Lecture Notes (Ho Chi Minh City 2014) *

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Contents

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The aim of this course on control for Partial Differential Equation (PDE) is two-fold. In a first part we will present the concept of observability, controllability, stabilization and inverse source problem for a model linear PDE. In a second part we will introduce basic notions, tools, and results of existence and quantitative unique continuation for two linear PDEs: the wave equation and the heat equation.

The first model control problems concern the wave equation in a bounded domain.

1 Formulation of the problems

There are closed links between the following five problems: identification of solutions, observation, controllability, stabilization and inverse source problem.

1.1 Identification of solutions

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$. We consider the following wave equation with Dirichlet boundary condition

$$\left\{ \begin{array}{l} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ \quad u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ . \end{array} \right. \tag{1.1}$$

Let $T > 0$, ω be a nonempty open subset of Ω and Γ be a nonempty subset of $\partial\Omega$. Let n denote the outward unit normal vector to $\partial\Omega$. Here, we ask to answer the following two questions concerning identification of solutions: let u and v be two solutions of (1.1),

- does $u = v$ in $\omega \times (0,T)$ imply $u \equiv v$ in $\overline{\Omega} \times \mathbb{R}$?
- does $\partial_n u = \partial_n v$ on $\Gamma \times (0,T)$ imply $u \equiv v$ in $\overline{\Omega} \times \mathbb{R}$?

Introduce

$$\mathcal{N} = \left\{ u \in H^1 \left(\Omega \times (0,T) \right) \text{ being a solution of (1.1) such that } \partial_n u = 0 \text{ on } \Gamma \times (0,T) \right\} \ .$$

By linearity, the above two questions are reduced to the unique continuation property (UCP) for the wave equation, that is "does $\mathcal{N} = \{0\}$?". Due to finite speed of propagation, the UCP holds only for $T > 0$ large enough and Ω is supposed to be a connected domain. For instance, for a sufficiently smooth boundary, if $T > 2\max\left\{ \text{dist}\,(x,\Gamma)\,, x \in \overline{\Omega} \right\}$, then by Holmgren uniqueness theorem $\mathcal{N} = \{0\}$.

1.2 Observation and observability

When the UCP holds, we will have some interest to quantify it, or in other words, to get an observation estimate. Under some conditions on $T > 0$ and $\omega \subset \Omega$, the internal observability for the wave equation consists to establish the existence of a constant $c > 0$, such that any solution u of (1.1) with initial data $(u_0, u_1) = (u(\cdot,0)\,, \partial_t u(\cdot,0)) \in L^2(\Omega) \times H^{-1}(\Omega)$, satisfies

$$\left\| (u_0,u_1) \right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq c \int_0^T \int_\omega \left| u(x,t) \right|^2 dxdt \ .$$

Under some conditions on $T > 0$ and $\Gamma \subset \partial\Omega$, the boundary observability for the wave equation consists to establish the existence of a constant $c > 0$, such that any solution u of (1.1) with initial data $(u_0, u_1) = (u(\cdot,0)\,, \partial_t u(\cdot,0)) \in H_0^1(\Omega) \times L^2(\Omega)$, satisfies

$$\left\| (u_0,u_1) \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq c \int_0^T \int_\Gamma \left| \partial_n u(x,t) \right|^2 d\sigma dt \ .$$

The above two estimates give the stability of the observation. We may only have a quantitative unique continuation estimate as follows. Let Ω be a smooth connected open bounded set of $\mathbb{R}^N$, $N > 1$, with boundary $\partial\Omega$. Let $\omega \subset \Omega$ be a nonempty open subset of Ω, and $\Gamma \subset \partial\Omega$ be a nonempty subset of $\partial\Omega$. There exist $c>0$ and $T>0$ such that any solution u of (1.1) with initial data $(u_0,u_1)=(u(\cdot,0),\partial_t u(\cdot,0))\in H_0^1(\Omega)\times L^2(\Omega)$, $(u_0,u_1)\neq 0$, satisfies

$$\|(u_0,u_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}\leq e^{c\frac{\|(u_0,u_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}}\int_0^T\int_\omega |u(x,t)|^2\,dxdt\ ,$$

$$\|(u_0,u_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}\leq e^{c\frac{\|(u_0,u_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}}\int_0^T\int_\Gamma |\partial_n u(x,t)|^2\,d\sigma dt\ .$$

