Observability for parabolic equations from a measurable set in time

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EXACT CONTROL FOR THE HEAT EQUATION

Let Ω *be a bounded smooth domain in* $\mathbb{R}^n$, $\omega \subset \Omega$, $T > 0$. *Given* $y_0 \in L^2(\Omega)$, *there is* (y, f) *satisfying*

$$\begin{cases} \partial_t y - \Delta y = 1_{|\omega} f & \text{in } \Omega \times (0,T) \ , \\ y = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ y(\cdot, 0) = y_0 & \text{in } \Omega \ , \\ y(\cdot, T) = 0 & \text{in } \Omega \ , \\ \|f\|_{L^2(\Omega\times(0,T))} \leq c \, \|y_0\|_{L^2(\Omega)} & . \end{cases}$$

G. Lebeau, L. Robbiano, *CPDE (1995)*.

A. Fursikov, O. Imanuvilov, *Lecture Note Series (1996)*. Also OK for parabolic equations

EQUIVALENTLY

$$\begin{cases} \partial_t u - \Delta u = 0 & \textit{in } \Omega \times (0,T) \ , \\ u = 0 & \textit{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) \in L^2(\Omega) \ , \end{cases}$$

$$\boxed{\|u(\cdot,T)\|^2_{L^2(\Omega)} \leq c \int_\omega \int_0^T |u(x,t)|^2 \, dxdt \ .}$$

ORIGINAL LEBEAU-ROBBIANO STRATEGY

Hölder type Observation for elliptic

$$\|w\|_{H^1(\Omega\times(\gamma,T-\gamma))} \leq C \left(\|w\|_{L^2(\omega\times(0,T))}\right)^{1-\alpha} \left(\|w\|_{H^1(\Omega\times(0,T))}\right)^{\alpha}$$

$$for\ any\ w \in H^2\left(\Omega\times(0,T)\right),\ \ \left(\partial_t^2+\Delta\right) w = 0\ and\ w_{|\partial\Omega} = 0\ .$$

$\Longrightarrow$ Controllability in finite and infinite dim

$\Longrightarrow$ Observability

SHORTCUT OF LEBEAU-ROBBIANO STRATEGY

Hölder type Observation for elliptic

$\Longrightarrow$ Sum of eigenfunctions estimate

$\Longrightarrow$ Observability

G. Lebeau, E. Zuazua, *ARMA (1998)*.

L. Miller, *DCDS (2010)*.

PLAN & AIM

I. A similar way for parabolic equations

II. Replace $(0, T)$ by a measurable set E in time, $|E| > 0$

III. Time optimal bang-bang control

NEW OBSERVABILITY INEQUALITY

$a \in L^{\infty}(0,T;L^{q}(\Omega))$, $q \geq 2$ and $b \in L^{\infty}(\Omega \times (0,T))^{n}$

$$\begin{cases} \partial_t u - \Delta u + au + b \cdot \nabla u = 0 & \textit{in } \Omega \times (0,T) \ , \\ u = 0 & \textit{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) \in L^2(\Omega) \ , & \end{cases}$$

Ω convex, $\omega \subset \Omega$, $E \subset (0,T)$ and $|E| > 0$

$$\boxed{\|u(\cdot,T)\|_{L^2(\Omega)} \leq c \int_{\omega}\int_{E} |u(x,t)|\, dxdt \ .}$$

STRATEGY FOR PARABOLIC

Hölder type Observation from one point in time

$$\|u(\cdot,\ell)\|_{L^2(\Omega)} \leq Ce^{\frac{C}{\ell}} \left(\|u(\cdot,\ell)\|_{L^2(\omega)}\right)^{1-\alpha} \left(\|u(\cdot,0)\|_{L^2(\Omega)}\right)^{\alpha}$$

$$\Longrightarrow$$

Observability for a measurable set in time

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq c \int_{\omega}\int_{E} |u(x,t)|\, dxdt$$

