Doubling property for the Laplacian and its applications (Course Chengdu 2007)

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1 The original approach of N. Garofalo and F.H. Lin

For simplicity, we reproduce the proof of N. Garofalo and F.H. Lin in the simplest case of the Laplacian.

Let $B_{y_o,R} = \left\{y \in \mathbb{R}^{N+1},\ \ |y - y_o| < R\right\}$, with $y_o \in \mathbb{R}^{N+1}$ and $R > 0$. The unit sphere is defined by $S^N = \partial B_{0,1}$.

Theorem 4.1 .- *Let $D \subset \mathbb{R}^{N+1}$, $N \geq 1$, be a connected bounded open set such that $B_{0,1} \subseteq D$. If $v = v(y) \in H^2(D)$ is a solution of $\Delta_y v = 0$ in D, then for any $0 < M \leq 1/2$, we have*

$$\int_{B_{0,2M}} |v(y)|^2 \, dy \leq 2^{N+1} \exp\left((\ln 4) \frac{\int_{B_{0,1}} |\nabla v(y)|^2 \, dy}{\int_{\partial B_{0,1}} |v(y)|^2 \, d\sigma} \right) \int_{B_{0,M}} |v(y)|^2 \, dy \ .$$

Proof .- We will use the following three formulas. Let $R > 0$,

$$\int_{B_{0,R}} f(y) \, dy = \int_0^R \int_{S^N} f(rs) \, r^N dr d\sigma(s) \ , \tag{4.1}$$

$$\frac{d}{dR} \int_{B_{0,R}} f(y) \, dy = \int_{S^N} f(Rs) \, R^N d\sigma(s) = \int_{\partial B_{0,R}} f(y) \, d\sigma(y) \ , \tag{4.2}$$

$$\int_{B_{0,R}} \frac{\partial f}{\partial y_i}(y) \, dy = \int_{S^N} f(Rs) \, s_i \ R^N d\sigma(s) \ . \tag{4.3}$$

The identity (4.1) is the formula of change of variable in spherical coordinate. (4.2) comes from (4.1) when f is a continuous function. The identity (4.3), available when ∇f is integrable, traduces the Green formula. Indeed,

$$\begin{aligned} \int_{B_{0,R}} \frac{\partial f}{\partial y_i}(y) \, dy &= \int_{\partial B_{0,R}} f(y) \, \nu_i(y) \, d\sigma(y) \\ &= \int_{S^N} f(Rs) \, s_i \ R^N d\sigma(s) \quad \text{because on } \partial B_{0,R},\ \nu(x) = \tfrac{x}{R} \ . \end{aligned}$$

Now, we will introduce the following quantities. Let $r > 0$,

$$\begin{aligned} \mathcal{H}(r) &= \int_{\partial B_{0,r}} |v(y)|^2 \, d\sigma(y) \ , \\ \mathcal{D}(r) &= \int_{B_{0,r}} |\nabla v(y)|^2 \, dy \ , \end{aligned} \tag{4.4}$$

and

$$\mathcal{N}(r)=\frac{r\mathcal{D}(r)}{\mathcal{H}(r)}\ . \tag{4.5}$$

The goal is to show that $\mathcal{N}$ is a non-decreasing function for $0<r\leq 1$. To this end, we will compute the derivatives of $\mathcal{H}$ and $\mathcal{D}$.

The computation of the derivative of $\mathcal{H}(r)=\int_{S^N}|v(rs)|^2 r^N d\sigma(s)$ gives

$$\begin{array}{ll}\mathcal{H}'(r) & =Nr^{N-1}\int_{S^N}|v(rs)|^2 d\sigma(s)+\int_{S^N}2v(rs)(\nabla v)_{|rs}\cdot sr^N d\sigma(s)\\ & =N\frac{1}{r}\mathcal{H}(r)+\int_{S^N}\left(\nabla v^2\right)_{|rs}\cdot sr^N d\sigma(s)\\ & =N\frac{1}{r}\mathcal{H}(r)+\int_0^r\int_{S^N}div\left(\nabla v^2\right)r^N drd\sigma(s)\\ & =N\frac{1}{r}\mathcal{H}(r)+\int_{B_{0,r}}\Delta\left(v^2\right)dy\ ,\end{array}$$

but $\Delta\left(v^2\right)=2v\Delta v+2|\nabla v|^2$. Since $\Delta_y v=0$, one finds

$$\frac{\mathcal{H}'(r)}{\mathcal{H}(r)}=N\frac{1}{r}+2\frac{\int_{B_{0,r}}|\nabla v(y)|^2 dy}{\mathcal{H}(r)}\ . \tag{4.6}$$

Next, one remarks that

$$\int_{B_{0,r}}|\nabla v(y)|^2 dy=\int_{\partial B_{0,r}}v(y)\nabla v(y)\cdot\frac{y}{|y|}d\sigma(y)\ , \tag{4.7}$$

indeed,

$$\begin{array}{ll}\int_{B_{0,r}}\partial_{y_i}v\partial_{y_i}vdy & =\int_{B_{0,r}}\partial_{y_i}\left[v\partial_{y_i}v\right]dy-\int_{B_{0,r}}v\partial_{y_i}^2 vdy\\ & =\int_{\partial B_{0,r}}\left[v\partial_{y_i}v\right]\frac{y_i}{|y|}d\sigma(y)\quad\text{because }\Delta_y v=0\text{ and on }\partial B_{0,R},\ \nu(y)=\frac{y}{|y|}\ .\end{array}$$

Consequently, (4.6) becomes from (4.7)

$$\frac{\mathcal{H}'(r)}{\mathcal{H}(r)}=N\frac{1}{r}+2\frac{\int_{\partial B_{0,r}}v(y)\nabla v(y)\cdot\frac{y}{|y|}d\sigma(y)}{\mathcal{H}(r)}\ . \tag{4.8}$$

On another hand, the derivative of $\mathcal{D}(r)=\int_0^r\int_{S^N}\left|(\nabla v)_{|\rho s}\right|^2\rho^N d\rho d\sigma(s)$ is

$$\begin{array}{ll}\mathcal{D}'(r) & =\int_{S^N}\left|(\nabla v)_{|rs}\right|^2 r^N d\sigma(s)\\ & =\frac{1}{r}\int_{S^N}\left|(\nabla v)_{|rs}\right|^2 rs\cdot sr^N d\sigma(s)\\ & =\frac{1}{r}\int_0^r\int_{S^N}div\left(\left|(\nabla v)_{|rs}\right|^2 rs\right)r^N drd\sigma(s)\\ & =\frac{1}{r}\int_{B_{0,r}}div\left(|\nabla v|^2 y\right)dy\ ,\end{array}$$

but $div\left(|\nabla v|^2 y\right)=|\nabla v|^2 divy+\nabla\left(|\nabla v|^2\right)\cdot y$, with $divy=N+1$. It remains to compute $\int_{B_{0,r}}\nabla\left(|\nabla v|^2\right)\cdot ydy$, when $v=v(y)\in H^2(D)$. One have

$$\begin{array}{l}\int_{B_{0,r}}\partial_{y_i}\left(\left(\partial_{y_j}v\right)^2\right)\cdot y_i dy\\ =2\int_{B_{0,r}}\partial_{y_j}v\partial_{y_iy_j}^2 vy_i dy\\ =2\int_{B_{0,r}}\partial_{y_j}\left[\partial_{y_j}v\partial_{y_i}vy_i\right]dy-2\int_{B_{0,r}}\partial_{y_j}^2 v\partial_{y_i}vy_i dy-2\int_{B_{0,r}}\partial_{y_j}v\partial_{y_i}v\partial_{y_j}y_i dy\\ =2r\int_{\partial B_{0,r}}\left(\partial_{y_j}v\frac{y_j}{|y|}\right)\left(\partial_{y_i}v\frac{y_i}{|y|}\right)d\sigma(y)-2\int_{B_{0,r}}\left|\partial_{y_i}v\right|^2 dy\\ \quad\text{because }\Delta_y v=0\text{ and on }\partial B_{0,R},\ \nu(y)=\frac{y}{|y|}=\frac{y}{R}\ .\end{array}$$

Consequently, $\mathcal{D}'(r)=\frac{N-1}{r}\mathcal{D}(r)+2\int_{\partial B_{0,r}}\left|\nabla v(y)\cdot\frac{y}{|y|}\right|^2 d\sigma(y)$, that is

$$\frac{\mathcal{D}'(r)}{\mathcal{D}(r)}=(N-1)\frac{1}{r}+2\frac{\int_{\partial B_{0,r}}\left|\nabla v(y)\cdot\frac{y}{|y|}\right|^2 d\sigma(y)}{\mathcal{D}(r)}\ . \tag{4.9}$$

