\Gamma \Phi^2 \left(\partial_t E \wedge n\right) H \\ &= \int_0^T \int_\Omega \mu \partial_t \left(\Phi^2 H\right) \partial_t H + z \int_0^T \int_\Gamma \Phi^2 \left(n \wedge H\right) \left(\partial_t H \wedge n\right) \ .\end{aligned}$$

We claim that

$$\left\|(E,H)\right\|_{L^2(\Omega)} \leq c \left\|(curl E, curl H)\right\|_{L^2(\Omega)} \ .$$

Indeed, by a contradiction argument, if it is false then there exists a sequence (E_n, H_n) bounded in $\left(H^1(\Omega)\right)^6$ such that

$$\left\|(E_n,H_n)\right\|_{L^2(\Omega)} = 1 \text{ and } \lim_{n\to+\infty} \left\|(curl E_n, curl H_n)\right\|_{L^2(\Omega)} = 0, \ \lim_{n\to+\infty} \left\|E_n \wedge n\right\|_{L^2(\Gamma)} = 0$$

$$div E_n = div H_n = 0, \ E_n \wedge n_{|\Gamma_o} = H_n . n_{|\Gamma_o} = 0, (E_n \wedge n) \wedge n + z \left(H_n \wedge n\right)_{|\Gamma} = 0$$

then, by compact injection there exists $(E,H) \in \mathcal{N} \cup \mathcal{N}^\perp$ where $\mathcal{N}^\perp = \mathcal{M}_\Gamma$ such that $\left\|(E,H)\right\|_{L^2(\Omega)} = 1$, which is impossible.

By Cauchy-Schwarz inequality, we conclude that

$$\int_0^T \int_\Omega \left|\Phi \partial_t E\right|^2 + \int_0^T \int_\Omega \left|\Phi \partial_t H\right|^2 \leq c \left(\int_0^T \int_\Omega \left|\partial_t H\right|^2 + \int_0^T \int_\Gamma \left|\partial_t H \wedge n\right|^2 \right) \ .$$

Choosing $\Phi \in C_0^\infty(]0,T[)$ such that $\Phi = 1$ in $[T/3, 2T/3]$, we have using the decreasing property of the energy,

$$\begin{aligned}\varepsilon \int_0^T \int_\Omega \left|\partial_t E\right|^2 + \mu \int_0^T \int_\Omega \left|\partial_t H\right|^2 &\leq T\mathcal{E}'(0) = T\mathcal{E}'(T) + Tz \int_0^T \int_\Gamma \left|\partial_t H \wedge n\right|^2 \\ &\leq 3 \left(\tfrac{T}{3} \mathcal{E}'\left(\tfrac{2T}{3}\right)\right) + Tz \int_0^T \int_\Gamma \left|\partial_t H \wedge n\right|^2 \\ &\leq c \left(\int_0^T \int_\Omega \left|\partial_t H\right|^2 + \int_0^T \int_\Gamma \left|\partial_t H \wedge n\right|^2 \right) \ .\end{aligned} \tag{4.3}$$

Conclusion.- We see that it is sufficient to study the observability problem only for the magnetic field.

Step 2 :

We begin to establish preliminary estimates on the boundary Γ.

First, we have $H = n \wedge (H \wedge n) + (H.n)\, n$ and $|H| \leq c\left(|H \wedge n| + |H.n|\right)$. Supposing the domain Ω in the form $\Omega = \{(x_1,x_2,x_3) \setminus x_3 - g(x_1,x_2) > 0\}$, we recall that the outward normal vector at the boundary $\partial\Omega$ is defined by $n = (\nu m_1, \nu m_2, \nu)^t$ where $\nu^2\left(1 + m_1^2 + m_2^2\right) = 1$.

Consequently, H can be written $H = (m_1, m_2, 1)^t H_3 + F$ where $|F| \leq \frac{1}{\nu} |H \wedge n|$. We deduce that $\exists c > 0$

$$\left|(H \wedge n) \wedge n\right| \leq c \left|H \wedge n\right| \text{ et } \left|H.n\right| \leq c\left(|H_3| + |H \wedge n|\right) \ . \tag{4.4}$$

In particular, the Silver-Müller's boundary condition implies that $|E \wedge n| \leq c\, |H \wedge n|$.

Now we study $\partial_t H.n$: $\mu \partial_t H.n = -rot E.n = -div_{|\Gamma}(E \wedge n)$, where $div_{|\Gamma}$ is a tangential differential operator of order 1, thus

$$\left\|\partial_t H.n\right\|_{H^{-1}(\Gamma)} \leq c \left\|E \wedge n\right\|_{L^2(\Gamma)} \ .$$

Consequently,

$$\exists c>0 \qquad \|\partial_t H\|_{H^{-1}(\Gamma\times]0,T[)} \leq c\,\|H\wedge n\|_{L^2(\Gamma\times]0,T[)} \ . \tag{4.5}$$

Concerning the normal derivative of each component of H, we introduce, using the change of coordinates (1.1)

$$\widetilde{H}(s,y_1,y_2,t)=H(\Psi(s,y_1,y_2),t)=H(x_1,x_2,x_3,t)$$

in order that

- the divergence becomes

$$\begin{pmatrix} m_1\\ m_2\\ 1\end{pmatrix}\partial_s\widetilde{H}+\nu\begin{pmatrix}1+m_2^2\\ -m_1m_2\\ -m_1\end{pmatrix}\partial_{y_1}\widetilde{H}+\nu\begin{pmatrix}-m_1m_2\\ 1+m_1^2\\ -m_2\end{pmatrix}\partial_{y_2}\widetilde{H}=0$$

- the Silver-Müller's boundary condition becomes

$$\begin{pmatrix}1&0&-m_1\\0&1&-m_2\end{pmatrix}\left(\partial_s\widetilde{H}+\varepsilon z\partial_t\widetilde{H}\right)=\nu\begin{pmatrix}m_1&m_2&1\\0&0&0\end{pmatrix}\partial_{y_1}\widetilde{H}+v\begin{pmatrix}0&0&0\\m_1&m_2&1\end{pmatrix}\partial_{y_2}\widetilde{H}\ .$$

Finally, the boundary condition becomes

$$\begin{pmatrix}1&0&-m_1\\0&1&-m_2\\m_1&m_2&1\end{pmatrix}\partial_s\widetilde{H}+\varepsilon z\begin{pmatrix}1&0&-m_1\\0&1&-m_2\\m_1&m_2&1\end{pmatrix}\partial_t\widetilde{H}+...$$

$$...-\nu\begin{pmatrix}m_1&m_2&1\\0&0&0\\-1-m_2^2&m_1m_2&m_1\end{pmatrix}\partial_{y_1}\widetilde{H}-\nu\begin{pmatrix}0&0&0\\m_1&m_2&1\\m_1m_2&-1-m_1^2&m_2\end{pmatrix}\partial_{y_2}\widetilde{H}=0\ .$$

Remark that the matrix in front of $\partial_s\widetilde{H}=\partial_n H$ is invertible. Consequently,

$$\partial_s\widetilde{H}+\varepsilon z\begin{pmatrix}1-\nu m_1^2&-\nu m_1m_2&m_1\left(-1+\nu m_1^2+\nu m_2^2\right)\\-\nu m_1m_2&1-\nu m_2^2&m_2\left(-1+\nu m_1^2+\nu m_2^2\right)\\-\nu m_1&-\nu m_2&\nu\left(m_1^2+m_2^2\right)\end{pmatrix}\partial_t\widetilde{H}+...$$

$$...-\nu\begin{pmatrix}0&m_2&1\\-m_2&0&0\\-1&0&0\end{pmatrix}\partial_{y_1}\widetilde{H}-\nu\begin{pmatrix}0&-m_1&0\\m_1&0&1\\0&-1&0\end{pmatrix}\partial_{y_2}\widetilde{H}=0\ .$$

It follows the following estimates

$$\begin{aligned}\exists c>0 \qquad \|\partial_n H\|_{H^{-1}(\Gamma\times]0,T[)} &\leq c\,\|H\|_{L^2(\Gamma\times]0,T[)}\\ &\leq c\left(\|H_3\|_{L^2(\Gamma\times]0,T[)}+\|H\wedge n\|_{L^2(\Gamma\times]0,T[)}\right)\ .\end{aligned}\tag{4.6}$$

Step 3 :
We estimate H_3 on the boundary Γ .

Recall that $\widetilde{H}_3(s,y_1,y_2,t)$ solves

$$\begin{cases} P\left(\widetilde{H}_3\right)=\left(-\partial_s^2+R(s,y,D_{y,t})\right)\left(\widetilde{H}_3\right)=0 & \text{in } \{s>0\}\\ \partial_s\widetilde{H}_3+\nu\partial_{y_1}\widetilde{H}_3+\nu\partial_{y_2}\widetilde{H}_3+\varepsilon z\partial_t F=0 & \text{on } \{s=0\}\\ F=m_1\left(\widetilde{H}\wedge n\right)_2-m_2\left(\widetilde{H}\wedge n\right)_1 & \text{on } \{s=0\}\end{cases}$$

and the boundary conditions can be replaced by

$$\begin{aligned} B\left(y_1,y_2,t,D_{y,t}\right)\left(\widetilde{H}_3\right) &= \partial_s \widetilde{H}_3 + \nu m_1 \partial_{y_1} \widetilde{H}_3 + \nu m_2 \partial_{y_2} \widetilde{H}_3 + \nu\left(\partial_{y_1} m_1 + \partial_{y_2} m_2\right) \widetilde{H}_3 \\ &= \nu\left(\partial_{y_1} - m_1 \partial_t\right)\left(\widetilde{H}\wedge n\right)_2 - \nu\left(\partial_{y_2} - m_2\partial_t\right) m_2 \left(\widetilde{H}\wedge n\right)_1 \ . \end{aligned}$$

Microlocally, in a neighborhood of the directions where the operator ∂_t is not characteristic, $\left(\widetilde{H}_3\right)_{|s=0} \in L^2_\rho$ as soon as $\left(\partial_t \widetilde{H}_3\right)_{|s=0} \in H^{-1}_\rho$. But such neighborhood is the elliptic region for the hyperbolic operator. We conclude by a standard microlocal way that there exists, in the elliptic region, a tangential pseudo-differential operator of order 1, A_1, elliptic, such that $\partial_s \widetilde{H}_3 = A_1\left(\widetilde{H}_3\right)$ on $s=0$. Thus, in the elliptic region, the operator B becomes elliptic of order 1 and its principal symbol is

$$\sigma(B) = A\left(y_1,y_2,t,\zeta_1,\zeta_2,\tau\right) + i\nu\left(y_1,y_2\right)\left[m_1\left(y_1,y_2\right)\zeta_1 + m_2\left(y_1,y_2\right)\zeta_2\right] \ .$$

Going back to the solution H_3, we have ([BHLRZ], [BLR]) $\quad \exists c,d>0$

$$\left\|H_3\right\|_{L^2(\Gamma\times]0,T[)} \leq c\left(\left\|H\wedge n\right\|_{L^2(\Gamma\times]0,T[)} + \left\|\partial_t H_3\right\|_{H^{-1}(\Gamma\times]0,T[)}\right) + d\left\|H_3\right\|_{H^{-1}(\Gamma\times]0,T[)} \ .$$

Conclusion, from (4.5), $\quad \exists c,d>0$

$$\left\|H_3\right\|_{L^2(\Gamma\times]0,T[)} \leq c\left\|H\wedge n\right\|_{L^2(\Gamma\times]0,T[)} + d\left\|H_3\right\|_{H^{-1}(\Gamma\times]0,T[)} \ . \tag{4.7}$$

Step 4 :
We go back to the wave equation.

$$\left\{\begin{array}{c} \varepsilon\mu\partial_t^2 H - \Delta H = 0 \quad \text{in } \Omega\times]0,T[\\ H.n = 0, \quad curl H\wedge n = 0 \quad \text{on } \Gamma_o\times]0,T[\\ \left(\partial_t H\wedge n\right)\wedge n - \frac{1}{\varepsilon z}\left(curl H\wedge n\right) = 0 \quad \text{on } \Gamma\times]0,T[\ . \end{array}\right.$$

