# Lecture Notes (Wuhan 2006)

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First, I formulate the different notions of control and of stabilization on a model equation. Next, I recall the different works and tools which play an important role in the development of control theory for PDE. Finally, I describe my personal contribution about the polynomial decay rate of the damped wave equation.

The model control problems concern the wave equation in a bounded domain.

## 1 Formulation of the problems

There are closed links between the following four problems: identification of solutions, observation, controllability and stabilization.

### 1.1 Identification of solutions

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$. We consider the following wave equation

$$\left\{ \begin{array}{l} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ \quad u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ . \end{array} \right. \tag{1.1}$$

Let $T > 0$, $\omega$ be a nonempty open subset of $\Omega$ and $\Gamma$ be a nonempty subset of $\partial\Omega$. Here, we ask to answer the following two questions concerning identification of solutions: let $u$ and $v$ be two solutions of (1.1),

- does $u = v$ in $\omega \times (0, T)$ imply $u \equiv v$ ?
- does $\partial_n u = \partial_n v$ on $\Gamma \times (0, T)$ imply $u \equiv v$ ?

By linearity, the above two questions are reduced to the unique continuation property $(UCP)$ for the wave equation. Due to finite speed of propagation, the $(UCP)$ holds only for $T > 0$ large enough and $\Omega$ will be supposed a connected domain. For instance, for a sufficiently smooth boundary, if $T > 2 \max \left\{ dist\,(x, \Gamma)\,, x \in \overline{\Omega} \right\}$, then by Holmgren uniqueness theorem

$$\mathcal{N} = \left\{ u \in H^1\left(\Omega \times (0, T)\right) \text{ being a solution of (1.1) such that } \partial_n u = 0 \text{ on } \Gamma \times (0, T) \right\} = \{0\} \ .$$

## 1.2 Observation

When the *UCP* holds, we will have some interest to quantify it, or in other words, to get an observation estimate. Under some conditions on $T>0$ and $\omega\subset\Omega$, the internal observability for the wave equation consists to establish the existence of a constant $c>0$, such that any solution $u$ of (1.1) with initial data $(u_0,u_1)=(u(\cdot,0),\partial_t u(\cdot,0))\in L^2(\Omega)\times H^{-1}(\Omega)$, satisfies

$$\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\leq c\int_0^T\int_\omega\left|u(x,t)\right|^2dxdt\ .$$

Under some conditions on $T>0$ and $\Gamma\subset\partial\Omega$, the boundary observability for the wave equation consists to establish the existence of a constant $c>0$, such that any solution $u$ of (1.1) with initial data $(u_0,u_1)=(u(\cdot,0),\partial_t u(\cdot,0))\in H^1_0(\Omega)\times L^2(\Omega)$, satisfies

$$\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\leq c\int_0^T\int_\Gamma\left|\partial_n u(x,t)\right|^2d\sigma dt\ .$$

The above two estimates give the stability of the observation. We may only have a quantitative unique continuation estimate as follows. Let $\Omega$ be a smooth connected open bounded set of $\mathbb{R}^N$, $N>1$, with boundary $\partial\Omega$. Let $\omega\subset\Omega$ be a nonempty open subset of $\Omega$, and $\Gamma\subset\partial\Omega$ be a nonempty subset of $\partial\Omega$. There exist $c>0$ and $T>0$ such that any solution $u$ of (1.1) with initial data $(u_0,u_1)=(u(\cdot,0),\partial_t u(\cdot,0))\in H^1_0(\Omega)\times L^2(\Omega)$, $(u_0,u_1)\neq 0$, satisfies

$$\left\|(u_0,u_1)\right\|^2_{H^1(\Omega)\times L^2(\Omega)}\leq e^{c\frac{\left\|(u_0,u_1)\right\|^2_{H^1(\Omega)\times L^2(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}}\int_0^T\int_\omega\left|u(x,t)\right|^2dxdt\ ,$$

$$\left\|(u_0,u_1)\right\|^2_{H^1(\Omega)\times L^2(\Omega)}\leq e^{c\frac{\left\|(u_0,u_1)\right\|^2_{H^1(\Omega)\times L^2(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}}\int_0^T\int_\Gamma\left|\partial_n u(x,t)\right|^2d\sigma dt\ .$$

More generally, we may look for the following kind of interpolation inequality.

$$\left\|(u_0,u_1)\right\|^2_{H^1(\Omega)\times L^2(\Omega)}\leq f\left(\frac{\left\|(u_0,u_1)\right\|^2_{H^1(\Omega)\times L^2(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}\right)\int_0^T\int_\omega\left|u(x,t)\right|^2dxdt\ ,$$

for some positive increasing function $f:\mathbb{R}^+\rightarrow\mathbb{R}^+$.

## 1.3 Controllability

Let $T>0$ and $\Gamma\subset\partial\Omega$. Let $\mathcal{G}$ be a mapping from $L^2(\partial\Omega\times(0,T))$ to $L^2(\Omega)\times H^{-1}(\Omega)$ defined by

$$\mathcal{G}(f)=(v(\cdot,0),\partial_t v(\cdot,0))\quad\text{in }\Omega\ ,$$

where

$$\begin{cases}\partial_t^2 v-\Delta v=0 & \text{in }\Omega\times(0,T)\ ,\\ v=f_{|\Gamma\times(0,T)} & \text{on }\partial\Omega\times(0,T)\ ,\\ (v(\cdot,T),\partial_t v(\cdot,T))=(0,0) & \text{in }\Omega\ .\end{cases}$$

Let us introduce $C_{ad}\subseteq L^2(\Gamma\times(0,T))$ and $D_{ad}\subseteq L^2(\Omega)\times H^{-1}(\Omega)$, we choose $(v_0,v_1)\in D_{ad}$. We consider the map $J_{(v_0,v_1)}$ on $C_{ad}$ defined by

$$J_{(v_0,v_1)}(f)=\left\|\mathcal{G}(f)-(v_0,v_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\ .$$

The exact boundary controllability for the wave equation is equivalent to the surjectivity of the map $\mathcal{G}$ (recall that wave have the time reversibility property). According to physical properties of waves, the

natural question is: what geometrical situations and in particular, what hypothesis on $(\Gamma, T)$ should we impose to have the surjectivity of $\mathcal{G}$?.
In the case where such geometrical hypothesis are not satisfied, we will look for an adequate functional space $D_{ad}$ in which $\operatorname{Im}(\mathcal{G})$ is dense. Thus, the problem of approximate controllability can be rewritten as follows: for all $\epsilon > 0$, for all $(v_0, v_1) \in D_{ad}$, does exist an approximate control function $f \in L^2(\Gamma \times (0,T))$ such that $J_{(v_0,v_1)}(f) \leq \epsilon$?. Furthermore, are we able to estimate the cost of such approximate control $f$ with respect to $\epsilon$?. Of course, the choice of the control function is connected with the cost.
Eventually (and this correspond to the notion of optimal control) one try to minimize over all possible control $f$ , the map $J_{(v_0,v_1)}(f)$, when $(v_0, v_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. This leads to the following question: does exist an admissible optimal control function $f \in C_{ad}$ such that $J_{(v_0,v_1)}(f) = \inf_{g \in C_{ad}} J_{(v_0,v_1)}(g)$?.

Similar questions appear in the context of internal controllability. Let $T > 0$ and $\omega \subset \Omega$. What are the data $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ for which there exists a control function $f \in L^2(\omega \times (0,T))$ such that the solution $v \in C^0\left(\mathbb{R}, H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}, L^2(\Omega)\right)$ of

$$\begin{cases} \partial_t^2 v - \Delta v = f_{|\omega \times (0,T)} & \text{in } \Omega \times \mathbb{R} \ , \\ v = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (v(\cdot,0), \partial_t v(\cdot,0)) = (v_0, v_1) & \text{in } \Omega \ , \end{cases}$$

satisfies $v(\cdot,T) = \partial_t v(\cdot,T) = 0$, that is $v_{|t \geq T} \equiv 0$?.