Finally, the computation of the derivative of $\mathcal{N}(r)=\frac{r\mathcal{D}(r)}{\mathcal{H}(r)}$ gives

$$\mathcal{N}'(r)=\mathcal{N}(r)\left[\frac{1}{r}+\frac{\mathcal{D}'(r)}{\mathcal{D}(r)}-\frac{\mathcal{H}'(r)}{\mathcal{H}(r)}\right] \ , \tag{4.10}$$

and one conclude from (4.4), (4.8), (4.9) and (4.10) that

$$\mathcal{N}'(r)=\mathcal{N}(r)\left[2\frac{\left(\int_{\partial B_{0,r}}\left|\nabla v\cdot\frac{y}{|y|}\right|^2 dy\right)\left(\int_{\partial B_{0,r}}|v|^2 dy\right)-\left(\int_{\partial B_{0,r}}v\nabla v\cdot\frac{y}{|y|}dy\right)^2}{\mathcal{D}(r)\mathcal{H}(r)}\right] \ .$$

Thanks to the Cauchy-Schwarz inequality, one deduce that $\mathcal{N}'(r)\geq 0$ i.e., $\mathcal{N}$ is non-decreasing on $]0,1]$. Therefore, for $r\leq 1$, $\mathcal{N}(r)\leq\mathcal{N}(1)$ that is $\frac{\mathcal{D}(r)}{\mathcal{H}(r)}\leq\frac{1}{r}\mathcal{N}(1)$. Thus, from (4.6) and (4.5), one have

$$\frac{\mathcal{H}'(r)}{\mathcal{H}(r)}-N\frac{1}{r}\leq 2\mathcal{N}(1)\frac{1}{r} \ .$$

Consequently, $\forall r\leq 1$,

$$\frac{d}{dr}\left(\ln\left(\frac{\mathcal{H}(r)}{r^N}\right)\right)\leq 2\mathcal{N}(1)\frac{d}{dr}\left(\ln\frac{1}{r}\right) \ . \tag{4.11}$$

By integrating (4.11) between $R>0$ and $2R\geq 1$, one finds

$$\ln\left(\frac{\mathcal{H}(2R)}{\mathcal{H}(R)}\frac{1}{2^N}\right)\leq 2\mathcal{N}(1)(\ln 2) \ ,$$

that is $\forall 0<R\leq 1/2$,

$$\int_{S^N}|v(2Rs)|^2(2R)^N d\sigma(s)\leq 2^N e^{\mathcal{N}(1)\ln 4}\int_{S^N}|v(Rs)|^2 R^N d\sigma(s) \ .$$

One conclude that for any $M\leq 1/2$,

$$\begin{aligned}\int_{B_{0,2M}}|v(y)|^2 dy &= \int_0^{2M}\int_{S^N}|v(rs)|^2 r^N dr d\sigma(s)\\ &= 2\int_0^M\int_{S^N}|v(2Rs)|^2(2R)^N dR d\sigma(s)\\ &\leq 2^{N+1}e^{\mathcal{N}(1)\ln 4}\int_0^M\int_{S^N}|v(Rs)|^2 R^N dR d\sigma(s)\\ &\leq 2^{N+1}e^{\mathcal{N}(1)\ln 4}\int_{B_{0,M}}|v(y)|^2 dy \ .\end{aligned}$$

Comment .- The above computations can be generalized to an elliptic operator of second order (see [GaL], [Ku2]).

2 The improved approach of I. Kukavica

It seems more natural to consider the monotonicity properties of the frequency function defined by

$$\frac{\int_{B_{0,r}}|\nabla v(y)|^2\left(r^2-|y|^2\right)dy}{\int_{B_{0,r}}|v(y)|^2 dy}\quad\text{instead of}\quad\frac{r\int_{B_{0,r}}|\nabla v(y)|^2 dy}{\int_{\partial B_{0,r}}|v(y)|^2 d\sigma(y)} \ .$$

Following the ideas of Kukavica ([Ku], [KN], see also [AE]), one obtains the following three lemmas.

2.1 Monotonicity formula

We present the following lemmas.

Lemma A .- *Let $D\subset\mathbb{R}^{N+1}$, $N\geq 1$, be a connected bounded open set such that $\overline{B_{y_o,R_o}}\subset D$ with $y_o\in D$ and $R_o>0$. If $v=v(y)\in H^2(D)$ is a solution of $\Delta_y v=0$ in D, then*

$$\Phi(r)=\frac{\int_{B_{y_o,r}}\left|\nabla v(y)\right|^2\left(r^2-\left|y-y_o\right|^2\right)dy}{\int_{B_{y_o,r}}\left|v(y)\right|^2dy}\quad \text{is non-decreasing on } 0<r<R_o\ ,$$

and

$$\frac{d}{dr}\ln\int_{B_{y_o,r}}\left|v(y)\right|^2dy=\frac{1}{r}\left(N+1+\Phi(r)\right)\ .$$

Lemma B .- *Let $D\subset\mathbb{R}^{N+1}$, $N\geq 1$, be a connected bounded open set such that $\overline{B_{y_o,R_o}}\subset D$ with $y_o\in D$ and $R_o>0$. Let r_1, r_2, r_3 be three real numbers such that $0<r_1<r_2<r_3<R_o$. If $v=v(y)\in H^2(D)$ is a solution of $\Delta_y v=0$ in D, then*

$$\int_{B_{y_o,r_2}}\left|v(y)\right|^2dy\leq\left(\int_{B_{y_o,r_1}}\left|v(y)\right|^2dy\right)^{\alpha}\left(\int_{B_{y_o,r_3}}\left|v(y)\right|^2dy\right)^{1-\alpha}\quad ,$$

where $\alpha=\frac{1}{\ln\frac{r_2}{r_1}}\left(\frac{1}{\ln\frac{r_2}{r_1}}+\frac{1}{\ln\frac{r_3}{r_2}}\right)^{-1}\in\left]0,1\right[$.

The above two results are still available when we are closed to a part Γ of the boundary $\partial\Omega$ under the homogeneous Dirichlet boundary condition on Γ.

Lemma C .- *Let $D\subset\mathbb{R}^{N+1}$, $N\geq 1$, be a connected bounded open set with boundary ∂D. Let Γ be a non-empty Lipschitz open subset of ∂D. Let r_o, r_1, r_2, r_3, R_o be five real numbers such that $0<r_1<r_o<r_2<r_3<R_o$. Suppose that $y_o\in D$ satisfies the following three conditions:*
i). $B_{y_o,r}\cap D$ is star-shaped with respect to y_o $\quad\forall r\in\left]0,R_o\right[$,
ii). $B_{y_o,r}\subset D$ $\quad\forall r\in\left]0,r_o\right[$,
iii). $B_{y_o,r}\cap\partial D\subset\Gamma$ $\quad\forall r\in\left[r_o,R_o\right[$.
If $v=v(y)\in H^2(D)$ is a solution of $\Delta_y v=0$ in D and $v=0$ on Γ, then

$$\int_{B_{y_o,r_2}\cap D}\left|v(y)\right|^2dy\leq\left(\int_{B_{y_o,r_1}}\left|v(y)\right|^2dy\right)^{\alpha}\left(\int_{B_{y_o,r_3}\cap D}\left|v(y)\right|^2dy\right)^{1-\alpha}\quad ,$$

where $\alpha=\frac{1}{\ln\frac{r_2}{r_1}}\left(\frac{1}{\ln\frac{r_2}{r_1}}+\frac{1}{\ln\frac{r_3}{r_2}}\right)^{-1}\in\left]0,1\right[$.

2.2 Proof of Lemma B

Let

$$H(r)=\int_{B_{y_o,r}}\left|v(y)\right|^2dy\ .$$

By applying Lemma A, we know that

$$\frac{d}{dr}\ln H(r)=\frac{1}{r}\left(N+1+\Phi(r)\right)\ .$$

Next, from the monotonicity property of Φ, one deduces that the following two inequalities

$$\begin{array}{ll} \ln\left(\frac{H(r_2)}{H(r_1)}\right) & = \int_{r_1}^{r_2} \frac{N+1+\Phi(r)}{r} dr \\ & \leq (N+1+\Phi(r_2)) \ln\frac{r_2}{r_1} \ , \\ \ln\left(\frac{H(r_3)}{H(r_2)}\right) & = \int_{r_2}^{r_3} \frac{N+1+\Phi(r)}{r} dr \\ & \geq (N+1+\Phi(r_2)) \ln\frac{r_3}{r_2} \ . \end{array}$$

Consequently,

$$\frac{\ln\left(\frac{H(r_2)}{H(r_1)}\right)}{\ln\frac{r_2}{r_1}} \leq (N+1) + \Phi(r_2) \leq \frac{\ln\left(\frac{H(r_3)}{H(r_2)}\right)}{\ln\frac{r_3}{r_2}} \ ,$$

and therefore the desired estimate holds

$$H(r_2) \leq (H(r_1))^{\alpha} (H(r_3))^{1-\alpha} \ ,$$

where $\alpha = \frac{1}{\ln\frac{r_2}{r_1}} \left(\frac{1}{\ln\frac{r_2}{r_1}} + \frac{1}{\ln\frac{r_3}{r_2}}\right)^{-1}$.

