Quantification of unique continuation for the wave equation with Dirichlet boundary condition

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1 Introduction and main result

The purpose of this note is to prove the following result:

Theorem .- *Let Ω be a bounded connected C^2 domain in $\mathbb{R}^n$, $n>1$, and ω be a non-empty open subset of Ω. Let us consider the following wave equation in $\Omega\times\mathbb{R}$, with initial data and homogeneous Dirichlet boundary condition :*

$$\left\{\begin{array}{c} \partial_t^2 u-\Delta u=0 \quad in\ \Omega\times\mathbb{R} \\ u=0 \quad on\ \partial\Omega\times\mathbb{R} \\ u(\cdot,0)=u_0,\, \partial_t u(\cdot,0)=u_1 \quad in\ \Omega\ . \end{array}\right. \tag{1.1}$$

Then, there exist a constant $c>0$ and a time $T>0$ such that for all initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, $(u_0,u_1)\neq 0$, the solution u of (1.1) satisfies the following estimate :

$$\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq e^{c\frac{\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}} \int_0^T\int_\omega \left|u(x,t)\right|^2 dxdt\ . \tag{1.2}$$

Remark that estimate (1.2) is equivalent to

$$\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \frac{c}{\ln\left(2+\frac{\left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}}{\left\|u\right\|_{L^2(\omega\times(0,T))}}\right)} \left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\ , \tag{1.3}$$

or equivalently,

$$\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq ce^{c/\varepsilon}\int_0^T\int_\omega \left|u(x,t)\right|^2 dxdt + \varepsilon\ \left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\ ,\ \forall\varepsilon>0\ , \tag{1.4}$$

where the value of the constant $c>0$ may changed from line (1.2) to line (1.4) but not its dependence.

Similar kind of estimates already appears in [R], [LR] in the framework of boundary control and stabilization for hyperbolic equations. Here, we use the ideas from [R].

2 Proof of Theorem

2.1 step 1: Energy estimates

First, recall the conservation identity of the energy for the solution u of

$$\left\{ \begin{array}{c} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R} \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \\ u\left(\cdot,0\right) = u_0,\ \partial_t u\left(\cdot,0\right) = u_1 \quad \text{in } \Omega \ , \end{array} \right. \tag{2.1}$$

which is, for all $t \in \mathbb{R}$

$$\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} = \int_\Omega \left(\left|\partial_t u\left(x,t\right)\right|^2 + \left|\nabla u\left(x,t\right)\right|^2 \right) dx \ . \tag{2.2}$$

Furthermore, we have the following inequality: there exists a constant $c > 0$ such that for all $T \geq 1$, we get

$$T\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq c \int_0^T \int_\Omega \left|u\left(x,t\right)\right|^2 dx \ . \tag{2.3}$$

Indeed, let $\Psi \in C_0^\infty\left((0,T)\right)$ such that $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ on $\left(\frac{T}{4},\frac{3T}{4}\right)$ and $\left|\Psi'\right| \leq \frac{8}{T}$. Multiplying (2.1) by Ψu and integrating over $\Omega \times (0,T)$ by parts yields:

$$\begin{array}{ll} \left\|\nabla u\right\|^2_{L^2\left(\Omega\times\left(\frac{T}{4},\frac{3T}{4}\right)\right)} & \leq \int_0^T \int_\Omega \Psi\left(t\right)\left|\nabla u\left(x,t\right)\right|^2 dxdt \\ & = -\int_0^T \int_\Omega \partial_t^2 u\left(x,t\right)\Psi\left(t\right)u\left(x,t\right) dxdt \\ & = \int_0^T \int_\Omega \partial_t u\left(x,t\right)\Psi'\left(t\right)u\left(x,t\right) dxdt + \int_0^T \int_\Omega \Psi\left(t\right)\left|\partial_t u\left(x,t\right)\right|^2 dxdt \\ & \leq \int_0^T \int_\Omega \left|\partial_t u\left(x,t\right)\right|^2 dxdt + \left\|\Psi'\partial_t u\right\|_{L^2(\Omega\times(0,T))}\left\|u\right\|_{L^2(\Omega\times(0,T))} \\ & \leq \int_0^T \int_\Omega \left|\partial_t u\left(x,t\right)\right|^2 dxdt + \frac{8}{T}\left\|\partial_t u\right\|_{L^2(\Omega\times(0,T))}\left\|u\right\|_{L^2(\Omega\times(0,T))} \end{array} \tag{2.4}$$