## 1.4 Stabilization

When the control function $f$ depends on the solution $v$ (closed-loop problems) and when the system becomes dissipative (for instance if absorbing boundary conditions or damped terms are involved), the energy is a positive time decreasing function. Therefore, we study the long time asymptotic behavior of the energy. In particular, the choice of different Cauchy data and/or geometrical hypothesis gives different estimates for the decreasing rate of the energy. The strong stabilization consists obtaining an uniform time exponential rate of decay.

For example, we study the following systems

$$\begin{cases} \partial_t^2 w - \Delta w = 0 \quad \text{in } \Omega \times (0,+\infty) \ , \\ \partial_n w + \lambda(x)\, \partial_t w = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \ , \\ (w, \partial_t w)(\cdot,0) = (w_0, w_1) \quad \text{in } \Omega \ , \end{cases}$$

or

$$\begin{cases} \partial_t^2 w - \Delta w + a(x)\, \partial_t w = 0 \quad \text{in } \Omega \times (0,+\infty) \ , \\ w = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \ , \\ (w, \partial_t w)(\cdot,0) = (w_0, w_1) \quad \text{in } \Omega \ , \end{cases}$$

where $a \in L^\infty(\Omega)$, $a \geq 0$, $\lambda \in L^\infty(\partial\Omega)$, $\lambda \geq 0$ and $(w_0, w_1) \in H^1(\Omega) \times L^2(\Omega)$ with their associated compatibility conditions. Denote by $\mathcal{E}(w,t)$ the energy of the solution $w$:

$$\mathcal{E}(w,t) = \frac{1}{2} \int_\Omega \left( |\partial_t w(x,t)|^2 + |\nabla w(x,t)|^2 \right) dx \ .$$

The weak stabilization consists to prove that for any $(w_0, w_1)$ in a suitable space, $\lim_{t \to +\infty} \mathcal{E}(w,t) = 0$. The strong stabilization consists to prove, under suitable conditions, the existence of $c > 0$ and $\beta > 0$ such that for any $(w_0, w_1)$ in a suitable space, we have a uniform and exponential decay rate

$$\mathcal{E}(w,t) \leq c e^{-\beta t} \mathcal{E}(w,0) \ .$$

We are also interested in the decay rate of the energy for more smooth initial data. In particular, we may only get the existence of a function $f : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$, $\lim\limits_{t \rightarrow +\infty} f(t) = 0$, such that for any regular initial data $(w_0, w_1)$ in a suitable space, we have

$$\mathcal{E}(w,t) \leq f(t)\left[\mathcal{E}(w,0) + \mathcal{E}(\partial_t w, 0)\right] \ .$$

# 2 Background

## References


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### 2.1 Control on the whole boundary

It is one of the first result on control for the wave. Russell [ Ru] uses Huygens' principle to give a control result for the wave equation when the control acts on the whole boundary (see also [ L]).

## 2.2 Dimension $N = 1$

The particular case of the one dimension is well-understood for the nonlinear wave equation thanks to [ Z]. Usually, we used the following two ideas: $\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$; we interchange the time variable with the space variable. In particular, the solutions of

$$\begin{cases} \partial_t^2 p - \partial_x^2 p = 0 & \text{for } (x,t) \in (0,1) \times \mathbb{R} , \\ p = 0 & \text{for } (x,t) \in \{0,1\} \times \mathbb{R} , \end{cases}$$

are of the form $p(x,t) = g(t+x) - g(t-x)$ with $g$ being a 2-periodic function.

## 2.3 Geometric condition

There are in the literature two ways to get an observability estimate for the wave equation in a bounded open set $\Omega$ in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$:

- multipliers method [ Li] (see also [ Li2])-[ K] ($\partial\Omega$ of class $C^2$);
- geometric optics [ BLR] (see also [ Le])-[ BG] ($\partial\Omega$ of class $C^\infty$, later reduced to $C^3$ domains by Burq).

The above two techniques come from scattering problems (study of hyperbolic systems in exterior domains) (see e.g. PhD of Pauen). The multiplier techniques can be seen as a generalization of the Morawetz energy method. The geometric optic techniques are based on microlocal analysis and the theorem of propagation of singularities of Melrose and Sjöstrand which allows to answer the conjecture of Lax and Phillips. More recently, using defect measure, Burq and Gérard established that the geometric control condition of Bardos, Lebeau and Rauch is a necessary and sufficient condition for the exact controllability of the wave equations with Dirichlet boundary conditions.

When no geometric condition is required, Robbiano [ R] proves a quantitative unique continuation estimate for hyperbolic equations from a local Carleman inequality for elliptic operators and a Fourier-Bros-Iagolnitzer transform (see also [ Be]). Then, the cost of the approximate controllability for hyperbolic equations is deduced. Application to boundary stabilization without geometric control condition is established in [ LR]. The optimal result without geometrical hypothesis is given in [ B].

## 2.4 Potential

The study of hyperbolic equations with a potential in $(x,t)$-variable is done by Zhang [ Zh] using a global Carleman inequality.

## 2.5 Other papers

About the heat equation

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About the doubling property for elliptic operators and for parabolic operators

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# 3 Wave equation in the whole space

- When dimension $N = 1$, we have the D'Alembert's formula. If $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, then

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$$

is $C^2$ and solves

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{on } \mathbb{R}^{1+1} , \\ u_{|t=0} = f, \quad \partial_t u_{|t=0} = g . \end{cases}$$

- When dimension $N = 2$, we have

$$u(x,t) = \partial_t \left( \frac{t}{2\pi} \int_{|y|\leq 1} f(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \right) + \frac{t}{2\pi} \int_{|y|\leq 1} g(x+ty) \frac{dy}{\sqrt{1-|y|^2}} .$$

- When dimension $N = 3$, if $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$, then

$$u(x,t) = \frac{1}{4\pi} \int_{y \in \mathbb{S}^2} [f(x+ty) + ty \cdot \nabla f(x+ty) + tg(x+ty)]\, d\sigma(y)$$

is $C^2$ and solves

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{on } \mathbb{R}^{3+1} \ , \\ u_{|t=0} = f, \quad \partial_t u_{|t=0} = g \ . \end{cases} \tag{3.1}$$

### 3.1 Huygens principle

When dimension $N = 3$, if $f$ and $g$ are smooth and compactly supported, say $f = g = 0$ for $|x| \geq R$ for some $R > 0$, then $u(x,t) = 0$ unless $t - R \leq |x| \leq t + R$.

The above result is still true for $N \geq 3$ odd. A weaker version also exists for $N \geq 2$ even.