More generally, we may look for the following kind of interpolation inequality.

$$\|(u_0,u_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}\leq f\left(\frac{\|(u_0,u_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}}{\|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}\right)\int_0^T\int_\omega |u(x,t)|^2\,dxdt\ ,$$

for some positive increasing function $f:\mathbb{R}^+\to\mathbb{R}^+$.

1.3 Controllability

Let $T>0$ and $\Gamma\subset\partial\Omega$. Let $\mathcal{G}$ be a mapping from $L^2(\partial\Omega\times(0,T))$ to $L^2(\Omega)\times H^{-1}(\Omega)$ defined by

$$\mathcal{G}(f)=(v(\cdot,0),\partial_t v(\cdot,0))\quad\text{in }\Omega\ ,$$

where

$$\begin{cases}\partial_t^2 v-\Delta v=0 & \text{in }\Omega\times(0,T)\ ,\\ v=f_{|\Gamma\times(0,T)} & \text{on }\partial\Omega\times(0,T)\ ,\\ (v(\cdot,T),\partial_t v(\cdot,T))=(0,0) & \text{in }\Omega\ .\end{cases}$$

Let us introduce $C_{ad}\subseteq L^2(\Gamma\times(0,T))$ and $D_{ad}\subseteq L^2(\Omega)\times H^{-1}(\Omega)$, we choose $(v_0,v_1)\in D_{ad}$. We consider the map $J_{(v_0,v_1)}$ on C_{ad} defined by

$$J_{(v_0,v_1)}(f)=\|\mathcal{G}(f)-(v_0,v_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\ .$$

The exact boundary controllability for the wave equation is equivalent to the surjectivity of the map $\mathcal{G}$ (recall that wave have the time reversibility property). According to physical properties of waves, the natural question is: what geometrical situations and in particular, what hypothesis on (Γ,T) should we impose to have the surjectivity of $\mathcal{G}$?
In the case where such geometrical hypothesis are not satisfied, we will look for an adequate functional space D_{ad} in which Im($\mathcal{G}$) is dense. Thus, the problem of approximate controllability can be rewritten as follows: for all $\epsilon>0$, for all $(v_0,v_1)\in D_{ad}$, does exist an approximate control function $f\in L^2(\Gamma\times(0,T))$ such that $J_{(v_0,v_1)}(f)\leq\epsilon$? Furthermore, are we able to estimate the cost of such approximate control f with respect to ϵ ? Of course, the choice of the control function is connected with the cost.
Eventually (and this correspond to the notion of optimal control) one try to minimize over all possible control f , the map $J_{(v_0,v_1)}(f)$, when $(v_0,v_1)\in L^2(\Omega)\times H^{-1}(\Omega)$. This leads to the following question: does exist an admissible optimal control function $f\in C_{ad}$ such that $J_{(v_0,v_1)}(f)=\inf\limits_{g\in C_{ad}}J_{(v_0,v_1)}(g)$?

Similar questions appear in the context of internal controllability. Let $T>0$ and $\omega\subset\Omega$. What are the data $(v_0,v_1)\in H_0^1(\Omega)\times L^2(\Omega)$ for which there exists a control function $f\in L^2(\omega\times(0,T))$ such that the solution $v\in C^0\left(\mathbb{R},H_0^1(\Omega)\right)\cap C^1\left(\mathbb{R},L^2(\Omega)\right)$ of

$$\begin{cases}\partial_t^2 v-\Delta v=f_{|\omega\times(0,T)} & \text{in }\Omega\times\mathbb{R}\ ,\\ v=0\quad\text{on }\partial\Omega\times\mathbb{R}\ ,\\ (v(\cdot,0),\partial_t v(\cdot,0))=(v_0,v_1) & \text{in }\Omega\ ,\end{cases}$$

satisfies $v(\cdot,T)=\partial_t v(\cdot,T)=0$ in Ω, that is $v_{|t\geq T}\equiv 0$?

