Quantitative uniqueness for some PDE's

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QUCP for the heat equation

Ω smooth bounded domain in $\mathbb{R}^d$, $E \subset \Omega$, $|E| > 0$, $T > 0$

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \frac{1}{T}e^{C\left(1+\frac{1}{T}\right)}\left\|e^{t\Delta}u_0\right\|_{L^1(E\times(0,T))}$$

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq e^{C\left(1+\frac{1}{T}\right)}\left\|e^{T\Delta}u_0\right\|_{L^1(E)}^{\mu}\left\|u_0\right\|_{L^2(\Omega)}^{1-\mu}$$

$$\left\|u_0\right\|_{L^2(\Omega)} \leq e^{C\left(1+\frac{1}{T}+T\left(\frac{\|u_0\|_{L^2(\Omega)}}{\|u_0\|_{H^{-1}(\Omega)}}\right)^2\right)}\left\|e^{T\Delta}u_0\right\|_{L^1(E)}, \quad u_0 \neq 0$$

Interpolation estimate

For any $h > 0$,

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq e^{C/h}e^{C\left(1+\frac{1}{T}\right)}\left\|e^{T\Delta}u_0\right\|_{L^1(E)} + e^{-1/h}\left\|u_0\right\|_{L^2(\Omega)}$$

$$\left\|u_0\right\|_{H^{-1}(\Omega)} \leq e^{CT^2/h}e^{C\left(1+\frac{1}{T}\right)}\left\|e^{T\Delta}u_0\right\|_{L^1(E)} + \sqrt{h}\left\|u_0\right\|_{L^2(\Omega)}$$

references

Observability: Zuazua, Fernandez Cara, ...
Measurable sets: Apraiz-Escauriaza, Vessella, ...
Backward estimate: Bardos-Tartar, ...

QUCP for the Schrödinger equation

Ω smooth bounded domain in $\mathbb{R}^d$, $E \subset \Omega$, $|E| > 0$, $T > 0$

$$\begin{cases} D(A) = \left\{ u_0 \in H_0^1(\Omega) \,;\, \Delta u_0 \in L^2(\Omega) \right\} \\ Au_0 = -i\Delta u_0 \quad \forall u_0 \in D(A) \ . \end{cases}$$

$$\boxed{\|u_0\|_{D(A)} \leq e^{C\left(1+\frac{1}{T}\right)\left(\frac{\|u_0\|_{D(A)}}{\|u_0\|_{L^2(\Omega)}}\right)^{1/\beta}} \left\|e^{it\Delta}u_0\right\|_{L^1(E\times(0,T))}}$$

$\forall \beta \in [1/2, 1)$.

Fourier-Bros-Iagolnitzer transform

Let $\beta \in [1/2, 1)$, and choose $N \in \mathbb{N}^*$ such that $\beta + \frac{1}{2N} \leq 1$. Put $\gamma = 1 - \frac{1}{2N}$. For any $\lambda \geq 1$, the function $F_\lambda(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\tau} e^{-\left(\frac{\tau}{\lambda^\gamma}\right)^{2N}} d\tau$ is holomorphic in $\mathbb{C}$, and there exist four positive constants C, c_0, c_1 and c_2 (independent on λ) such that

$$\begin{cases} \forall z \in \mathbb{C} \quad |F_\lambda(z)| \leq C\lambda^\gamma e^{c_0 \lambda |\mathrm{Im} z|^{1/\gamma}} , \\ |\mathrm{Im} z| \leq c_2 |\mathrm{Re} z| \Rightarrow |F_\lambda(z)| \leq C\lambda^\gamma e^{-c_1 \lambda |\mathrm{Re} z|^{1/\gamma}} . \end{cases}$$

Let $s, \ell_o \in \mathbb{R}$, we introduce the following FBI transform

$$\boxed{W_{\ell_o,\lambda}(x,s) = \int_{\mathbb{R}} F_\lambda(\ell_o + is - \ell)\Phi(\ell)u(x,\ell)d\ell}$$

where $\mathcal{F}(F_\lambda)(\tau) = e^{-\left(\frac{\tau}{\lambda^\gamma}\right)^{2N}}$, $\Phi \in C_0^\infty(\mathbb{R})$ and $u(x,t) = e^{it\Delta}u_0$.