### 3.2 Application to control [ Ru] by acting on the whole boundary

Let $\Omega$ be a bounded set in $\mathbb{R}^3$, with boundary $\partial\Omega$ of class $C^\infty$. Let $T_o > diam\Omega$. Let $v_0 \in C^3(\overline{\Omega})$ and $v_1 \in C^2(\overline{\Omega})$. Now we introduce $\delta > 0$ such that $T_o > 2\delta + diam\Omega$, and

$$\Omega_\delta = \left\{x \in \mathbb{R}^3, \exists \widehat{x} \in \Omega \text{ with } |x - \widehat{x}| < \delta\right\} \ .$$

Consider $f_\delta \in C^3(\mathbb{R}^3)$ and $g_\delta \in C^2(\mathbb{R}^3)$ be such that

$$\begin{cases} f_\delta = v_0 \quad \text{in } \Omega, \quad f_\delta(x) = 0 \quad \text{for } x \notin \Omega_\delta \ , \\ g_\delta = v_1 \quad \text{in } \Omega, \quad g_\delta(x) = 0 \quad \text{for } x \notin \Omega_\delta \ . \end{cases}$$

Therefore, the solution $u_\delta$ of (3.1) with initial data $(f_\delta, g_\delta)$ satisfies $u_\delta(x,t) = 0$ for $(x,t) \in \Omega \times [T_0, +\infty)$. Indeed, we only need to see that for $x \in \Omega$ and $t \geq T_o$, we get $x + ty \notin \Omega_\delta$ $\forall y \in \mathbb{S}^2$. (Let $\widehat{x} \in \Omega$,

$$\begin{aligned} T_o &\leq t = t\|y\| = \|ty\| \\ &\leq \|ty + x - \widehat{x}\| + \|x - \widehat{x}\| \\ &\leq \|ty + x - \widehat{x}\| + diam\Omega \end{aligned}$$

Thus $2\delta < T_o - diam\Omega \leq \|ty + x - \widehat{x}\|$, which implies $x + ty \notin \Omega_\delta$).
Finally, consider $v$ the restriction to $u_\delta$ on $\Omega \times (0, T_o)$. Then for any smooth $(v_0, v_1)$, there exists $\chi$ such that

$$\begin{cases} \partial_t^2 v - \Delta v = 0 \quad \text{in } \Omega \times (0, T_o) \ , \\ v = \chi \quad \text{on } \partial\Omega \times (0, T_o) \ , \\ (v(\cdot, 0), \partial_t v(\cdot, 0)) = (v_0, v_1) \quad \text{in } \Omega \ , \end{cases}$$

and $v_{|t \geq T_o} \equiv 0$.

## 4 Wave equation in a bounded domain

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $N > 1$, with boundary $\partial\Omega$ of class $C^2$ (or $\Omega$ be a convex domain in order that $H^2(\Omega) \cap H_0^1(\Omega) = \left\{u \in H_0^1(\Omega),\ \Delta u \in L^2(\Omega)\right\}$). Let $T > 0$.

## 4.1 Well-posedness

- $\forall (u_0,u_1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad \exists! u \in C\left(\mathbb{R}, L^2(\Omega)\right) \cap C^1\left(\mathbb{R}, H^{-1}(\Omega)\right)$ weak solution of

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R} \ , \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0,u_1) & \text{in } \Omega \ , \end{cases}$$

in the distribution sense (see also weak solution in the transposition sense when $u = f$ on $\partial\Omega \times (0,T)$ for some $f \in L^1\left(0,T;L^2(\partial\Omega)\right)$)

$$\begin{cases} 0 = \left\langle \partial_t^2 u - \Delta u, \Psi(t)\Phi(x) \right\rangle & \forall \Psi \in C_0^1(\mathbb{R}), \forall \Phi \in C_0^\infty(\Omega) \ , \\ \int_{-\infty}^{+\infty} \Psi(t) u(\cdot,t)\, dt \in H_0^1(\Omega) & \forall \Psi \in C_0^1(\mathbb{R}) \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0,u_1) & \text{in } \Omega \ . \end{cases}$$

- $\forall f \in L^1\left(0,T;L^2(\Omega)\right), \forall (u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega) \quad \exists! u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right)$ solution of

$$\begin{cases} \partial_t^2 u - \Delta u = f & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0,u_1) & \text{in } \Omega \ , \end{cases}$$

Moreover, $\exists c > 0$

$$\|u\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} \leq c\left( \|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))} \right) \ .$$

Also,

$$u(x,t) = \sum_{j\geq 1} \left\{ a_j^0 \cos\left(t\sqrt{\lambda_j}\right) + a_j^1 \frac{1}{\sqrt{\lambda_j}} \sin\left(t\sqrt{\lambda_j}\right) + \frac{1}{\sqrt{\lambda_j}} \int_0^t \sin\left((t-s)\sqrt{\lambda_j}\right) f_j(s)\, ds \right\} e_j(x)$$

where

$$\begin{cases} u_0(x) = \sum_{j\geq 1} a_j^0 e_j(x) \ , & \sum_{j\geq 1} \lambda_j \left|a_j^0\right|^2 < +\infty \ , \\ u_1(x) = \sum_{j\geq 1} a_j^1 e_j(x) \ , & \sum_{j\geq 1} \left|a_j^1\right|^2 < +\infty \ , \\ f(x,t) = \sum_{j\geq 1} f_j(t) e_j(x) \ , \end{cases}$$

the $\{e_j\}_{j\geq 1}$ is a Hilbert basis in $L^2(\Omega)$ formed by the eigenfunctions of the operator $-\Delta$, i.e.,

$$\begin{cases} -\Delta e_j = \lambda_j e_j & \text{in } \Omega \ , \\ e_j = 0 & \text{on } \partial\Omega \ . \end{cases}$$

- $\forall (u_0,u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \quad \exists! u \in C\left(\mathbb{R}, H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}, H_0^1(\Omega)\right) \cap C^2\left(\mathbb{R}, L^2(\Omega)\right)$ strong solution of

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R} \ , \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0,u_1) & \text{in } \Omega \ . \end{cases} \tag{4.1}$$

## 4.2 Energy

Denote by $\mathcal{E}(u,t)$ the energy of the solution $u$ of (4.1) with initial data $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$:

$$\mathcal{E}(u,t) = \frac{1}{2}\int_\Omega \left( |\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx \ .$$

Proposition 1.- $\mathcal{E}(u,t) = \mathcal{E}(u,0)\ \forall t \in \mathbb{R}$ .

Proof.- First, we consider a smooth solution with initial data in $(u_0,u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Therefore, for any $T>0$, $u \in C^1\left(0,T; H_0^1(\Omega)\right)$ and $\Delta u \in C\left(0,T; L^2(\Omega)\right)$. Clearly, by integrations by parts,

$$\frac{d}{dt}\mathcal{E}(u,t) = 0 \text{ in } [0,T] \ .$$

Then, in the general case $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and by density, there exists $(u_{0,n},u_{1,n}) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ such that $(u_{0,n},u_{1,n}) \underset{n\to+\infty}{\longrightarrow} (u_0,u_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$. Thus, $u_n \underset{n\to+\infty}{\longrightarrow} u$ in $C\left(0,T; H_0^1(\Omega)\right) \cap C^1\left(0,T; L^2(\Omega)\right)$ where $u_n$ is solution of (4.1) with initial data $(u_{0,n},u_{1,n})$. Finally, $\mathcal{E}(u_n,t) \underset{n\to+\infty}{\longrightarrow} \mathcal{E}(u,t)$ in $C[0,T]$.

Proposition 2.-

$$\|\partial_t u(\cdot,t)\|^2_{H^{-1}(\Omega)} + \|u(\cdot,t)\|^2_{L^2(\Omega)} = \|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ .$$

Proof.- Recall that $\|u\|^2_{H^{-1}(\Omega)} = \left((-\Delta)^{-1} u, u\right)_{H_0^1(\Omega),H^{-1}(\Omega)}$.

## 4.3 The normal derivative

For the solutions of the wave equation, we have a better result than the classical trace theorem.