2.3 Proof of Lemma A

We introduce the following two functions H and D for $0 < r < R_o$:

$$\begin{array}{ll} H(r) = & \int_{B_{y_o,r}} |v(y)|^2 dy \ , \\ D(r) = & \int_{B_{y_o,r}} |\nabla v(y)|^2 \left(r^2 - |y-y_o|^2\right) dy \ . \end{array}$$

First, the derivative of $H(r) = \int_0^r \int_{S^N} |v(\rho s + y_o)|^2 \rho^N d\rho d\sigma(s)$ is $H'(r) = \int_{\partial B_{y_o,r}} |v(y)|^2 d\sigma(y)$. Next, recall the Green formula

$$\int_{\partial B_{y_o,r}} |v|^2 \partial_\nu G d\sigma(y) - \int_{\partial B_{y_o,r}} \partial_\nu \left(|v|^2\right) G d\sigma(y) = \int_{B_{y_o,r}} |v|^2 \Delta G dy - \int_{B_{y_o,r}} \Delta\left(|v|^2\right) G dy \ .$$

We apply it with $G(y) = r^2 - |y-y_o|^2$ where $G_{|\partial B_{y_o,r}} = 0$, $\partial_\nu G_{|\partial B_{y_o,r}} = -2r$, and $\Delta G = -2(N+1)$. It gives

$$\begin{array}{ll} H'(r) & = \frac{1}{r} \int_{B_{y_o,r}} (N+1)|v|^2 dy + \frac{1}{2r} \int_{B_{y_o,r}} \Delta\left(|v|^2\right) \left(r^2 - |y-y_o|^2\right) dy \\ & = \frac{N+1}{r} H(r) + \frac{1}{r} \int_{B_{y_o,r}} div(v\nabla v) \left(r^2 - |y-y_o|^2\right) dy \\ & = \frac{N+1}{r} H(r) + \frac{1}{r} \int_{B_{y_o,r}} \left(|\nabla v|^2 + v\Delta v\right) \left(r^2 - |y-y_o|^2\right) dy \ . \end{array}$$

Consequently, when $\Delta_y v = 0$,

$$H'(r) = \frac{N+1}{r} H(r) + \frac{1}{r} D(r) \ , \qquad \text{(A.1)}$$

that is $\frac{H'(r)}{H(r)} = \frac{N+1}{r} + \frac{1}{r}\frac{D(r)}{H(r)}$ the second equality in Lemma A.

Now, we compute the derivative of $D(r) = \int_0^r \int_{S^N} \left|(\nabla v)_{|\rho s + y_o}\right|^2 \left(r^2 - \rho^2\right) \rho^N d\rho d\sigma(s)$:

$$\begin{array}{ll} D'(r) & = \frac{d}{dr}\left(r^2 \int_0^r \int_{S^N} \left|(\nabla v)_{|\rho s + y_o}\right|^2 \rho^N d\rho d\sigma(s)\right) - \int_{S^N} r^2 \left|(\nabla v)_{|rs+y_o}\right|^2 r^N d\sigma(s) \\ & = 2r \int_0^r \int_{S^N} \left|(\nabla v)_{|\rho s + y_o}\right|^2 \rho^N d\rho d\sigma(s) \\ & = 2r \int_{B_{y_o,r}} |\nabla v|^2 dy \ . \end{array} \qquad \text{(A.2)}$$

Here, we remark that

$$\begin{aligned} 2r \int_{B_{y_o,r}} |\nabla v|^2 \, dy &= \tfrac{N+1}{r} D(r) + \tfrac{4}{r} \int_{B_{y_o,r}} |(y-y_o)\cdot \nabla v|^2 \, dy \\ &\quad - \tfrac{1}{r} \int_{B_{y_o,r}} \nabla v \cdot (y-y_o) \, \Delta v \left(r^2 - |y-y_o|^2 \right) dy \ , \end{aligned} \tag{A.3}$$

indeed,

$$\begin{aligned} &(N+1) \int_{B_{y_o,r}} |\nabla v|^2 \left(r^2 - |y-y_o|^2 \right) dy \\ &= \int_{B_{y_o,r}} div \left(|\nabla v|^2 \left(r^2 - |y-y_o|^2 \right) (y-y_o) \right) dy - \int_{B_{y_o,r}} \nabla \left(|\nabla v|^2 \left(r^2 - |y-y_o|^2 \right) \right) \cdot (y-y_o) \, dy \\ &= - \int_{B_{y_o,r}} \partial_{y_i} \left(|\nabla v|^2 \left(r^2 - |y-y_o|^2 \right) \right) (y_i - y_{oi}) \, dy \quad \text{because on } \partial B_{y_o,r},\ r = |y-y_o| \\ &= - \int_{B_{y_o,r}} 2 \nabla v \partial_{y_i} \nabla v \left(r^2 - |y-y_o|^2 \right) (y_i - y_{oi}) \, dy - \int_{B_{y_o,r}} |\nabla v|^2 \left(-2 (y_i - y_{oi}) \right) (y_i - y_{oi}) \, dy \\ &= - \int_{B_{y_o,r}} 2 \nabla v \partial_{y_i} \nabla v \left(r^2 - |y-y_o|^2 \right) (y_i - y_{oi}) \, dy + 2 \int_{B_{y_o,r}} |\nabla v|^2 |y-y_o|^2 \, dy \ , \end{aligned}$$

and

$$\begin{aligned} &- \int_{B_{y_o,r}} \partial_{y_j} v \partial^2_{y_i y_j} v \left(r^2 - |y-y_o|^2 \right) (y_i - y_{oi}) \, dy \\ &= - \int_{B_{y_o,r}} \partial_{y_j} \left((y_i - y_{oi}) \partial_{y_j} v \partial_{y_i} v \left(r^2 - |y-y_o|^2 \right) \right) dy \\ &\quad + \int_{B_{y_o,r}} \partial_{y_j} (y_i - y_{oi}) \partial_{y_j} v \partial_{y_i} v \left(r^2 - |y-y_o|^2 \right) dy \\ &\quad + \int_{B_{y_o,r}} (y_i - y_{oi}) \partial^2_{y_j} v \partial_{y_i} v \left(r^2 - |y-y_o|^2 \right) dy \\ &\quad + \int_{B_{y_o,r}} (y_i - y_{oi}) \partial_{y_j} v \partial_{y_i} v \partial_{y_j} \left(r^2 - |y-y_o|^2 \right) dy \\ &= 0 \quad \text{because on } \partial B_{y_o,r},\ r = |y-y_o| \\ &\quad + \int_{B_{y_o,r}} |\nabla v|^2 \left(r^2 - |y-y_o|^2 \right) dy \\ &\quad + \int_{B_{y_o,r}} (y-y_o) \cdot \nabla v \Delta v \left(r^2 - |y-y_o|^2 \right) dy \\ &\quad -2 \int_{B_{y_o,r}} |(y-y_o) \cdot \nabla v|^2 \, dy \ . \end{aligned}$$

Therefore,

$$\begin{aligned} (N+1) \int_{B_{y_o,r}} |\nabla v|^2 \left(r^2 - |y-y_o|^2 \right) dy &= 2r^2 \int_{B_{y_o,r}} |\nabla v|^2 \, dy - 4 \int_{B_{y_o,r}} |(y-y_o) \cdot \nabla v|^2 \, dy \\ &\quad +2 \int_{B_{y_o,r}} (y-y_o) \cdot \nabla v \Delta v \left(r^2 - |y-y_o|^2 \right) dy \ , \end{aligned}$$

and this is the desired estimate (A.3).

Consequently, from (A.2) and (A.3), we obtain, when $\Delta_y v = 0$, the following formula

$$D'(r) = \frac{N+1}{r} D(r) + \frac{4}{r} \int_{B_{y_o,r}} |(y-y_o) \cdot \nabla v|^2 \, dy \ . \tag{A.4}$$

The computation of the derivative of $\Phi(r) = \frac{D(r)}{H(r)}$ gives

$$\Phi'(r) = \frac{1}{H^2(r)} \left[D'(r) H(r) - D(r) H'(r) \right] \ ,$$

which implies using (A.1) and (A.4) that

$$\begin{aligned} H^2(r) \Phi'(r) &= \left[\tfrac{N+1}{r} D(r) + \tfrac{4}{r} \int_{B_{y_o,r}} |(y-y_o) \cdot \nabla v|^2 \, dy \right] H(r) - \left[\tfrac{N+1}{r} H(r) + \tfrac{1}{r} D(r) \right] D(r) \\ &= \tfrac{1}{r} \left(4 \int_{B_{y_o,r}} |(y-y_o) \cdot \nabla v|^2 \, dy H(r) - D^2(r) \right) \geq 0 \ , \end{aligned}$$

indeed, thanks to an integration by parts and using Cauchy-Schwarz inequality, we have

$$\begin{aligned} D^2(r) &= 4 \left(\int_{B_{y_o,r}} v \nabla v \cdot (y-y_o) \, dy \right)^2 \\ &\leq 4 \left(\int_{B_{y_o,r}} |(y-y_o) \cdot \nabla v|^2 \, dy \right) \left(\int_{B_{y_o,r}} |v|^2 \, dy \right) \\ &\leq 4 \left(\int_{B_{y_o,r}} |(y-y_o) \cdot \nabla v|^2 \, dy \right) H(r) \ . \end{aligned}$$

Therefore, we have proved the desired monotonicity for Φ and this completes the proof of Lemma A.