1.4 Stabilization

When the control function f depends on the solution v (closed-loop problems) and when the system becomes dissipative (for instance if absorbing boundary conditions or damped terms are involved), the energy is a positive time decreasing function. Therefore, we study the long time asymptotic behavior of the energy. In particular, the choice of different Cauchy data and/or geometrical hypothesis gives different estimates for the decreasing rate of the energy. The strong stabilization consists obtaining an uniform time exponential rate of decay.

For example, we study the following systems

$$\left\{\begin{array}{l} \partial_t^2 w - \Delta w = 0 \quad \text{in } \Omega \times (0,+\infty) \ , \\ \partial_n w + \lambda(x)\,\partial_t w = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \ , \\ (w, \partial_t w)(\cdot,0) = (w_0, w_1) \quad \text{in } \Omega \ , \end{array}\right.$$

or

$$\left\{\begin{array}{l} \partial_t^2 w - \Delta w + a(x)\,\partial_t w = 0 \quad \text{in } \Omega \times (0,+\infty) \ , \\ w = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \ , \\ (w, \partial_t w)(\cdot,0) = (w_0, w_1) \quad \text{in } \Omega \ , \end{array}\right.$$

where $a \in L^\infty(\Omega)$, $a \geq 0$, $\lambda \in L^\infty(\partial\Omega)$, $\lambda \geq 0$ and $(w_0, w_1) \in H^1(\Omega) \times L^2(\Omega)$ with their associated compatibility conditions. Denote by $E(w,t)$ the energy of the solution w:

$$E(w,t) = \frac{1}{2}\int_\Omega \left(|\partial_t w(x,t)|^2 + |\nabla w(x,t)|^2 \right) dx \ .$$

The weak stabilization consists to prove that for any (w_0, w_1) in a suitable space, $\lim_{t\to+\infty} E(w,t) = 0$. The strong stabilization consists to prove, under suitable conditions, the existence of $c > 0$ and $\beta > 0$ such that for any (w_0, w_1) in a suitable space, we have a uniform and exponential decay rate

$$E(w,t) \leq c e^{-\beta t} E(w,0) \ .$$

We are also interested in the decay rate of the energy for more smooth initial data. In particular, we may only get the existence of a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, $\lim_{t\to+\infty} f(t) = 0$, such that for any regular initial data (w_0, w_1) in a suitable space, we have

$$E(w,t) \leq f(t)\left[E(w,0) + E(\partial_t w, 0)\right] \ .$$

1.5 Inverse source problem

We consider the wave equation with Dirichlet boundary condition in the solution $u = u(x,t)$

$$\left\{\begin{array}{l} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ \quad u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \end{array}\right.$$

living in a bounded open set Ω in $\mathbb{R}^N$, $N \geq 1$, either convex or C^2 and connected, with boundary $\partial\Omega$. The resolution of such equation is well-known when initial data is known, e.g. when $(u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ then the above problem is well-posed and have a unique solution $u \in C\left(\mathbb{R}; H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}; L^2(\Omega)\right)$. This is a direct problem, e.g. a Cauchy problem. On the other hand, one can ask to recover the initial data (u_0, u_1) from the knowledge of u in $\omega \times (0,T)$ where $T > 0$ and ω is a nonempty open subset of Ω. Similarly, one can ask to recover the initial data from the knowledge of $\partial_n u$ on $\Gamma \times (0,T)$ where $T > 0$ and Γ is a nonempty subset of $\partial\Omega$. These are inverse problems.