From (1.39), we have the following estimate $\quad \exists c>0$

$$\left\|H\right\|_{L^2(\Omega\times]0,T[)} \leq c\left(\left\|\partial_n H\right\|_{H^{-1}(\Gamma\times]0,T[)} + \left\|H\right\|_{L^2(\Gamma\times]0,T[)}\right) \ . \tag{4.8}$$

The inequalities (4.4), (4.6),(4.7) allow to get $\quad \exists c,d>0$

$$\left\|\partial_n H\right\|_{H^{-1}(\Gamma\times]0,T[)} + \left\|H\right\|_{L^2(\Gamma\times]0,T[)} \leq c\left\|H\wedge n\right\|_{L^2(\Gamma\times]0,T[)} + d\left\|(E,H)\right\|_{H^{-1}(\Gamma\times]0,T[)} \ .$$

Such result is still true if we replace (E,H) by $(\partial_t E,\partial_t H)$. By the trace theorem, (4.3) and (4.8), we deduce the following estimate $\quad \exists c,d>0$

$$\left\|(\partial_t E,\partial_t H)\right\|_{L^2(\Omega\times]0,T[)} \leq c\left\|\partial_t H\wedge n\right\|_{L^2(\Gamma\times]0,T[)} + d\left\|(E,H)\right\|_{H^{1/2+\epsilon}(\Omega\times]0,T[)} \ .$$

The regularity results in ([Ba]($p.76$)) allow to conclude that $\quad \exists c,d>0$

$$\left\|(E,H)\right\|_{H^1(\Omega\times]0,T[)} \leq c\left\|\partial_t H\wedge n\right\|_{L^2(\Gamma\times]0,T[)} + d\left\|(E,H)\right\|_{L^2(\Omega\times]0,T[)} \ . \tag{4.9}$$

Step 5 :
It remains to prove that $\left\|(E,H)\right\|_{L^2(\Omega\times]0,T[)} \leq c\left\|\partial_t H\wedge n\right\|_{L^2(\Gamma\times]0,T[)}$. We use a contradiction argument.

Indeed, suppose that there exists a sequence (E_n, H_n) solution of system (4.1) such that

$$\|(E_n, H_n)\|_{L^2(\Omega\times]0,T[)} = 1 \text{ and } \lim_{n\to+\infty} \|\partial_t H \wedge n\|_{L^2(\Gamma\times]0,T[)} = 0$$

then, (E_n, H_n) satisfies (4.9) : $\|(E_n, H_n)\|_{H^1(\Omega\times]0,T[)} \leq c\,\|\partial_t H_n \wedge n\|_{L^2(\Gamma\times]0,T[)} + d$. Then (E_n, H_n) is bounded in $H^1(\Omega \times]0,T[)$. By compact injection of $H^1(\Omega \times]0,T[)$ in $L^2(\Omega \times]0,T[)$, we can extract a convergent subsequence. And there exists $(E, H) = \lim_{n\to+\infty} (E_n, H_n)$ in $L^2(\Omega \times]0,T[)$ such that

$$\begin{cases} \varepsilon\partial_t E - curl H = 0,\ \mu\partial_t H + curl E = 0 \quad \text{in } \Omega\times]0,T[\\ div\,(\varepsilon E) = 0,\ div\,(\mu H) = 0 \quad \text{in } \Omega\times]0,T[\\ E\wedge n = 0,\ H.n = 0 \quad \text{on } \Gamma_o\times]0,T[\\ E\wedge n = 0,\ H\wedge n = 0 \quad \text{on } \Gamma\times]0,T[\\ \|(E,H)\|_{L^2(\Omega\times]0,T[)} = 1\ . \end{cases}$$

We introduce $\mathcal{N}$ the space of such solutions which is a nonempty closed subspace in $H^1(\Omega \times]0,T[)$ and such that $\|(E,H)\|_{H^2(\Omega\times]0,T[)} \leq d\,\|(E,H)\|_{H^1(\Omega\times]0,T[)}$. Thus, by compact injection of $H^2(\Omega \times]0,T[)$ in $H^1(\Omega \times]0,T[)$, we deduce that $\mathcal{N}$ is a space of finite dimension.

As ∂_t is a linear application from $\mathcal{N}$ to $\mathcal{N}$, there exist λ and $(E,H) \in \mathcal{N}$ such that $\partial_t (E,H) = \lambda (E,H)$. We conclude that $(E,H) \equiv (0,0)$, because $(E(\cdot,0), H(\cdot,0)) \in \mathcal{W} \cap \mathcal{M}_\Gamma$, where $\mathcal{M}_\Gamma$ is the orthogonal space of steady states for the norm $\mathcal{V}$, which contradicts the fact that $\|(E,H)\| = 1$.

This completes the proof of theorem 4.2.

5 Internal stabilization of the Maxwell equations

We study the asymptotic behaviour in time of the electromagnetic field of the Maxwell equations with Ohm's law. This work completes some previous results on the internal dissipation for the Maxwell equations [Ph2]. We also recall that a low frequency study of the dissipative Maxwell equations was done by N.Weck and K.Witsch [WW].

Here, we will easily see that when the dissipation acts on the electric field with a positive conductivity then the energy decays exponentially and uniformly. More interesting is the case when we choose the conductivity $\sigma \geq 0$. Let $\omega_- = \Omega \backslash (\mathrm{supp}\sigma \cap \Omega)$ and ω_+ be two open connected sets such that $\omega_- = \Omega \backslash (\mathrm{supp}\sigma \cap \Omega)$ and $\Omega = \omega_+ \cup \Gamma \cup \omega_-$ where Γ denotes the interface surface between ω_+ and ω_-. Under the geometric control condition, we show that the decay rate of the energy (exponential or polynomial) depends on the way σ grows at the interface Γ. More precisely, we investigate two cases. First we consider $\sigma \in L^\infty(\Omega)$ such that σ is bounded by two positive constants in ω_+, and prove the uniform and exponential decay. Second, we consider $\sigma \in C(\Omega)$, vanishing on Γ with the same order than $dist\,(x,\Gamma)$, and prove the polynomial decay of the energy.

5.1 Presentation of the Maxwell equations with Ohm's law

Let Ω be a bounded open connected region in $\mathbb{R}^3$, with a smooth boundary $\partial\Omega$ (C^∞ having no contacts of infinite order with its tangents). The domain Ω is occupied by an electromagnetic medium of constant electric permittivity ε and constant magnetic permeability μ. Let E and H denote the electric field and the magnetic field respectively. The Maxwell equations with Ohm's law are described by

$$\left\{\begin{array}{c} \varepsilon\partial_t E - curl H + \sigma E = 0,\ \mu\partial_t H + curl E = 0 \quad \text{in } \Omega\times[0,+\infty[\\ E(\cdot,0)=E_o,\ H(\cdot,0)=H_o \quad \text{in } \Omega \\ div(\mu H)=0 \quad \text{in } \Omega\times[0,+\infty[\\ E\wedge n = 0,\ H.n=0 \quad \text{on } \partial\Omega\times[0,+\infty[\ . \end{array}\right. \tag{5.1}$$

The conductivity is such that $\sigma\in L^\infty(\Omega)$ and $\sigma\geq 0$. We begin to establish the main well-posedness results. The asymptotic behaviour in time of the energy of the electromagnetic field is then studied.

5.1.1 Existence and uniqueness

Let $\mathcal{A}_o$ be an unbounded operator on the Hilbert space $\mathcal{H}_o = \left(L^2(\Omega)\right)^6$ with domain $D(\mathcal{A}_o)$, defined as follows.

$$\begin{array}{l} \|(f,g)\|^2_{\mathcal{H}_o} = \varepsilon\|f\|^2_{L^2(\Omega)} + \mu\|g\|^2_{L^2(\Omega)} \\ \mathcal{A}_o = \left(\begin{array}{cc} 0 & -\varepsilon^{-1}curl \\ \mu^{-1}curl & 0 \end{array}\right) \\ D(\mathcal{A}_o) = \left\{(f,g)\in\mathcal{H}_o \setminus\ (curl f, curl g)\in\mathcal{H}_o,\ f\wedge n_{|\partial\Omega}=0\right\}\ . \end{array}$$

We recall that $-\mathcal{A}_o$ is the infinitesimal generator of an unitary group in $\mathcal{H}_o$ of class $\mathcal{C}^o$ ([DL]($8, p.519$) or [C]($p.255$)).

Let $\mathcal{A}$ be an unbounded operator on the Hilbert space $\mathcal{H}_o$ defined as follows.

$$\mathcal{A} = \mathcal{A}_o + \mathcal{B} \quad \text{,with } \mathcal{B} = \left(\begin{array}{cc} \varepsilon^{-1}\sigma & 0 \\ 0 & 0 \end{array}\right)\ .$$

We have

$D(\mathcal{A}_o+\mathcal{B})$ is dense in $\mathcal{H}_o$ (because $D(\mathcal{A}_o+\mathcal{B}) = D(\mathcal{A}_o)$ is dense in $\mathcal{H}_o$).

$(\mathcal{A}_o+\mathcal{B})$ is closed (because $\mathcal{B}$ is a bounded operator).

We check that $-(\mathcal{A}_o+\mathcal{B})$ and $-(\mathcal{A}_o+\mathcal{B})^* = (\mathcal{A}_o-\mathcal{B})$ are dissipatives and that for all $\lambda>0$, $(Id+\lambda(\mathcal{A}_o+\mathcal{B}))$ is a bijection from $D(\mathcal{A}_o)$ to $\mathcal{H}_o$. Consequently, $-\mathcal{A}$ is the generator of an infinitesimal semi-group of class $\mathcal{C}^o$.

On another hand, let

$$\begin{array}{r} \mathcal{V} = \left\{f\in\left(L^2(\Omega)\right)^3\right\}\times\left\{g\in\left(L^2(\Omega)\right)^3 \setminus\ divg=0,\ g.n_{|\partial\Omega}=0\right\} \\ \mathcal{W} = \left\{(f,g)\in\left(L^2(\Omega)\right)^3\times\left(L^2(\Omega)\right)^3 \setminus\ curl f\in\left(L^2(\Omega)\right)^3,\ f\wedge n=0,\right. \\ \left. divg=0,\ g.n_{|\partial\Omega}=0,\ curl g\in\left(L^2(\Omega)\right)^3\right\}\ . \end{array}$$

Remark that the space $\mathcal{V}$ is stable for the semi-group and we get the following well-posedness result.

$$\forall (E_o, H_o) \in \mathcal{W} \quad \exists! \, (E, H) \in C^0 \left( ]0, +\infty[\, , \mathcal{W} \right) \cap C^1 \left( ]0, +\infty[\, , \mathcal{V} \right) \quad \text{solution of system (5.1)} \ .$$

Moreover, posing $\mathcal{E}(t) = \frac{1}{2} \int_\Omega \left(\varepsilon |E|^2 + \mu |H|^2 \right)$ the energy of the system (5.1), we have

$$\forall (E_o, H_o) \in \mathcal{W} \qquad \frac{d}{dt} \mathcal{E}(t) + \int_\Omega \sigma |E|^2 = 0 \ .$$

This last relation implies that the energy is a decreasing function of time.

On another hand, posing $\mathcal{E}'(t)$ a functional defined by $\mathcal{E}'(t) = \frac{1}{2} \int_\Omega \left(\varepsilon |\partial_t E|^2 + \mu |\partial_t H|^2 \right)$, we have $\frac{d}{dt} \mathcal{E}'(t) + \int_\Omega \sigma |\partial_t E|^2 = 0$.

5.1.2 Stable sub-spaces

We will make the following geometrical assumptions ([DL]($5, p.252$)):

- Let Γ_0,..,Γ_m be the connected components of $\partial\Omega$.
- Let Σ_1,..,Σ_N be the cuttings of Ω so that the domain $\Omega \setminus \cup \Sigma_j$ is simply connected.

Let us introduce $\mathcal{M}_H$ the orthogonal space to $\mathbb{H}_1(\Omega)$ for the $\left(L^2(\Omega) \right)^3$ norm. Thus, $\mathcal{M}_H = \left(L^2(\Omega) \right)^3$, if the assumptions of the Poincaré lemma hold. Let $\mathcal{S}_H = \left(L^2(\Omega) \right)^3 \times \mathcal{M}_H$.

The space $\mathcal{W} \cap \mathcal{S}_H$ is stable for the system (5.1). Indeed, it is enough to multiply the equation $\mu \partial_t H + curl E = 0$ by $h_1 \in \mathbb{H}_1(\Omega)$ and to integrate by parts over Ω in order to check it. Consequently, we obtain the following well-posedness results.