2.4 Proof of Lemma C

Under the assumption $B_{y_o,r}\cap\partial D\subset\Gamma$ for any $r\in[r_o,R_o)$, we extend v by zero in $\overline{B_{y_o,R_o}\backslash D}$ and denote by $\overline{v}$ its extension. Since $v=0$ on Γ, we have

$$\begin{cases}\overline{v}=v1_D & \text{in } \overline{B_{y_o,R_o}}\ ,\\ \overline{v}=0 & \text{on } B_{y_o,R_o}\cap\partial D\ ,\\ \nabla\overline{v}=\nabla v1_D & \text{in } B_{y_o,R_o}\ .\end{cases}$$

Now, we denote $\Omega_r=B_{y_o,r}\cap D$, when $0<r<R_o$. In particular, $\Omega_r=B_{y_o,r}$, when $0<r<r_o$. We introduce the following three functions:

$$\begin{aligned} H(r)=&\ \int_{\Omega_r}|v(y)|^2\,dy\ ,\\ D(r)=&\ \int_{\Omega_r}|\nabla v(y)|^2\left(r^2-|y-y_o|^2\right)dy\ ,\end{aligned}$$

and

$$\Phi(r)=\frac{D(r)}{H(r)}\geq 0\ .$$

Our goal is to show that Φ is a non-decreasing function. Indeed, we will prove that the following equality holds

$$\frac{d}{dr}\ln H(r)=(N+1)\frac{d}{dr}\ln r+\frac{1}{r}\Phi(r)\ . \tag{C.1}$$

Therefore, from the monotonicity of Φ, we will deduce that (in a similar way than in the proof of Lemma A) that

$$\frac{\ln\left(\frac{H(r_2)}{H(r_1)}\right)}{\ln\frac{r_2}{r_1}}\leq(N+1)+\Phi(r_2)\leq\frac{\ln\left(\frac{H(r_3)}{H(r_2)}\right)}{\ln\frac{r_3}{r_2}}\ ,$$

and this will imply the desired estimate

$$\int_{\Omega_{r_2}}|v(y)|^2\,dy\leq\left(\int_{B_{y_o,r_1}}|v(y)|^2\,dy\right)^{\alpha}\left(\int_{\Omega_{r_3}}|v(y)|^2\,dy\right)^{1-\alpha}\ ,$$

where $\alpha=\frac{1}{\ln\frac{r_2}{r_1}}\left(\frac{1}{\ln\frac{r_2}{r_1}}+\frac{1}{\ln\frac{r_3}{r_2}}\right)^{-1}$.

First, we compute the derivative of $H(r)=\int_{B_{y_o,r}}|\overline{v}(y)|^2\,dy=\int_0^r\int_{S^N}|\overline{v}(\rho s+y_o)|^2\rho^N d\rho d\sigma(s)$.

$$\begin{aligned} H'(r) &= \int_{S^N}|\overline{v}(rs+y_o)|^2 r^N d\sigma(s)\\ &= \frac{1}{r}\int_{S^N}|\overline{v}(rs+y_o)|^2 rs\cdot sr^N d\sigma(s)\\ &= \frac{1}{r}\int_{B_{y_o,r}} div\left(|\overline{v}(y)|^2(y-y_o)\right)dy\\ &= \frac{1}{r}\int_{B_{y_o,r}}\left((N+1)|\overline{v}(y)|^2+\nabla|\overline{v}(y)|^2\cdot(y-y_o)\right)dy\\ &= \frac{N+1}{r}H(r)+\frac{2}{r}\int_{\Omega_r}v(y)\nabla v(y)\cdot(y-y_o)\,dy\ .\end{aligned} \tag{C.2}$$

Next, when $\Delta_y v=0$ in D and $v_{|\Gamma}=0$, we remark that

$$D(r)=2\int_{\Omega_r}v(y)\nabla v(y)\cdot(y-y_o)\,dy\ , \tag{C.3}$$

indeed,

$$\begin{array}{l}
\int_{\Omega_r} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right) dy \\
= \int_{\Omega_r} div \left(v\nabla v \left(r^2 - |y-y_o|^2\right)\right) dy - \int_{\Omega_r} v div \left(\nabla v \left(r^2 - |y-y_o|^2\right)\right) dy \\
= -\int_{\Omega_r} v\Delta v \left(r^2 - |y-y_o|^2\right) dy - \int_{\Omega_r} v\nabla v \cdot \nabla \left(r^2 - |y-y_o|^2\right) dy \\
\quad \text{because on } \partial B_{y_o,r},\ r = |y-y_o| \text{ and } v_{|\Gamma} = 0 \\
= 2\int_{\Omega_r} v\nabla v \cdot (y-y_o)\, dy \quad \text{because } \Delta_y v = 0 \text{ in } D\ .
\end{array}$$

Consequently, from (C.2) and (C.3), we obtain

$$H'(r) = \frac{N+1}{r} H(r) + \frac{1}{r} D(r)\ , \tag{C.4}$$

and this is (C.1).

On another hand, the derivative of $D(r) = \int_0^r \int_{S^N} \left|(\nabla \overline{v})_{|\rho s+y_o}\right|^2 (r^2-\rho^2)\, \rho^N d\rho d\sigma(s)$ is

$$\begin{array}{rl}
D'(r) & = 2r \int_0^r \int_{S^N} \left|(\nabla \overline{v})_{|\rho s+y_o}\right|^2 \rho^N d\rho d\sigma(s) \\
& = 2r \int_{\Omega_r} |\nabla v(y)|^2\, dy\ .
\end{array} \tag{C.5}$$

Here, when $\Delta_y v = 0$ in D and $v_{|\Gamma} = 0$, we will remark that

$$\begin{array}{rl}
2r\int_{\Omega_r} |\nabla v(y)|^2\, dy & = \frac{N+1}{r} D(r) + \frac{4}{r} \int_{B_{y_o,r}} |(y-y_o)\cdot \nabla v(y)|^2\, dy \\
& + \frac{1}{r} \int_{\Gamma\cap B_{y_o,r}} |\partial_\nu v|^2 \left(r^2 - |y-y_o|^2\right)(y-y_o)\cdot \nu d\sigma(y)\ ,
\end{array} \tag{C.6}$$

indeed,

$$\begin{array}{l}
(N+1)\int_{\Omega_r} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right) dy \\
= \int_{\Omega_r} div \left(|\nabla v|^2 \left(r^2 - |y-y_o|^2\right)(y-y_o)\right) dy - \int_{\Omega_r} \nabla \left(|\nabla v|^2 \left(r^2 - |y-y_o|^2\right)\right)\cdot (y-y_o)\, dy \\
= \int_{\Gamma\cap B_{y_o,r}} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right)(y-y_o)\cdot \nu d\sigma(y) - \int_{\Omega_r} \partial_{y_i} \left(|\nabla v|^2 \left(r^2 - |y-y_o|^2\right)\right)(y_i - y_{oi})\, dy \\
= \int_{\Gamma\cap B_{y_o,r}} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right)(y-y_o)\cdot \nu d\sigma(y) \\
\quad -\int_{\Omega_r} 2\nabla v \partial_{y_i} \nabla v \left(r^2 - |y-y_o|^2\right)(y_i - y_{oi})\, dy + 2\int_{\Omega_r} |\nabla v|^2 |y-y_o|^2\, dy\ ,
\end{array}$$

and

$$\begin{array}{l}
-\int_{\Omega_r} \partial_{y_j} v \partial^2_{y_i y_j} v \left(r^2 - |y-y_o|^2\right)(y_i - y_{oi})\, dy \\
= -\int_{\Omega_r} \partial_{y_j} \left((y_i - y_{oi})\partial_{y_j} v \partial_{y_i} v \left(r^2 - |y-y_o|^2\right)\right) dy \\
\quad + \int_{\Omega_r} \partial_{y_j} (y_i - y_{oi})\partial_{y_j} v \partial_{y_i} v \left(r^2 - |y-y_o|^2\right) dy \\
\quad + \int_{\Omega_r} (y_i - y_{oi})\partial^2_{y_j} v \partial_{y_i} v \left(r^2 - |y-y_o|^2\right) dy \\
\quad + \int_{\Omega_r} (y_i - y_{oi})\partial_{y_j} v \partial_{y_i} v \partial_{y_j} \left(r^2 - |y-y_o|^2\right) dy \\
= -\int_{\Gamma\cap B_{y_o,r}} \nu_j \left((y_i - y_{oi})\partial_{y_j} v \partial_{y_i} v \left(r^2 - |y-y_o|^2\right)\right) d\sigma(y) \\
\quad + \int_{\Omega_r} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right) dy \\
\quad + 0 \quad \text{because } \Delta_y v = 0 \text{ in } D \\
\quad - \int_{\Omega_r} 2\,|(y-y_o)\cdot \nabla v|^2\, dy\ .
\end{array}$$

