Quantification of unique continuation for an elliptic equation with Dirichlet boundary condition

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1 Introduction and main result

The purpose of this note is to prove the following result:

Theorem .- *Let Ω be a bounded connected C^2 domain in $\mathbb{R}^n$, $n > 1$. We choose $T > 0$ and $\delta \in (0, T/2)$. Let us consider the following elliptic equation of second order in $\Omega \times (0,T)$ with the Dirichlet boundary condition :*

$$\left\{ \begin{array}{c} \partial_t^2 v + \Delta_x v = 0 \quad \text{in } \Omega \times (0,T) \\ v = 0 \quad \text{on } \partial\Omega \times (0,T) \\ v = v(x,t) \in H^2(\Omega \times (0,T)) \ . \end{array} \right. \tag{1.1}$$

Then, for all $\varphi \in C_0^\infty(\Omega \times (0,T))$, $\varphi \neq 0$, there exist $C > 0$ and $\mu \in (0,1)$ such that for all v solution of (1.1), we have :

$$\int_\delta^{T-\delta} \int_\Omega |v(x,t)|^2 \, dxdt \leq C \left(\int_0^T \int_\Omega |v(x,t)|^2 \, dxdt \right)^\mu \left(\int_0^T \int_\Omega |\varphi v(x,t)|^2 \, dxdt \right)^{1-\mu} . \tag{1.2}$$

Similar kind of estimates already appears in [LR] from techniques of Carleman inequality (see also [FI]). Here, we will use the method described by [E] (see also [AE], [GL]).

2 Proof of Theorem, the results of [E]

Let Ω be an open, bounded, connected subset of $\mathbb{R}^n$, $n > 1$, with a C^2 boundary $\partial\Omega$. Let Γ be a non-empty subset of $\partial\Omega$. Let us consider the following elliptic equation of second order in $\Omega \times \mathbb{R}_t$, with the Dirichlet boundary condition :

$$\left\{ \begin{array}{c} \partial_t^2 u + \Delta_x u = 0 \quad \text{in } \Omega \times \mathbb{R} \\ u = 0 \quad \text{on } \Gamma \times \mathbb{R} \\ u = u(x,t) \in H^2(\Omega \times \mathbb{R}) \ . \end{array} \right. \tag{2.1}$$

The goal of this Section is to derive interpolation inequalities associated with the solutions u of (2.1).

Let B_r denote the open ball of center $(x_o, t_o) \in \Omega \times \mathbb{R}^+$ with radius r.

We propose to prove the two following lemmas : (the two following lemmas are stated in only one in [E])

Lemma 1 .- *Let r_1, r_2, r_3, R_o be four real numbers such that $0 < r_1 < r_2 < r_3 < R_o$. Let us consider $(x_o, t_o) \in \Omega \times R^+$ such that $\overline{B_{R_o}} \subset \Omega \times R$, then there exists $\alpha \in (0,1)$ such that for all u solution of (2.1), we have :*

$$\int_{B_{r_2}} |u(x,t)|^2 \, dxdt \leq \left(\int_{B_{r_1}} |u(x,t)|^2 \, dxdt \right)^{\alpha} \left(\int_{B_{r_3}} |u(x,t)|^2 \, dxdt \right)^{1-\alpha} . \tag{2.2}$$

Lemma 2 .- *Let r_o, r_1, r_2, r_3, R_o be five real numbers such that $0 < r_1 < r_o < r_2 < r_3 < R_o$. Let us consider $(x_o, t_o) \in \Omega \times R^+$ satisfying the three following conditions :*

i. $B_r \cap (\Omega \times \mathbb{R})$ is star-shaped with center (x_o, t_o) $\quad \forall r \in (0, R_o)$,

ii. $B_r \subset (\Omega \times \mathbb{R})$ $\quad \forall r \in (0, r_o)$,

iii. $B_r \cap^c (\Omega \times \mathbb{R}) \neq \emptyset$ and $B_r \cap (\partial\Omega \times \mathbb{R}) \subset (\Gamma \times \mathbb{R})$ $\quad \forall r \in [r_o, R_o)$.