Application of Fourier-Bros-Iagolnitzer transform

$$\begin{cases} \partial_s W_{\ell_o,\lambda}(x,s) + \Delta W_{\ell_o,\lambda}(x,s) = i \displaystyle\int_{\mathbb{R}} F_\lambda(\ell_o + is - \ell)\Phi'(\ell)u(x,\ell)d\ell \ , \\ W_{\ell_o,\lambda}(x,s) = 0 \quad \text{for } x \in \partial\Omega \ , \\ W_{\ell_o,\lambda}(x,0) = \left(F_\lambda * \Phi u(x,\cdot)\right)(\ell_o) \quad \text{for } x \in \Omega \ . \end{cases}$$

Application of Parseval's identity

Let $K_o = \left[\frac{3T}{8}, \frac{5T}{8}\right]$,

$$\begin{aligned}
&\|\Phi u\,(x,\cdot)\|_{L^2(K_o)} \\
&\leq \|\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\|_{L^2(K_o)} + \|F_\lambda * \Phi u(x,\cdot)\|_{L^2(K_o)} \\
&\leq \|\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\|_{L^2(\mathbb{R})} + \left(\int_{\ell_o \in K_o} |W_{\ell_o,\lambda}(x,0)|^2 \, d\ell_o\right)^{1/2} .
\end{aligned}$$

But

$$\begin{aligned}
&\|\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\|_{L^2(\mathbb{R})} \\
&= \frac{1}{\sqrt{2\pi}} \|\mathcal{F}\left(\Phi u(x,\cdot) - F_\lambda * \Phi u(x,\cdot)\right)\|_{L^2(\mathbb{R})} \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} \left|\left(1 - e^{-\left(\frac{\tau}{\lambda^\gamma}\right)^{2N}}\right) \mathcal{F}\left(\Phi u(x,\cdot)\right)(\tau)\right|^2 d\tau\right)^{1/2} \\
&\leq C \frac{1}{\lambda^\beta} \|\partial_t \left(\Phi u(x,\cdot)\right)\|_{L^2(\mathbb{R})} .
\end{aligned}$$

Application of observability for heat

$$\int_\Omega |W_{\ell_o,\lambda}(x,0)|^2\,dx$$
$$\leq Ce^{C/a}\left(\int_0^a\int_E |W_{\ell_o,\lambda}(x,s)|\,dxds\right)^2$$
$$+Ce^{C/a}\int_0^a\int_\Omega\left|i\int_{\mathbb{R}} F_\lambda(\ell_o+is-\ell)\Phi'(\ell)u(x,\ell)d\ell\right|^2 dxds\ .$$

Let $K=\left[0,\frac{T}{4}\right]\cup\left[\frac{3T}{4},T\right]$,

$$\forall\ell\in K\quad |F_\lambda(\ell_o+is-\ell)|\leq C\lambda^\gamma e^{-c_1\lambda\left(\frac{T}{8}\right)^{1/\gamma}}\ .$$

Take $a=\frac{c_2T}{8}$ and $\lambda=\frac{8^{1/\gamma}C}{c_1}\frac{\mu}{T^{1+1/\gamma}}$. For any $\mu>0$,

$$\|u_0\|^2_{L^2(\Omega)}\leq \frac{C}{\mu^{2\beta}}T^{2[\beta(1+1/\gamma)]}\|\Delta u_0\|^2_{L^2(\Omega)}$$
$$+Ce^{C\mu/T}\left(\int_E\int_0^T|u(x,t)|\,dtdx\right)^2\ .$$

Wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \ , & \end{cases}$$

$$\begin{cases} D(A) = \left\{ (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \, ; \Delta u_0 \in L^2(\Omega) \, , u_1 \in H_0^1(\Omega) \right\} \\ A(u_0, u_1) = (-u_1, -\Delta u_0,) \quad \forall (u_0, u_1) \in D(A) \ . \end{cases}$$

Observability for $d = 1$ (Russell, Privat-Trélat-Zuazua)
Observation for $d > 1$

QUCP for the wave equation

$$\lambda\left(u_0, u_1\right) = \frac{\left\|\left(u_0, u_1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}}{\left\|\left(u_0, u_1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)}} .$$

$$\exists C > 0,\ \exists T > 0,$$
$$\left\|\left(u_0, u_1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq C e^{C[\lambda(u_0,u_1)]^{1/\beta}} \left\|u\right\|_{L^1(E\times(0,T))}, \forall \beta \in (0,1)$$

$$\left\|\left(u_0, u_1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq C \left\|u\right\|_{L^2\left(E\times\left(0,e^{C\lambda(u_0,u_1)}\right)\right)}$$

Application of FBI transform

Let $s, \ell_o \in \mathbb{R}$,

$$\boxed{W_{\ell_o,\lambda}(x,s) = \int_{\mathbb{R}} F_\lambda(\ell_o + is - \ell)\Phi(\ell)u(x,\ell)d\ell}$$

where $\mathcal{F}\left(F_\lambda\right)(\tau) = e^{-\left(\frac{\tau}{\lambda^\gamma}\right)^{2N}}$, $\Phi \in C_0^\infty(\mathbb{R})$ and $u\left(x,t\right)$ solves the wave equation.