$\forall (E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_H \quad \exists! \, (E, H) \in C^0 \left( ]0, +\infty[\, , \mathcal{W} \cap \mathcal{S}_H \right) \cap C^1 \left( ]0, +\infty[\, , \mathcal{V} \cap \mathcal{S}_H \right)$ solution of the system (5.1).

In order to study the range of the curl, we add the following geometrical assumptions on ω_- ([DL]($5, p.252$)).

- Let γ_0,..,γ_r be the connected components of $\partial\omega_-$.
- Let s_1,..,s_ℓ be the cuttings of ω_- so that the domain $\omega_- \setminus \cup s_j$ is simply connected.

We recall that the range of the curl, $curl \left(H^1(\omega_-) \right)^3$, is closed in $\left(L^2(\omega_-) \right)^3$ ([DL]($5, p.257$)), and

$$curl \left(H^1(\omega_-) \right)^3 = \left\{ U \in \left(L^2(\omega_-) \right)^3 \setminus div U = 0, \ \int_{\gamma_i} U.n = 0 \quad \text{for } i = 0 \text{ to } r \right\} \ .$$

Its orthogonal space for the $\left(L^2(\omega_-) \right)^3$ norm is

$$\begin{aligned} \left(curl \left(H^1(\omega_-) \right)^3 \right)^\perp &= \left\{ V \in \left(L^2(\omega_-) \right)^3 \setminus curl V = 0, \ V \wedge n_{|\partial\omega_-} = 0 \right\} \\ &= \left\{ V = \nabla\varphi \setminus \varphi \in H^1(\omega_-), \ \varphi_{|\gamma_i} = constant \text{ for } i = 0 \text{ to } r \right\} \ . \end{aligned}$$

Under the assumption $\omega_- \neq \emptyset$, let us introduce $\mathcal{M}_\omega = \left(curl\left(H^1(\omega_-)\right)^3 \cap \left(L^2(\Omega)\right)^3\right) \times \mathcal{M}_H$. The space $\mathcal{W} \cap \mathcal{M}_\omega$ is stable for the system (5.1), which can be seen by multiplying by $V \in \left(curl\left(H^1(\omega_-)\right)^3\right)^\perp$ the equation $\varepsilon \partial_t E - curl H + \sigma E = 0$. Then, we have the following well-posedness result.

$\forall (E_o, H_o) \in \mathcal{W} \cap \mathcal{M}_\omega \quad \exists! \, (E,H) \in C^0\left(]0,+\infty[\,, \mathcal{W}\cap\mathcal{M}_\omega\right) \cap C^1\left(]0,+\infty[\,, \mathcal{V}\cap\mathcal{M}_\omega\right)$ solution of the system (5.1).

5.1.3 Orthogonal decomposition

We will decompose the electric field in two parts: one part with free divergence and another part with free curl.

We need to get the following decomposition for the electromagnetic field.

Lemma 5.1
$\forall (E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_H \quad \exists! \, (p, h_2, A) \in C^1\left(]0,+\infty[\,, H_0^1(\Omega) \times \mathbb{H}_2(\Omega)\right) \times C^2\left(]0,+\infty[\,, \left(H^1(\Omega)\right)^3\right)$ such that

$$\begin{cases} \quad (E,H) \text{ solution of the the system (5.1)} \\ E = -\nabla p - \partial_t A + h_2 \\ \mu H = curl A \\ div A = 0, \ A \wedge n_{|\partial\Omega} = 0, \ \int_{\Gamma_i} A.n d\Gamma = 0 \quad \text{for } i = 0 \text{ to } m \ . \end{cases}$$

Moreover, we have the following relations

$$\|E\|^2_{L^2(\Omega)} = \|\nabla p\|^2_{L^2(\Omega)} + \|\partial_t A\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Omega)} \tag{5.2}$$

$$\exists c > 0 \qquad \|A\|^2_{L^2(\Omega)} \leq c \, \|curl A\|^2_{L^2(\Omega)} \ . \tag{5.3}$$

Proof of Lemma 5.1.- The magnetic field $H \in \left(L^2(\Omega)\right)^3$ can be uniquely written as follows

$$\mu H = \nabla q + h_1 + curl A, \ o\grave{u} \ q \in H^1(\Omega), \ h_1 \in \mathbb{H}_1(\Omega), \ A \in \left(H^1(\Omega)\right)^3$$

such that $div A = 0, \ A \wedge n_{|\partial\Omega} = 0, \ \int_{\Gamma_i} A.n d\Gamma = 0 \quad$ for $i = 0$ to m ([DL](5, p.261)).

But $div H = 0$ and $H.n_{|\partial\Omega} = 0$, and consequently, $\Delta q = 0$ and $\partial_n q = 0$, which implies that q is a constant function. Moreover, $H \in \mathcal{M}_H$, and then $h_1 = 0$.

Going back to the system (5.1), we have

$$\begin{cases} curl\left(\partial_t A + \nabla p + E\right) = 0 \\ div\left(\partial_t A + \nabla p + E\right) = 0 \\ \left(\partial_t A + \nabla p + E\right) \wedge n_{|\partial\Omega} = 0 \end{cases}$$

which implies the desired decomposition for the electromagnetic field, with $h_2 = \partial_t A + \nabla p + E$. By the second definition of $\mathbb{H}_2(\Omega)$, A is orthogonal to h_2, $\int_\Omega A h_2 = \int_\Omega A.\nabla\varphi = -\int_\Omega \varphi div A + \int_{\partial\Omega} \varphi A.n = 0$, and we deduce the equality (5.2).

The inequality (5.3) is obtained by a contradiction argument. Indeed, if it is false, then there exists a sequence (A_n) bounded in $\left(H^1(\Omega)\right)^3$ such that

$$\|A_n\|_{L^2(\Omega\times]0,T[)} = 1 \text{ and } \lim_{n\to+\infty} \|curlA_n\|_{L^2(\Omega)} = 0,\ divA_n = 0,\ A_n \wedge n_{|\partial\Omega} = 0\ .$$

By compact injection of $H^1(\Omega)$ in $L^2(\Omega)$, there exists $A \in \mathbb{H}_2(\Omega)$ such that $\|A\|_{L^2(\Omega)} = 1$ and $\int_\Omega Ah_2 = 0 \quad \forall h_2 \in \mathbb{H}_2(\Omega)$, which is impossible.

This ends the proof of lemma 5.1.

5.1.4 Applications

We begin to link locally in time the electric field to the magnetic field.

Proposition 5.1 Let $T > 0$, there exists $c > 0$ such that for any initial data $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_H$ of the system (5.1), we have

$$\int_0^T \int_\Omega \varepsilon |E|^2 + \int_0^T \int_\Omega \mu |H|^2 \leq c \int_0^T \int_\Omega |E|^2\ . \tag{5.4}$$

Proof of proposition 5.1.- It comes from the lemma below

Lemma 5.2 Let $T > 0$ and $\Phi \in C_0^\infty(]0,T[)$, there exists $c > 0$ such that for any initial data $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_H$ of the the system (5.1), we have

$$\int_0^T \int_\Omega \varepsilon |\Phi E|^2 + \int_0^T \int_\Omega \mu |\Phi H|^2 \leq c \int_0^T \int_\Omega |E|^2 \tag{5.5}$$

where the constant c depends on $\sup_{[0,T]} |\Phi|$ and $\sup_{[0,T]} |\partial_t \Phi|$.

Proof of lemma 5.2.- It comes from lemma 5.1.
We decompose the electric field E and the magnetic field H as follows

$$\left\{\begin{array}{l} E = -\nabla p - \partial_t A + h_2 \\ \mu H = curlA \end{array}\right. \quad \text{where} \quad \left\{\begin{array}{l} p \in H_0^1(\Omega)\,,\ h_2 \in \mathbb{H}_2(\Omega)\,,\ A \in \left(H^1(\Omega)\right)^3 \\ divA = 0,\ A \wedge n_{|\partial\Omega} = 0,\ \int_{\Gamma_i} A.nd\Gamma = 0 \quad \text{for } i = 0 \text{ to } m. \end{array}\right.$$

We have by integration by parts the following relations

$$\begin{array}{rl} \int_0^T \int_\Omega |\Phi rotA|^2 & = \int_0^T \int_\Omega \Phi^2 AcurlcurlA \\ & = \int_0^T \int_\Omega \Phi^2 A\mu\left(\varepsilon\partial_t E + \sigma E\right) \\ & = -\int_0^T \int_\Omega \partial_t\left(\Phi^2 A\right)\mu\varepsilon E + \int_0^T \int_\Omega \Phi^2 A\mu\sigma E \\ & = -\int_0^T \int_\Omega 2\Phi\partial_t\Phi A\mu\varepsilon E - \int_0^T \int_\Omega \Phi^2 \partial_t A\mu\varepsilon E + \int_0^T \int_\Omega \Phi^2 A\mu\sigma E\ . \end{array}$$

From the relations (5.2) and (5.3) of lemma 5.1 and using Cauchy-Schwarz inequality, we conclude that $\exists c > 0$

$$\int_0^T \int_\Omega |\Phi curlA|^2 \leq c\left(\int_0^T \int_\Omega |E|^2 + \|\Phi\partial_t A\|_{L^2(\Omega\times]0,T[)} \cdot \|E\|_{L^2(\Omega\times]0,T[)}\right) \tag{5.6}$$

and (5.5) follows.

Choosing $\Phi \in C_0^\infty(]0,T[)$ such that $\Phi = 1$ in $[T/3, 2T/3]$, we have with the decreasing property of the energy

$$\begin{array}{ll} \int_0^T \int_\Omega \varepsilon |E|^2 + \int_0^T \int_\Omega \mu |H|^2 & \leq T\mathcal{E}(0) = T\mathcal{E}(T) + T \int_0^T \int_\Omega \sigma |E|^2 \\ & \leq 3\left(\frac{T}{3}\mathcal{E}\left(\frac{2T}{3}\right)\right) + T\int_0^T \int_\Omega \sigma |E|^2 \\ & \leq (3c + T\sup|\sigma|) \int_0^T \int_\Omega |E|^2 \ . \end{array} \tag{5.7}$$

Remark 5.1 We also have the following assertion.

Let $T > 0$, there exists $c > 0$ such that for all $\zeta > 0$ and $(E_o, H_o) \in \mathcal{W}$ initial data of the system (5.1), we get

$$\int_\zeta^{\zeta+T} \int_\Omega \varepsilon |\partial_t E|^2 + \int_\zeta^{\zeta+T} \int_\Omega \mu |\partial_t H|^2 \leq c \int_\zeta^{\zeta+T} \int_\Omega |\partial_t E|^2 \ . \tag{5.8}$$

Indeed, we decompose $(\partial_t E, \partial_t H) \in \mathcal{W} \cap \mathcal{S}_H$ as follows

$$\left\{ \begin{array}{l} \partial_t E = -\nabla p - \partial_t A + h_2 \\ \mu \partial_t H = curl A \end{array} \right. \quad \text{where} \quad \left\{ \begin{array}{l} p \in H_0^1(\Omega),\ h_2 \in \mathbb{H}_2(\Omega),\ A \in \left(H^1(\Omega)\right)^3 \\ div A = 0,\ A \wedge n_{|\partial\Omega} = 0,\ \int_{\Gamma_i} A.nd\Gamma = 0 \quad \text{for } i = 0 \text{ to } m \end{array} \right.$$

We have by integration by parts, with $I =]\zeta, \zeta + T[$ and $\Phi \in C_0^\infty(I)$, the following relations

$$\begin{array}{ll} \int_I \int_\Omega |\Phi rot A|^2 & = \int_I \int_\Omega \Phi^2 A curl curl A \\ & = -\int_I \int_\Omega \Phi^2 A \mu \left(\varepsilon \partial_t^2 E + \sigma \partial_t E\right) \\ & = \int_I \int_\Omega \partial_t\left(\Phi^2 A\right) \mu\varepsilon \partial_t E - \int_I \int_\Omega \Phi^2 A \mu \sigma \partial_t E \\ & = \int_I \int_\Omega 2\Phi \partial_t \Phi A \mu \varepsilon \partial_t E + \int_I \int_\Omega \Phi^2 \partial_t A \mu \varepsilon \partial_t E + \int_I \int_\Omega \Phi^2 A \mu \sigma \partial_t E \ . \end{array}$$

From the relations (5.2), (5.3) of lemma 5.1 and using Cauchy-Schwarz inequality, we obtain that $\exists c > 0$

$$\int_\zeta^{\zeta+T} \int_\Omega \varepsilon |\Phi \partial_t E|^2 + \mu |\Phi \partial_t H|^2 \leq c \int_\zeta^{\zeta+T} \int_\Omega |\partial_t E|^2 \ . \tag{5.9}$$

We conclude in the same way than in the proof of proposition 5.1.

