Asymptotic control of chaos for a partial differential equation

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Abstract .- We control asymptotically the chaotic behaviour of a infinite-dimensional dynamical system generated by a first-order linear partial differential equation described in [1].

1 Introduction and main result

For $n \in \mathbb{N}\backslash\{0\}$, let X_n be the space of functions defined by

$$X_n = \left\{ v \in C^n [0,1] ; v(0) = v'(0) = \cdots = v^{(n)}(0) = 0 \right\} ,$$

and equipped with the topology of uniform convergence with derivatives of order $\leq n$ and with the norm

$$\|v\| = \max_{x \in [0,1]} \left| v^{(n)}(x) \right| .$$

Let $\{S_t\}_{t \geq 0}$ be the semiflow on the space X_n generated by the first-order partial differential equation

$$u_t + x u_x = \lambda u \quad \text{for } (x,t) \in [0,1] \times [0,+\infty) , \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and with initial condition $u(\cdot, t=0)$ in X_n

$$u(x,0) = v(x) \quad \text{for } x \in [0,1] .$$

Then

$$S_t v(x) = u(x,t) = e^{\lambda t} v\left(x e^{-t}\right) .$$

In [1], the authors prove that if $\lambda \leq n$, then for all $v \in X_n$, $\lim_{t \to +\infty} \|S_t v\| = 0$ and that if $\lambda > n$, then for almost every $v \in X_n$, the trajectory starting from v is strongly turbulent. Our motivation is to control the semiflow $\{S_t\}_{t \geq 0}$ when $\lambda > n$.

Let $f = f(x,t) \in C([0,+\infty); X_n)$ denote the additive control action. Introducing f locally in space-time variables into the infinite-dimensional dynamic system (1.1) yields

$$u_t(x,t) + x u_x(x,t) = \lambda u(x,t) + f(x,t) \cdot \chi(x)_{[0,2\varepsilon]} \cdot 1(t)_{[0,T]} \tag{1.2}$$

for $(x,t) \in [0,1] \times [0,+\infty)$ and with initial condition $u(\cdot, t=0)$ in X_n

$$u(x,0) = v(x) \quad \text{for } x \in [0,1] , \tag{1.3}$$

where $T > 0$, $\varepsilon \in (0,1/2)$, $1(x)_{[0,R]}$ is the characteristic function on $[0,R]$ for some $R > 0$, i.e. $1(x)_{[0,R]} = 1$ if $x \in [0,R]$, $1(x)_{[0,R]} = 0$ if $x \notin [0,R]$, and $\chi(x)_{[0,R]}$ denotes a smooth function $C^\infty(\mathbb{R})$ such that $\chi(x)_{[0,R]} = 1$ if $x \in [-R/2, R/2]$, $\chi(x)_{[0,R]} = 0$ if $x \notin [-R,R]$.

The control problem is to find a control function f such that the solution u of system (1.2) satisfies $\lim_{t\to+\infty} \|u(\cdot,t)\| = 0$.

The main result of this note is as follows.

Theorem .- *For all $v \in X_n$ the control function f given by*

$$f(x,t) = \frac{-1}{T} e^{\lambda t} v\left(xe^{-t}\right)$$

implies that the solution u of the infinite-dimensional dynamical system (1.2)-(1.3) satisfies

$$\lim_{t\to+\infty} \|u(\cdot,t)\| = 0 \ .$$

2 Proof of Theorem

Consider the first-order partial differential equation with second member $F = F(x,t) \in C([0,T];X_n)$

$$u_t + xu_x = \lambda u + F \quad \text{for } (x,t) \in [0,1] \times [0,T] \ ,$$

with initial condition $u(\cdot, t=0)$ in X_n

$$u(x,0) = v(x) \quad \text{for } x \in [0,1] \ .$$

Then

$$u(x,t) = S_t v(x) + \int_0^t S_{t-s} F(x,s)\, ds = e^{\lambda t} v\left(xe^{-t}\right) + \int_0^t e^{\lambda(t-s)} F\left(xe^{-(t-s)}, s\right) ds$$

and

$$u(x,T)\cdot 1(x)_{[0,\varepsilon]} = e^{\lambda T}\left(v\left(xe^{-T}\right)\cdot 1(x)_{[0,\varepsilon]} + \int_0^T e^{-\lambda s} F\left(xe^{-(T-s)}, s\right)\cdot 1(x)_{[0,\varepsilon]}\, ds\right) \ .$$

If $F(x,t) = \frac{-1}{T} e^{\lambda t} v\left(xe^{-t}\right) \cdot \chi(x)_{[0,2\varepsilon]}$ then $F\left(xe^{-(T-s)}, s\right) = \frac{-1}{T} e^{\lambda s} v\left(xe^{-(T-s)}e^{-s}\right) \cdot \chi\left(xe^{-(T-s)}\right)_{[0,2\varepsilon]}$, and

$$\begin{aligned} e^{-\lambda s} F\left(xe^{-(T-s)}, s\right)\cdot 1(x)_{[0,\varepsilon]} &= \frac{-1}{T} v\left(xe^{-T}\right)\cdot \chi\left(xe^{-(T-s)}\right)_{[0,2\varepsilon]} \cdot 1(x)_{[0,\varepsilon]} \\ &= \frac{-1}{T} v\left(xe^{-T}\right)\cdot 1(x)_{[0,\varepsilon]} \ , \end{aligned}$$

because when $(x,s) \in [0,\varepsilon] \times (0,T)$, $xe^{-(T-s)} \in [0,\varepsilon]$. We conclude that $u(x,T) = 0$ for all $x \in [0,\varepsilon]$.

Now, consider the first-order partial differential equation for times $t \in (T, +\infty)$

$$u_t + xu_x = \lambda u \quad \text{for } (x,t) \in [0,1] \times (T,+\infty) \ ,$$

with initial condition $u(\cdot, t=T)$ in X_n

$$u(x,T) = w(x) \quad \text{for } x \in [0,1] \ ,$$

such that $w(x) = 0$ for $x \in [0,\varepsilon]$. Then

$$u(x,t) = e^{\lambda(t-T)} w\left(xe^{-(t-T)}\right)$$

and consequently, for all $(x,t) \in [0,1] \times \left(\ln \frac{e^T}{\varepsilon}, +\infty\right)$, $xe^{-(t-T)} \in [0,\varepsilon]$ and $u(x,t) = 0$.

This completes the proof.

References

[1] J. Myjak and R. Rudnicki, Stability versus chaos for a partial differential equation. Chaos, Solitons and Fractals 14 (2002) 607-612.