Proof

Step I. Observation from one point in time

$\longrightarrow$here Ω convex or star-shaped domain

$\longrightarrow$interpolation inequality, parameter $\varepsilon > 0$

Step II. Usage of set of positive measure, $E \subset (0, T)$ and $|E| > 0$

$\longrightarrow$Construction of a sequence of points $\{\ell_j\}_{j \geq 1}$

$\longrightarrow$parameter $z > 1$

Step III. Choice of the parameters: ε and z

Observation from a point in time

$$\boxed{\|u(\cdot,\ell)\|_{L^2(\Omega)} \leq C e^{\frac{C}{\ell}} \left(\|u(\cdot,\ell)\|_{L^2(\omega)}\right)^{1-\alpha} \left(\|u(\cdot,0)\|_{L^2(\Omega)}\right)^{\alpha}}$$

$$\Longrightarrow$$

$$\boxed{\begin{array}{r} \|u(\cdot,\ell)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon^{\gamma}} C e^{\frac{C}{\ell}} \|u(\cdot,\ell)\|_{L^1(\omega)} + \varepsilon \|u(\cdot,0)\|_{L^2(\Omega)} \\ \forall \varepsilon > 0 \end{array}}$$

Set of positive measure

Let $T > 0$ *and* $E \subset (0, T)$ *be a set of positive measure. Let* L *be a point of density for* E *. Then for any* $z > 1$*, there exists* $\ell_1 \in (L, T)$ *such that* the decreasing sequence $\{\ell_j\}_{j\geq 1}$ *given by*

$$\boxed{\ell_{j+1} = L + \frac{1}{z^j}\left(\ell_1 - L\right)}$$

converges to L *and satisfies*

$$\boxed{\ell_j - \ell_{j+1} \leq 3\left|E \cap \left(\ell_{j+1}, \ell_j\right)\right| \ .}$$

End of proof 1/2

Take $0 < \ell_{j+2} < \ell_{j+1} \leq t \leq \ell_j < T$,

$$\begin{aligned} \|u(\cdot,t)\|_{L^2(\Omega)} &\leq \varepsilon \|u(\cdot,\ell_{j+2})\|_{L^2(\Omega)} \\ &\quad + \frac{1}{\varepsilon^\gamma} C e^{C/(t-\ell_{j+2})} \|u(\cdot,t)\|_{L^1(\omega)} \end{aligned}$$

We know that $\|u(\cdot,\ell_j)\|_{L^2(\Omega)} \leq c \|u(\cdot,t)\|_{L^2(\Omega)}$.

We integrate over $E \cap (\ell_{j+1},\ell_j)$

$$\begin{aligned} &|E \cap (\ell_{j+1},\ell_j)| \, c \|u(\cdot,\ell_j)\|_{L^2(\Omega)} \\ &\leq |E \cap (\ell_{j+1},\ell_j)| \, \varepsilon \|u(\cdot,\ell_{j+2})\|_{L^2(\Omega)} \\ &\quad + \frac{C}{\varepsilon^\gamma} e^{C/(\ell_{j+1}-\ell_{j+2})} \int_{E\cap(\ell_{j+1},\ell_j)} \|u(\cdot,t)\|_{L^1(\omega)} \, dt \end{aligned}$$

End of proof 2/2

We use ℓ_j

$$\varepsilon^{\gamma} e^{-C\left[\frac{1}{\ell_1-L}\frac{z^{j+2}}{z(z-1)}\right]} \left\| u\left(\cdot,\ell_j\right)\right\|_{L^2(\Omega)} - \varepsilon^{\gamma+1} e^{-C\left[\frac{1}{\ell_1-L}\frac{z^{j+2}}{z(z-1)}\right]} \left\| u\left(\cdot,\ell_{j+2}\right)\right\|_{L^2(}$$
$$\leq c\int_{E\cap(\ell_{j+1},\ell_j)} \left\| u\left(\cdot,t\right)\right\|_{L^1(\omega)} dt$$

Set $d = C\left[\frac{1}{\ell_1-L}\frac{1}{z(z-1)}\right]$.