Then there exists $\alpha \in (0,1)$ such that for all u solution of (2.1), we have :

$$\int_{B_{r_2} \cap (\Omega \times \mathbb{R})} |u(x,t)|^2 \, dxdt \leq \left(\int_{B_{r_1}} |u(x,t)|^2 \, dxdt \right)^{\alpha} \left(\int_{B_{r_3} \cap (\Omega \times \mathbb{R})} |u(x,t)|^2 \, dxdt \right)^{1-\alpha} . \tag{2.3}$$

The proof of Theorem is deduced from Lemma 1 and Lemma 2 by an adequate partition of unity. To apply Lemma 2, we will choose x_o in a neighborhood of the boundary such that the conditions *i, ii, iii*, hold.

3 Proof of Lemma 1

Let us denote $y = (x,t) \in \mathbb{R}^{n+1}$, $y_o = (x_o, t_o)$ and $\nabla = (\nabla_x, \partial_t)$. We introduce the three following functions H, D, N, for $0 < r < R_o$:

$$\begin{array}{rl} H(r) = & \int_{B_r} |u(y)|^2 \, dy \\ D(r) = & \frac{1}{2} \int_{B_r} |\nabla u(y)|^2 \left(r^2 - |y - y_o|^2 \right) dy \end{array} \tag{3.1}$$

and

$$N(r) = \frac{D(r)}{H(r)} \geq 0 . \tag{3.2}$$

Our goal is to prove that $N = N(r)$ is a non-decreasing function for $r \in (0, R_o)$. Indeed, we will prove that the following equality holds :

$$\frac{d}{dr} \ln H(r) = (n+1) \frac{d}{dr} \ln r + \frac{2}{r} N(r) . \tag{3.3}$$

So, from the non-decreasing property of N, we deduce that :

$$\begin{array}{rl} \ln \left(\frac{H(r_2)}{H(r_1)} \right) & = (n+1) \ln \frac{r_2}{r_1} + 2 \int_{r_1}^{r_2} \frac{N(r)}{r} dr \\ & \leq (n+1) \ln \frac{r_2}{r_1} + 2N(r_2) \ln \frac{r_2}{r_1} , \end{array} \tag{3.4}$$

$$\begin{array}{rl} \ln \left(\frac{H(r_3)}{H(r_2)} \right) & = (n+1) \ln \frac{r_3}{r_2} + 2 \int_{r_2}^{r_3} \frac{N(r)}{r} dr \\ & \geq (n+1) \ln \frac{r_3}{r_2} + 2N(r_2) \ln \frac{r_3}{r_2} . \end{array} \tag{3.5}$$

Consequently,

$$\frac{\ln\left(\frac{H(r_2)}{H(r_1)}\right)}{\ln\frac{r_2}{r_1}} \leq (n+1) + 2N(r_2) \leq \frac{\ln\left(\frac{H(r_3)}{H(r_2)}\right)}{\ln\frac{r_3}{r_2}} \ , \tag{3.6}$$

and we get what we wanted :

$$\int_{B_{r_2}} |u(y)|^2\, dy \leq \left(\int_{B_{r_1}} |u(y)|^2\, dy\right)^{\alpha} \left(\int_{B_{r_3}} |u(y)|^2\, dy\right)^{1-\alpha} \ , \tag{3.7}$$

with $\alpha = \frac{1}{\ln\frac{r_2}{r_1}}\left(\frac{1}{\ln\frac{r_2}{r_1}} + \frac{1}{\ln\frac{r_3}{r_2}}\right)^{-1}$.

First, we compute the derivative of $H(r) = \int_0^r \int_{S^n} |u(\rho s + y_o)|^2 \rho^n d\rho d\sigma(s)$:

$$\begin{aligned} H'(r) &= \int_{S^n} |u(rs+y_o)|^2 r^n d\sigma(s) \\ &= \frac{1}{r}\int_{S^n} |u(rs+y_o)|^2 rs \cdot s r^n d\sigma(s) \\ &= \frac{1}{r}\int_{B_r} div\left(|u(y)|^2 (y-y_o)\right) dy \\ &= \frac{1}{r}\int_{B_r} \left((n+1)|u(y)|^2 + \nabla |u(y)|^2 \cdot (y-y_o)\right) dy \\ &= \frac{n+1}{r} H(r) + \frac{2}{r}\int_{B_r} u\nabla u \cdot (y-y_o)\, dy \ . \end{aligned} \tag{3.8}$$