Proposition 3.- *For any $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the unique solution $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right)$ of (4.1) satisfies $\partial_n u \in L^2(\partial\Omega \times (0,T))$ and*

$$\exists c>0 \quad \int_0^T \int_{\partial\Omega} |\partial_n u(x,t)|^2 \, d\sigma dt \leq c\,\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \quad \forall (u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega) \ .$$

Proof.- Let $H \in C^1\left(\overline{\Omega}\right)$ be a vector field. For a strong solution $u$, we multiply the equation $\partial_t^2 u - \Delta u = 0$ by $H \cdot \nabla u$ and integrate over $\Omega \times (0,T)$. It comes by integrations by parts

$$\begin{array}{rl} 0 & = \int_0^T \int_\Omega \left(\partial_t^2 u - \Delta u\right) H \cdot \nabla u \\ & = \left[\int_\Omega \partial_t u H \cdot \nabla u\right]_0^T + \frac{1}{2}\int_0^T \int_\Omega divH \left(|\partial_t u|^2 - |\nabla u|^2\right) + \int_0^T \int_\Omega \partial_{x_i} u \partial_{x_i} H_j \partial_{x_j} u \\ & \quad - \int_0^T \int_{\partial\Omega} \partial_n u H \cdot \nabla u + \frac{1}{2}\int_0^T \int_{\partial\Omega} H \cdot n\, |\nabla u|^2 \ . \end{array} \tag{4.2}$$

We choose $H$ be such that $H = n$ on $\partial\Omega$. And recall that $\nabla u = \partial_n u n$ on $\partial\Omega$ because $u = 0$ on $\partial\Omega$. Consequently,

$$\frac{1}{2}\int_0^T \int_{\partial\Omega} |\partial_n u|^2 = \left[\int_\Omega \partial_t u H \cdot \nabla u\right]_0^T + \frac{1}{2}\int_0^T \int_\Omega divH \left(|\partial_t u|^2 - |\nabla u|^2\right) + \int_0^T \int_\Omega \partial_{x_i} u \partial_{x_i} H_j \partial_{x_j} u \ .$$

Now, by a density argument, the above equality is still true for any $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ which gives the desired inequality where $c>0$ depends only on $\Omega$ and $T>0$.

## 4.4 Preliminary properties

Here, we establish three properties concerning the observation for the wave equation. Proposition 4 below shows that observability holds for different norms. Proposition 5 below says that observability can dealt with a term in a lower norm. Proposition 6 says that it still has a interest to get an observation with a small term in a higher norm.

Proposition 4.- . *Let $\omega \subset \Omega$ be a nonempty open subset of $\Omega$. Let $u$ be a solution of (4.1) with initial data $(u_0, u_1)$. The following two statements are equivalent:*

*i) there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq c \int_0^T \int_\omega |u(x,t)|^2 \, dxdt \ ,$$

*for any $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ ;*

*ii) there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq c \int_0^T \int_\omega |\partial_t u(x,t)|^2 \, dxdt \ ,$$

*for any $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ .*

Proof of $i) \Rightarrow ii)$.- We fix $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$. Then, we apply $i)$ to $\varphi(x,t) = \partial_t u(x,t)$.

Proof of $ii) \Rightarrow i)$.- We fix $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. Then, we apply $ii)$ to $\varphi(x,t) = \int_0^t u(x,s)\, ds - (-\Delta)^{-1} u_1(x)$.

Proposition 5.- "a standard uniqueness-compactness argument". *Let $\omega \subset \Omega$ be a nonempty open subset of $\Omega$. Let $T > 0$ be such that*

$$\mathcal{N} = \left\{ u \in L^2(\Omega \times (0,T)) \text{ being a solution of } (4.1) \text{ such that } \partial_t u = 0 \text{ on } \omega \times (0,T) \right\} = \{0\} \ .$$

*Let $u$ be a solution of (4.1) with initial data $(u_0, u_1)$. The following two statements are equivalent:*

*i) there exist $c > 0$ and $d > 0$ such that*

$$\|(u_0, u_1)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq c \int_0^T \int_\omega |\partial_t u(x,t)|^2 \, dxdt + d \|(u_0, u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ ,$$

*for any $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ ;*

*ii) there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq c \int_0^T \int_\omega |\partial_t u(x,t)|^2 \, dxdt \ ,$$

*for any $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ .*

Proof.- Suppose that $ii)$ is false. Then there exists a sequence $(u_{0,n}, u_{1,n})_{n\in\mathbb{N}} \in H^1_0(\Omega) \times L^2(\Omega)$ such that

$$\|(u_{0,n}, u_{1,n})\|^2_{H^1_0(\Omega)\times L^2(\Omega)} = 1 \quad \text{and} \quad \int_0^T \int_\omega |\partial_t u_n|^2 \, dxdt \underset{n\to+\infty}{\longrightarrow} 0 \ ,$$

where $u_n$ is the solution of (4.1) with initial data $(u_{0,n}, u_{1,n})$. By Rellich compactness theorem, there exists a subsequence still denoted by $(u_{0,n}, u_{1,n})_{n\in\mathbb{N}} \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\|(u_{0,n}, u_{1,n})\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \underset{n\to+\infty}{\longrightarrow} \|(u_0, u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} ,$$

where $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. By applying $i)$ to $(u_{0,n}, u_{1,n})_{n\in\mathbb{N}} \in H_0^1(\Omega) \times L^2(\Omega)$, we get

$$1 \leq d\,\|(u_0, u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \quad \text{and } \partial_t u = 0 \text{ on } \omega \times (0, T) .$$

This contradicts $\mathcal{N} = \{0\}$. A method to get $\mathcal{N} = \{0\}$ is as follows. We check that $\mathcal{N}$ is finite dimensional and prove that $\mathcal{N}$ is stable by $\partial_t$ (in particular, we need $\mathcal{N} \subset H^1(\Omega \times (0,T))$). Then, if $\mathcal{N} \neq \{0\}$, there will exist an eigenfunction $u$ for $\partial_t$ on $\mathcal{N}$ associated to the eigenvalue $\lambda$. Finally, we get a contradiction using uniqueness theorem for second order elliptic operator in a connected domain $\Omega$ with $\omega \subset \Omega$.

Proposition 6.- *Let $\omega \subset \Omega$ be a nonempty open subset of $\Omega$. Let $u$ be a solution of (4.1) with initial data $(u_0, u_1)$. The following two statements are equivalent:*

*i) there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq e^{c/\varepsilon} \int_0^T \int_\omega |u(x,t)|^2 \, dxdt + \varepsilon \|(u_0, u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} ,$$

*for any $\varepsilon > 0$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ ;*

*ii) there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq c e^{c \frac{\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}} \int_0^T \int_\omega |u(x,t)|^2 \, dxdt ,$$

*for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ .*

Proof of $i) \Rightarrow ii)$.- We chose $\varepsilon = \frac{1}{2} \frac{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}{\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}$.

Proof of $ii) \Rightarrow i)$.- When $\frac{\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}} \leq \frac{1}{\varepsilon}$, then $\frac{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}{\int_0^T \int_\omega |u(x,t)|^2 dxdt} \leq ce^{c/\varepsilon}$. When $\frac{\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}} > \frac{1}{\varepsilon}$, then $\|(u_0, u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \varepsilon \|(u_0, u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}$.

## 4.5 Gaussian beam

We present a numerical approximation of a solution of the wave equation in a square domain in $\mathbb{R}^2$ with the Dirichlet boundary condition. From the computations done by Ralston (see http://math.ucla.edu /˜ralston/pub/Gaussnotes.pdf) the solution $u(x_1, x_2, t) = a_0(x_1, x_2, t)\, e^{ik\Phi(x_1,x_2,t)}$ given below solves $\partial_t^2 u - \Delta u = O\left(1/\sqrt{k}\right)$ and is concentrated on the curve $\gamma = \left\{\left(x_1^0, x_2^0 + t\right), t \geq 0\right\}$ where $\left(x_1^0, x_2^0\right) \in \mathbb{R}^2$: for $\alpha > 0$, $\beta > 0$,

$$a_0(x_1, x_2, t) = \frac{1}{\sqrt{1 + 2i\alpha t}} ,$$

$$\Phi(x_1, x_2, t) = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ t \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ t \end{pmatrix} \cdot \begin{pmatrix} \frac{i\alpha}{1+2i\alpha t} & 0 & 0 \\ 0 & i\beta & -i\beta \\ 0 & -i\beta & i\beta \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ t \end{pmatrix} ,$$

therefore

$$\begin{aligned} u\left(x_1,x_2,t\right) &= \tfrac{1}{\sqrt{1+2i\alpha t}}\exp\left(i\tfrac{k}{2}\left[\left(x_2-x_2^0\right)-t\right]+i\tfrac{k\alpha^2 t}{1+(2\alpha t)^2}\left(x_1-x_1^0\right)^2\right) \\ &\quad \exp\left(-\tfrac{k\alpha}{2\left(1+(2\alpha t)^2\right)}\left(x_1-x_1^0\right)^2-\tfrac{k\beta}{2}\left[\left(x_2-x_2^0\right)-t\right]^2\right)\ . \end{aligned}$$