$$\begin{cases} \partial_s^2 W_{\ell_o,\lambda}(x,s) + \Delta W_{\ell_o,\lambda}(x,s) \\ \qquad\qquad = -\displaystyle\int_{\mathbb{R}} F_\lambda(\ell_o + is - \ell)\left[\Phi''(\ell)u(x,\ell) + 2\Phi'(\ell)\partial_t u(x,\ell)\right] d\ell \\ W_{\ell_o,\lambda}(x,s) = 0 \quad \text{for } x \in \partial\Omega\ , \\ W_{\ell_o,\lambda}(x,0) = \left(F_\lambda * \Phi u(x,\cdot)\right)(\ell_o) \quad \text{for } x \in \Omega\ . \end{cases}$$

QUCP for second order elliptic equation

Proposition .- *Let* $f = f(x,s) \in L^2(\Omega \times (-1,1))$. *Then there exists* $c > 0$ *such that for all* $w = w(x,s) \in H^2(\Omega \times (-1,1))$ *solution of*

$$\begin{cases} \partial_s^2 w + \Delta w = f & \text{in } \Omega \times (-1,1) \ , \\ w = 0 & \text{on } \partial\Omega \times (-1,1) \ , \end{cases}$$

for all $h > 0$, *we have :*

$$\int_{|s|\leq 1/2} \int_\Omega |w(x,s)|^2 \, dxds$$
$$\leq c e^{c/h} \left(\left(\int_{|s|\leq 1} \int_E |w(x,s)| \, dxds \right)^2 + \int_{|s|\leq 1} \int_\Omega |f(x,s)|^2 \, dxds \right)$$
$$+ e^{-1/h} \int_{|s|\leq 1} \int_\Omega |w(x,s)|^2 \, dxds \ .$$

Resolvent approach

Another strategy:

QUCP for elliptic

$\Longrightarrow$ Resolvent estimate for the damped wave operator

$\Longrightarrow$ Energy decay rate for the damped wave equation

$\Longrightarrow$ Observation estimate for the damped wave equation

$\Longrightarrow$ Observation estimate for the wave equation

The damped wave operator

We consider the damped wave equation

$$\begin{cases} \partial_t^2 y - \Delta y + 1_E \partial_t y = 0 & \text{in } \Omega \times (0, +\infty) \ , \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty) \ , \\ (y, \partial_t y)(\cdot, 0) = (u_0, u_1) \ , \end{cases}$$

which can be rewritten as follows

$$\begin{cases} \dot{x}(t) = Bx(t) \ , \ t \geq 0 \ , \\ x(0) = x_0 \ , \ x_0 \in X \ , \end{cases}$$

where B is the operator on $X = H_0^1(\Omega) \times L^2(\Omega)$ defined by

$$\begin{cases} D(B) = \left\{ (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \, ; \Delta u_0 \in L^2(\Omega) \, , u_1 \in H_0^1(\Omega) \right\} \\ B(u_0, u_1) = (u_1, \Delta u_0 - 1_E u_1) \quad \forall (u_0, u_1) \in D(B) \ . \end{cases}$$

Resolvent estimate and logarithmic decay rate

The resolvent operator of B is defined by

$$\begin{array}{rcl} \rho(B) \subset \mathbb{C} & \longrightarrow & \mathcal{L}(B) \\ \lambda & \longmapsto & R(\lambda, B) = (\lambda - B)^{-1} \end{array} .$$

Denote

$$R(\lambda, B) = (\lambda - B)^{-1}$$

$$M(\xi) = \max_{\tau \in [-\xi, \xi]} \|R(i\tau, B)\|, \ \xi \geq 0 .$$

Batty-Duyckaerts proves that $i\mathbb{R} \subset \rho(B)$ is a necessary and sufficient condition to get $\left\|T(t)B^{-1}\right\| \to 0, t \to \infty$. Further, if $M(\xi) \leq C\exp(C\xi)$ for some constant $C > 0$, then

$$\forall k \geq 1, \ \exists C_k > 0, \ \|x(t)\| \leq \frac{C_k}{(\log(2+t))^k} \|x_0\|_{D(B^k)} .$$

See also Burq-Lebeau.