Adding $\left\|\partial_t u\right\|^2_{L^2\left(\Omega\times\left(\frac{T}{4},\frac{3T}{4}\right)\right)}$, using Poincaré inequality and conservation of energy (2.2), then there exists a constant $c > 0$ such that for all $T \geq 1$,

$$T\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq \sqrt{cT}\left\|\partial_t u\right\|_{L^2(\Omega\times(0,T))}\left\|(u_0,u_1)\right\|_{H^1_0(\Omega)\times L^2(\Omega)} \ , \tag{2.5}$$

that is

$$T\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq c \int_0^T \int_\Omega \left|\partial_t u\left(x,t\right)\right|^2 dx \ . \tag{2.6}$$

Now, we deduce (2.3) from (2.6). Indeed, let $v\left(x,t\right) = \int_0^t u\left(x,\ell\right) d\ell - \left(-\Delta^{-1}\right)u_1\left(x\right)$, then $\Delta v\left(x,t\right) = \int_0^t \Delta u\left(x,\ell\right) d\ell + u_1\left(x\right)$ and $\partial_t v\left(x,t\right) = u\left(x,t\right)$,

$$\begin{array}{ll} \partial_t^2 v\left(x,t\right) & = \partial_t u\left(x,t\right) \\ & = \partial_t u\left(x,0\right) + \int_0^t \partial_t^2 u\left(x,\ell\right) d\ell \\ & = u_1\left(x\right) + \int_0^t \Delta u\left(x,\ell\right) d\ell \\ & = \Delta v\left(x,t\right) \ . \end{array} \tag{2.7}$$

Also, $v\left(x,0\right) = -\left(-\Delta^{-1}\right)u_1\left(x\right)$, $\partial_t v\left(x,0\right) = u_0\left(x\right)$, and the homogeneous Dirichlet boundary condition is satisfied. Consequently,

$$\begin{array}{ll} \left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} & = \left\|\left(\left(-\Delta^{-1}\right)u_1,u_0\right)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \\ & = \left\|\left(v\left(x,0\right),\partial_t v\left(x,0\right)\right)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq \frac{c}{T}\int_0^T \int_\Omega \left|\partial_t v\left(x,t\right)\right|^2 dxdt \\ & \leq \frac{c}{T}\int_0^T \int_\Omega \left|u\left(x,t\right)\right|^2 dxdt \ , \end{array} \tag{2.8}$$

that is (2.3).

2.2 step 2: FBI transform

Let $F(z)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau}e^{-\tau^2}d\tau$, then $F(z)=\frac{\sqrt{\pi}}{2\pi}e^{\frac{1}{4}\left(|\mathrm{Im}\,z|^2-|\mathrm{Re}\,z|^2\right)}e^{-\frac{i}{2}(\mathrm{Im}\,z\,\mathrm{Re}\,z)}$. Also, with $\lambda>0$, let us consider

$$F_\lambda(z)=\lambda F(\lambda z)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau}e^{-\left(\frac{\tau}{\lambda}\right)^2}d\tau\ . \tag{2.9}$$

We have,

$$|F_\lambda(z)|=\frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}\left(|\mathrm{Im}\,z|^2-|\mathrm{Re}\,z|^2\right)}\ , \tag{2.10}$$

and, denoting $\widehat{F}$ the Fourier transform of F,

$$\widehat{F_\lambda}\,(\tau)=e^{-\left(\frac{\tau}{\lambda}\right)^2}\ . \tag{2.11}$$