We choose $\varepsilon = e^{-dz^{j+2}}$ and $(\gamma+1)\, z^2 = (\gamma+2)$

$$e^{-d(\gamma+2)z^j} \left\| u\left(\cdot,\ell_j\right)\right\|_{L^2(\Omega)} - e^{-d(\gamma+2)z^{j+2}} \left\| u\left(\cdot,\ell_{j+2}\right)\right\|_{L^2(\Omega)}$$
$$\leq c\int_{E\cap(\ell_{j+1},\ell_j)} \left\| u\left(\cdot,t\right)\right\|_{L^1(\omega)} dt$$

We take $j = 2j'$ and make the sum for $j' = 1$ until infinity.

Proof of Observation from a point in time 1/3

$$G_{\lambda}(x,t)=\frac{e^{-\frac{|x-x_0|^2}{4(\ell-t+\lambda)}}}{(\ell-t-\lambda)^{n/2}}$$

$$N_{\lambda,u}(t)=\frac{\int_{\Omega}|\nabla u(x,t)|^2\,G_{\lambda}(x,t)\,dx}{\int_{\Omega}|u(x,t)|^2\,G_{\lambda}(x,t)\,dx}$$

Proof of Observation from a point in time 2/3

Ω convex

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2 G_\lambda dx + \int_{\Omega}|\nabla u|^2 G_\lambda dx = \int_{\Omega} u\left(\partial_t - \Delta\right) u G_\lambda dx$$

$$\frac{d}{dt}N_{\lambda,u}\left(t\right) \leq \frac{1}{\ell - t + \lambda}N_{\lambda,u}\left(t\right) + \frac{\int_{\Omega}\left|\left(\partial_t - \Delta\right)u\right|^2 G_\lambda dx}{\int_{\Omega}|u|^2 G_\lambda dx}$$

Proof of Observation from a point in time 3/3

Step I. Make appear $B_r \subset \omega$

$$\left[-\frac{16\lambda}{r^2}\left(\lambda N_{\lambda,u}(\ell)+\frac{n}{4}\right)+1\right]\int_\Omega |u(\ell)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx \leq \int_{B_r}|u(\ell)|^2 dx$$

Step II. Estimate $\lambda N_{\lambda,u}(\ell)+\frac{n}{4}$

$$\lambda N_{\lambda,u}(\ell)+\frac{n}{4} \leq C\left(\frac{\lambda}{\ell}+1\right)e^{C\ell}\ln\left[e^{C\left(1+\frac{1}{\ell}\right)}\frac{\int_\Omega |u(0)|^2 dx}{\int_\Omega |u(\ell)|^2 dx}\right]$$

Step III. Choice of λ

$$\frac{1}{2}=\frac{16\lambda}{r^2}C\left(\frac{\lambda}{\ell}+1\right)e^{C\ell}\ln\left[e^{C\left(1+\frac{1}{\ell}\right)}\frac{\int_\Omega |u(0)|^2 dx}{\int_\Omega |u(\ell)|^2 dx}\right]$$

Exact control and cost imply Bang-bang control

For any $y_0 \in L^2(\Omega)$, *there exists a control* $f \in L^\infty(\Omega \times E)$ *such that the unique solution* y *of*

$$\begin{cases} \partial_t y - \Delta y + a(x,t)\,y = 1_{|\omega\times E}\, f & \text{in } \Omega \times (0,T)\ , \\ y = 0 & \text{on } \partial\Omega \times (0,T)\ , \\ y(\cdot,0) = y_0 & \text{in } \Omega\ , \end{cases}$$

satisfies

$$\boxed{\quad y(\cdot,T) = 0 \quad \text{and} \quad \|f\|_{L^\infty(\Omega\times E)} \le c\,\|y_0\|_{L^2(\Omega)}\ . \quad}$$

Known result

Let Ω *be a bounded smooth domain in* $\mathbb{R}^n$, $\omega \subset \Omega$. *Given* $y_0 \in L^2(\Omega)$ *and* $M > 0$, *if* $\left(y^*, 1_{|(0,T^*)} f^*\right)$ *is a couple* (y, f) *satisfying*

$$\begin{cases} \partial_t y - \Delta y = 1_{|\omega} f & \text{in } \Omega \times (0, +\infty) \ , \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty) \ , \\ y(\cdot, 0) = y_0 & \text{in } \Omega \ , \\ y(\cdot, T^*) = 0 & \text{in } \Omega \ , \\ \| f(\cdot, t) \|_{L^2(\Omega)} \leq M & \text{a.e. } t \in (0, +\infty) \ , \end{cases}$$

and $T^* > 0$ *is the smallest of the admissible time among all the couples* (y, f), *then*

$$\boxed{\| f^*(\cdot, t) \|_{L^2(\Omega)} = M \quad \text{a.e. } t \in (0, T^*) \ .}$$