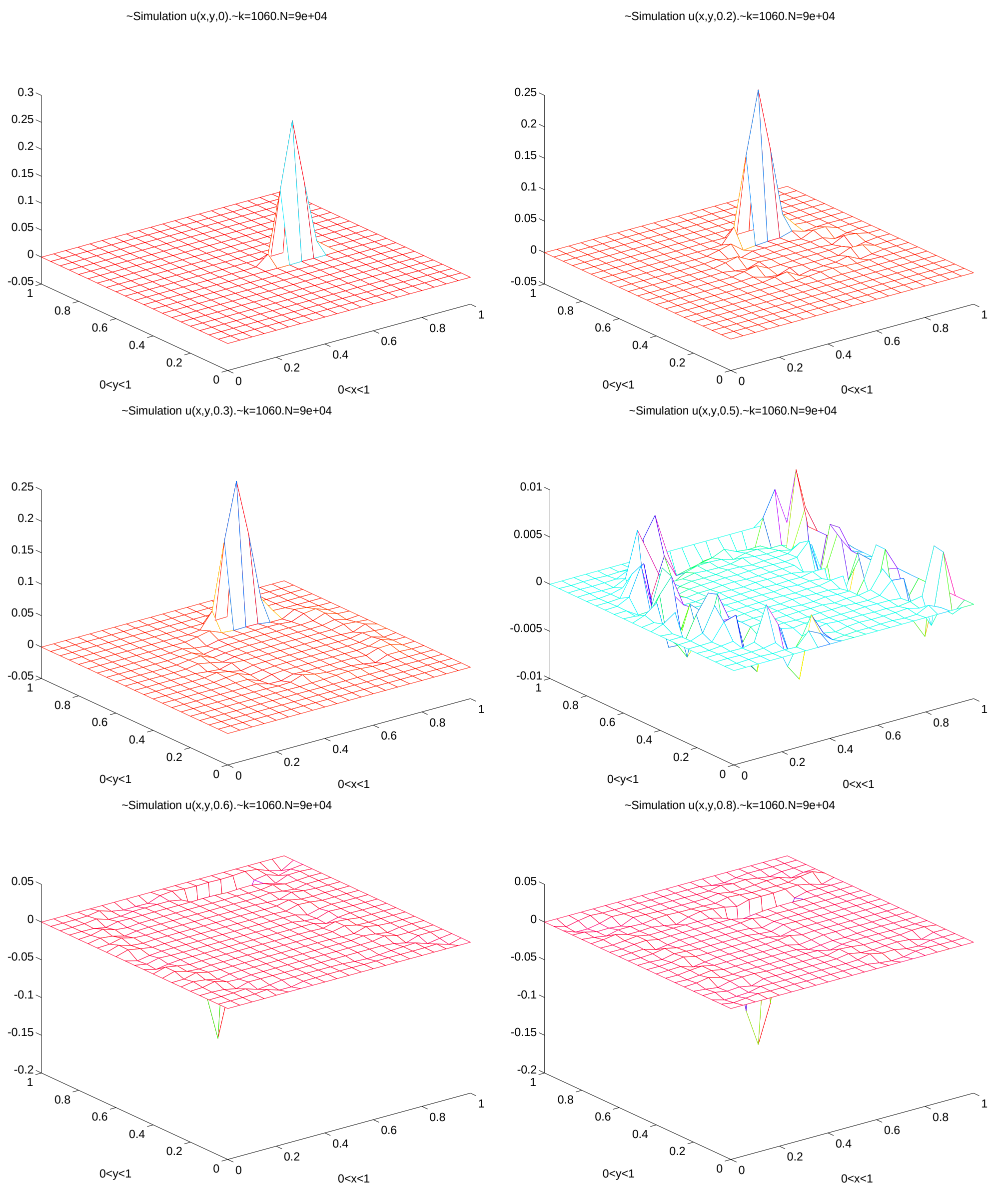
By taking its real part, we get

$$\begin{aligned} u\left(x_1,x_2,t\right) &= \left(\tfrac{1}{1+(2\alpha t)^2}\right)^{1/4}\exp\left(-\tfrac{k\alpha}{2\left(1+(2\alpha t)^2\right)}\left(x_1-x_1^0\right)^2-\tfrac{k\beta}{2}\left[\left(x_2-x_2^0\right)-t\right]^2\right) \\ &\quad \cos\left(\tfrac{k\alpha^2 t}{1+(2\alpha t)^2}\left(x_1-x_1^0\right)^2+\tfrac{k}{2}\left[\left(x_2-x_2^0\right)-t\right]-\tfrac{1}{2}\arctan\left(2\alpha t\right)\right)\ . \end{aligned}$$

Now, the initial data are

$$\begin{aligned} u_0\left(x_1,x_2\right) &= \exp\left(-\tfrac{k\alpha}{2}\left(x_1-x_1^0\right)^2-\tfrac{k\beta}{2}\left(x_2-x_2^0\right)^2\right)\cos\left(\tfrac{k}{2}\left(x_2-x_2^0\right)\right)\\ u_1\left(x_1,x_2\right) &= \exp\left(-\tfrac{k\alpha}{2}\left(x_1-x_1^0\right)^2-\tfrac{k\beta}{2}\left(x_2-x_2^0\right)^2\right)\\ &\qquad \left[k\beta\left(x_2-x_2^0\right)\cos\left(\tfrac{k}{2}\left(x_2-x_2^0\right)\right)-\left(k\alpha^2\left(x_1-x_1^0\right)^2-\tfrac{k}{2}-\alpha\right)\sin\left(\tfrac{k}{2}\left(x_2-x_2^0\right)\right)\right]\ , \end{aligned}$$

and using a Galerkin approximation, we may get a visual idea of a localized solution of the wave equation for $(x,t)\in(0,1)^2\times[0,0.8]$ :




Reflection of a gaussian beam at the boundary under homogeneous Dirichlet boundary conditions.

# 5 Controllability

Two methods allow to get both internal and boundary exact controllability for wave equations from an observability inequality: the HUM method (Hilbert Uniqueness Method) of Lions; the variational approach.

## 5.1 HUM method

We apply the HUM method of Lions to get boundary controllability for the wave equation.

Step 1.- Let us introduce the operator

$$\mathcal{C} : f \in L^2(\Gamma \times (0,T)) \longrightarrow (\partial_t v(\cdot,0), -v(\cdot,0)) \in H^{-1}(\Omega) \times L^2(\Omega) \ ,$$

where $v \in C\left([0,T], L^2(\Omega)\right) \cap C^1\left([0,T], H^{-1}(\Omega)\right)$ is the solution of

$$\begin{cases} \partial_t^2 v - \Delta v = 0 & \text{in } \Omega \times (0,T) \ , \\ v = f_{|\Gamma \times (0,T)} & \text{on } \partial\Omega \times (0,T) \ , \\ (v(\cdot,T), \partial_t v(\cdot,T)) = (0,0) & \text{in } \Omega \ . \end{cases}$$

Then $\mathcal{C}$ is a linear continuous operator. We define $H^{-1}(\Omega) \times L^2(\Omega) \supset \mathcal{F} = Im\mathcal{C}$, be the range of $\mathcal{C}$, the space of exact controllable data at time $T$ by acting on $\Gamma$. Now, we will need to construct the dual operator of $\mathcal{C}$. Let us introduce the operator

$$\mathcal{K} : (u_1, u_0) \in H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \partial_n u_{|\Gamma \times (0,T)} \in L^2(\Gamma \times (0,T)) \ ,$$

where $u \in C\left([0,T], H_0^1(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right)$ is the solution of

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) & \text{in } \Omega \ . \end{cases}$$

Then $\mathcal{K}$ is a linear continuous operator.