Let $s,\ell_o\in\mathbb{R}$, we introduce the following gaussian transformation which is a particular case of the Fourier-Bros-Iagolnitzer transformation in [LR] :

$$W_{\ell_o,\lambda}(x,s)=\int_{\mathbb{R}} F_\lambda(\ell_o+is-\ell)\Phi(\ell)u(x,\ell)d\ell \tag{2.12}$$

where $\Phi\in C_0^\infty(\mathbb{R})$. We remark that $\partial_s^2F_\lambda(\ell_o+is-\ell)=-\partial_\ell^2F_\lambda(\ell_o+is-\ell)$, so

$$\begin{array}{ll}\partial_s^2W_{\ell_o,\lambda}(x,s) & =\int_{\mathbb{R}}-\partial_\ell^2F_\lambda(\ell_o+is-\ell)\Phi(\ell)u(x,\ell)d\ell\\ & =\int_{\mathbb{R}}-F_\lambda(\ell_o+is-\ell)\left[\Phi''(\ell)u(x,\ell)+2\Phi'(\ell)\partial_tu(x,\ell)+\Phi(\ell)\partial_t^2u(x,\ell)\right]d\ell.\end{array} \tag{2.13}$$

As u is solution of the wave equation, $W_{\ell_o,\lambda}$ satisfies:

$$\left\{\begin{array}{c}\partial_s^2W_{\ell_o,\lambda}(x,s)+\Delta W_{\ell_o,\lambda}(x,s)=\int_{\mathbb{R}}-F_\lambda(\ell_o+is-\ell)\left[\Phi''(\ell)u(x,\ell)+2\Phi'(\ell)\partial_tu(x,\ell)\right]d\ell\\ W_{\ell_o,\lambda}(x,s)=0\quad\text{for } x\in\partial\Omega\\ W_{\ell_o,\lambda}(x,0)=\left(F_\lambda*\Phi u(x,\cdot)\right)(\ell_o)\quad\text{for } x\in\Omega.\end{array}\right. \tag{2.14}$$

In another hand, we also have for some $T>0$,

$$\begin{array}{ll}\|\Phi u\,(x,\cdot)\|_{L^2\left(\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\right)} & \leq\|\Phi u(x,\cdot)-F_\lambda*\Phi u(x,\cdot)\|_{L^2\left(\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\right)}\\ & \quad+\|F_\lambda*\Phi u(x,\cdot)\|_{L^2\left(\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\right)}\\ & \leq\|\Phi u(x,\cdot)-F_\lambda*\Phi u(x,\cdot)\|_{L^2(\mathbb{R})}\\ & \quad+\left(\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}|W_{t,\lambda}(x,0)|^2\,dt\right)^{1/2}\ .\end{array} \tag{2.15}$$

But using Parseval equality and (2.11),

$$\begin{array}{ll}\|\Phi u(x,\cdot)-F_\lambda*\Phi u(x,\cdot)\|_{L^2(\mathbb{R})} & =\frac{1}{\sqrt{2\pi}}\left\|\widehat{\Phi u(x,\cdot)-F_\lambda*\Phi u(x,\cdot)}\right\|_{L^2(\mathbb{R})}\\ & =\frac{1}{\sqrt{2\pi}}\left(\int_{\mathbb{R}}\left|\left(1-e^{-\left(\frac{\tau}{\lambda}\right)^2}\right)\widehat{\Phi u(x,\cdot)}\,(\tau)\right|^2d\tau\right)^{1/2}\\ & \leq\frac{1}{\sqrt{2\pi}}\left(\int_{\mathbb{R}}\left|\left(\sqrt{2}\frac{\tau}{\lambda}\right)\widehat{\Phi u(x,\cdot)}\,(\tau)\right|^2d\tau\right)^{1/2}\\ & \leq\frac{\sqrt{2}}{\sqrt{2\pi}}\frac{1}{\lambda}\left(\int_{\mathbb{R}}\left|\partial_t\left(\widehat{\Phi u(x,}\cdot)\right)(\tau)\right|^2d\tau\right)^{1/2}\\ & \leq\sqrt{2}\frac{1}{\lambda}\|\partial_t\left(\Phi u(x,\cdot)\right)\|_{L^2(\mathbb{R})}\ ,\end{array} \tag{2.16}$$

so,

$$\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|\Phi(t)u(x,t)\right|^2 dt \leq 4\frac{1}{\lambda^2}\left\|\partial_t\left(\Phi u(x,\cdot)\right)\right\|^2_{L^2(\mathbb{R})}+2\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|W_{t,\lambda}(x,0)\right|^2 dt \qquad (2.17)$$