2 The wave equation

2.1 Wave equation in $\mathbb{R}^N$

2.1.1 $N = 1$

When dimension $N = 1$, we have the D'Alembert's formula. If $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, then

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

is C^2 and solves

$$\left\{\begin{array}{l} \partial_t^2 u - \Delta u = 0 \quad \text{on } \mathbb{R}^{1+1} \ , \\ u_{|t=0} = f, \quad \partial_t u_{|t=0} = g \ . \end{array}\right.$$

Application to control: the particular case of the one dimension is well-understood for the nonlinear wave equation thanks to [Z]. Usually, we used the following two ideas: $\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$; we interchange the time variable with the space variable. In particular, the solutions of

$$\left\{\begin{array}{c} \partial_t^2 p - \partial_x^2 p = 0 \quad \text{for } (x,t) \in (0,1) \times \mathbb{R} \ , \\ p = 0 \quad \text{for } (x,t) \in \{0,1\} \times \mathbb{R} \ , \end{array}\right.$$

are of the form $p(x,t) = h(t+x) - h(t-x)$ with h being a 2-periodic function.

Proof .- Let $(x,t) \in (0,1) \times \mathbb{R}$, we interchange the t-variable with the x-variable. Consider the following 1-d wave equation

$$\left\{\begin{array}{c} \partial_t^2 u - \partial_x^2 u = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,1) \ , \\ u = 0 \quad \text{for } (x,t) \in \mathbb{R} \times \{0,1\} \ , \end{array}\right.$$

then, the solutions u are of the form

$$u(x,t) = \frac{1}{2}\int_0^{x+t} f(\tau)\,d\tau - \frac{1}{2}\int_0^{x-t} f(\tau)\,d\tau$$

with f being a 2-periodic function. Indeed, we will use the method of characteristics: first, notice that

$$(\partial_t - \partial_x)(\partial_t u + \partial_x u) = 0 \ .$$

Therefore, let $v(x,t) = \partial_t u + \partial_x u$. Now, from the method of characteristics, recall that

$$\left\{\begin{array}{l} \partial_t v - \partial_x v = 0 \\ v(\cdot, t=0) = f \end{array}\right. \quad \Leftrightarrow v(x,t) = f(x+t)$$

and

$$\left\{\begin{array}{l} \partial_t u + \partial_x u = v \\ u(\cdot, t=0) = 0 \end{array}\right. \quad \Leftrightarrow u(x,t) = \int_0^t v(x-t+s, s)\,ds \ .$$

Consequently, for any $(x,t) \in \mathbb{R} \times (0,1)$,

$$u(x,t) = \int_0^t f(x-t+2s)\,ds = \frac{1}{2}\int_{x-t}^{x+t} f(\tau)\,d\tau = \frac{1}{2}\int_0^{x+t} f(\tau)\,d\tau - \frac{1}{2}\int_0^{x-t} f(\tau)\,d\tau$$

and the condition $u(x, t=1) = 0 \ \forall x \in \mathbb{R}$ implies that $\frac{d}{dx}\left(\int_{x-1}^{x+1} f(\tau)\,d\tau\right) = 0 \ \forall x \in \mathbb{R}$, which gives $f(x+1) = f(x-1) \ \forall x \in \mathbb{R}$, that is the desired periodicity property for f.

Finally, for any $(x,t) \in (0,1) \times \mathbb{R}$, $p(x,t) = u(t,x)$ completes the proof.

2.1.2 $N = 3$ and Huyghens' principle

When dimension $N = 3$, if $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$, then we have the Kirchoff's formula

$$u(x,t) = \frac{1}{4\pi}\int_{y\in\mathbb{S}^2}\left[f(x+ty) + ty\cdot\nabla f(x+ty) + tg(x+ty)\right]d\sigma(y)$$

which is $C^2(\mathbb{R}^{3+1})$ and solves

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{on } \mathbb{R}^{3+1}, \\ u_{|t=0} = f, \quad \partial_t u_{|t=0} = g. \end{cases} \tag{2.1}$$

Proof .- $u \in C^2(\mathbb{R}^{3+1})$ by classical theorem of calculus. First we solve