Next we have to remark that

$$D(r) = \int_{B_r} u\nabla u \cdot (y-y_o)\, dy \ , \tag{3.9}$$

indeed,

$$\begin{aligned} \int_{B_r} u\nabla u \cdot (y-y_o)\, dy &= -\frac{1}{2}\int_{B_r} u\nabla u \cdot \nabla\left(r^2 - |y-y_o|^2\right) dy \\ &= -\frac{1}{2}\int_{B_r} div\left(\left(r^2 - |y-y_o|^2\right) u\nabla u\right) dy + \frac{1}{2}\int_{B_r}\left(r^2 - |y-y_o|^2\right) div(u\nabla u)\, dy \\ &= \frac{1}{2}\int_{B_r}\left(r^2 - |y-y_o|^2\right) div(u\nabla u)\, dy \quad \text{because on } \partial B_r,\ r = |y-y_o| \\ &= \frac{1}{2}\int_{B_r} |\nabla u(y)|^2 \left(r^2 - |y-y_o|^2\right) dy \quad \text{because } \Delta_y u = 0 \ . \end{aligned}$$

Consequently, from (3.8) and (3.9), we prove : $\frac{H'(r)}{H(r)} = \frac{n+1}{r} + \frac{2}{r}\frac{D(r)}{H(r)}$ and this is (3.3).

Now, we compute the derivative of $D(r) = \frac{1}{2}\int_0^r \int_{S^n} \left|(\nabla u)_{|\rho s + y_o}\right|^2 (r^2 - \rho^2) \rho^n d\rho d\sigma(s)$:

$$\begin{aligned} D'(r) &= \frac{1}{2}\frac{d}{dr}\left(r^2 \int_0^r \int_{S^n} \left|(\nabla u)_{|\rho s+y_o}\right|^2 \rho^n d\rho d\sigma(s)\right) - \frac{1}{2}\int_{S^n} r^2 \left|(\nabla u)_{|rs+y_o}\right|^2 r^n d\sigma(s) \\ &= r\int_0^r \int_{S^n} \left|(\nabla u)_{|\rho s+y_o}\right|^2 \rho^n d\rho d\sigma(s) \\ &= r\int_{B_r} |\nabla u(y)|^2\, dy \ . \end{aligned} \tag{3.10}$$

The computation of the derivative of $N(r) = \frac{D(r)}{H(r)}$ gives :

$$N'(r) = \frac{1}{H^2(r)}\left[D'(r) H(r) - D(r) H'(r)\right] \ .$$

As the desired non-decreasing monotonicity of N depends on the positivity of $D'(r)H(r) - D(r)H'(r) = \left[D'(r) - D(r)\frac{H'(r)}{H(r)}\right] H(r)$, we are reduced from (3.3) to prove that

$$\frac{2}{r} D^2(r) \leq \left[D'(r) - \frac{n+1}{r} D(r)\right] H(r) \ . \tag{3.11}$$

By Cauchy-Schwarz inequality, we have :

$$\begin{aligned} D^2(r) &= \left(\int_{B_r} u\nabla u \cdot (y-y_o)\, dy \right)^2 \\ &\leq \left(\int_{B_r} |(y-y_o)\cdot \nabla u|^2\, dy \right) \left(\int_{B_r} |u|^2\, dy \right) \\ &\leq \left(\int_{B_r} |(y-y_o)\cdot \nabla u|^2\, dy \right) H(r) \ . \end{aligned} \tag{3.12}$$

Consequently, from (3.11) and (3.12), N is a non-decreasing function if

$$\frac{2}{r}\left(\int_{B_r} |(y-y_o)\cdot \nabla u|^2\, dy \right) \leq D'(r) - \frac{n+1}{r} D(r) \ . \tag{3.13}$$

Our goal is now reduced to prove that for all u solution of (2.1), if $0 < r < R_o$, then

$$\frac{2}{r}\int_{B_r} |(y-y_o)\cdot \nabla u|^2\, dy \leq D'(r) - \frac{n+1}{r} D(r) \ . \tag{3.14}$$