Now, recall that from the Cauchy theorem we have:

Proposition .- *Let f be a holomorphic function in a domain $D\subset\mathbb{C}$. Let $a,b>0$, $z\in\mathbb{C}$. We suppose that*

$$D_o=\left\{(x,y)\in\mathbb{R}^2\simeq\mathbb{C}\backslash\ \left|x-\mathrm{Re}\,(z)\right|\leq a,\ \ \left|y-\mathrm{Re}\,(z)\right|\leq b\right\}\subset D\ ,$$

then

$$f\left(z\right)=\frac{1}{\pi ab}\int\int_{\left|\frac{x-\mathrm{Re}(z)}{a}\right|^2+\left|\frac{y-\mathrm{Im}(z)}{b}\right|^2\leq 1} f\left(x+iy\right)dxdy\ .$$

Choosing $z=t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)\subset\mathbb{R}$ and $x+iy=\ell_o+is$, we deduce that

$$\begin{array}{ll}\left|W_{t,\lambda}(x,0)\right| & \leq\frac{1}{\pi ab}\int_{\left|\ell_o-t\right|\leq a}\int_{\left|s\right|\leq b}\left|W_{\ell_o+is,\lambda}(x,0)\right|d\ell_o ds\\ & \leq\frac{1}{\pi ab}\int_{\left|\ell_o-t\right|\leq a}\int_{\left|s\right|\leq b}\left|W_{\ell_o,\lambda}(x,s)\right|dsd\ell_o\\ & \leq\frac{1}{\pi ab}\left(\int_{\left|\ell_o-t\right|\leq a}\int_{\left|s\right|\leq b}\left|W_{\ell_o,\lambda}(x,s)\right|^2 dsd\ell_o\right)^{1/2}\left(\int_{\left|\ell_o-t\right|\leq a}\int_{\left|s\right|\leq b}dsd\ell_o\right)^{1/2}\\ & \leq\frac{2}{\pi\sqrt{ab}}\left(\int_{\left|\ell_o-t\right|\leq a}\int_{\left|s\right|\leq b}\left|W_{\ell_o,\lambda}(x,s)\right|^2 dsd\ell_o\right)^{1/2}\quad ,\end{array} \qquad (2.18)$$

and with $a=b=1$,

$$\begin{array}{ll}\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|W_{t,\lambda}(x,0)\right|^2 dt & \leq\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left(\int_{\left|\ell_o-t\right|\leq 1}\int_{\left|s\right|\leq 1}\left|W_{\ell_o,\lambda}(x,s)\right|^2 dsd\ell_o\right)dt\ .\\ & \leq\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\int_{\left|s\right|\leq 1}\left|W_{\ell_o,\lambda}(x,s)\right|^2 dsd\ell_o dt\\ & \leq 2\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\int_{\left|s\right|\leq 1}\left|W_{\ell_o,\lambda}(x,s)\right|^2 dsd\ell_o\ .\end{array} \qquad (2.19)$$

Consequently, from (2.17), (2.19) and integrating over Ω, we get

$$\begin{array}{ll}\int_\Omega\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|\Phi(t)u(x,t)\right|^2 dtdx & \leq 4\frac{1}{\lambda^2}\int_\Omega\int_{\mathbb{R}}\left|\Phi'\left(t\right)u(x,t)+\Phi\left(t\right)\partial_t u(x,t)\right|^2 dtdx\\ & \quad +4\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_\Omega\int_{\left|s\right|\leq 1}\left|W_{\ell_o,\lambda}(x,s)\right|^2 dsdx\right)d\ell_o\ .\end{array} \qquad (2.20)$$