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = 0 \quad \text{on } \mathbb{R}^{3+1}, \\ u_{1|t=0} = 0, \quad \partial_t u_{1|t=0} = g. \end{cases}$$

We claim

$$u_1(x,t) = \frac{t}{4\pi}\int_{y\in\mathbb{S}^2} g(x+ty)\,d\sigma(y) .$$

Next, let $u_2 = \partial_t u_1$. Then

$$\begin{cases} \partial_t^2 u_2 - \Delta u_2 = 0 \quad \text{on } \mathbb{R}^{3+1}, \\ u_{2|t=0} = g, \quad \partial_t u_{2|t=0} = \partial_t^2 u_{1|t=0} = \Delta u_{1|t=0} = 0. \end{cases}$$

We conclude by linearity that

$$u(x,t) = \frac{d}{dt}\left(\frac{t}{4\pi}\int_{y\in\mathbb{S}^2} f(x+ty)\,d\sigma(y)\right) + \frac{t}{4\pi}\int_{y\in\mathbb{S}^2} g(x+ty)\,d\sigma(y)$$

which gives the desired formula. It remains to prove our claim. Let

$$v(x,t) = \frac{1}{4\pi}\int_{y\in\mathbb{S}^2} g(x+ty)\,d\sigma(y)$$

then

$$\begin{aligned} \partial_t v(x,t) &= \tfrac{1}{4\pi}\int_{y\in\mathbb{S}^2} y\cdot(\nabla g)(x+ty)\,d\sigma(y) \\ &= \tfrac{1}{4\pi}\int_{B(0,1)} div\left[(\nabla g)(x+tz)\right]dz \\ &= \tfrac{t}{4\pi}\int_{B(0,1)} (\Delta g)(x+tz)\,dz \\ &= \tfrac{t}{4\pi}\int_{B(0,t)} (\Delta g)(x+w)\,\frac{dw}{t^3} \\ &= \tfrac{1/t^2}{4\pi}\int_0^t r^2 dr\int_{\mathbb{S}^2}(\Delta g)(x+r\theta)\,d\theta . \end{aligned}$$

Finally, since $u_1 = tv$,

$$\begin{aligned} \partial_t^2 u_1(x,t) &= \tfrac{d}{dt}\left(v(x,t) + t\partial_t v(x,t)\right) \\ &= \partial_t v(x,t) + \tfrac{d}{dt}\left(\tfrac{1/t}{4\pi}\int_0^t r^2 dr\int_{\mathbb{S}^2}(\Delta g)(x+r\theta)\,d\theta\right) \\ &= \partial_t v(x,t) - \tfrac{1/t^2}{4\pi}\int_0^t r^2 dr\int_{\mathbb{S}^2}(\Delta g)(x+r\theta)\,d\theta \\ &+ \tfrac{1/t}{4\pi}t^2\int_{\mathbb{S}^2}(\Delta g)(x+t\theta)\,d\theta \\ &= \tfrac{t}{4\pi}\int_{\mathbb{S}^2}(\Delta g)(x+t\theta)\,d\theta = \Delta u_1(x,t) . \end{aligned}$$

This completes the proof.

By a change of variable,

$$u(x,t) = \frac{1/t^2}{4\pi} \int_{\|x-y\|=t} \left[f(y) + (y-x) \cdot \nabla f(y) + t g(y) \right] dy \ .$$

Consequently, the value of u at a point (x,t), $t > 0$ only depends on the values of the data f, g on the set $\{y; \|x-y\| = t\}$ (or more precisely in an infinitesimal neighborhood of this sphere, since the formula involves $(y-x) \cdot \nabla f(y)$). As a consequence, an initial disturbance at the origin, say a flash of light, propagates with unit speed and can only be seen on the forward light cone with vertex at the origin, namely the set $\{(x,t); t = \|x\|\}$. This is known as the (strong) Huygens' principle. This is still valid in any odd dimension). In dimensions $N = 1$ or 2 (and in any even dimension $N > 2$) a weaker version of Huygens' principle holds. Then u at (x,t) depends on the values of f, g in the ball $\{y; \|x-y\| \leq t\}$. Consequently, a flash of light at the origin will be visible to an observer at a point x_o in space, at times $t \geq \|x_o\|$, and not just at $t = \|x_o\|$ as in dimensions $N = 3, 5, \ldots$

When dimension $N = 3$, if f and g are smooth and compactly supported functions, say $f = g = 0$ for $|x| \geq R$ for some $R > 0$, then $u(x,t) = 0$ unless $t - R \leq |x| \leq t + R$.