Further, $\left(y^*, 1_{|(0,T^*)} f^*\right)$ *is unique.*

NEW RESULT

Let Ω *be a convex or star-shaped domain in* $\mathbb{R}^n$, $\omega \subset \Omega$, $T > 0$, $a \in L^\infty(\Omega \times (0,T))$. *Given* $y_0 \in L^2(\Omega)$ *and* $M > 0$, *if* $\left(y^*, 1_{|(\tau^*,T)} f^*\right)$ *is a couple* (y, f) *satisfying*

$$\begin{cases} \partial_t y - \Delta y + a(x,t)\, y = 1_{|\omega} f & \text{in } \Omega \times (0,T) \ , \\ y = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ y(\cdot, 0) = y_0 & \text{in } \Omega \ , \\ y(\cdot, T) = 0 & \text{in } \Omega \ , \\ \|f(\cdot,t)\|_{L^2(\Omega)} \leq M & \text{a.e. } t \in (0,T) \ , \end{cases}$$

and $\tau^* \in [0,T)$ *is the largest admissible time among all the couples* (y, f), *then*

$$\boxed{\|f^*(\cdot,t)\|_{L^2(\Omega)} = M \quad \text{a.e. } t \in (\tau^*, T) \ .}$$

Further, $\left(y^*, 1_{|(\tau^*,T)} f^*\right)$ *is unique.*

Proof by contradiction

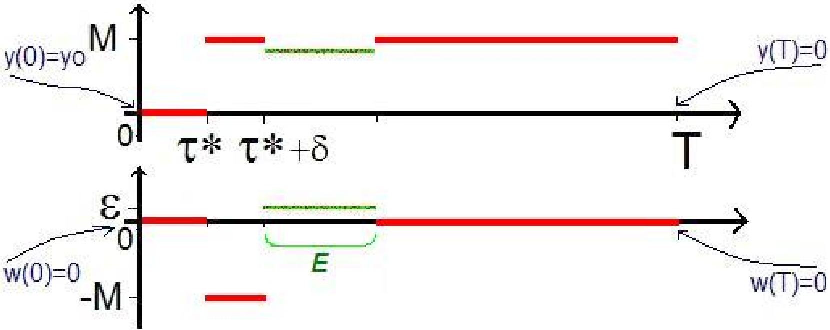
If $\|f^*(\cdot,t)\|_{L^2(\Omega)} = M$ a.e. $t \in (\tau^*, T)$ is false,
then $\exists\ \varepsilon > 0$ and $E \subset (\tau^*, T)$, $|E| > 0$ such that

$$\|f^*(\cdot,t)\|_{L^2(\Omega)} \leq M - \varepsilon \quad \text{a.e. } t \in E\ .$$

Aim : Contradiction with the optimality of τ^*
i.e., construction of a couple $\left(z, 1_{|(\tau^*+\delta,T)}v\right)$ solution of

$$\begin{cases} \partial_t z - \Delta z + az = 1_{|\omega\times(\tau^*+\delta,T)}v & \text{in } \Omega\times(0,T)\ , \\ z = 0 & \text{on } \partial\Omega\times(0,T)\ , \\ z(\cdot,0) = y_0 & \text{in } \Omega\ , \\ z(\cdot,T) = 0 & \text{in } \Omega\ , \\ \|v(\cdot,t)\|_{L^2(\Omega)} \leq M & \text{a.e. } t\in(0,T)\ , \end{cases}$$

for a small $\delta > 0$ with $E \subset (\tau^* + \delta_o, T)$ for a sufficiently small $\delta_o > 0$.



y(0)=yo
M
y(T)=0
0
τ* τ*+δ
T
ε
0
w(0)=0
E
w(T)=0
-M

Thank You!