Now recall the following quantification result for unique continuation of elliptic equation with Dirichlet boundary condition:

Proposition .- *Let Ω be a bounded connected C^2 domain in $\mathbb{R}^n$, $n>1$, and ω be a non-empty open subset of Ω. Let $f=f\left(x,s\right)\in L^2\left(\Omega\times\left(-2,2\right)\right)$. Then there exists $C>0$ such that for all $w=w\left(x,s\right)\in H^2\left(\Omega\times\left(-2,2\right)\right)$ solution of*

$$\left\{\begin{array}{c}\partial_s^2 w+\Delta_x w=f\quad \text{in }\Omega\times\left(-2,2\right)\\ w=0\quad \text{on }\partial\Omega\times\left(-2,2\right)\end{array}\right.$$

for all $\varepsilon>0$, we have :

$$\begin{array}{ll}\int_{\left|s\right|\leq 1}\int_\Omega\left|w\left(x,s\right)\right|^2 dxds & \leq Ce^{C/\varepsilon}\left(\int_{\left|s\right|\leq 2}\int_\omega\left|w\left(x,s\right)\right|^2 dxds+\int_{\left|s\right|\leq 2}\int_\Omega\left|f\left(x,s\right)\right|^2 dxds\right)\\ & \quad +e^{-4/\varepsilon}\int_{\left|s\right|\leq 2}\int_\Omega\left|w\left(x,s\right)\right|^2 dxds\ .\end{array}$$

Applying to $W_{\ell_0,\lambda}$, we deduce that for all $\varepsilon > 0$,

$$\begin{array}{l}\int_{|s|\leq 1}\int_{\Omega}|W_{\ell_0,\lambda}(x,s)|^2\,dxds\\ \leq e^{-4/\varepsilon}\int_{|s|\leq 2}\int_{\Omega}|W_{\ell_0,\lambda}(x,s)|^2\,dxds\\ \quad +Ce^{C/\varepsilon}\int_{|s|\leq 2}\int_{\omega}|W_{\ell_0,\lambda}(x,s)|^2\,dxds\\ \quad +Ce^{C/\varepsilon}\int_{|s|\leq 2}\int_{\Omega}\left|\int_{\mathbb{R}}-F_\lambda(\ell_0+is-\ell)\left[\Phi''(\ell)u(x,\ell)+2\Phi'(\ell)\partial_t u(x,\ell)\right]d\ell\right|^2 dxds\ .\end{array}\tag{2.21}$$

Consequently, from (2.20) and (2.21), there exists a constant $C > 0$, such that for all $\varepsilon > 0$,

$$\begin{array}{l}\int_{\Omega}\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}|\Phi(t)u(x,t)|^2\,dtdx\\ \leq 4\frac{1}{\lambda^2}\int_{\Omega}\int_{\mathbb{R}}|\Phi'(t)\,u(x,t)+\Phi(t)\,\partial_t u(x,t)|^2\,dtdx\\ \quad +4e^{-4/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\Omega}|W_{\ell_0,\lambda}(x,s)|^2\,dxds\right)d\ell_o\\ \quad +4Ce^{C/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\omega}|W_{\ell_0,\lambda}(x,s)|^2\,dxds\right)d\ell_o\\ \quad +4Ce^{C/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\Omega}\left|\int_{\mathbb{R}}-F_\lambda(\ell_0+is-\ell)\left[\Phi''(\ell)u(x,\ell)+2\Phi'(\ell)\partial_t u(x,\ell)\right]d\ell\right|^2 dxds\right)d\ell_o\ .\end{array}\tag{2.22}$$

Let define $\Phi \in C_0^\infty(\mathbb{R})$ more precisely now: let $T > 16$, we choose $\Phi \in C_0^\infty((0,T))$, $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on $\left(\frac{T}{4},\frac{3T}{4}\right)$. Furthermore, let $K = \left[0,\frac{T}{4}\right]\cup\left[\frac{3T}{4},T\right]$ such that $\mathrm{supp}(\Phi') = K$ and $\mathrm{supp}(\Phi'') \subset K$. Let $K_0 = \left[\frac{3T}{8},\frac{5T}{8}\right]$ such that $\mathrm{dist}(K,K_o) = \frac{T}{8}$. We will choose $\ell_0 \in \left(\frac{T}{2}-2,\frac{T}{2}+2\right) \subset K_0$.