We begin to recall the following Rellich-Necas identity with vector field $(y-y_o)$ for all u solution of (2.1) :

$$2div\left(\left((y-y_o)\cdot\nabla u\right)\nabla u\right) = div\left((y-y_o)|\nabla u|^2\right) - (n-1)|\nabla u|^2 \ , \tag{3.15}$$

indeed,

$$\begin{aligned} div\left(\left((y-y_o)\cdot\nabla u\right)\nabla u\right) &= \left((y-y_o)\cdot\nabla u\right)\Delta u + \nabla\left((y-y_o)\cdot\nabla u\right)\cdot\nabla u \\ &= \partial_{y_i}\left((y-y_o)_j\,\partial_{y_j}u\right)\partial_{y_i}u \\ &= \partial_{y_i}(y-y_o)_j\,\partial_{y_j}u\partial_{y_i}u + (y-y_o)_j\,\partial^2_{y_iy_j}u\partial_{y_i}u \\ &= |\nabla u|^2 + \tfrac{1}{2}\nabla\left(|\nabla u|^2\right)\cdot(y-y_o) \\ div\left((y-y_o)|\nabla u|^2\right) &= |\nabla u|^2\, div\,(y-y_o) + \nabla\left(|\nabla u|^2\right)\cdot(y-y_o) \\ &= (n+1)|\nabla u|^2 + 2\partial_{y_j}u\partial^2_{y_iy_j}u\,(y-y_o)_i \ . \end{aligned}$$

Consequently,

$$2\int_{B_r} div\left(\left((y-y_o)\cdot\nabla u\right)\nabla u\right) dy = \int_{B_r} div\left((y-y_o)|\nabla u|^2\right) dy - (n-1)\int_{B_r}|\nabla u|^2\, dy \ . \tag{3.16}$$

By multiplying (3.16) by r , and from (3.10) we deduce that

$$2r\int_{S^{n-1}}\left(rs\cdot(\nabla u)_{|rs+y_o}\right)(\nabla u)_{|rs+y_o}\cdot sr^n d\sigma(s) = r\int_{S^{n-1}} rs\left|(\nabla u)_{|rs+y_o}\right|^2 \cdot sr^n d\sigma(s) - (n-1)D'(r) \ . \tag{3.17}$$

By integrating (3.17) on $(0,r)$, we have

$$2\int_{B_r}|(y-y_o)\cdot\nabla u|^2\, dy = \int_{B_r}|y-y_o|^2|\nabla u|^2\, dy - (n-1)D(r) \ . \tag{3.18}$$

Finally, from (3.1) and (3.18), we obtain :

$$\begin{aligned} 2\int_{B_r}|(y-y_o)\cdot\nabla u|^2\, dy &= r^2\int_{B_r}|\nabla u|^2\, dy - \int_{B_r}\left(r^2 - |y-y_o|^2\right)|\nabla u|^2\, dy - (n-1)D(r) \\ &= r^2\int_{B_r}|\nabla u|^2\, dy + [-2-(n-1)]\,D(r) \\ &= rD'(r) - (n+1)D(r) \ , \end{aligned} \tag{3.19}$$

and this is (3.14). The proof of Lemma 1 is now complete.

4 Proof of Lemma 2

Let us denote $y = (x,t) \in \mathbb{R}^{n+1}$, $y_o = (x_o,t_o)$ and $\nabla = (\nabla_x, \partial_t)$. As $B_r \cap^c (\Omega \times \mathbb{R}) \neq \emptyset$ and $B_r \cap (\partial\Omega \times \mathbb{R}) \subset (\Gamma \times \mathbb{R})$ for all $r \in [r_o, R_o[$, we extend u to be zero outside $\Omega \times \mathbb{R}$. As $u = 0$ on $\Gamma \times \mathbb{R}$, we deduce that

$$\left\{\begin{array}{ll} \overline{u} & = u_{|\overline{\Omega}} \quad \text{in } \overline{B_{R_o}} \\ \overline{u} & = 0 \quad \text{on } B_{R_o} \cap \partial\Omega \times \mathbb{R} \\ \nabla\overline{u} & = \nabla u_{|\Omega} \quad \text{in } B_{R_o} \\ \Delta\overline{u} & = 0 \quad \text{in } \Omega \times \mathbb{R} \ . \end{array}\right. \tag{4.1}$$

Let us denote $\Omega_r = B_r \cap (\Omega \times \mathbb{R})$, when $0 < r < R_o$. We introduce the three following functions :

$$\begin{array}{ll} H(r) = & \int_{\Omega_r} |u(y)|^2 \, dy \\ D(r) = & \frac{1}{2} \int_{\Omega_r} |\nabla u(y)|^2 \left(r^2 - |y - y_o|^2 \right) dy \end{array} \tag{4.2}$$

and

$$N(r) = \frac{D(r)}{H(r)} \geq 0 \ . \tag{4.3}$$

Our goal is to prove that N is a non-decreasing function. Indeed, we will prove that the following equality holds :

$$\frac{d}{dr} \ln H(r) = (n+1) \frac{d}{dr} \ln r + \frac{2}{r} N(r) \ . \tag{4.4}$$