Therefore, when $\Delta_y v = 0$ in D, we have

$$\begin{array}{rl}
(N+1)\int_{\Omega_r} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right) dy & = \int_{\Gamma\cap B_{y_o,r}} |\nabla v|^2 \left(r^2 - |y-y_o|^2\right)(y-y_o)\cdot \nu d\sigma(y) \\
& -2\int_{\Gamma\cap B_{y_o,r}} \partial_{y_j} v \nu_j \left((y_i - y_{oi})\partial_{y_i} v\right)\left(r^2 - |y-y_o|^2\right) d\sigma(y) \\
& +2r^2 \int_{\Omega_r} |\nabla u|^2\, dy - 4\int_{\Omega_r} |(y-y_o)\cdot \nabla v|^2\, dy\ .
\end{array}$$

By using the fact that $v_{|\Gamma}=0$, we get $\nabla v=(\nabla v\cdot\nu)\,\nu$ on Γ and we deduce that

$$\begin{array}{ll}(N+1)\int_{\Omega_r}|\nabla v|^2\left(r^2-|y-y_o|^2\right)dy & =-\int_{\Gamma\cap B_{y_o,r}}|\partial_\nu v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y)\\ & \quad+2r^2\int_{\Omega_r}|\nabla v|^2\,dy-4\int_{\Omega_r}|(y-y_o)\cdot\nabla v|^2\,dy\ ,\end{array}$$

and this is (C.6).

Consequently, from (C.5) and (C.6), when $\Delta_y v=0$ in D and $v_{|\Gamma}=0$, we have

$$D'(r)=\frac{N+1}{r}D(r)+\frac{4}{r}\int_{\Omega_r}|(y-y_o)\cdot\nabla v(y)|^2\,dy+\frac{1}{r}\int_{\Gamma\cap B_{y_o,r}}|\partial_\nu v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y)\ . \tag{C.7}$$

The computation of the derivative of $\Phi(r)=\frac{D(r)}{H(r)}$ gives

$$\Phi'(r)=\frac{1}{H^2(r)}\left[D'(r)\,H(r)-D(r)\,H'(r)\right]\ ,$$

which implies from (C.4) and (C.7), that

$$\begin{array}{ll}H^2(r)\,\Phi'(r) & =\frac{1}{r}\left(4\int_{\Omega_r}|(y-y_o)\cdot\nabla v(y)|^2\,dy\ H(r)-D^2(r)\right)\\ & \quad+\frac{1}{r}\int_{\Gamma\cap B_{y_o,r}}|\partial_\nu v|^2\left(r^2-|y-y_o|^2\right)(y-y_o)\cdot\nu d\sigma(y)\ \ H(r)\ .\end{array}$$

Thanks to (C.3) and Cauchy-Schwarz inequality, we obtain that

$$0\leq 4\int_{\Omega_r}|(y-y_o)\cdot\nabla v(y)|^2\,dy\ H(r)-D^2(r)\ .$$

the inequality $0\leq(y-y_o)\cdot\nu$ on Γ hols when $B_{y_o,r}\cap D$ is star-shaped with respect to y_o for any $r\in\,]0,R_o[$. Therefore, we get the desired monotonicity for Φ which completes the proof of Lemma C.

3 Quantitative unique continuation property for the Laplacian

Let $D\subset\mathbb{R}^{N+1}$, $N\geq 1$, be a connected bounded open set with boundary ∂D. Let Γ be a non-empty Lipschitz open part of ∂D. We consider the Laplacian in D, with a homogeneous Dirichlet boundary condition on $\Gamma\subset\partial\Omega$:

$$\left\{\begin{array}{ll}\Delta_y v=0 & \text{in } D\ ,\\ v=0 & \text{on } \Gamma\ ,\\ v=v(y)\in H^2(D)\ . & \end{array}\right. \tag{4.12}$$

The goal of this section is to describe interpolation inequalities associated to solutions v of (4.12).

Theorem 4.2 .- *Let ω be a non-empty open subset of D. Then, for any $D_1\subset D$ such that $\partial D_1\cap\partial D\Subset\Gamma$ and $\overline{D_1}\backslash(\Gamma\cap\partial D_1)\subset D$, there exist $C>0$ and $\mu\in\,]0,1[$ such that for any v solution of (4.12), we have*

$$\int_{D_1}|v(y)|^2\,dy\leq C\left(\int_\omega|v(y)|^2\,dy\right)^\mu\left(\int_D|v(y)|^2\,dy\right)^{1-\mu}\ .$$

Or in a equivalent way,

Theorem 4.3 .- *Let ω be a non-empty open subset of D. Then, for any $D_1 \subset D$ such that $\partial D_1 \cap \partial D \Subset \Gamma$ and $\overline{D_1} \backslash (\Gamma \cap \partial D_1) \subset D$, there exist $C > 0$ and $\mu \in]0,1[$ such that for any v solution of* (4.12)*, we have*

$$\int_{D_1} |v(y)|^2 dy \leq C \left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}} \int_\omega |v(y)|^2 dy + \varepsilon \int_D |v(y)|^2 dy \quad \forall \varepsilon > 0 \ .$$

Proof of Theorem 4.2 .- We divide the proof into two steps.

Step 1 .- We apply Lemma B, and use a standard argument (voir e.g., [Ro]) which consists to construct a sequence of balls chained along a curve. More precisely, we claim that for any non-empty compact sets in D, K_1 and K_2, such that meas$(K_1) > 0$, there exists $\mu \in]0,1[$ such that for any $v = v(y) \in H^2(D)$, solution of $\Delta_y v = 0$ in D, we have

$$\int_{K_2} |v(y)|^2 dy \leq \left(\int_{K_1} |v(y)|^2 dy\right)^\mu \left(\int_D |v(y)|^2 dy\right)^{1-\mu} . \tag{4.13}$$

Indeed, let $\delta > 0$ and $q_j \in \mathbb{R}^N$ for $j = 0, 1, \cdots, m$, one can construct a sequence of balls $\left\{B_{q_j,\delta}\right\}_{j=0,..,m}$, such that the following inclusion hold

$$\begin{cases} K_1 \supset B_{q_0,\delta} \\ K_2 \subset B_{q_m,\delta_o} \quad \text{for some } \delta_o > 0 \\ B_{q_{j+1},\delta} \subset B_{q_j,2\delta} \quad \forall j = 0,..,m-1 \\ B_{q_j,3\delta} \subset D \quad \forall j = 0,..,m \ . \end{cases}$$

Then, thanks to Lemma B, there exist $\alpha, \alpha_1, \mu \in]0,1[$, such that

$$\begin{array}{ll} \int_{K_2} |v(y)|^2 dy & \leq \int_{B_{q_m,\delta_o}} |v(y)|^2 dy \\ & \leq \left(\int_{B_{q_m,\delta}} |v(y)|^2 dy\right)^\alpha \left(\int_{B_{q_m,3\delta}} |v(y)|^2 dy\right)^{1-\alpha} \\ & \leq \left(\int_{B_{q_{m-1},2\delta}} |v(y)|^2 dy\right)^\alpha \left(\int_D |v(y)|^2 dy\right)^{1-\alpha} \\ & \leq \left(\left(\int_{B_{q_{m-1},\delta}} |v(y)|^2 dy\right)^{\alpha_1} \left(\int_D |v(y)|^2 dy\right)^{1-\alpha_1}\right)^\alpha \left(\int_D |v(y)|^2 dy\right)^{1-\alpha} \\ & \leq \cdots \\ & \leq \left(\int_{B_{q_0,\delta}} |v(y)|^2 dy\right)^\mu \left(\int_D |v(y)|^2 dy\right)^{1-\mu} \ , \end{array}$$

which implies the desired inequality (4.13).

Step 2 .- We apply Lemma C, and choose y_o in a neighborhood of the part Γ such that the conditions *i, ii, iii*, hold. Next, by an adequate partition of D, we deduce from (4.13) that for any $D_1 \subset D$ such that $\partial D_1 \cap \partial D \Subset \Gamma$ and $\overline{D_1} \backslash (\Gamma \cap \partial D_1) \subset D$, there exist $C > 0$ and $\mu \in]0,1[$ such that for any $v = v(y) \in H^2(D)$ such that $\Delta_y v = 0$ on D and $v = 0$ on Γ, we have

$$\int_{D_1} |v(y)|^2 dy \leq C \left(\int_\omega |v(y)|^2 dy\right)^\mu \left(\int_D |v(y)|^2 dy\right)^{1-\mu} .$$

This completes the proof.