Till the end of the proof, C_T will denote a generic positive constant independent of ε and λ but dependent on (Ω,T), whose value may change from line to line.

The first term of the second member of (2.22) becomes, using (2.2),

$$\frac{1}{\lambda^2}\int_{\Omega}\int_{\mathbb{R}}|\Phi'(t)\,u(x,t)+\Phi(t)\,\partial_t u(x,t)|^2\,dtdx \leq \frac{C_T}{\lambda^2}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\tag{2.23}$$

The second term of the second member of (2.22) becomes, using (2.10),

$$\begin{array}{l}e^{-4/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\Omega}|W_{\ell_0,\lambda}(x,s)|^2\,dxds\right)d\ell_o\\ = e^{-4/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\Omega}\left|\int_{\mathbb{R}}F_\lambda(\ell_o+is-\ell)\Phi(\ell)u(x,\ell)d\ell\right|^2 dxds\right)d\ell_o\\ \leq e^{-4/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\Omega}\left|\int_0^T\frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}\left(|s|^2-|\ell_o-\ell|^2\right)}|u(x,\ell)|\,d\ell\right|^2 dxds\right)d\ell_o\\ \leq \left(\frac{\sqrt{\pi}}{2\pi}\lambda e^{\lambda^2}\right)^2 e^{-4/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\Omega}\left|\int_0^T|u(x,\ell)|\,d\ell\right|^2 dxds\right)d\ell_o\\ \leq \left(\frac{\sqrt{\pi}}{2\pi}\lambda e^{\lambda^2}\right)^2 e^{-4/\varepsilon}4\left(4T\int_{\Omega}\int_0^T|u(x,\ell)|^2\,d\ell dx\right)\\ \leq C_T\lambda^2 e^{2\lambda^2}e^{-4/\varepsilon}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\end{array}\tag{2.24}$$

The third term of the second member of (2.22) becomes, using (2.10),

$$\begin{array}{l}e^{C/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\omega}|W_{\ell_0,\lambda}(x,s)|^2\,dxds\right)d\ell_o\\ = e^{C/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\omega}\left|\int_{\mathbb{R}}F_\lambda(\ell_o+is-\ell)\Phi(\ell)u(x,\ell)d\ell\right|^2 dxds\right)d\ell_o\\ \leq e^{C/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\omega}\left|\int_0^T\frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}\left(|s|^2-|\ell_o-\ell|^2\right)}|\Phi(\ell)u(x,\ell)|\,d\ell\right|^2 dxds\right)d\ell_o\\ \leq \left(\frac{\sqrt{\pi}}{2\pi}\lambda e^{\lambda^2}\right)^2 e^{C/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_{\omega}\left|\int_0^T|u(x,\ell)|\,d\ell\right|^2 dxds\right)d\ell_o\\ \leq 4\lambda^2 Te^{2\lambda^2}e^{C/\varepsilon}\int_{\omega}\int_0^T|u(x,t)|^2\,dtdx\end{array}\tag{2.25}$$

The fourth term of the second member of (2.22) becomes, using (2.10) and the choice of Φ,