So, from the non-decreasing property of N, we deduce that :

$$\begin{array}{ll} \ln\left(\frac{H(r_2)}{H(r_1)}\right) & = (n+1) \ln \frac{r_2}{r_1} + 2 \int_{r_1}^{r_2} \frac{N(r)}{r} dr \\ & \leq (n+1) \ln \frac{r_2}{r_1} + 2N(r_2) \ln \frac{r_2}{r_1} \ , \end{array} \tag{4.5}$$

$$\begin{array}{ll} \ln\left(\frac{H(r_3)}{H(r_2)}\right) & = (n+1) \ln \frac{r_3}{r_2} + 2 \int_{r_2}^{r_3} \frac{N(r)}{r} dr \\ & \geq (n+1) \ln \frac{r_3}{r_2} + 2N(r_2) \ln \frac{r_3}{r_2} \ . \end{array} \tag{4.6}$$

Consequently,

$$\frac{\ln\left(\frac{H(r_2)}{H(r_1)}\right)}{\ln \frac{r_2}{r_1}} \leq (n+1) + 2N(r_2) \leq \frac{\ln\left(\frac{H(r_3)}{H(r_2)}\right)}{\ln \frac{r_3}{r_2}} \ , \tag{4.7}$$

and we get the desired result :

$$\int_{\Omega_{r_2}} |u(y)|^2 \, dy \leq \left(\int_{B_{r_1}} |u(y)|^2 \, dy \right)^{\alpha} \left(\int_{\Omega_{r_3}} |u(y)|^2 \, dy \right)^{1-\alpha} \ , \tag{4.8}$$

with $\alpha = \frac{1}{\ln \frac{r_2}{r_1}} \left(\frac{1}{\ln \frac{r_2}{r_1}} + \frac{1}{\ln \frac{r_3}{r_2}} \right)^{-1}$.

First, we compute the derivative of $H(r) = \int_{B_r} |\overline{u}(y)|^2 \, dy = \int_0^r \int_{S^n} |\overline{u}(\rho s + y_o)|^2 \rho^n d\rho d\sigma(s)$:

$$\begin{array}{ll} H'(r) & = \int_{S^n} |\overline{u}(rs + y_o)|^2 r^n d\sigma(s) \\ & = \frac{1}{r} \int_{S^n} |\overline{u}(rs + y_o)|^2 rs \cdot sr^n d\sigma(s) \\ & = \frac{1}{r} \int_{B_r} div \left(|\overline{u}(y)|^2 (y - y_o) \right) dy \\ & = \frac{1}{r} \int_{B_r} \left((n+1) |\overline{u}(y)|^2 + \nabla |\overline{u}(y)|^2 \cdot (y - y_o) \right) dy \\ & = \frac{n+1}{r} H(r) + \frac{2}{r} \int_{\Omega_r} u \nabla u \cdot (y - y_o) \, dy \ . \end{array} \tag{4.9}$$

Next we have to remark that

$$D(r) = \int_{\Omega_r} u\nabla u\cdot(y-y_o)\,dy, \tag{4.10}$$

indeed,

$$\begin{aligned}
\int_{\Omega_r} u\nabla u\cdot(y-y_o)\,dy &= -\tfrac{1}{2}\int_{B_r}\overline{u}\nabla\overline{u}\cdot\nabla\left(r^2-|y-y_o|^2\right)dy\\
&= -\tfrac{1}{2}\int_{B_r} div\left(\left(r^2-|y-y_o|^2\right)\overline{u}\nabla\overline{u}\right)dy+\tfrac{1}{2}\int_{B_r}\left(r^2-|y-y_o|^2\right)div\left(\overline{u}\nabla\overline{u}\right)dy\\
&= \tfrac{1}{2}\int_{B_r}\left(r^2-|y-y_o|^2\right)div\left(\overline{u}\nabla\overline{u}\right)dy \quad \text{because on } \partial B_r,\ r=|y-y_o|\\
&= \tfrac{1}{2}\int_{B_r}\left(r^2-|y-y_o|^2\right)\left(\overline{u}\Delta_y\overline{u}+|\nabla\overline{u}|^2\right)dy\\
&= \tfrac{1}{2}\int_{\Omega_r}|\nabla u|^2\left(r^2-|y-y_o|^2\right)dy \quad \text{because } \Delta_y u=0 \text{ in } \Omega\times\mathbb{R} \text{ and } u_{|\Gamma}=0.
\end{aligned}$$

Consequently, from (4.9) and (4.10), we prove : $\frac{H'(r)}{H(r)} = \frac{n+1}{r} + \frac{2}{r}\frac{D(r)}{H(r)}$ and this is (4.4).