Remark .- From standard minimization technique, the above inequality implies

$$\int_{D_1} |v(y)|^2 dy \leq C^{\frac{1}{\mu}} \left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}} \int_\omega |v(y)|^2 dy + \varepsilon \int_D |v(y)|^2 dy \quad \forall \varepsilon > 0 \ .$$

Indeed, we denote $A = \int_{D_1} |v(y)|^2 dy \neq 0$, $B = \int_\omega |v(y)|^2 dy$ and $E = \int_D |v(y)|^2 dy$. We know that there exist $C > 0$ and $\mu \in]0,1[$ such that $A \leq CB^\mu E^{1-\mu}$. Therefore,

$$A \leq C^{\frac{1}{\mu}} B \left(\frac{E}{A}\right)^{\frac{1-\mu}{\mu}} .$$

Now, if $\frac{E}{A} \leq \frac{1}{\varepsilon}$, then $A \leq C^{\frac{1}{\mu}} B \left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}}$. And, if $\frac{E}{A} > \frac{1}{\varepsilon}$, then $A \leq \varepsilon E$. Consequently, one obtain the desired interpolation inequality. Conversely, suppose that there exist $C > 0$ and $\mu \in]0,1[$ such that

$$\int_{D_1} |v(y)|^2 dy \leq C^{\frac{1}{\mu}} \left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}} \int_\omega |v(y)|^2 dy + \varepsilon \int_D |v(y)|^2 dy \quad \forall \varepsilon > 0 ,$$

then, we choose $\varepsilon = \frac{1}{2}\frac{A}{E}$ in order to get $A \leq 2CB^\mu E^{1-\mu}$.

Comment .- The above computations can be generalized to solutions of the following elliptic system

$$\begin{cases} \Delta_y u = f & \text{in } D , \\ u = 0 & \text{on } \Gamma , \\ u = u(y) \in H^2(D) , & \\ (-\Delta_y)^{-1} f \in H^2 \cap H_0^1(D) , & \end{cases}$$

in order to get the following estimate

$$\int_{D_1} |u(y)|^2 dy \leq C \left(\left\| (-\Delta_y)^{-1} f \right\|^2_{L^2(D)} + \int_\omega |u(y)|^2 dy \right)^\mu \left(\int_D |u(y)|^2 dy \right)^{1-\mu} .$$

4 Quantitative unique continuation property for the elliptic operator $\partial_t^2 + \Delta$

In this section, we present the following result (to be compared to [LeR]).

Theorem 4.4 .- *Let Ω be a Lipschitz connected bounded open set of $\mathbb{R}^N$, $N \geq 1$. We choose $T > 0$ and $\delta \in]0, T/2[$. We consider the elliptic operator of second order in $\Omega \times]0,T[$ with a homogeneous Dirichlet boundary condition on $\partial\Omega \times (0,T)$,*

$$\begin{cases} \partial_t^2 u + \Delta u = 0 & \text{on } \Omega \times]0,T[, \\ u = 0 & \text{in } \partial\Omega \times]0,T[, \\ u = u(x,t) \in H^2(\Omega \times]0,T[) . & \end{cases} \tag{4.14}$$

Then, for any $\varphi \in C_0^\infty(\Omega \times (0,T))$, $\varphi \neq 0$, there exist $C > 0$ and $\mu \in]0,1[$ such that for any u solution of (4.14), we have

$$\int_\delta^{T-\delta} \int_\Omega |u(x,t)|^2 dxdt \leq C \left(\int_0^T \int_\Omega |u(x,t)|^2 dxdt \right)^{1-\mu} \left(\int_0^T \int_\Omega |\varphi u(x,t)|^2 dxdt \right)^\mu .$$

Proof .- We apply Theorem 4.2 with $D = \Omega \times]0,T[$, $\Omega \times]\delta, T-\delta[\subset D_1$, $y = (x,t)$, $\Delta_y = \partial_t^2 + \Delta$.

5 Quantitative unique continuation property for the sum of eigenfunctions

The goal of this section is to obtain the following results (to be compared to [LZ] or [JL]).

Theorem 4.5 .- *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, either convex or C^2 and connected. Let ω be a non-empty open subset in Ω. Then, there exists $C > 0$ such that for any sequence $\{a_j\}_{j\geq 1}$ of real numbers and any integer $M > 1$, we have*

$$\sum_{1\leq j\leq M} a_j^2 \leq Ce^{C\sqrt{\lambda_M}} \int_\omega \left| \sum_{1\leq j\leq M} a_j e_j(x) \right|^2 dx \ ,$$

where $\{\lambda_j\}_{j\geq 1}$ and $\{e_j\}_{j\geq 1}$ are the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, constituting an orthonormal basis in $L^2(\Omega)$.

Or in an equivalent way,

Theorem 4.6 .- *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, either convex or C^2 and connected. Let ω be a non-empty open subset in Ω. Then, there exists $C > 0$ such that for any sequence $\{a_j\}_{j\geq 1}$ of real numbers and any $R > \lambda_1$, we have*

$$\sum_{\{j;\lambda_j\leq R\}} a_j^2 \leq Ce^{C\sqrt{R}} \int_\omega \left| \sum_{\{j;\lambda_j\leq R\}} a_j e_j(x) \right|^2 dx \ ,$$

where $\{\lambda_j\}_{j\geq 1}$ and $\{e_j\}_{j\geq 1}$ are the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, constituting an orthonormal basis in $L^2(\Omega)$.

Proof of Theorem 4.5 .- We divide the proof into three steps.

Step 1 .- For any $a_j \in \mathbb{R}$, we introduce the solution

$$w(x,t) = \sum_{j\leq M} a_j e_j(x) \operatorname{ch}\left(\sqrt{\lambda_j} t\right) - \chi(x) \sum_{j\leq M} a_j e_j(x) \ ,$$

where $\chi \in C_0^\infty(\omega)$, $\chi = 1$ in $\widetilde{\omega} \Subset \omega$. Recall that $\operatorname{ch} t = (e^t + e^{-t})/2$. Therefore, w solves

$$\begin{cases} \partial_t^2 w + \Delta w = \Delta f & \text{in } \Omega \times]0,T[\ , \\ w = 0 & \text{on } \partial\Omega \times]0,T[\ , \\ w = \partial_t w = 0 & \text{on } \widetilde{\omega} \times \{0\} \ , \\ w = w(x,t) \in H^2(\Omega \times]0,T[) \ , & \end{cases}$$

for any $T > 0$, where $f = -\chi \sum_{j\leq M} a_j e_j \in H_0^2(\Omega)$. We denote by $\overline{w}$ the extension of w by zero in $\overline{\widetilde{\omega} \times]-T,0[}$. Therefore, $\overline{w}$ solves

$$\begin{cases} \partial_t^2 \overline{w} + \Delta \overline{w} = \Delta f 1_{\Omega\times(0,T)} & \text{in } \Omega \times]0,T[\cup \widetilde{\omega} \times]-T,0[\ , \\ \overline{w} = 0 & \text{in } \widetilde{\omega} \times]-T,0[\ , \\ \overline{w} = 0 & \text{on } \partial\Omega \times]0,T[\ . \end{cases}$$

At present, we define D, a connected open set in $\mathbb{R}^{N+1}$, satisfying the following six conditions:
i). $\Omega \times]\delta, T-\delta[\subset D$ for some $\delta \in]0,T/2[$;
ii). $\partial\Omega \times]\delta, T-\delta[\subset \partial D$;
iii). $D \subset \Omega \times]0,T[\cup \widetilde{\omega} \times]-T,0[$;

iv). there exists a non-empty open subset $\omega_o \Subset D \cap \widetilde{\omega} \times]-T_o, 0[$ for some $T_o \in]0, T[$;
v). $D \in C^2$ if Ω is C^2 and connected ;
vi). D is convex with an adequate choice of (δ, T_o) if Ω is convex .

In particular, $\overline{w} \in H^2(D)$.

Step 2 .- We claim that there exists $g \in H^2(\Omega \times]-T,T[) \cap H_0^1(\Omega \times]-T,T[) \subset H^2(D)$ such that

$$\begin{cases} \partial_t^2 g + \Delta g = \Delta f 1_{\Omega \times (0,T)} & \text{in } \Omega \times]-T,T[\ , \\ g = 0 & \text{on } \partial(\Omega \times]-T,T[) \ , \end{cases}$$

and

$$\|g\|_{L^2(D)} \leq \|f\|_{L^2(\Omega \times]0,T[)} \ . \tag{4.15}$$

Indeed, we will proceed with six substeps when Ω is C^2 and connected (the case where Ω is convex is well-known since then $\Omega \times]-T,T[$ is convex). We denote $h = \Delta f 1_{\Omega \times (0,T)} \in L^2(\Omega \times]-T,T[)$.

Substep 1: one recall that $h \in L^2(\Omega \times]-T,T[)$ implies the existence of $g \in H_0^1(\Omega \times]-T,T[)$.

Substep 2: thanks to the interior regularity for elliptic systems, for any $D_0 \Subset \Omega \times]-T,T[$, $g \in H^2(D_0)$.

Substep 3: thank to the boundary regularity for elliptic systems, but not closed to the boundary $\Omega \times \{-T,T\}$, g is also locally in H^2 because Ω is C^2.