Remark 5.2 The study of the electric field $E \in C^1\left([0,+\infty[; \left(L^2(\Omega)\right)^3\right)$ will be made with the following orthogonal decomposition : $E = -\nabla p + W$, where $p \in C^1\left([0,+\infty[; H_0^1(\Omega)\right)$ and $W \in C^1\left([0,+\infty[; \left(L^2(\Omega)\right)^3,\ \text{with free divergence}\right)$.

Thus,

$$\left\{ \begin{array}{c} div E = -\Delta p \\ curl E = curl W \\ E \wedge n_{|\partial\Omega} = W \wedge n_{|\partial\Omega} \end{array} \right. \tag{5.10}$$

(indeed, $\nabla p \wedge n$ is a tangential derivative of p on $\partial\Omega$, but $p_{|\partial\Omega} = 0$).

The equations of the system (5.1) become

$$\left\{ \begin{array}{c} \varepsilon \partial_t W - \varepsilon \partial_t \nabla p - curl H + \sigma E = 0 \quad \text{in } \Omega \times [0,+\infty[\\ \mu \partial_t H + curl E = 0 \quad \text{in } \Omega \times [0,+\infty[\\ div W = div H = 0 \quad \text{in } \Omega \times [0,+\infty[\\ W \wedge n = 0,\ H.n = 0,\ p = 0 \quad \text{on } \partial\Omega \times [0,+\infty[\ . \end{array} \right. \tag{5.11}$$

Then, the curl part of the electric field, W, solves the hyperbolic system

$$\left\{ \begin{array}{c} \varepsilon \partial_t^2 W + \mu^{-1} curl curl W + \sigma \partial_t W - \sigma \partial_t \nabla p - \varepsilon \partial_t^2 \nabla p = 0,\ div W = 0 \quad \text{in } \Omega \times [0,+\infty[\\ W \wedge n = 0 \quad \text{on } \partial\Omega \times [0,+\infty[\end{array} \right.$$

or also

$$\begin{cases} \varepsilon\partial_t^2 W - \mu^{-1}\Delta W + \sigma\partial_t W - \sigma\partial_t \nabla p - \varepsilon\partial_t^2 \nabla p = 0 \quad \text{in} \quad \Omega\times[0,+\infty[\\ W\wedge n = 0, \ divW = 0 \quad \text{on } \partial\Omega\times[0,+\infty[\ . \end{cases}$$

Concerning the divergence part of the electric field, we have the following result.

Proposition 5.2 For any initial data $(E_o, H_o) \in \mathcal{W}$ of system (5.1), we have

$$\|\varepsilon\partial_t\nabla p\|_{L^2(\Omega)} \le \|\sigma E\|_{L^2(\Omega)} \tag{5.12}$$

$$\left\|\varepsilon\partial_t^2\nabla p\right\|_{L^2(\Omega)} \le \|\sigma\partial_t E\|_{L^2(\Omega)} \ . \tag{5.13}$$

Proof of proposition 5.2.- We multiply by $\varepsilon\partial_t\nabla p$ the equation $\varepsilon\partial_t W - \varepsilon\partial_t\nabla p - curlH + \sigma E = 0$ and we integrate by parts over Ω.

5.2 Asymptotic behaviour

5.2.1 Dissipation on all the domain

We easily deduce from proposition 5.1 the following result.

Theorem 5.1 Let $C > 0$. Suppose that $\sigma \in L^\infty(\Omega)$ is such that $\sigma(x) \ge C$ for all $x \in \Omega$. Then there exist $c > 0$ and $\beta > 0$ such that for any initial data $(E_o, H_o) \in \mathcal{V}\cap\mathcal{S}_H$ of the system (5.1), we have

$$\forall t \ge 0 \quad \mathcal{E}(t) \le ce^{-\beta t}\ \mathcal{E}(0) \ .$$

5.2.2 Dissipation on a part of the domain

Consider the case where $\omega_- = \Omega\setminus(\mathrm{supp}\sigma\cap\Omega)$ is a non-empty connected open set and $\sigma \in L^\infty(\Omega)$. We begin to establish the asymptotic behaviour in time of $(\partial_t E, \partial_t H)$.

Theorem 5.2 $\forall (E_o, H_o) \in \mathcal{W}$ initial data of the system (5.1) $\lim_{t\to+\infty} \mathcal{E}'(t) = 0.$

We deduce the following theorem

Theorem 5.3 Suppose that $\partial\Omega$ has only one connected component. $\forall (E_o, H_o) \in \mathcal{V}\cap\mathcal{M}_\omega$ initial data of the system (5.1) $\lim_{t\to+\infty} \mathcal{E}(t) = 0.$

Also, we look for a sufficient condition in order to bound continuously the energy of the electromagnetic field by the functional $\mathcal{E}'$.

Theorem 5.4 Suppose that $\partial\Omega$ has only one connected component and that $\omega_+ = \Omega\setminus(\overline{\omega_-}\cap\Omega)$ is a connected open set. Let $C > 0$. If $\sigma \in L^\infty(\Omega)$ such that $\sigma(x) \ge C$ for all $x \in \omega_+$, then there exists $c > 0$ such that for all initial data $(E_o, H_o) \in \mathcal{V}\cap\mathcal{M}_\omega$ of the system (5.1), we have

$$\forall t \ge 0 \quad \mathcal{E}(t) \le c\mathcal{E}'(t) \ .$$

Proof of theorem 5.2.- It comes from the following lemma.

Lemma 5.3 $\lim_{t\to+\infty} \int_\Omega |\partial_t \nabla p|^2 = 0.$

Proof of lemma 5.3.-

$$\begin{aligned}(t-T)\|\partial_t\nabla p\|^2_{L^2(\Omega)} &= \int_T^t \tfrac{d}{ds}(s-T)\|\partial_t\nabla p\|^2_{L^2(\Omega)}\,ds + \int_T^t (s-T)\tfrac{d}{ds}\left(\|\partial_t\nabla p\|^2_{L^2(\Omega)}\right)ds \\ &\le \int_T^t \|\partial_t\nabla p\|^2_{L^2(\Omega)}\,ds + 2(t-T)\int_T^t \|\partial_t\nabla p\|_{L^2(\Omega)}\left\|\partial_t^2\nabla p\right\|_{L^2(\Omega)}ds\ .\end{aligned}$$

Choosing $0 < T < 1+T < t$ and multiplying this last inequality by $\frac{1}{(t-T)}$, we have

$$\|\partial_t\nabla p\|^2_{L^2(\Omega)} \le 2\int_T^t \|\partial_t\nabla p\|^2_{L^2(\Omega)}\,ds + \int_T^t \left\|\partial_t^2\nabla p\right\|_{L^2(\Omega)}ds\ .$$

On another hand, from the inequalities (5.12) and (5.13) of proposition 5.2, we get

$$\int_T^t |\partial_t\nabla p|^2 \le \varepsilon^{-2}(\sup|\sigma|)\int_T^t\left(-\frac{d}{dt}\mathcal{E}\right) \le \varepsilon^{-2}(\sup|\sigma|)\,|\mathcal{E}(T)-\mathcal{E}(t)|$$

$$\int_T^t \left|\partial_t^2\nabla p\right|^2 \le \varepsilon^{-2}(\sup|\sigma|)\int_T^t\left(-\frac{d}{dt}\mathcal{E}'\right) \le \varepsilon^{-2}(\sup|\sigma|)\,|\mathcal{E}'(T)-\mathcal{E}'(t)|\ .$$

As the functions $\mathcal{E}(t)$ and $\mathcal{E}'(t)$ are continuous, decreasing and positives, they are convergent

$$\forall\epsilon>0\quad \exists T_1>0\quad \forall s,s>T_1\quad |\mathcal{E}(s)-\mathcal{E}(t)|+|\mathcal{E}'(s)-\mathcal{E}'(t)|<\epsilon\ .$$

Let $\epsilon > 0$, choosing $T = 1 + T_1$, we conclude that

$$\forall t>(1+T)\quad \|\partial_t\nabla p(\cdot,t)\|^2_{L^2(\Omega)} \le 3\varepsilon^{-2}(\sup|\sigma|)\,\epsilon\ .$$

Now, we go back to the proof of theorem 5.2.

We check that $(\partial_t W, \partial_t H)$ is uniformly bounded in $\left(H^1(\Omega)\right)^6$ as soon as $(E_o, H_o) \in D\left(\mathcal{A}^2\right)$, and more precisely, we have the following estimate

$$\|(\partial_t W,\partial_t H)\|_{H^1(\Omega)} \le \|(E_o,H_o)\|_{D(\mathcal{A}^2)}\ . \tag{5.14}$$

The injection of $H^1(\Omega)$ in $L^2(\Omega)$ is compact and then we deduce that there exists a subsequence $(\partial_t W(t_k), \partial_t H(t_k))$ such that

$$\lim_{k\to+\infty}(\partial_t W(t_k),\partial_t H(t_k)) = (W'_\infty, H'_\infty)\quad \text{in } \left(L^2(\Omega)\right)^6\ .$$

Remark also that

$$(\partial_t E,\partial_t H) = Z(t)(E'_o,H'_o)\quad \text{where } (\varepsilon E'_o,\mu H'_o) = (curl H_o-\sigma E_o, -curl E_o)\in D(\mathcal{A})\ .$$

From lemma 5.3, we deduce that there exists a subsequence $Z(t_k)(E'_o, H'_o)$ such that

$$\lim_{k\to+\infty} Z(t_k)(E'_o,H'_o) = (E'_\infty,H'_\infty)\quad \text{in } \left(L^2(\Omega)\right)^6\ .$$

Moreover, we notice that $divE'_\infty = 0$ in Ω, because $divW'_\infty = 0$ in Ω. On another hand, we check that $H'_\infty \in \mathcal{M}_H$, because $\int_\Omega H'_o h_1 = 0 \quad \forall h_1 \in \mathbb{H}_1(\Omega)$.
As $Z(t)$ is a semi-group, we also have $\quad \forall s > 0$

$$\lim_{t_k \to +\infty} \|Z(s) Z(t_k - s)(E'_o, H'_o)\|_{\mathcal{H}_o} = \lim_{t_k \to +\infty} \|Z(t_k)(E'_o, H'_o)\|_{\mathcal{H}_o} = \|(E'_\infty, H'_\infty)\|_{\mathcal{H}_o} .$$

Now, $2\mathcal{E}'(t) = \left\|\left(\sqrt{\varepsilon}\partial_t E, \sqrt{\mu}\partial_t H\right)\right\|^2_{L^2(\Omega)} = \|Z(t)(E'_o, H'_o)\|^2_{\mathcal{H}_o}$ is a decreasing positive function. Choosing a subsequence $t_{k'}$ such that $t_{k'} \leq t_k - s$, we obtain the relations

$$\begin{aligned} \|(E'_\infty, H'_\infty)\|_{\mathcal{H}_o} &\leq \lim_{t_{k'} \to +\infty} \|Z(s) Z(t_{k'})(E'_o, H'_o)\|_{\mathcal{H}_o} \\ &\leq \|Z(s)(E'_\infty, H'_\infty)\|_{\mathcal{H}_o} \leq \|(E'_\infty, H'_\infty)\|_{\mathcal{H}_o} . \end{aligned}$$

We conclude that $(E'_\infty, H'_\infty) = (0, 0)$.