Now, we compute the derivative of $D(r) = \frac{1}{2}\int_0^r\int_{S^n}\left|(\nabla\overline{u})_{|\rho s+y_o}\right|^2\left(r^2-\rho^2\right)\rho^n d\rho d\sigma(s)$:

$$\begin{aligned}
D'(r) &= \tfrac{1}{2}\tfrac{d}{dr}\left(r^2\int_0^r\int_{S^n}\left|(\nabla\overline{u})_{|\rho s+y_o}\right|^2\rho^n d\rho d\sigma(s)\right)-\tfrac{1}{2}\int_{S^n}r^2\left|(\nabla\overline{u})_{|rs+y_o}\right|^2 r^n d\sigma(s)\\
&= r\int_0^r\int_{S^n}\left|(\nabla\overline{u})_{|\rho s+y_o}\right|^2\rho^n d\rho d\sigma(s)\\
&= r\int_{\Omega_r}|\nabla u(y)|^2\,dy\ .
\end{aligned} \tag{4.11}$$

The computation of the derivative of $N(r) = \frac{D(r)}{H(r)}$ gives :

$$N'(r) = \frac{1}{H^2(r)}\left[D'(r)H(r)-D(r)H'(r)\right]\ .$$

As the desired non-decreasing monotonicity of N depends on the positivity of $D'(r)H(r)-D(r)H'(r)$, we are reduced from (4.4) to prove that

$$\frac{2}{r}D^2(r) \leq \left[D'(r)-\frac{n+1}{r}D(r)\right]H(r)\ . \tag{4.12}$$

By Cauchy-Schwarz inequality, we have :

$$\begin{aligned}
D^2(r) &= \left(\int_{\Omega_r}u\nabla u\cdot(y-y_o)\,dy\right)^2\\
&\leq \left(\int_{\Omega_r}|(y-y_o)\cdot\nabla u|^2\,dy\right)\left(\int_{\Omega_r}|u|^2\,dy\right)\\
&\leq \left(\int_{\Omega_r}|(y-y_o)\cdot\nabla u|^2\,dy\right)H(r)\ .
\end{aligned} \tag{4.13}$$

Consequently, from (4.12) and (4.13), N is a non-decreasing function if

$$\frac{2}{r}\left(\int_{\Omega_r}|(y-y_o)\cdot\nabla u|^2\,dy\right) \leq D'(r)-\frac{n+1}{r}D(r)\ . \tag{4.14}$$

Our goal is now reduced to prove that for all u solution of (2.1), if $r_o \leq r < R_o$ and the hypothesis *i,ii,iii,* of Lemma 2 hold, then

$$\frac{2}{r}\int_{\Omega_r}|(y-y_o)\cdot\nabla u|^2\,dy \leq D'(r)-\frac{n+1}{r}D(r)\ . \tag{4.15}$$

We note that the case $0<r<r_o$ is already studied in the proof of Lemma 1.
We begin to recall the following Rellich-Necas identity with vector field $(y-y_o)$ for all u solution of (2.1) :

$$2div\left(\left(\left(y-y_o\right)\cdot\nabla u\right)\nabla u\right)=div\left(\left(y-y_o\right)\left|\nabla u\right|^2\right)-\left(n-1\right)\left|\nabla u\right|^2\ . \tag{4.16}$$

Consequently,

$$2\int_{\Omega_r}div\left(\left(\left(y-y_o\right)\cdot\nabla u\right)\nabla u\right)dy=\int_{\Omega_r}div\left(\left(y-y_o\right)\left|\nabla u\right|^2\right)dy-\left(n-1\right)\int_{\Omega_r}\left|\nabla u\right|^2dy\ . \tag{4.17}$$