$$\begin{array}{l}
e^{C/\varepsilon}\int_{\ell_o\in\left(\frac{T}{2}-2,\frac{T}{2}+2\right)}\left(\int_{|s|\leq 2}\int_\Omega\left|\int_{\mathbb{R}}-F_\lambda(\ell_0+is-\ell)\left[\Phi''(\ell)u(x,\ell)+2\Phi'(\ell)\partial_t u(x,\ell)\right]d\ell\right|^2 dxds\right)d\ell_o\\
\leq e^{C/\varepsilon}\int_{K_o}\left(\int_{|s|\leq 2}\int_\Omega\left|\int_K\frac{\sqrt{\pi}}{2\pi}\lambda e^{\frac{\lambda^2}{4}\left(|s|^2-|\ell_o-\ell|^2\right)}\left|\Phi''(\ell)u(x,\ell)+2\Phi'(\ell)\partial_t u(x,\ell)\right|d\ell\right|^2 dxds\right)d\ell_o\\
\leq\left(\frac{\sqrt{\pi}}{2\pi}\lambda e^{\lambda^2}\right)^2 e^{C/\varepsilon}C_T\int_{K_o}\left(\int_{|s|\leq 2}\int_\Omega\left|\int_K e^{-\frac{\lambda^2}{4}|\ell_o-\ell|^2}\left(|u(x,\ell)|+|\partial_t u(x,\ell)|\right)d\ell\right|^2 dxds\right)d\ell_o\\
\leq C_T\left(\frac{\sqrt{\pi}}{2\pi}\lambda e^{\lambda^2}e^{-\frac{\lambda^2}{4}\mathrm{dist}(K,K_0)^2}\right)^2 e^{C/\varepsilon}\int_\Omega\left|\int_K\left(|u(x,\ell)|+|\partial_t u(x,\ell)|\right)d\ell\right|^2 dx\\
\leq C_T\lambda^2 e^{2\lambda^2}e^{-\frac{\lambda^2}{2}\mathrm{dist}(K,K_0)^2}e^{C/\varepsilon}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\
\leq C_T\lambda^2 e^{\frac{\lambda^2}{2}\left(4-\frac{T^2}{8^2}\right)}e^{C/\varepsilon}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}
\end{array}\qquad(2.26)$$

We finally obtain from (2.23), (2.24), (2.25), (2.26) and (2.22), that

$$\begin{array}{ll}
\int_\Omega\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|\Phi(t)u(x,t)\right|^2 dtdx & \leq\frac{C_T}{\lambda^2}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\
& +C_T\lambda^2 e^{2\lambda^2}e^{-4/\varepsilon}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\
& +4c\lambda^2 Te^{2\lambda^2}e^{C/\varepsilon}\int_\omega\int_0^T|u(x,t)|^2 dtdx\\
& +C_T\lambda^2 e^{\frac{\lambda^2}{2}\left(4-\frac{T^2}{8^2}\right)}e^{C/\varepsilon}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\ .
\end{array}\qquad(2.27)$$

We begin to choose $\lambda^2=\frac{1}{\varepsilon}$ in order that

$$\begin{array}{ll}
\int_\Omega\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|\Phi(t)u(x,t)\right|^2 dtdx & \leq\varepsilon C_T\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\
& +e^{-1/\varepsilon}C_T\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\
& +e^{(3+C)/\varepsilon}C_T\int_\omega\int_0^T|u(x,t)|^2 dtdx\\
& +C_T\exp\left(\left(3-\frac{1}{2}\frac{T^2}{8^2}+C\right)\frac{1}{\varepsilon}\right)\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\ .
\end{array}\qquad(2.28)$$

We finally choose $T>16$ large enough such that $\left(3-\frac{1}{2}\frac{T^2}{8^2}+C\right)\leq -1$ that is $8^2\left(8+2C\right)\leq T^2$, to deduce that

$$\begin{array}{ll}
\int_\Omega\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}|u(x,t)|^2 dtdx & \leq\int_\Omega\int_{t\in\left(\frac{T}{2}-1,\frac{T}{2}+1\right)}\left|\Phi(t)u(x,t)\right|^2 dtdx\\
& \leq\varepsilon C_T\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}+e^{(3+C)/\varepsilon}C_T\int_\omega\int_0^T|u(x,t)|^2 dtdx\ .
\end{array}\qquad(2.29)$$

Now we conclude from (2.3), that there exist a constant $c>0$ and a time $T>0$ large enough such that for all $\varepsilon\in(0,1)$ small enough, we have

$$\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\leq e^{c/\varepsilon}\int_\omega\int_0^T|u(x,t)|^2 dtdx+\varepsilon\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\qquad(2.30)$$

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