Indeed, (E'_∞, H'_∞) is initial data of the following system

$$\begin{cases} \varepsilon\partial_t E - curlH = 0, \ \mu\partial_t H + curlE = 0 \quad \text{in } \Omega \times [0, +\infty[\\ divH = 0 \quad \text{in } \Omega \times [0, +\infty[\\ E \wedge n = 0, \ H.n = 0 \quad \text{on } \partial\Omega \times [0, +\infty[\\ E = 0 \quad \text{in } \omega_+ \times [0, +\infty[\end{cases}$$

with $(E'_\infty, H'_\infty) \in \left(L^2(\Omega)\right)^6$ such that $H'_\infty \in \mathcal{M}_H$ and $divE'_\infty = 0$. Consequently, $divE \equiv 0$. The electric field E is then solution of a hyperbolic system. By Holmgren theorem or by the unique continuation result of J.Rauch and M.Taylor [RT], we get $E \equiv 0$. Then, $H \in \mathbb{H}_1(\Omega) \cap \mathcal{M}_H$, and also $H \equiv 0$. In particular, $(E'_\infty, H'_\infty) = (0, 0)$.

Finally, by density of $D\left(\mathcal{A}^2\right)$ in $D(\mathcal{A})$, we have

$$\forall (E_o, H_o) \in D(\mathcal{A}) \text{ initial data of the system (5.1)} \quad \lim_{t \to +\infty} \mathcal{E}'(t) = 0 .$$

Proof of theorem 5.3.- We deduce the asymptotic behaviour of the electromagnetic field by a density argument.

Let $\mathcal{A}_\omega$ be the restriction of the operator $\mathcal{A}$ on the space $\mathcal{V} \cap \mathcal{M}_\omega$. We check that $-\mathcal{A}_\omega$ is dissipative, $\mathcal{A}_\omega$ is closed and $D(\mathcal{A}_\omega) = \mathcal{W} \cap \mathcal{M}_\omega$ is dense in $\mathcal{V} \cap \mathcal{M}_\omega$. As $\overline{Im\mathcal{A}_\omega} = (Ker\mathcal{A}^*_\omega)^\perp$ with $Ker\mathcal{A}^*_\omega = (Ker\mathcal{A}^*) \cap \mathcal{M}_\omega$, we claim that $Im\mathcal{A}_\omega$ is dense in $\mathcal{V} \cap \mathcal{M}_\omega$ if $\partial\Omega$ has only one connected component.

Indeed, we are reduce to solve the following static system

$$\begin{cases} curlH + \sigma E = 0, \ rotE = 0 \quad \text{in } \Omega \\ divH = 0 \quad \text{in } \Omega \\ E \wedge n = 0, \ H.n = 0 \quad \text{on } \partial\Omega \\ (E, H) \in \mathcal{M}_\omega, \ \mathbb{H}_2(\Omega) = \{0\} . \end{cases}$$

We claim that $(E, H) \equiv 0$. Indeed, from Green formula $E = 0$ in ω_+. We recall that $\omega_- \neq \emptyset$, and Ω is divided by two non-empty open sets ω_+ and ω_- such that $\Omega = \omega_+ \cup \Gamma \cup \omega_-$ where Γ is a surface and $\omega_- = \Omega \backslash (\text{supp}\sigma \cap \Omega)$ with $\sigma > 0$ in ω_+.

From the decomposition of the electric field as $E = -\nabla p + W$, it becomes

$$\begin{cases} divH = 0 \\ curlH = 0 \\ H.n = 0 \end{cases} \quad \text{and} \quad \begin{cases} divW = 0 \\ curlW = 0 \\ W \wedge n = 0 \end{cases}$$

which implies that $(W,H)\equiv 0$, because $H\in(\mathbb{H}_1(\Omega))^{\perp}$ and $\mathbb{H}_2(\Omega)=\{0\}$.
We deduce that $\nabla p=0$ in ω_+ and $p=constant$ on Γ. On another hand, from Green formula, we have

$$\int_{\omega_-}|\nabla p|^2=\int_{\omega_-}divEp-\int_{\partial\omega_-}E.np=\int_{\omega_-}divEp-\sum_{i=0}^{r}p_{|\gamma_i}\int_{\gamma_i}E.n\ .$$

But, $E_{|\omega_-}\in curl\left(H^1(\omega_-)\right)^3$, where

$$curl\left(H^1(\omega_-)\right)^3=\left\{E\in\left(L^2(\omega_-)\right)^3\setminus divE=0 \text{ in } \omega_-,\ \int_{\gamma_i}E.n=0 \quad \text{for } i=0 \text{ to } r\right\}\ .$$

Consequently, $E=0$ in ω_-. This ends the resolution of the static system.

Finally, if $\partial\Omega$ has only one connected component, then $\overline{Im\mathcal{A}_\omega}=\mathcal{V}\cap\mathcal{M}_\omega$. This can be written as follows

$$\forall\epsilon>0\quad\forall(E_o,H_o)\in\mathcal{V}\cap\mathcal{M}_\omega\quad\exists(U_o,V_o)\in\mathcal{W}\cap\mathcal{M}_\omega\qquad\|(E_o,H_o)-\mathcal{A}(U_o,V_o)\|_{L^2(\Omega)}\le\epsilon\ .$$

Then, $\quad\forall\epsilon,t>0$

$$\|Z(t)(E_o,H_o)\|\le\epsilon+2\mathcal{E}'(t;(U_o,V_o))\ . \tag{5.15}$$

We conclude that if $\partial\Omega$ has only one connected component, then

$$\forall(E_o,H_o)\in\mathcal{V}\cap\mathcal{M}_\omega \text{ initial data of the system (5.1)}\quad\lim_{t\to+\infty}\mathcal{E}(t)=0\ .$$

Proof of theorem 5.4.- We want to prove the following estimate $\exists c>0\quad\forall t\ge 0$

$$\mathcal{E}(t)=\frac{1}{2}\int_\Omega\left(\varepsilon|E|^2+\mu|H|^2\right)\le c\frac{1}{2}\int_\Omega\left(\varepsilon|\partial_tE|^2+\mu|\partial_tH|^2\right)=c\mathcal{E}'(t)\ .$$

From the equations (5.1) and the positivity of σ in ω_+, we obtain

$$(\inf|\sigma|)\int_{\omega_+}|E|^2\le\int_\Omega\sigma|E|^2=-\int_\Omega\varepsilon\partial_tEE-\int_\Omega\mu\partial_tHH\le 8\sqrt{\mathcal{E}(t)}\sqrt{\mathcal{E}'(t)}\ . \tag{5.16}$$

It follows from the assumptions $H\in(\mathbb{H}_1(\Omega))^{\perp}$ and $\mathbb{H}_2(\Omega)=\{0\}$, the following inequalities hold $\exists c,d>0$

$$\|H\|_{L^2(\Omega)}\le c\|curlH\|_{L^2(\Omega)}\le d\left(\|\varepsilon\partial_tE\|_{L^2(\Omega)}+\|\sigma E\|_{L^2(\Omega)}\right) \tag{5.17}$$

$$\|W\|_{L^2(\Omega)}\le c\|curlW\|_{L^2(\Omega)}\le d\|\mu\partial_tH\|_{L^2(\Omega)}\ . \tag{5.18}$$

On another hand, we have

$$\|\nabla p\|_{L^2(\omega_+)}\le\|E\|_{L^2(\omega_+)}+\|W\|_{L^2(\Omega)}\ . \tag{5.19}$$

Conclusion, from the inequalities (5.16), (5.17), (5.18) and (5.19), $\exists c>0$

$$\|(H,W)\|^2_{L^2(\Omega)}+\|\nabla p\|^2_{L^2(\omega_+)}\le c\left(\mathcal{E}'(t)+\sqrt{\mathcal{E}(t)}\sqrt{\mathcal{E}'(t)}\right)\ . \tag{5.20}$$

It remains to estimate ∇p in ω_- , this will be done from the result below.

Lemma 5.4 Suppose that ω_+ is a non-empty connected open set. Then there exists $c>0$ such that for any initial data $(E_o,H_o)\in\mathcal{V}\cap\left(curl\left(H^1(\omega_-)\right)^3\cap\left(L^2(\Omega)\right)^3\right)\times\left(L^2(\Omega)\right)^3$ of the system (5.1), we have

$$\|\nabla p\|^2_{L^2(\omega_-)}\leq c\left(\|\nabla p\|^2_{L^2(\omega_+)}+\|W\|^2_{L^2(\omega_-)}\right)\ . \tag{5.21}$$

Proof of lemma 5.4.- We divide the proof into two cases.

Case where $\partial\omega_+\cap\partial\Omega\neq\emptyset$: We recall that $-\Delta p=divE$ and then p solves the following elliptic system

$$\left\{\begin{array}{c}\Delta p=0\quad\text{in }\omega_-\\ p=0\quad\text{on }\partial\Omega\cap\partial\omega_-\\ p\in H^{1/2}(\partial\omega_+)\quad\text{on }\partial\omega_+\cap\partial\omega_-\ .\end{array}\right.$$

Thus, from Poincaré inequality, we have the following estimate $\exists c,d>0$

$$\|\nabla p\|_{L^2(\omega_-)}\leq c\,\|p\|_{H^{1/2}(\partial\omega_+)}\leq d\,\|\nabla p\|_{L^2(\omega_+)}\ . \tag{5.22}$$

Case where $\partial\omega_+\cap\partial\Omega=\emptyset$: we decompose $p\in H^1_0(\Omega)$ as follows. Let

$$\delta\widetilde{e}=\left|\begin{array}{l}\widetilde{e}=p_{|\omega_+}-\frac{1}{mes(\omega_+)}\int_{\omega_+}p\quad\text{in }\omega_+\\ \widehat{e}\quad\text{in }\omega_-,\quad\text{solution of }\left\{\begin{array}{l}\Delta\widehat{e}=0\quad\text{in }\omega_-\\ \widehat{e}=0\quad\text{on }\partial\Omega\\ \widehat{e}=\widetilde{e}\quad\text{on }\partial\omega_+\end{array}\right.\end{array}\right.$$

We check that $\delta\widetilde{e}\in H^1_0(\Omega)$ and $\int_{\omega_+}\widetilde{e}=0$. By a trace theorem and the Poincaré inequality ([DL](3, p.922)) $\exists c,d>0$

$$\|\nabla\widehat{e}\|_{L^2(\omega_-)}\leq c\,\|\widetilde{e}\|_{H^{1/2}(\partial\omega_+)}\leq d\,\|\nabla\widetilde{e}\|_{L^2(\omega_+)}=d\,\|\nabla p\|_{L^2(\omega_+)}\ . \tag{5.23}$$

Let $\overline{e}=p-\delta\widetilde{e}$. We check that $\overline{e}\in H^1_0(\Omega)$ and $\overline{e}$ is a constant function on $\partial\omega_+$. Consequently, $\nabla\overline{e}\in\left(curl\left(H^1(\omega_-)\right)^3\right)^\perp$. But, with $E\in\left(curl\left(H^1(\omega_-)\right)^3\right)$, we deduce that

$$\|\nabla\overline{e}\|^2_{L^2(\omega_-)}=\int_{\omega_-}E\nabla\overline{e}-\int_{\omega_-}\nabla\delta\widetilde{e}\nabla\overline{e}-\int_{\omega_-}W\nabla\overline{e}\ . \tag{5.24}$$

Finally, from (5.23) and (5.24), $\exists c>0$

$$\|\nabla p\|_{L^2(\omega_-)}\leq\|\nabla\delta\widetilde{e}\|_{L^2(\omega_-)}+\|\nabla\overline{e}\|_{L^2(\omega_-)}\leq c\left(\|\nabla p\|_{L^2(\omega_+)}+\|W\|_{L^2(\omega_-)}\right)\ .$$

This ends the proof of lemma 5.4 and completes the proof of theorem 5.4.

5.3 Exponential decay

We complete the asymptotic behaviour of the Maxwell equations with Ohm's law by studying the decay rate of the energy of the electromagnetic field.