As $B_r\cap\left(\partial\Omega\times\mathbb{R}\right)\subset\left(\Gamma\times\mathbb{R}\right)$ and $u_{|\Gamma}=0$, we have $\nabla u\left(y\right)=\left(\partial_\nu u\left(y\right)\right)\nu$ on $B_r\cap\partial\Omega\times\mathbb{R}$ with $\nu=\left(\nu_x,0\right)\in\mathbb{R}^n\times\mathbb{R}$ on $\partial\Omega$. We obtain :

$$\begin{array}{ll}\int_{\Omega_r}div\left(\left(\left(y-y_o\right)\cdot\nabla u\right)\nabla u\right)dy&=\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}\frac{1}{r}\left(\left(y-y_o\right)\cdot\nabla u\right)^2d\sigma+\int_{B_r\cap\left(\partial\Omega\times\mathbb{R}\right)}\left(\left(y-y_o\right)\cdot\nabla u\right)\frac{\partial u}{\partial\nu}d\sigma\\ &=\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}\frac{1}{r}\left|\left(y-y_o\right)\cdot\nabla u\right|^2d\sigma+\int_{B_r\cap\left(\partial\Omega\times\mathbb{R}\right)}\left(\left(x-x_o\right)\cdot\nu_x\right)\left|\partial_\nu u\right|^2d\sigma\ ,\end{array} \tag{4.18}$$

$$\begin{array}{ll}\int_{\Omega_r}div\left(\left(y-y_o\right)\left|\nabla u\right|^2\right)dy&=\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}r\left|\nabla u\right|^2d\sigma+\int_{B_r\cap\left(\partial\Omega\times\mathbb{R}\right)}\left(\left(y-y_o\right)\left|\nabla u\right|^2\right)\cdot\nu d\sigma\\ &=\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}r\left|\nabla u\right|^2d\sigma+\int_{B_r\cap\left(\partial\Omega\times\mathbb{R}\right)}\left(\left(x-x_o\right)\cdot\nu_x\right)\left|\partial_\nu u\right|^2d\sigma\ .\end{array} \tag{4.19}$$

Consequently, from (4.11), (4.17), (4.18) and (4.19), we have

$$\begin{array}{l}2\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}\frac{1}{r}\left|\left(y-y_o\right)\cdot\nabla u\right|^2d\sigma+\int_{B_r\cap\left(\partial\Omega\times\mathbb{R}\right)}\left(\left(x-x_o\right)\cdot\nu_x\right)\left|\partial_\nu u\right|^2d\sigma\\ =\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}r\left|\nabla u\right|^2d\sigma-\frac{1}{r}\left(n-1\right)D'\left(r\right)\ .\end{array} \tag{4.20}$$

If $\left(\left(x-x_o\right)\cdot\nu_x\right)\geq0$ on Γ, then

$$2\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}\left|\left(y-y_o\right)\cdot\nabla u\right|^2d\sigma\leq\int_{\partial B_r\cap\left(\Omega\times\mathbb{R}\right)}r^2\left|\nabla u\right|^2d\sigma-\left(n-1\right)D'\left(r\right)\ . \tag{4.21}$$

So, by integrating (4.21), we deduce that

$$2\int_{\Omega_r}\left|\left(y-y_o\right)\cdot\nabla u\right|^2dy\leq\int_{\Omega_r}\left|y-y_o\right|^2\left|\nabla u\right|^2dy-\left(n-1\right)D\left(r\right)\ . \tag{4.22}$$

From (4.2), (4.11) and (4.22), we obtain

$$\begin{array}{ll}2\int_{\Omega_r}\left|\left(y-y_o\right)\cdot\nabla u\right|^2dy&\leq r^2\int_{\Omega_r}\left|\nabla u\right|^2dy-\int_{\Omega_r}\left(r^2-\left|y-y_o\right|^2\right)\left|\nabla u\right|^2dy-\left(n-1\right)D\left(r\right)\\ &\leq r^2\int_{\Omega_r}\left|\nabla u\right|^2dy+\left(-2-\left(n-1\right)\right)D\left(r\right)\\ &\leq rD'\left(r\right)-\left(n+1\right)D\left(r\right)\ ,\end{array} \tag{4.23}$$

and this is (4.14). The hypothesis $\left(\left(x-x_o\right)\cdot\nu_x\right)\geq0$ on Γ is of course true when $B_r\cap\left(\Omega\times\mathbb{R}\right)$ is star-shaped with center $\left(x_o,t_o\right)$ for all $r\in\left(0,R_o\right)$. The proof of Lemma 2 is now complete.

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