Link between QUCP and resolvent estimate

Fix $f = (f_0, f_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and write $u = (u_0, u_1) \in D(B)$. The following equation

$$u = R(i\tau, B) f$$

is equivalent to

$$i\tau(u_0, u_1) - (u_1, \Delta u_0 - 1_E u_1) = (f_0, f_1)$$

that is

$$u_1 = i\tau u_0 - f_0 \text{ and } -\Delta u_0 + i\tau u_1 + 1_E u_1 = f_1 \ .$$

Then, $u_0 \in H^2 \cap H_0^1(\Omega)$ and

$$-\Delta u_0 - \tau^2 u_0 + i\tau 1_E u_0 = f_1 + 1_E f_0 + i\tau f_0 \text{ in } \Omega \ .$$

Let $w(x, s) = e^{\tau s} u_0(x)$. Then $w \in H^2(\Omega \times (-1, 1))$ and solves

$$\begin{cases} \partial_s^2 w + \Delta w = e^{\tau s}(i\tau 1_E u_0 - i\tau f_0 - 1_E f_0 - f_1) & \text{in } \Omega \times (-1, 1) \ , \\ w = 0 & \text{on } \partial\Omega \times (-1, 1) \ . \end{cases}$$

Applications

We conclude that

$$\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq Ce^{C\xi}\,\|(f_0,f_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}\,,\ \tau\in[-\xi,\xi]$$

that is

$$M(\xi)\leq Ce^{C\xi}\ .$$

By application of Batty-Duyckaerts' Theorem

$$\|(y,\partial_t y)(\cdot,t)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \frac{C}{\log(2+t)}\,\|(u_0,u_1)\|_{D(B)}\ .$$

By energy estimate

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq 4\int_0^{\exp\left(C\frac{\|(u_0,u_1)\|_{D(B)}}{\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}}\right)}\int_E |\partial_t y|^2\,dxdt\ .$$

Some tricks

$$\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq 4\int_0^{\exp\left(C\frac{\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}}\right)}\int_E |y|^2\,dxdt$$

$$\Downarrow$$

$$\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq 4\int_0^{\exp\left(C\frac{\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}}\right)}\int_E |u|^2\,dxdt$$

$$\Downarrow$$

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C\int_0^{\exp\left(C\frac{\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}}\right)}\int_E |u|^2\,dxdt$$

Radiative transfert equation

Ω smooth bounded convex domain in $\mathbb{R}^d$,
$\Gamma_- = \left\{(x,v) \in \partial\Omega \times \mathbb{S}^{d-1}; v \cdot n_x < 0\right\}$, $S_d = \int_{\mathbb{S}^{d-1}} dv$,

$$\begin{cases} \varepsilon \partial_t f_\varepsilon + v \cdot \nabla f_\varepsilon + \frac{1}{\varepsilon}\left(f_\varepsilon - \frac{1}{S_d}\int_{\mathbb{S}^{d-1}} f_\varepsilon dv\right) = 0 & \\ & \text{in } \Omega \times \mathbb{S}^{d-1} \times (0,T) \ , \\ f_\varepsilon = 0 & \text{in } \Gamma_- \times (0,T) \ , \\ f_\varepsilon(x,v,0) = f^{in}(x) & \text{for } (x,v) \in \Omega \times \mathbb{S}^{d-1} \ . \end{cases}$$

$$f_\varepsilon \to u \text{ when } \varepsilon \to 0$$

$$\begin{cases} \partial_t u - \frac{1}{d}\Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) = f^{in} & \text{in } \Omega \ . \end{cases}$$

Main result

$$\text{If} \quad 0 < e^{c\left(1+\frac{1}{T}+T\left(\frac{\|f^{in}\|_{L^2(\Omega)}}{\|f^{in}\|_{H^{-1}(\Omega)}}\right)^2\right)}\sqrt{\varepsilon} \leq \frac{1}{2}\frac{\|f^{in}\|_{L^2(\Omega)}}{\|f^{in}\|_{H^3(\Omega)}}$$

then

$$\boxed{\|f_\varepsilon(\cdot,\cdot,T)\|_{L^2\left(\Omega\times\mathbb{S}^{d-1}\right)} \leq \frac{1}{T}e^{C\left(1+T+\frac{1}{T}\right)}\|f_\varepsilon\|_{L^1\left(E\times\mathbb{S}^{d-1}\times(0,T)\right)}}$$

$$\boxed{\|f^{in}\|_{L^2(\Omega)} \leq 2e^{c\left(1+\frac{1}{T}+T\left(\frac{\|f^{in}\|_{L^2(\Omega)}}{\|f^{in}\|_{H^{-1}(\Omega)}}\right)^2\right)}\|f_\varepsilon(\cdot,\cdot,T)\|_{L^1\left(E\times\mathbb{S}^{d-1}\right)}}$$

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Thank You