Theorem 5.5 Let ω be an open region in $\mathbb{R}^3$ such that ω geometrically controls Ω. Suppose that $\omega_+=(\omega\cap\Omega)$, $\omega_-=\Omega\setminus(\text{supp}\sigma\cap\Omega)$ are non-empty connected open sets and that $\partial\Omega$ has only one connected component. Let $C > 0$. If $\sigma \in L^\infty(\Omega)$ is such that $\sigma(x) = 0$ for all $x \in \omega_-$, and $\sigma(x) \geq C$ for all $x \in \omega_+$, then there exist $c > 0$ and $\beta > 0$ such that we have

$$\forall (E_o, H_o) \in \mathcal{V} \cap \mathcal{M}_\omega \text{ initial data of the system (5.1)} \qquad \forall t \geq 0 \quad \mathcal{E}(t) \leq ce^{-\beta t}\, \mathcal{E}(0).$$

Theorem 5.6 Let ω be an open region in $\mathbb{R}^3$ such that ω geometrically controls Ω. Suppose that $\omega_+ = (\omega \cap \Omega)$, $\omega_- = \Omega \setminus (\text{supp}\sigma \cap \Omega)$ are non-empty connected open sets and that $\partial\Omega$ has only one connected component. Let $C > 0$. If $\sigma \in L^\infty(\Omega)$ is such that $\sigma(x) = 0$ for all $x \in \omega_-$, and $\sigma(x) \geq C$ for all $x \in \omega_+$, then there exist $c > 0$ and $\beta > 0$ such that we have

$$\forall (E_o, H_o) \in \mathcal{W} \cap \mathcal{M}_\omega \text{ initial data of the system (5.1)} \qquad \forall t \geq 0 \quad (\mathcal{E} + \mathcal{E}')(t) \leq ce^{-\beta t}\, (\mathcal{E} + \mathcal{E}')(0).$$

Proof of theorem 5.6.- We divide the proof into two steps.

Step 1 :
We link $\partial_t H$ to $\partial_t E$: recall that from an orthogonal decomposition of the electric field as $E = -\nabla p + W$ and from proposition 5.1, we have the following estimate $\exists c > 0$

$$\int_0^T \mathcal{E}'(t)\, dt \leq c \int_0^T \int_\Omega |\partial_t E|^2 \ . \tag{5.25}$$

Step 2 :
We estimate $\partial_t W$, in order that under the geometric control condition, we have $\exists c > 0$

$$\int_0^T \int_\Omega |\partial_t E|^2 \leq c \left(\int_0^T \int_\Omega \sigma |\partial_t E|^2 + \int_0^T \int_\Omega \sigma |E|^2 \right) \ . \tag{5.26}$$

Indeed, the solution W solves the following hyperbolic system

$$\left\{ \begin{array}{c} \varepsilon\partial_t^2 W - \mu^{-1}\Delta W + \sigma\partial_t E - \varepsilon\partial_t^2 \nabla p = 0 \quad \text{in } \Omega \times [0, +\infty[\\ divW = 0, \ W \wedge n = 0 \quad \text{on } \partial\Omega \times [0, +\infty[\ . \end{array} \right.$$

By applying (1.31) of theorem 1.4, to the hyperbolic system for $\partial_t W$, we get under the geometric control condition $\exists c > 0 \ \exists \widehat{\omega} \subset \Omega$

$$\|\partial_t W\|_{L^2(\Omega \times]0,T[)} \leq c \left(\|\partial_t W\|_{L^2(\widehat{\omega} \times]0,T[)} + \left\|\sigma \partial_t E - \varepsilon \partial_t^2 \nabla p\right\|_{L^2(\Omega \times]0,T[)} \right)$$

with

$$\|\partial_t W\|^2_{L^2(\widehat{\omega} \times]0,T[)} \leq c \int_0^T \int_\Omega \sigma |\partial_t W|^2 \ .$$

Consequently, from (5.12) and (5.13) of proposition 5.2, we obtain $\exists c > 0$

$$\int_0^T \int_\Omega |\partial_t E|^2 = \|(\partial_t W, \partial_t \nabla p)\|^2_{L^2(\Omega \times]0,T[)} \leq c \left(\int_0^T \int_\Omega \sigma |\partial_t E|^2 + \int_0^T \int_\Omega \sigma |E|^2 \right)$$

and finally, from (5.25), $\exists c > 0$

$$\int_0^T \mathcal{E}'(t)\, dt \leq c \left(\int_0^T \int_\Omega \sigma |\partial_t E|^2 + \int_0^T \int_\Omega \sigma |E|^2 \right) \ . \tag{5.27}$$

By applying theorem 5.4, which links $\mathcal{E}'$ to $\mathcal{E}$, we deduce that if ω_+ is connected and $\mathbb{H}_2(\Omega) = \{0\}$, then the following estimate holds

$$\exists c, T > 0 \qquad \{\mathcal{E} + \mathcal{E}'\}(T) \leq c \int_0^T -\frac{d}{dt}\{\mathcal{E} + \mathcal{E}'\}(s)\, ds \ . \tag{5.28}$$

Conclusion,

$$\exists \delta \in \left]0,1\right[, T>0 \qquad \{\mathcal{E}+\mathcal{E}'\}(T) \leq \delta \{\mathcal{E}+\mathcal{E}'\}(0) \ . \tag{5.29}$$

The properties of the semi-group [Pa] allow to complete the proof of theorem 5.6.

The proof of theorem 5.5 comes from the lemma below.

Lemma 5.5 Let $\mathcal{H}$ be a Hilbert space, $-\mathcal{A}$ be the generator of a infinitesimal semi-group $Z(t)$ of class C^0 on $\mathcal{H} \supset D(\mathcal{A})$. If there exist $c, \beta > 0$, such that

$$\forall U_o \in D(\mathcal{A}) \quad \forall t \geq 0 \qquad \|Z(t)U_o\|_{\mathcal{H}}^2 + \|\mathcal{A}Z(t)U_o\|_{\mathcal{H}}^2 \leq ce^{-\beta t} \ \left(\|U_o\|_{\mathcal{H}}^2 + \|\mathcal{A}U_o\|_{\mathcal{H}}^2\right)$$

then we have

$$\forall U_o \in \mathcal{H} \quad \forall t \geq 0 \qquad \|Z(t)U_o\|_{\mathcal{H}}^2 \leq 10ce^{-\beta t} \ \|U_o\|_{\mathcal{H}}^2 \ .$$

Proof of lemma 5.5.- The unbounded operator $-\mathcal{A}$ with domain $D(\mathcal{A}) \subset \mathcal{H}$ is dissipative and

$$\forall V_o \in \mathcal{H} \quad \exists U_o \in D(\mathcal{A}) \qquad U_o + \mathcal{A}U_o = V_o \ .$$

Consequently,

$$\|Z(t)V_o\|_{\mathcal{H}} \leq \|Z(t)U_o\|_{\mathcal{H}} + \|\mathcal{A}Z(t)U_o\|_{\mathcal{H}}$$

and by monotonicity of $\mathcal{A}$, we have

$$\|U_o\|_{\mathcal{H}}^2 \leq \|U_o\|_{\mathcal{H}}^2 + (\mathcal{A}U_o, U_o)_{\mathcal{H}} = (V_o, U_o)_{\mathcal{H}} \ .$$

Thus,

$$\|U_o\|_{\mathcal{H}}^2 + \|\mathcal{A}U_o\|_{\mathcal{H}}^2 \leq 5\|V_o\|_{\mathcal{H}}^2 \ .$$

Conclusion,

$$\forall V_o \in \mathcal{H} \quad \forall t \geq 0 \qquad \|Z(t)V_o\|_{\mathcal{H}}^2 \leq 10ce^{-\beta t} \ \|V_o\|_{\mathcal{H}}^2 \ .$$

This completes the proof of lemma 5.5.

5.4 Lemma of polynomial decay

We need the following result.

Lemma 5.6 Let $\mathcal{F}$ be a positive decreasing functional. Suppose that $\exists c, \beta, T > 0,\ c_o \geq 0 \quad \forall \zeta > 0\ \forall \epsilon > 0$

$$\int_{\zeta}^{\zeta+T} \mathcal{F}(t)\,dt \leq c\left(\epsilon\left(c_o + \mathcal{F}(0)\right) + \frac{1}{\epsilon^{\beta}}\int_{\zeta}^{\zeta+T} -\frac{d}{dt}\mathcal{F}(s)\,ds\right)$$

then there exist $c > 0$ and $\gamma = \frac{1}{\beta+1} \in \left]0,1\right[$ such that

$$\forall t \geq 0 \qquad \mathcal{F}(t) \leq c\left(\frac{1}{t+1}\right)^{\gamma} \ \left(c_o + \mathcal{F}(0)\right) \ .$$

The proof follows the same strategy than the one for the logarithmic decay for the boundary damped wave equation done by G.Lebeau and L.Robbiano [LR].

Proof of lemma 5.6.- The decreasing property of the functional $\mathcal{F}$ implies that
$\exists c, \beta, T > 0$, $\exists c_o \geq 0 \quad \forall \zeta > 0 \; \forall \epsilon > 0$

$$T\mathcal{F}(\zeta + T) \leq c\left(\epsilon\left[c_o + \mathcal{F}(0)\right] + \frac{1}{\epsilon^\beta}\left[\mathcal{F}(\zeta) - \mathcal{F}(\zeta + T)\right]\right) .$$

Let us introduce $\mathcal{L}(t) = \frac{\mathcal{F}(t)}{c_o + \mathcal{F}(0)}$. We minimize the last inequality with respect to ϵ and we get, with $\gamma = \frac{1}{\beta+1}$, the following estimate $\quad \exists c > 1 \quad \forall \zeta > 0$

$$\mathcal{L}(\zeta + T) \leq c\left(\mathcal{L}(\zeta) - \mathcal{L}(\zeta + T)\right)^\gamma .$$

Let $t \in \mathbb{R}$, we write $t = nT + s$ with $0 \leq s < T$, and then $nT \leq t < (n+1)T$. The decreasing property of the functional $\mathcal{L}$ allows to get $\mathcal{L}(t) \leq \mathcal{L}(nT)$, and on another $\mathcal{L}(t) \leq 1$ for all $t \geq 0$. Let $\zeta = (n-1)T > 0$. We obtain the following iterative relation $\quad \forall n > 1$

$$\mathcal{L}(nT) \leq c\left(\mathcal{L}((n-1)T) - \mathcal{L}(nT)\right)^\gamma .$$

Let $\alpha_n = \mathcal{L}(nT)$. The sequence α_n is decreasing, bounded by $\mathcal{L}(0) \leq 1$, and satisfies when $n > 1$

$$\begin{cases} \alpha_n \leq c\left(\alpha_{n-1} - \alpha_n\right)^\gamma \\ \alpha_{n-1} \leq 1 \end{cases}$$

Thus,

if $0 < \alpha_{n-1} - \alpha_n \leq \frac{1}{n}$, then $\alpha_n \leq c\frac{1}{n^\gamma}$,

if $\alpha_{n-1} - \alpha_n > \frac{1}{n}$, then $n^\gamma \alpha_n < (n-1)^\gamma \alpha_{n-1}\left(\frac{n-1}{n}\right)\left(\frac{n}{n-1}\right)^\gamma = (n-1)^\gamma \alpha_{n-1}\left(\frac{n-1}{n}\right)^{1-\gamma}$.

Conclusion

either $\quad n^\gamma \alpha_n \leq c$

either $\quad n^\gamma \alpha_n < (n-1)^\gamma \alpha_{n-1} \quad$ where $\alpha_1 \leq 1$.

We conclude that for all $n > 1$, $n^\gamma \alpha_n \leq \max(c; 1) = c$. Going back to the functional $\mathcal{F}$, we obtain

$$\begin{cases} \mathcal{F}(t) \leq c\left(\frac{T}{t-T}\right)^\gamma \left(c_o + \mathcal{F}(0)\right) \quad \forall t > T \\ \mathcal{F}(t) \leq \mathcal{F}(0) \quad \forall t \leq T \end{cases}$$

This completes the proof of lemma 5.6.

5.5 Polynomial decay

Now, we are interested in the Maxwell equations with Ohm's law with a smooth conductivity: $\sigma \in C(\Omega)$ and $\nabla\sigma \in L^\infty(\Omega)$.