Substep 4: we extend the solution at $t = T$ as follows.
Let $\overline{h}(x,t) = h(x,t)$ for $(x,t) \in \Omega \times]-T,T[$ and $\overline{h}(x,t) = -h(x, 2T - t)$ for $(x,t) \in \Omega \times]T, 3T[$. Thus $\overline{h} \in L^2(\Omega \times]-T, 3T[)$. Let $\overline{g}(x,t) = g(x,t)$ for $(x,t) \in \Omega \times [-T,T[$ and $\overline{g}(x,t) = -g(x, 2T - t)$ for $(x,t) \in \Omega \times [T, 3T]$. Thus, $\overline{g}$ solves

$$\begin{cases} \partial_t^2 \overline{g} + \Delta \overline{g} = \overline{h} & \text{in } \Omega \times]-T, 3T[\ , \\ \overline{g} = 0 & \text{on } \partial(\Omega \times]-T, 3T[) \ . \end{cases}$$

By applying the boundary regularity as in substep 3, one obtain that $\overline{g} \in H^2(\Omega \times]0, 2T[)$. In particular, $g \in H^2(\Omega \times]0,T[)$.

Substep 5: we extend in a similar way at $t = -T$ in order to conclude that $g \in H^2(\Omega \times]-T, 0[)$.

Substep 6: finally, we multiply $\partial_t^2 g + \Delta g = h$ by $(-\Delta)^{-1} g$ and integrate by parts over $\Omega \times]-T,T[$, in order to obtain

$$\begin{array}{ll} \int_{-T}^{T} \|\partial_t g\|_{H^{-1}(\Omega)}^2 dt + \|g\|_{L^2(\Omega \times]-T,T[)}^2 & = \int_0^T \int_\Omega -\Delta f(x) (-\Delta)^{-1} g(x,t)\, dxdt \\ & = \int_0^T \int_\Omega f(x) (-\Delta)(-\Delta)^{-1} g(x,t)\, dxdt \quad \text{because } f \in H_0^2(\Omega) \\ & \leq \|f\|_{L^2(\Omega \times]0,T[)} \|g\|_{L^2(\Omega \times]-T,T[)} \quad \text{from Cauchy-Schwarz} \ . \end{array}$$

which gives the desired inequality (4.15).

Step 3 .- Finally, we apply Theorem 4.3 with $\Delta_y = \partial_t^2 + \Delta$, $v = \overline{w} - g$ in D with $\Gamma = \partial\Omega \times]0,T[$ and $\Omega \times]\delta, T - \delta[\subset D_1 \subset D$ such that $\partial D_1 \cap \partial D \Subset \Gamma$ and $\overline{D_1} \backslash (\Gamma \cap \partial D_1) \subset D$ in order that

$$\int_{D_1} |\overline{w} - g|^2 dy \leq C \left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}} \int_{\omega_o} |\overline{w} - g|^2 dy + \varepsilon \int_D |\overline{w} - g|^2 dy \quad \forall \varepsilon > 0 \ ,$$

which implies that

$$\int_{D_1} |\overline{w}|^2 dy \leq C \left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}} \int_D |g|^2 dy + \varepsilon \int_D |\overline{w}|^2 dy \quad \forall \varepsilon \in]0,1[\ ,$$

where we have used that $\overline{w} = 0$ in ω_o. From (4.15), we conclude that there exist $C > 0$ and $\mu \in]0,1[$ such that

$$\int_\delta^{T-\delta} \int_\Omega |w(x,t)|^2 dxdt \leq C \left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}} \int_0^T \int_\Omega |f(x)|^2 dxdt + \varepsilon \int_0^T \int_\Omega |w(x,t)|^2 dxdt \quad \forall \varepsilon > 0 \ .$$

On another hand, we have the following inequalities

$$\begin{aligned}\int_0^T \int_\Omega |f(x)|^2 \,dxdt &= \int_0^T \int_\Omega \left|\chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dxdt \\ &\leq T \int_\omega \left|\chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dx \ ,\end{aligned}$$

$$\begin{aligned}\int_0^T \int_\Omega |w(x,t)|^2 \,dxdt &= \int_0^T \int_\Omega \left|\sum_{j\leq M} a_j e_j(x) \operatorname{ch}\left(\sqrt{\lambda_j} t\right) - \chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dxdt \\ &\leq 2T e^{2\sqrt{\lambda_M} T} \sum_{j\leq M} a_j^2 + 2T \int_\omega \left|\chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dx \ ,\end{aligned}$$

$$\int_\delta^{T-\delta} \int_\Omega \left|\sum_{j\leq M} a_j e_j(x) \operatorname{ch}\left(\sqrt{\lambda_j} t\right)\right|^2 dxdt \leq 2 \int_\delta^{T-\delta} \int_\Omega |w(x,t)|^2 \,dxdt + 2T \int_\omega \left|\chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dx \ .$$

Consequently, from the last four inequalities, we deduce that for any $\varepsilon > 0$,

$$\begin{aligned}(T-2\delta) \sum_{j\leq M} a_j^2 &\leq \int_\delta^{T-\delta} \int_\Omega \left|\sum_{j\leq M} a_j e_j(x) \operatorname{ch}\left(\sqrt{\lambda_j} t\right)\right|^2 dxdt \\ &\leq 2C \left(\tfrac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}} T \int_\omega \left|\chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dx \\ &\quad + 4\varepsilon \left(T e^{2\sqrt{\lambda_M} T} \sum_{j\leq M} a_j^2 + T \int_\omega \left|\chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dx \right) \\ &\quad + 2T \int_\omega \left|\chi(x) \sum_{j\leq M} a_j e_j(x)\right|^2 dx \ .\end{aligned}$$

Choosing $\varepsilon = \frac{1}{8} \frac{(T-2\delta)}{T e^{2\sqrt{\lambda_M} T}}$, we obtain the existence of $C > 0$ such that

$$\sum_{j\leq M} a_j^2 \leq C e^{C\sqrt{\lambda_M}} \int_\omega \left|\sum_{j\leq M} a_j e_j(x)\right|^2 dx \ .$$

6 Application to the wave equation

From the idea of L. Robbiano which consists to use an interpolation inequality of Hölder type for the elliptic operator $\partial_t^2 + \Delta$ and the FBI transform introduced by G. Lebeau et L. Robbiano, we obtain the following estimate of logarithmic type.

Theorem 4.9 .- *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, either convex or C^2 and connected. Let ω be a non-empty open subset in Ω. Then, for any $\beta \in]0,1[$, there exist $C > 0$ and $T > 0$ such that for any solution u of*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times]0,T[\ , \\ u = 0 & \text{on } \partial\Omega \times]0,T[\ , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \ , & \end{cases}$$

with non-identically zero initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$*, we have*

$$\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq e^{\left(C\frac{\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}}\right)^{1/\beta}} \|u\|_{L^2(\omega\times]0,T[)} \ .$$

7 Application to the heat equation

In this section, we propose the following result.

Theorem 4.7 .- *Let* Ω *be a bounded open set in* $\mathbb{R}^N$*,* $N \geq 1$*, either convex or* C^2 *and connected. Let* ω *be a non-empty open subset in* Ω*. Then, for any* $T > 0$*, there exists* $C > 0$ *such that for any* u *solution of*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times]0,T[\ , \\ u = 0 & \text{on } \partial\Omega \times]0,T[\ , \\ u(\cdot,0) = u_o\ , & \end{cases} \qquad (2.3)$$

with non-identically zero initial data $u_o \in H_0^1(\Omega)$*, and for any* $t_o \in]0,T[$*, we have*

$$\|u_o\|_{L^2(\Omega)} \leq Ce^{C\left(\frac{1}{t_o}+t_o\frac{\|u_o\|^2_{H_0^1(\Omega)}}{\|u_o\|^2_{L^2(\Omega)}}\right)} \|u(\cdot,t_o)\|_{L^2(\omega)} \ .$$

Comment .- We also have that

$$\|u_o\|_{H^{-1}(\Omega)} \leq Ce^{C\left(\frac{1}{t_o}+t_o\frac{\|u_o\|^2_{L^2(\Omega)}}{\|u_o\|^2_{H^{-1}(\Omega)}}\right)} \|u(\cdot,t_o)\|_{L^2(\omega)} \ .$$

Remark .- The quantitative unique continuation property from $\omega \times \{t_o\}$ for parabolic operator with space-time coefficients was established by L. Escauriaza, F.J. Fernandez and S. Vessella [EFV]).