Step 2.- We have the following duality result between $\mathcal{C}$ and $\mathcal{K}$.

Theorem 1.- *For all* $f \in L^2(\Gamma \times (0,T))$ *and* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\int_0^T \int_\Gamma f \mathcal{K}(u_0, u_1)\, d\sigma dt = \langle (u_0, u_1), \mathcal{C}(f) \rangle_{H_0^1(\Omega) \times L^2(\Omega), H^{-1}(\Omega) \times L^2(\Omega)} \ ,$$

*where* $\langle (u_0, u_1), \mathcal{C}(f) \rangle_{L^2(\Omega) \times H_0^1(\Omega), L^2(\Omega) \times H^{-1}(\Omega)} := (u(\cdot,0), \partial_t v(\cdot,0))_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega \partial_t u(\cdot,0)\, v(\cdot,0)\, dx.$

Step 3.- Now we have the following approximate controllability result.

Theorem 2.- $\mathcal{F} = Im\mathcal{C}$ *is dense in* $H^{-1}(\Omega) \times L^2(\Omega)$ *if and only if* $Ker\mathcal{K} = \{(0,0)\}$.

Proof.- We use the formula $\overline{Im\mathcal{C}} = H^{-1}(\Omega) \times L^2(\Omega) \Leftrightarrow \mathcal{F}^\perp = \{(0,0)\}$ and $\mathcal{F}^\perp = Ker\mathcal{K}$ where $\mathcal{F}^\perp$ denotes the orthogonal to $\mathcal{F}$ in $H^{-1}(\Omega) \times L^2(\Omega)$.

Step 4.- Now we have the following exact controllability result.

Theorem 3.- *The following two statements are equivalent.*

*i)* $\mathcal{F} = Im\mathcal{C} = H^{-1}(\Omega) \times L^2(\Omega)$ ;

*ii) there exists $c>0$ such that $\|(u_0,u_1)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq c\int_0^T\int_\Gamma |\partial_n u(x,t)|^2 d\sigma dt$ for any $(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega)$ where $u$ is the solution of (4.1).*

Proof of $i)\Rightarrow ii)$.- First, remark that by a classical functional analysis theorem, if $Im\mathcal{C} = H^{-1}(\Omega)\times L^2(\Omega)$, then $\exists\eta>0$ such that

$$B(0,\eta)_{H^{-1}(\Omega)\times L^2(\Omega)} \subset \mathcal{C}\left(B(0,1)_{L^2(\Gamma\times(0,T))}\right) .$$

Next, let $(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega)$. We construct $(v_0,v_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ such that

$$\left\{\begin{array}{l} \|(v_0,v_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} = 1 \\ (u_0,v_1)_{H^1_0(\Omega),H^{-1}(\Omega)} - \int_\Omega u_1 v_0 dx = \|(u_0,u_1)\|_{H^1_0(\Omega)\times L^2(\Omega)} \end{array}\right. .$$

Then, we take $f\in L^2(\Gamma\times(0,T))$ be such that $\mathcal{C}(f) = (v_1,-v_0)$ and $\|f\|_{L^2(\Gamma\times(0,T))}\leq\frac{1}{\eta}$ in order to get

$$\begin{array}{ll} \|(u_0,u_1)\|_{H^1_0(\Omega)\times L^2(\Omega)} & = (u_0,\partial_t v(\cdot,0))_{H^1_0(\Omega),H^{-1}(\Omega)} - \int_\Omega u_1 v(\cdot,0)\,dx \\ & = \int_0^T\int_\Gamma f\mathcal{K}(u_0,u_1)\,d\sigma dt \\ & \leq \|f\|_{L^2(\Gamma\times(0,T))}\|\mathcal{K}(u_0,u_1)\|_{L^2(\Gamma\times(0,T))} \\ & \leq \frac{1}{\eta}\|\mathcal{K}(u_0,u_1)\|_{L^2(\Gamma\times(0,T))} . \end{array}$$

Proof of $ii)\Rightarrow i)$.- We look for a control $f$ in the following particular form. Denote $\mathcal{B}=\mathcal{C}\circ\mathcal{K}$ and suppose that $f=\mathcal{K}(\varphi_0,\varphi_1)$ for some $(\varphi_0,\varphi_1)\in H^1_0(\Omega)\times L^2(\Omega)$, then for all $(\varphi_0,\varphi_1)\in H^1_0(\Omega)\times L^2(\Omega)$ and $(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega)$,

$$\int_0^T\int_\Gamma \mathcal{K}(\varphi_0,\varphi_1)\,\mathcal{K}(u_0,u_1)\,d\sigma dt = \langle (u_0,u_1),\mathcal{B}(\varphi_0,\varphi_1)\rangle_{H^1_0(\Omega)\times L^2(\Omega),H^{-1}(\Omega)\times L^2(\Omega)} .$$

In particular,

$$\int_0^T\int_\Gamma |\mathcal{K}(u_0,u_1)|^2\,d\sigma dt = \langle (u_0,u_1),\mathcal{B}(u_0,u_1)\rangle_{H^1_0(\Omega)\times L^2(\Omega),H^{-1}(\Omega)\times L^2(\Omega)} .$$

If we can prove that the bilinear form given by

$$((u_0,u_1),(\varphi_0,\varphi_1)) \longmapsto \langle (u_0,u_1),\mathcal{B}(\varphi_0,\varphi_1)\rangle_{H^1_0(\Omega)\times L^2(\Omega),H^{-1}(\Omega)\times L^2(\Omega)}$$

is coercive then by Lax-Milgram theorem, $\forall(v_1,v_0)\in H^{-1}(\Omega)\times L^2(\Omega)\quad\exists(\varphi_0,\varphi_1)\in H^1_0(\Omega)\times L^2(\Omega)$ such that $\mathcal{B}(\varphi_0,\varphi_1)=(v_1,-v_0)$ that is $(v(\cdot,0),\partial_t v(\cdot,0))=(v_0,v_1)$. Now, it is sufficient to see that the coercivity of the bilinear form is deduced from the observability estimate $ii)$.

Further comment.- The cost of the approximate control for can be deduced from a spectral analysis of the operator $\mathcal{B}$ (see [ R]).

## 5.2 Variational approach

We apply the variational approach to get the internal controllability for the wave equation (see course of Micu and Zuazua at http://math.univ-lille1.fr/˜jfcoulom/Journees/Zuazua/notas.pdf).

Step 1.- Let us define the duality product between $L^2(\Omega)\times H^{-1}(\Omega)$ and $H^1_0(\Omega)\times L^2(\Omega)$ by

$$\langle (u_0,u_1),(v_0,v_1)\rangle = (u_1,v_0)_{H^{-1}(\Omega),H^1_0(\Omega)} - \int_\Omega u_0 v_1 dx ,$$

for all $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and $(v_0,v_1)\in H^1_0(\Omega)\times L^2(\Omega)$. Then we have

Theorem 4.- *The initial data* $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *may be driven to zero at time* $T > 0$ *if and only if there exists* $f \in L^2(\omega \times (0,T))$ *such that*

$$0 = \int_0^T \int_\omega uf dxdt - \langle (u_0, u_1), (v_0, v_1) \rangle \ ,$$

*for any* $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *where* $u$ *is the corresponding solution of (4.1).*

Step 2.- Introduce the functional $\mathcal{J} : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$, by

$$\mathcal{J}(u_0, u_1) = \frac{1}{2} \int_0^T \int_\omega |u|^2 dxdt - \langle (u_0, u_1), (v_0, v_1) \rangle \ ,$$

where $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $u$ is the solution of (4.1) with initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. Then we have