Then, applying the divergence to the fist equation of the system (5.1), we have

$$\varepsilon \frac{d}{dt} divE + \sigma divE + \nabla\sigma.E = 0 \ . \tag{5.30}$$

By using Gronwall lemma, we see that the divergence of the electric field is locally in time defined in $L^2(\Omega)$. Moreover, we have the following assertion

$$divE(\cdot, 0) = 0 \quad \text{in } \omega_- \quad \Longrightarrow \quad divE = 0 \quad \text{in } \omega_- \times]0, +\infty[\ .$$

Recall that the kernel of the divergence in ω_- can be decomposed orthogonally as follows $\ker div = \mathbb{H}_2(\omega_-) \oplus curlH^1(\omega_-)$, ([DL](5, p.257)). Let us introduce $\mathcal{M}_s = \mathcal{M}_{E,\omega} \cap \left(L^2(\Omega)\right)^3 \times \mathcal{M}_H$, where $\mathcal{M}_{E,\omega}$ is the orthogonal space to $\mathbb{H}_2(\omega_-)$ for the $\left(L^2(\omega_-)\right)^3$ norm. We deduce from theorem 5.3 the following result:

Suppose that $\partial\Omega$ has only one connected component. If $\sigma \in C(\Omega)$ and $\nabla\sigma \in L^\infty(\Omega)$, then for any initial data $(E_o, H_o) \in \mathcal{V} \cap \mathcal{M}_\omega$ of the system (5.1) such that $divE_o = 0$ in ω_-, we have $\lim_{t\to+\infty} \mathcal{E}(t) = 0$.

Notice that $\mathcal{M}_s = \left(L^2(\Omega)\right)^6$, if the assumptions of the Poincaré lemma are satisfied and if $\partial\omega_+ \cap \partial\Omega \neq \emptyset$, with ω_+, ω_- connected open sets.

We complete the asymptotic behaviour of the Maxwell equations with Ohm's law with smooth conductivity by establishing the polynomial decay when initial data belongs to a sub-space of $\mathcal{V} \cap \mathcal{M}_\omega$ and under the geometric control condition.

5.5.1 Maxwell equations with a charge density square integrable

Let Ω be a bounded open connected region in $\mathbb{R}^3$, with a smooth boundary $\partial\Omega$ (C^∞ having no contacts of infinite order with its tangents). The domain Ω is occupied by an electromagnetic medium of constant electric permittivity ε and constant magnetic permeability μ. We divide the domain Ω as follows $\Omega = \omega_+ \cup \Gamma \cup \omega_-$ where Γ is the interface surface between ω_+ and ω_- two non-empty connected open sets. The conductivity σ is a continuous function in $\mathbb{R}^3$, null in ω_- , positive in ω_+. Let E and H denote the electric field and the magnetic field respectively. The Maxwell equations with Ohm's law are described by

$$\left\{ \begin{array}{c} \varepsilon\partial_t E - curlH + \sigma E = 0, \ \mu\partial_t H + curlE = 0 \quad \text{in } \Omega \times [0, +\infty[\\ E(\cdot, 0) = E_o, \ H(\cdot, 0) = H_o \quad \text{in } \Omega \\ divE = 0 \quad \text{in } \omega_- \times [0, +\infty[, \ divH = 0 \quad \text{in } \Omega \times [0, +\infty[\\ E \wedge n = 0, \ H.n = 0 \quad \text{on } \partial\Omega \times [0, +\infty[\ . \end{array} \right. \tag{5.31}$$

Let $\mathcal{H}$ be the Hilbert space given by

$$\mathcal{H} = \left\{ (E_o, H_o) \in \left(L^2(\Omega)\right)^6 \setminus \ \sigma divE_o \in L^2(\Omega), \ (divE_o)_{|\omega_-} = 0, \ divH_o = 0, \ H_o.n_{|\partial\Omega} = 0 \right\}$$

equipped with the norm

$$\|(E_o, H_o)\|^2_{\mathcal{H}} = \varepsilon \|E_o\|^2_{L^2(\Omega)} + \mu \|H_o\|^2_{L^2(\Omega)} + \varepsilon \|\sigma divE_o\|^2_{L^2(\Omega)} \ .$$

Let $\mathcal{A}$ be the unbounded operator on $\mathcal{H}$ with domain $D(\mathcal{A})$, defined as follows

$$\mathcal{A} = \begin{pmatrix} \varepsilon^{-1}\sigma & -\varepsilon^{-1}curl \\ \mu^{-1}curl & 0 \end{pmatrix} .$$

We check that $(Id + \lambda\mathcal{A})$ is a bijection from $D(\mathcal{A}_o) \cap \mathcal{H}$ to $\mathcal{H}$. And there exists $\beta > 0$, such that $-(\mathcal{A} + \beta Id)$ is dissipative with the $D(\mathcal{A}_o) \cap \mathcal{H}$ norm, because σ is smooth and bounded in Ω. Now, we will denote $D(\mathcal{A}) = D(\mathcal{A}_o) \cap \mathcal{H} \subset \mathcal{H}$. We conclude, by applying Hille-Yosida theorem, that $-\mathcal{A}$ generates an infinitesimal semi-group $Z(t) : \mathcal{H} \to \mathcal{H}$. The system (5.31) is well-posed and more precisely, we have the following results.

$\forall (E_o, H_o) \in D(\mathcal{A}) \quad \exists! (E, H) \in C^0(]0, +\infty[, D(\mathcal{A})) \cap C^1(]0, +\infty[, \mathcal{H})$ solution of the system (5.31) .

$\forall (E_o, H_o) \in D(\mathcal{A}^2) \quad \exists! (E, H) \in C^0(]0, +\infty[, D(\mathcal{A}^2)) \cap C^1(]0, +\infty[, D(\mathcal{A})) \cap C^2(]0, +\infty[, \mathcal{H})$ solution of the system (5.31) .

Let us introduce $\mathcal{E}'(t)$ defined by $\mathcal{E}'(t) = \frac{1}{2}\int_\Omega \left(\varepsilon |\partial_t E|^2 + \mu |\partial_t H|^2\right)$.

5.5.2 Energy estimates

We need the following energy estimates.

$$\frac{d}{dt}\mathcal{E}(t) + \int_\Omega \sigma |E|^2 = 0 \tag{5.32}$$

$$\frac{d}{dt}\mathcal{E}'(t) + \int_\Omega \sigma |\partial_t E|^2 = 0 \tag{5.33}$$

$$\frac{d}{dt}\int \varepsilon |\sigma div E|^2 + \left(\sup |\nabla\sigma|^2\right)\frac{d}{dt}\mathcal{E}(t) + \int_\Omega \sigma^3 |div E|^2 \leq 0 \tag{5.34}$$

$$\frac{d}{dt}\int \varepsilon |\sigma div \partial_t E|^2 + \left(\sup |\nabla\sigma|^2\right)\frac{d}{dt}\mathcal{E}'(t) + \int_\Omega \sigma^3 |div \partial_t E|^2 \leq 0 \ . \tag{5.35}$$

The proof of the first two equality (5.32) and (5.33) is clear. The third relation (5.34) comes as follows.

We check that

$$\varepsilon\frac{d}{dt}(\sigma div E) + \sigma^2 div E + \sigma\nabla\sigma.E = 0 \ . \tag{5.36}$$

Multiplying the equation (5.36) by $\sigma div E$, we obtain

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\textstyle\int \varepsilon |\sigma div E|^2 + \int_\Omega \sigma^3 |div E|^2 &\leq \left|\textstyle\int_\Omega \sigma^2 div E \nabla\sigma.E\right| \\ &\leq (\sup|\nabla\sigma|) \left\|\sqrt{\sigma}E\right\|_{L^2(\Omega)} \left\|\sigma\sqrt{\sigma} div E\right\|_{L^2(\Omega)} \ . \end{aligned}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int \varepsilon |\sigma div E|^2 + \int_\Omega \sigma^3 |div E|^2 \leq \frac{1}{2}\left((\sup|\nabla\sigma|)^2 \left\|\sqrt{\sigma}E\right\|^2_{L^2(\Omega)} + \left\|\sigma\sqrt{\sigma} div E\right\|^2_{L^2(\Omega)}\right) \ .$$

The relation (5.32) allows to conclude.

5.5.3 Polynomial decay

Suppose $\partial\omega_+$ is C^∞. Let $\widetilde{\rho} \in C^\infty\left(\overline{\omega_+}\right)$, be a positive function in ω_+, vanishing on Γ with the same order that $dist\,(x,\Gamma)$.

Now, we choose the conductivity as follows

$$\sigma(x) = \left| \begin{array}{ll} \widetilde{\rho}^s(x) & \text{if } x \in \omega_+ \quad \text{with } s \geq 1 \\ 0 & \text{if } x \in \omega_- \cup \Gamma \end{array} \right.$$

Theorem 5.7 Let ω be an open set in $\mathbb{R}^3$ such that ω geometrically controls Ω. Suppose that $\omega_+ = (\omega \cap \Omega)$, $\omega_- = \Omega \backslash (\overline{\omega_+} \cap \Omega)$ are non-empty connected open sets. Then there exist $\gamma \in]0,1[$ and $c > 1$ such that we have

$\forall (E_o, H_o) \in D(\mathcal{A})$ initial data of the system (5.31)

$$\forall t \geq 0 \qquad \mathcal{E}'(t) + \int_\Omega \sigma \left|div \partial_t E(\cdot,t)\right|^2 \leq c\left(\frac{1}{t+1}\right)^{\gamma/s} \left(\|\sigma div E_o\|^2_{L^2(\Omega)} + \mathcal{E}(0) + \mathcal{E}'(0)\right) .$$

Theorem 5.8 Let ω be an open set in $\mathbb{R}^3$ such that ω geometrically controls Ω. Suppose that $\omega_+ = (\omega \cap \Omega)$, $\omega_- = \Omega \backslash (\overline{\omega_+} \cap \Omega)$ are non-empty connected sets and that $\partial\Omega$ has only one connected component. Then, there exist $\gamma \in]0,1[$ and $c > 1$ such that we have

$\forall (E_o, H_o) \in \mathcal{H} \cap \mathcal{S}_H$ initial data of the system (5.31)

$$div(\varepsilon E_0) + \nabla\sigma.(-curl_\Omega)^{-1}(\mu H_o) = 0 \quad \text{in } \Omega \quad \Longrightarrow \quad \forall t \geq 0 \qquad \mathcal{E}(t) \leq c\left(\frac{1}{t+1}\right)^{\gamma/s} \mathcal{E}(0) .$$

Remark that the condition $div(\varepsilon E_0) + \nabla\sigma.(-curl_\Omega)^{-1}(\mu H_o) = 0$ in $H^{-1}(\Omega)$ is not stable for the semi-group.

Proof of theorem 5.8.- The proof of theorem 5.8 comes from theorem 5.7. We claim that

$$\forall (E_o, H_o) \in \mathcal{H} \cap \mathcal{S}_H \qquad div(\varepsilon E_0) + \nabla\sigma.(-curl_\Omega)^{-1}(\mu H_o) = 0 \quad \text{in } \omega_+$$

$$\Longrightarrow \quad \exists!\,(U_o, V_o) \in H_0(rot,\Omega) \times H_0(div0,\Omega)$$

$$\left\{ \begin{array}{c} curl V_o - \sigma U_o = \varepsilon E_o \quad \text{in } \Omega \\ -curl U_o = \mu H_o \quad \text{in } \Omega \\ div U_o = 0 \quad \text{in } \Omega \end{array} \right.$$

where $H_0(curl,\Omega) = \left\{U \in \left(L^2(\Omega)\right)^3 \setminus\ curl U \in \left(L^2(\Omega)\right)^3,\ U \wedge n_{|\partial\Omega} = 0\right\}$
and $H_0(div0,\Omega) = \left\{V \in \left(L^2(\Omega)\right)^3 \setminus\ div V = 0,\ V.n_{|\partial\Omega} = 0\right\}$.

Indeed, there exists $U_o \in H_0(curl,\Omega)$ such that $\mu H_o = -curl U_o$. Moreover, U_o is uniquely given such that $div U_o = 0$, $\int_{\Gamma_i} U_o.n d\Gamma = 0$ for $i = 0$ to m ([C]($p.53$)). Consequently, $U_0 \in \left(H^1(\Omega)\right)^3$ can be written $U_o = (-curl_\Omega)^{-1}(\mu H_o)$.

On another hand, $(\varepsilon E_o + \sigma U_o) \in \left(L^2(\Omega)\right)^3$ is such that $div\,(\varepsilon E_o + \sigma U_o) = 0$. We conclude that there exists $V_0 \in \left(H^1(\Omega)\right)^3$ such that $(\varepsilon E_o + \sigma U_o) = curl V_o$ with the assumption $\mathbb{H}_2(\Omega) = \{0\}$ ([C](p.53)).

The initial data (U_o, V_o) satisfies the assumptions of theorem 5.7 and we deduce that

$$\frac{d}{dt} Z(t)(U_o, V_o) = Z(t)(-\mathcal{A}(U_o, V_o)) = Z(t)(E_o, H_o) \ .$$

This completes the proof.