Proof of Theorem 4.7 .- We decompose the proof into two steps. First, in step 1, we will prove that the solution u of (2.3) satisfies the following estimate

$$\|u_o\|^2_{L^2(\Omega)} \leq \exp\left(2t_o\frac{\|u_o\|^2_{H_0^1(\Omega)}}{\|u_o\|^2_{L^2(\Omega)}}\right)\|u(\cdot,t_o)\|^2_{L^2(\Omega)} \ .$$

Next, in step 2, we will prove that the solution u of (2.3) satisfies the following estimate

$$\int_\Omega |u(x,t_o)|^2\,dx \leq C\left(e^{\frac{C}{t_o}}\int_\Omega |u(x,0)|^2\,dx\right)^{1-\mu}\left(\int_\omega |u(x,t_o)|^2\,dx\right)^{\mu} \ .$$

Finally, the above inequalities imply the existence of $C > 0$ such that

$$\|u_o\|^2_{L^2(\Omega)} \leq C^{\frac{1}{\mu}}e^{\frac{C/\mu}{t_o}}\exp\left(2t_o\frac{1}{\mu}\frac{\|u_o\|^2_{H_0^1(\Omega)}}{\|u_o\|^2_{L^2(\Omega)}}\right)\int_\omega |u(x,t_o)|^2\,dx \ .$$

Proof of step 1 .- Let us introduce for almost $t \in [0,T]$ such that the solution of (2.3) satisfies $u(\cdot,t) \neq 0$, the following quantity.

$$\Phi(t) = \frac{\|u(x,t)\|^2_{H_0^1(\Omega)}}{\|u(x,t)\|^2_{L^2(\Omega)}} \ .$$

We begin to check that Φ is a non-increasing function on $[0,T]$. This monotonicity property holds because for any initial data in a dense set of $H_0^1(\Omega)$, we have that $\frac{d}{dt}\Phi(t) \geq 0$. Indeed, from the following two equalities

$$\left\{\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left\|u(x,t)\right\|_{L^2(\Omega)}^2 + \left\|u(x,t)\right\|_{H_o^1(\Omega)}^2 = 0 \ , \\ \frac{1}{2}\frac{d}{dt}\left\|u(x,t)\right\|_{H_0^1(\Omega)}^2 + \left\|\Delta u(x,t)\right\|_{L^2(\Omega)}^2 = 0 \ , \end{array}\right.$$

we can deduce that

$$\frac{d}{dt}\Phi(t) = \frac{2}{\left\|u(x,t)\right\|_{L^2(\Omega)}^4}\left[-\left\|\Delta u(x,t)\right\|_{L^2(\Omega)}^2\left\|u(x,t)\right\|_{L^2(\Omega)}^2 + \left\|u(x,t)\right\|_{H_0^1(\Omega)}^4\right] \ .$$

Therefore, we get by classical density argument and Cauchy-Schwarz inequality that for any solution u of (2.3), $u(\cdot,t) \neq 0$ a.e., and any $t \in [0,T]$, $\Phi(t) \leq \Phi(0)$. On another hand, we also have that

$$\frac{1}{2}\frac{d}{dt}\left\|u(x,t)\right\|_{L^2(\Omega)}^2 + \Phi(t)\left\|u(x,t)\right\|_{L^2(\Omega)}^2 = 0 \ ,$$

which implies that

$$0 \leq \frac{1}{2}\frac{d}{dt}\left\|u(x,t)\right\|_{L^2(\Omega)}^2 + \Phi(0)\left\|u(x,t)\right\|_{L^2(\Omega)}^2 \ .$$

Therefore, by Gronwall Lemma, we get the desired estimate

Proof of step 2 .- Let λ_1,λ_2,$\cdots$ and e_1,e_2,$\cdots$ be the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, constituting an orthonormal basis in $L^2(\Omega)$. For any $u_o = u(\cdot,0) = \sum_{j\geq 1}\alpha_j e_j$ in $L^2(\Omega)$ where $\alpha_j = \int_\Omega u_o e_j dx$, the solution u of (2.3), can be written $u(x,t) = \sum_{j\geq 1}\alpha_j e_j(x) e^{-\lambda_j t}$. Let $t_o \in]0,T[$. We introduce (see [Lin] or [CRV]) the solution

$$w(x,t) = \sum_{j\geq 1}\alpha_j e_j(x) e^{-\lambda_j t_o}\mathrm{ch}\left(\sqrt{\lambda_j}t\right) - \chi(x)\sum_{j\geq 1}\alpha_j e_j(x) e^{-\lambda_j t_o},$$

where $\chi \in C_0^\infty(\omega)$, $\chi = 1$ in $\widetilde{\omega} \Subset \omega$. Therefore, w solves

$$\left\{\begin{array}{ll} \partial_t^2 w + \Delta w = \Delta f & \text{in } \Omega\times]0,T[\ , \\ w = 0 & \text{on } \partial\Omega\times]0,T[\ , \\ w = \partial_t w = 0 & \text{on } \widetilde{\omega}\times\{0\} \ , \\ w = w(x,t) \in H^2(\Omega\times]0,T[) \ , & \end{array}\right.$$

where $f = -\chi u(\cdot,t_o) \in H_0^2(\Omega)$. Consequently, in a similar way than in the proof of Theorem 4.5, for any $\delta \in]0,T/2[$, there exist $C > 0$ and $\mu \in]0,1[$ such that we have

$$\int_\delta^{T-\delta}\int_\Omega |w(x,t)|^2 dxdt \leq C\left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}}\int_0^T\int_\Omega |f(x)|^2 dxdt + \varepsilon\int_0^T\int_\Omega |w(x,t)|^2 dxdt \quad \forall\varepsilon > 0$$

On another hand, the following inequalities hold.

$$\int_0^T\int_\Omega |f(x)|^2 dxdt \leq T\int_\omega |\chi(x) u(x,t_o)|^2 dx \ ,$$

$$\int_0^T\int_\Omega |w(x,t)|^2 dxdt \leq 2T\sum_{j\geq 1}\alpha_j^2 e^{-2\left(\lambda_j t_o - \sqrt{\lambda_j}T\right)} + 2T\int_\omega |\chi(x) u(x,t_o)|^2 dx \ ,$$

$$\begin{array}{ll} \sum_{j\geq 1}\alpha_j^2 e^{-2\left(\lambda_j t_o - \sqrt{\lambda_j}T\right)} & = \sum_{\left\{j\geq 1;\sqrt{\lambda_j}\leq\frac{T}{t_o}\right\}}\alpha_j^2 e^{-2\left(\lambda_j t_o - \sqrt{\lambda_j}T\right)} + \sum_{\left\{j\geq 1;\sqrt{\lambda_j}>\frac{T}{t_o}\right\}}\alpha_j^2 e^{-2\left(\lambda_j t_o - \sqrt{\lambda_j}T\right)} \\ & \leq e^{2\frac{T^2}{t_o}}\sum_{j\geq 1}\alpha_j^2 \ , \end{array}$$

$$\int_{\delta}^{T-\delta}\int_{\Omega}\left|\sum_{j\geq1}\alpha_j e_j\left(x\right)e^{-\lambda_j t_o}\mathrm{ch}\left(\sqrt{\lambda_j}t\right)\right|^2 dxdt\leq 2\int_{\delta}^{T-\delta}\int_{\Omega}\left|w\left(x,t\right)\right|^2 dxdt+2T\int_{\omega}\left|\chi\left(x\right)u\left(x,t_o\right)\right|^2 dx\ ,$$

Consequently, from the fast five inequalities, we deduce that for any $\varepsilon>0$,

$$\begin{array}{ll}(T-2\delta)\sum_{j\geq1}\alpha_j^2 e^{-2\lambda_j t_o} & \leq (T-2\delta)\sum_{j\geq1}\alpha_j^2 e^{-2\left(\lambda_j t_o-\sqrt{\lambda_j}\delta\right)} \\ & \leq \int_{\delta}^{T-\delta}\int_{\Omega}\left|\sum_{j\geq1}\alpha_j e_j\left(x\right)e^{-\lambda_j t_o}\mathrm{ch}\left(\sqrt{\lambda_j}t\right)\right|^2 dxdt \\ & \leq 2TC\left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}}\int_{\omega}\left|\chi\left(x\right)u\left(x,t_o\right)\right|^2 dx \\ & \quad +4T\varepsilon\left(e^{2\frac{T^2}{t_o}}\sum_{j\geq1}\alpha_j^2+\int_{\omega}\left|\chi\left(x\right)u\left(x,t_o\right)\right|^2 dx\right) \\ & \quad +2T\int_{\omega}\left|\chi\left(x\right)u\left(x,t_o\right)\right|^2 dx\ .\end{array}$$

Finally, there exists $C>0$ such that for any $t_o>0$,

$$\begin{array}{ll}\int_{\Omega}\left|u\left(x,t_o\right)\right|^2 dx & =\sum_{j\geq1}\alpha_j^2 e^{-2\lambda_j t_o} \\ & \leq C\left(\frac{1}{\varepsilon}\right)^{\frac{1-\mu}{\mu}}\int_{\omega}\left|u\left(x,t_o\right)\right|^2 dx+\varepsilon e^{\frac{C}{t_o}}\int_{\Omega}\left|u\left(x,0\right)\right|^2 dx \quad \forall\varepsilon\in\left]0,1\right[\ ,\end{array}$$

which implies the desired estimate ,

$$\int_{\Omega}\left|u\left(x,t_o\right)\right|^2 dx\leq C^{\mu}e^{\frac{C(1-\mu)}{t_o}}\left(\int_{\Omega}\left|u\left(x,0\right)\right|^2 dx\right)^{1-\mu}\left(\int_{\omega}\left|u\left(x,t_o\right)\right|^2 dxdt\right)^{\mu}\ .$$

8 Notes on the papers in reference

... to be completed. (some reference to papers of Alessandrini have to be add too...)

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