Theorem 5.- *Let* $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *If* $(\varphi_0, \varphi_1)$ *is a minimizer of* $\mathcal{J}$, *then* $f = \varphi_{|\omega \times (0,T)}$ *is a control which leads* $(v_0, v_1)$ *to zero at time* $T$, *where* $\varphi$ *is the solution of*

$$\begin{cases} \partial_t^2 \varphi - \Delta \varphi = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ \varphi = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (\varphi(\cdot, 0), \partial_t \varphi(\cdot, 0)) = (\varphi_0, \varphi_1) \quad \text{in } \Omega \ . \end{cases}$$

Step 3.- It remains to prove that

Theorem 6.- . *If we have an observability estimate, that is*

$$\exists c > 0 \quad \|(u_0, u_1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq c \int_0^T \int_\omega |u(x,t)|^2 dxdt \ ,$$

*for any* $u$ *the solution of (4.1) with initial data* $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *then the functional* $\mathcal{J}$ *has an unique minimizer* $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

Step 4.- Moreover, we have

Theorem 7.- *Let* $f = \varphi_{|\omega \times (0,T)}$ *be the control given by minimizing the functional* $\mathcal{J}$. *If* $g \in L^2(\omega \times (0,T))$ *is any other control driving to zero the initial data* $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *then*

$$\|f\|_{L^2(\omega \times (0,T))} \leq \|g\|_{L^2(\omega \times (0,T))} \ .$$

Further comment.- The approximate controllability can be established by a variational approach as follows. Let $\varepsilon > 0$ and $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$. By time-reversibility, we are reduced to look for a control function $f_\varepsilon \in L^2(\omega \times (0,T))$ such that

$$\|(v(\cdot, T), \partial_t v(\cdot, T)) - (z_0, z_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \varepsilon \ ,$$

where

$$\begin{cases} \partial_t^2 v - \Delta v = f_{\varepsilon|\omega \times (0,T)} \quad \text{in } \Omega \times \mathbb{R} \ , \\ v = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (v(\cdot, 0), \partial_t v(\cdot, 0)) = (0, 0) \quad \text{in } \Omega \ . \end{cases}$$

Introduce the functional $\mathcal{J}_\varepsilon : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$, by

$$\mathcal{J}_\varepsilon(u_0, u_1) = \frac{1}{2} \int_0^T \int_\omega |u|^2 dxdt + \varepsilon \|(u_0, u_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} - \langle (u_0, u_1), (z_0, z_1) \rangle \ ,$$

where $u$ is the the solution of (4.1) with initial data $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$. Then under *UCP*, the functional $\mathcal{J}_\varepsilon$ has a minimizer $(\varphi_0,\varphi_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and $f_\varepsilon=\varphi_{|\omega\times(0,T)}$ is an approximate control which leads the solution of

$$\begin{cases} \partial_t^2 v-\Delta v=f_{\varepsilon|\omega\times(0,T)} \quad \text{in } \Omega\times\mathbb{R} \ , \\ v=0 \quad \text{on } \partial\Omega\times\mathbb{R} \ , \\ (v(\cdot,0),\partial_t v(\cdot,0))=(0,0) \quad \text{in } \Omega \ . \end{cases}$$

to $(v(\cdot,T),\partial_t v(\cdot,T))$ such that

$$\left\|(v(\cdot,T),\partial_t v(\cdot,T))-(z_0,z_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\leq\varepsilon \ ,$$

where $\varphi$ is the solution of

$$\begin{cases} \partial_t^2\varphi-\Delta\varphi=0 \quad \text{in } \Omega\times\mathbb{R} \ , \\ \varphi=0 \quad \text{on } \partial\Omega\times\mathbb{R} \ , \\ (\varphi(\cdot,0),\partial_t\varphi(\cdot,0))=(\varphi_0,\varphi_1) \quad \text{in } \Omega \ . \end{cases}$$

# 6 Stabilization

to be completed later (see http://freephung.free.fr/kimdang/pub.html).

# 7 Observability

## 7.1 Multipliers method [ Li2]

Let $x^o\in\mathbb{R}^N$, $\Gamma(x^o)=\{x\in\partial\Omega;(x-x^o)\cdot n(x)>0\}$ and $R(x^o)=\max_{x\in\overline{\Omega}}|x-x^o|$ .

Theorem 8.- *Assume that $T>2R(x^o)$. Then there exists $c>0$, such that any solution $u$ of (4.1) with initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, satisfies*

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\leq c\int_0^T\int_{\Gamma(x^o)}|\partial_n u(x,t)|^2\, d\sigma dt \ .$$

Proof.-

Step 1.- We apply the identity (4.2) with $H(x)=x-x^o$ in order to get

$$\begin{array}{l} \frac{N}{2}\int_0^T\int_\Omega\left(|\partial_t u|^2-|\nabla u|^2\right)+\int_0^T\int_\Omega|\nabla u|^2+\left[\int_\Omega\partial_t u\,(x-x^o)\cdot\nabla u\right]_0^T \\ =\frac{1}{2}\int_0^T\int_{\partial\Omega}(x-x^o)\cdot n\,|\partial_n u|^2 \\ \leq\frac{R(x^o)}{2}\int_0^T\int_{\Gamma(x^o)}|\partial_n u|^2 \ . \end{array}$$

Step 2.- We multiply the equation $\partial_t^2 u-\Delta u=0$ by $u$ and integrate over $\Omega\times(0,T)$. It comes by integrations by parts

$$\int_0^T\int_\Omega\left(|\partial_t u|^2-|\nabla u|^2\right)=\left[\int_\Omega\partial_t u u\right]_0^T \ .$$

Consequently,

$$\begin{aligned}&\tfrac{N}{2}\left[\int_\Omega \partial_t u u\right]_0^T+\tfrac{1}{2}\int_0^T\int_\Omega |\nabla u|^2+\tfrac{1}{2}\int_0^T\int_\Omega |\partial_t u|^2-\tfrac{1}{2}\left[\int_\Omega \partial_t u u\right]_0^T+\left[\int_\Omega \partial_t u\,(x-x^o)\cdot\nabla u\right]_0^T\\ &\leq \tfrac{R(x^o)}{2}\int_0^T\int_{\Gamma(x^o)}|\partial_n u|^2 \ .\end{aligned}$$

From the conservation of energy, we finally get

$$\begin{aligned}&T\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\ &\leq R(x^o)\int_0^T\int_{\Gamma(x^o)}|\partial_n u|^2-(N-1)\left[\int_\Omega \partial_t u u\right]_0^T+2\left|\left[\int_\Omega \partial_t u\,(x-x^o)\cdot\nabla u\right]_0^T\right|\\ &\leq R(x^o)\int_0^T\int_{\Gamma(x^o)}|\partial_n u|^2-(N-1)\left[\int_\Omega \partial_t u u\right]_0^T+2R(x^o)\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \ ,\end{aligned}$$

which implies, for $T>2R(x^o)$, the existence of $c>0$ and $d>0$ such that

$$\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\leq c\int_0^T\int_{\Gamma(x^o)}|\partial_n u|^2+d\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ .$$

We conclude by using a uniqueness-compactness argument.

Further comments.- We also can deduce from the above inequality the following internal observability estimate.

Theorem 9.- *Assume that $T>2R(x^o)$. Then there exists $c>0$ such that any solution $u$ of (4.1) with initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, satisfies*

$$\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\leq c\int_0^T\int_\omega |\partial_t u(x,t)|^2\,dxdt \ ,$$

*where $\omega=\vartheta(x^o)\cap\Omega$ and $\vartheta(x^o)$ is a neighborhood of $\Gamma(x^o)$ in $\mathbb{R}^N$.*

The choice of $T$ may be improved, depending on $\vartheta(x^o)$.