Proof of theorem 5.7.- We divide the proof into four steps.

Step 1 :
We link $\partial_t H$ to $\partial_t E$: recall that from the orthogonal decomposition of the electric field as $E = -\nabla p + W$ and from the remark 5.1, we have the following estimate $\exists c > 0 \quad \forall \zeta > 0$

$$\int_\zeta^{\zeta+T} \mathcal{E}'(t)\,dt \leq c \int_\zeta^{\zeta+T} \int_\Omega |\partial_t E|^2 \ . \tag{5.38}$$

Step 2 :
We estimate $\partial_t W$, in order the under the geometric control condition, we get $\exists c > 0 \quad \forall \zeta > 0$

$$\int_\zeta^{\zeta+T} \int_\Omega |\partial_t E|^2 \leq c \left(\int_\zeta^{\zeta+T} \int_\Omega \sigma |\partial_t E|^2 + \int_\zeta^{\zeta+T} \int_\Omega \left| \frac{d}{dt} \nabla p \right|^2 \right) \ .$$

Indeed, the solution W solves the following hyperbolic system

$$\left\{ \begin{array}{c} \varepsilon \partial_t^2 W - \mu^{-1} \Delta W + \sigma \partial_t E - \varepsilon \partial_t^2 \nabla p = 0 \quad \text{in } \Omega \times [0, +\infty[\\ div W = 0, \ W \wedge n = 0 \quad \text{on } \partial\Omega \times [0, +\infty[\ . \end{array} \right.$$

Applying (1.31) of theorem 1.4, to the hyperbolic system solved by $\partial_t W$, we obtain under the geometric control condition and where $I =]\zeta, \zeta + T[$, $\exists c > 0 \ \exists \widehat{\omega} \subset \Omega$

$$\|\partial_t W\|_{L^2(\Omega \times I)} \leq c \left(\|\partial_t W\|_{L^2(\widehat{\omega} \times I)} + \left\|\sigma \partial_t E - \varepsilon \partial_t^2 \nabla p\right\|_{L^2(\Omega \times I)} \right)$$

with

$$\|\partial_t W\|^2_{L^2(\widehat{\omega} \times I)} \leq c \int_I \int_\Omega \sigma |\partial_t W|^2 \ .$$

But from proposition 5.2, we have

$$\left\|\varepsilon \partial_t^2 \nabla p\right\|^2_{L^2(\Omega \times I)} \leq \|\sigma \partial_t E\|^2_{L^2(\Omega \times I)} \ .$$

Consequently

$$\int_\zeta^{\zeta+T} \int_\Omega |\partial_t E|^2 \leq c \left(\int_\zeta^{\zeta+T} \int_\Omega \sigma |\partial_t E|^2 + \int_\zeta^{\zeta+T} \int_\Omega |\nabla \partial_t p|^2 \right)$$

and finally, from (5.38), $\exists c > 0$

$$\int_\zeta^{\zeta+T} \mathcal{E}'(t)\,dt \leq c \left(\int_\zeta^{\zeta+T} \int_\Omega \sigma |\partial_t E|^2 + \int_\zeta^{\zeta+T} \int_\Omega \left| \frac{d}{dt} \nabla p \right|^2 \right) \ . \tag{5.39}$$

Step 3 :

We estimate $\partial_t \nabla p$, from the Hardy inequality $\quad \exists c > 0 \quad \forall \epsilon > 0$

$$\int_\zeta^{\zeta+T} \int_\Omega \left| \frac{d}{dt} \nabla p \right|^2 \leq \epsilon \int_\zeta^{\zeta+T} \int_{\omega_+} \left| \frac{d}{dt} \Delta p \right|^2 + \frac{c}{\epsilon^\beta} \int_\zeta^{\zeta+T} \int_{\omega_+} \sigma^3 \left| \frac{d}{dt} \Delta p \right|^2 \ . \tag{5.40}$$

Let $0 < h < \frac{1}{2}$ and $\rho \in C^\infty \left(\overline{\omega_+} \right)$, be a positive function in ω_+, vanishing on Γ with the same order that $dist\,(x, \Gamma)$. As $\frac{d}{dt} p \in H^2 (\Omega) \cap H_0^1 (\Omega)$, we have

$$\int_\Omega \left| \frac{d}{dt} \nabla p \right|^2 = - \int_{\omega_+} \Delta \partial_t p \partial_t p = - \int_{\omega_+} \rho^h \Delta \partial_t p \frac{\partial_t p}{\rho^h} \ .$$

The Hardy and Poincaré inequalities implies that $\quad \exists c, d > 0$

$$\int_\Omega \left| \frac{\partial_t p}{\rho^h} \right|^2 \leq c \left\| \partial_t p \right\|^2_{H^1(\omega_+)} \leq d \left\| \nabla \partial_t p \right\|^2_{L^2(\omega_+)} \ .$$

Thus,

$$\int_\Omega \left| \frac{d}{dt} \nabla p \right|^2 \leq c \int_{\omega_+} \rho^{2h} \left| \frac{d}{dt} \Delta p \right|^2 \ .$$

We divide ω_+ into two parts $\quad \omega_+ = D_\epsilon \cup \omega_\epsilon$ where $D_\epsilon = \left\{ x \in \omega_+ \setminus \sigma(x) \leq \epsilon^{\frac{s}{2h}} \right\}$ and $\omega_\epsilon = \omega_+ \setminus D_\epsilon$.We choose ρ such that $\rho \leq \widetilde{\rho}$, and we obtain that

$$\begin{aligned} \int_{\omega_+} \rho^{2h} \left| \tfrac{d}{dt} \Delta p \right|^2 &= \int_{D_\epsilon} \rho^{2h} \left| \tfrac{d}{dt} \Delta p \right|^2 + \int_{\omega_\epsilon} \rho^{2h} \left| \tfrac{d}{dt} \Delta p \right|^2 \\ &\leq \epsilon \int_\Omega \left| \tfrac{d}{dt} \Delta p \right|^2 + \tfrac{1}{\epsilon^\beta} \int_{\omega_\epsilon} \rho^3 \left| \tfrac{d}{dt} \Delta p \right|^2 \end{aligned}$$

where $\beta = \frac{-2h+3s}{2h} > 0$. This ends the step 3.

Step 4 :

We estimate $\frac{d}{dt} \Delta p$. From (5.34), we show that $\frac{d}{dt} divE$ is uniformly bounded $\quad \exists c > 0$

$$\int_\zeta^{\zeta+T} \int_\Omega \left| \frac{d}{dt} \Delta p \right|^2 \leq cT \left(\left\| \sigma divE_o \right\|^2_{L^2(\Omega)} + \mathcal{E}(0) \right) \ . \tag{5.41}$$

Indeed, it is clear because $\varepsilon \frac{d}{dt} \Delta p = -\sigma \Delta p + \nabla \sigma . E$ and $\sigma \Delta p$ is uniformly bounded from (5.34).

Conclusion,

let us introduce $\mathcal{F}(t) = \left(\sup \left| \nabla \sigma \right|^2 \right) \mathcal{E}'(t) + \varepsilon \left\| \sigma div \partial_t E(\cdot, t) \right\|^2_{L^2(\Omega)}$, we have $\quad \forall \zeta > 0 \ \forall \epsilon > 0$

$$\int_\zeta^{\zeta+T} \mathcal{F}(t)\, dt \leq c \left(\epsilon \left(\left\| \sigma divE_o \right\|^2_{L^2(\Omega)} + \mathcal{E}(0) + \mathcal{F}(0) \right) + \frac{1}{\epsilon^\beta} \int_\zeta^{\zeta+T} - \frac{d}{dt} \mathcal{F}(s)\, ds \right) \ .$$

We conclude with lemma 5.6 that

$$\forall t > T \qquad \mathcal{E}'(t) + \int_\Omega \sigma \left| div \partial_t E(\cdot, t) \right|^2 \leq c \left(\frac{T}{t-T} \right)^\gamma \ \left(\left\| \sigma divE_o \right\|^2_{L^2(\Omega)} + \mathcal{E}(0) + \mathcal{E}'(0) \right)$$

with $\gamma = \frac{2h}{3s}$ and $0 < h < \frac{1}{2}$.

References

[Ba] H.Barucq, Etude asymptotique du système de Maxwell avec conditions aux limites absorbantes, Thèse de l'université de Bordeaux I, 1993.

[Bu] N.Burq, Mesures semi-classiques et mesures de défaut, Séminaire Bourbaki,vol. 1997. Asterisque,245,Exp. n°826, 1997,p.167-195.

[BH] H.Barucq and B.Hanouzet, Etude asymptotique du système de Maxwell avec la condition aux limites absorbante de Silver-Müller II, C. R. Acad. Sci. Paris,316,série I, 1993,p.1019-1024.

[BHLRZ] C.Bardos, L.Halpern, G.Lebeau , J.Rauch and E.Zuazua, Stabilisation de l'équation des ondes au moyen d'un feedback portant sur la condition aux limites de Dirichlet, Asymptotic Analysis,4, 1991,p.285-291.

[BLR] C.Bardos, G.Lebeau and J.Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J.Control Optim.,30,n°5, Sept.1992,p.1024-1065.

[C] M.Cessenat, Mathematical method in electromagnetism - linear theory and applications, Series on Advances in Mathematics for Applied Science,41, 1996.

[DL] R.Dautray and J-L.Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, Masson, 1988.

[K] V.Komornik, Boundary stabilization, observation and control of Maxwell's equations, Panamer. Math. J.,4, 1994,p.47-61.

[La] J.Lagnese, Exact boundary controllability of Maxwell's equations in a general region, SIAM J. Control Opt.,27, 1989,p.374-388.

[Le] G.Lebeau, Contrôle et stabilisation hyperboliques, séminaire E.D.P. Ecole Polytechnique, 1990.

[Li] J.-L.Lions, Contrôlabilité exacte, stabilisation et perturbation des systèmes distribués, 1,coll.RMA, Masson, Paris.1988.

[LM] J.-L.Lions and E.Magenes, Problèmes aux limites non homogènes, Dunod,1, 1968.

[LR] G.Lebeau and L.Robbiano, Stabilisation de l'équation des ondes par le bord, Duke Math. J.,86, 1997,p.465-491.

[MS1] R.Melrose and J.Sjöstrand, Singularities of boundary value problems I, Comm. Pure Appl. Math.,31, 1978,p.593-617.

[MS2] R.Melrose and J.Sjöstrand, Singularities of boundary value problems II, Comm. Pure Appl. Math.,35, 1982,p.129-168.

[N] O.Nalin, Contrôlabilité exacte sur une partie du bord des équations de Maxwell, C. R. Acad. Sci. Paris,309,série I, 1989,p.811-815.

[Pa] A.Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[Ph1] K-D.Phung, Stabilisation frontière du système de Maxwell avec la condition aux limites absorbante de Silver-Müller, C. R. Acad. Sci. Paris,320,série I, 1995,p.187-192.

[Ph2] K-D.Phung, COntrolabilité exacte et stabilisation interne des équations de Maxwell, C. R. Acad. Sci.Paris,3233,série I, 1996,p.169-174.

[R] J.V.Ralston, Solutions of Wave equation with localized energy, Comm. Pure. Appl. Math.,22, 1969,p.807-823.

[RT] J.Rauch and M.Taylor, Penetration into shadow region and unique continuation properties in hyperbolic mixed problems, Indiana University Mathematics J.,22, 1972,p.277-284.

[T1] M.Taylor, Pseudodifferential operators, Princeton Univ. Press, Princeton, N.J., 1981.

[T2] M.Taylor, Reflection of singularities of solutions to systems of differential equations, Comm. Pure Appl. Math.,28, 1975,p.457-478.

[T3] M.Taylor, Grazing rays and reflection of singularities of solutions to wave equations II, Comm. Pure Appl. Math.,29, 1976,p.463-481.

[WW] N.Weck and K.J.Witsch, Low frequency asymptotics for dissipative Maxwell's equations in bounded domains, Math. Meth. in Appl. Sci,13, 1990,p81-93.

[Y] K.Yamamoto, Singularities of solutions to the boundary value problem for elastic and Maxwell's equations, Japan J. Math.,14,n°1, 1988,p.119-163.