Step 1.- If $\mu>0$ is such that $T-2\mu>2R(x^o)$, then we also have

$$\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\leq c\int_\mu^{T-\mu}\int_{\Gamma(x^o)}|\partial_n u|^2+d\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ .$$

Step 2.- It remains to prove that

$$\int_\mu^{T-\mu}\int_{\Gamma(x^o)}|\partial_n u|^2\leq c\int_0^T\int_{\vartheta(x^o)\cap\Omega}|\partial_t u(x,t)|^2\,dxdt+d\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ ,$$

by reproducing a identity like (4.2) with a suitable $H(x,t)\in C^1\left(\overline{\Omega}\times(0,T)\right)$ such that $H(x,\mu)=H(x,T-\mu)=0$. We will also need to check that

$$\int_0^T\int_{\vartheta(x^o)\cap\Omega}|\Phi(t)\nabla u(x,t)|^2\,dxdt\leq c\int_0^T\int_{\vartheta(x^o)\cap\Omega}|\partial_t u(x,t)|^2\,dxdt+d\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ ,$$

for some suitable $\Phi\in C_0^\infty(0,T)$.

## 7.2 Geometric control condition [ BLR]

### 7.2.1 Preliminary definitions

We begin to recall some definitions.

Definitions.- Let $P(x,D)$ be a differential operator in $\mathbb{R}^N$ with a real principal symbol $p(x,\xi)$. The Hamiltonian vector field of $p$ is given by

$$H_p(x,\xi)=\left(\frac{\partial p}{\partial \xi_1}(x,\xi),\cdots,\frac{\partial p}{\partial \xi_N}(x,\xi);-\frac{\partial p}{\partial x_1}(x,\xi),\cdots,-\frac{\partial p}{\partial x_N}(x,\xi)\right) .$$

A Hamiltonian curve of $p$ is an integral curve of $H_p$, that is a curve

$$\gamma(s)=\left(\begin{array}{c} x(s) \\ \xi(s) \end{array}\right) \quad \text{with } \frac{d}{ds}\gamma(s)=H_p\gamma(s) .$$

$x(s)$ and $\xi(s)$ are solutions of the system of ordinary differential equations

$$\left\{\begin{array}{l} \frac{d}{ds}x(s)=\nabla_\xi p(x(s),\xi(s)) , \\ \frac{d}{ds}\xi(s)=-\nabla_x p(x(s),\xi(s)) . \end{array}\right.$$

The integral curve on which $p(x(s),\xi(s))\equiv 0$, are called bicharacteristic of $p$. $\gamma(s)$ may be written as $e^{sH_p}\gamma(0)$ .

Application to the wave equation.- $P=\partial_t^2-\Delta$, its principal symbol is $p(x,t,\xi,\tau)=|\xi|^2-\tau^2$. The bicharacteristics associated to the wave equations are

$$\left\{\begin{array}{ll} \frac{d}{ds}x_j(s)=2\xi_j(s) , & \text{for } j=1,\cdots,N , \\ \frac{d}{ds}t(s)=-2\tau(s) , & \\ \frac{d}{ds}\xi_j(s)=0 , & \text{for } j=1,\cdots,N , \\ \frac{d}{ds}\tau(s)=0 , & \\ |\xi(s)|^2-\tau(s)^2=0 , & \end{array}\right.$$

which gives for any initial data $(x^o,t^o,\xi^o,\tau^o)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\backslash\{0\}$ such that $|\xi^o|^2=(\tau^o)^2$,

$$\left\{\begin{array}{l} x(s)=2\xi^o s+x^o , \\ t(s)=-2\tau^o s+t^o , \\ \xi(s)=\xi^o , \\ \tau(s)=\tau^o . \end{array}\right.$$

We end by giving an intuitive definition of a ray of geometric optics.

Definition.- A generalyzed bicharacteristic ray associated to $\partial_t^2-\Delta$ is a continuous trajectory

$$s\longmapsto\left(\begin{array}{c} x(s) \\ t(s) \end{array}\right) \text{ satisfying } x(s)\in\overline{\Omega},\ t(s)=s \text{ and}$$

when $x(t)\in\Omega$ then it propagates in straight line with speed one until it hits $\partial\Omega$ at time $t_0$. If it hits $\partial\Omega$ at time $t_0$ transversally, then the reflection off the boundary is subject to the optic geometric rules (the angle of incidence equals the angle of reflection) as light rays or billiard balls. If it hits $\partial\Omega$ at time $t_0$ tangentially, then either there exists another trajectory $\widetilde{x}(t)$ such that $x(t_0)=\widetilde{x}(t_0)$ and $\frac{d}{dt}x(t_0)=\frac{d}{dt}\widetilde{x}(t_0)$ living in $\Omega$ and then $x(t)=\widetilde{x}(t)$ for $t>t_0$ until it hits the boundary; or there is no such kind of $\widetilde{x}$ and then it glides along the boundary until it may branch onto a trajectory $\widetilde{x}(t)$ in $\Omega$.

The existence of a unique generalized bicharacteristic ray holds under one of the following assumptions: $\partial\Omega$ is analytic; $\partial\Omega$ is $C^\infty$ and $\partial\Omega$ has no contacts of infinite order with its tangents; $\partial\Omega$ is $C^k$ for some integer $k\geq 3$ and $\partial\Omega$ has no contacts of order $k-1$ with its tangents.

### 7.2.2 Necessary and sufficient condition

Here, we begin to describe the result of [ BG].

Let $\Theta \in C_0(\partial\Omega \times (0,T))$ a compactly supported continuous function. We say that $\Theta$ exactly controls $\Omega$ if for any $(v_0, v_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists $g \in L^2(\partial\Omega \times \mathbb{R})$ such that the solution of

$$\begin{cases} \partial_t^2 v - \Delta v = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ v = \Theta g \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (v(\cdot,0), \partial_t v(\cdot,0)) = (v_0, v_1) \quad \text{in } \Omega \ , \end{cases}$$

satisfies $v_{|t \geq T} \equiv 0$. We say that $\Theta$ geometrically controls $\Omega$ if any generalized bicharacteristic ray meets the set $\{(x,t) \in \partial\Omega \times \mathbb{R}; \Theta(x,t) \neq 0\}$ on a non-diffractive point.

Theorem ([ BG]).- *Assume that $\Omega$ is of class $C^\infty$ and $\partial\Omega$ has no contacts of infinite order with its tangents. Then the following statements are equivalent.*

*i) the function $\Theta$ exactly controls $\Omega$ ;*

*ii) the function $\Theta$ geometrically controls $\Omega$ ;*

*iii) there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq c \int_{\partial\Omega \times \mathbb{R}} |\Theta \partial_n u(x,t)|^2 \, d\sigma dt \ ,$$

*for any $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ where $u$ is the solution of (4.1).*

Previously, Bardos, Lebeau and Rauch proved that

Theorem ([ BLR]).- *Assume that $\Omega$ is of class $C^\infty$ and $\partial\Omega$ has no contacts of infinite order with its tangents. Let $T > 0$ and $\Gamma \subset \partial\Omega$ be a nonempty subset of $\partial\Omega$ such that any generalized bicharacteristic ray meets $\Gamma \times (0,T)$ in a non-diffractive point, then there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq c \int_0^T \int_\Gamma |\partial_n u(x,t)|^2 \, d\sigma dt \ ,$$

*for any $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ where $u$ is the solution of (4.1).*

A similar result holds for internal observability.

Theorem ([ BLR]).- *Assume that $\Omega$ is of class $C^\infty$ and $\partial\Omega$ has no contacts of infinite order with its tangents. Let $T > 0$ and $\omega \subset \Omega$ be a nonempty open subset of $\Omega$ such that any generalized bicharacteristic ray meets $\omega \times (0,T)$, then there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq c \int_0^T \int_\omega |u(x,t)|^2 \, dx dt \ ,$$

*for any $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ where $u$ is the solution of (4.1).*

# 8 Courses

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