The above result is still true for $N \geq 3$ odd. A weaker version also exists for $N \geq 2$ even.

2.1.3 Russell's method of controllability by acting on the whole boundary

It is one of the first results on control for the wave. Russell [Ru] uses Huygens' principle to give a control result for the wave equation when the control acts on the whole boundary (see also [L]).

Let Ω be a bounded set in $\mathbb{R}^3$, with boundary $\partial\Omega$ of class C^∞. Let $T_o >$diamΩ. Let $v_0 \in C^3\left(\overline{\Omega}\right)$ and $v_1 \in C^2\left(\overline{\Omega}\right)$. Now we introduce $\delta > 0$ such that $T_o > 2\delta+$diamΩ, and

$$\Omega_\delta = \left\{ x \in \mathbb{R}^3, \exists \widehat{x} \in \Omega \text{ with } |x - \widehat{x}| < \delta \right\} \ .$$

Consider $f_\delta \in C^3\left(\mathbb{R}^3\right)$ and $g_\delta \in C^2\left(\mathbb{R}^3\right)$ be such that

$$\begin{cases} f_\delta = v_0 \quad \text{in } \Omega, \quad f_\delta(x) = 0 \quad \text{for } x \notin \Omega_\delta \ , \\ g_\delta = v_1 \quad \text{in } \Omega, \quad g_\delta(x) = 0 \quad \text{for } x \notin \Omega_\delta \ . \end{cases}$$

Therefore, the solution u_δ of (2.1) with initial data (f_δ, g_δ) satisfies $u_\delta(x,t) = 0$ for $(x,t) \in \Omega \times [T_0, +\infty)$. Indeed, we only need to see that for $x \in \Omega$ and $t \geq T_o$, we get $x + ty \notin \Omega_\delta$ $\forall y \in \mathbb{S}^2$. (Let $\widehat{x} \in \Omega$,

$$\begin{aligned} T_o &\leq t = t\|y\| = \|ty\| \\ &\leq \|ty + x - \widehat{x}\| + \|x - \widehat{x}\| \\ &\leq \|ty + x - \widehat{x}\| + \text{diam}\Omega \end{aligned}$$

Thus $2\delta < T_o-$diam$\Omega \leq \|ty + x - \widehat{x}\|$, which implies $x + ty \notin \Omega_\delta$).
Finally, consider v the restriction to u_δ on $\Omega \times (0, T_o)$. Then for any smooth (v_0, v_1), there exists χ such that

$$\begin{cases} \partial_t^2 v - \Delta v = 0 \quad \text{in } \Omega \times (0, T_o) \ , \\ v = \chi \quad \text{on } \partial\Omega \times (0, T_o) \ , \\ (v(\cdot,0), \partial_t v(\cdot,0)) = (v_0, v_1) \quad \text{in } \Omega \ , \end{cases}$$

and $v_{|t \geq T_o} \equiv 0$.

2.2 Wave equation in bounded domain

Let Ω be a bounded open set of $\mathbb{R}^N$, $N > 1$, with boundary $\partial\Omega$ of class C^2 (or Ω be a convex domain in order that $H^2(\Omega) \cap H_0^1(\Omega) = \left\{u \in H_0^1(\Omega),\ \Delta u \in L^2(\Omega)\right\}$). Let $T > 0$.

2.2.1 Wellposedness (semigroup theory, spectral theory)

- $\forall (u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad \exists! u \in C\left(\mathbb{R}; L^2(\Omega)\right) \cap C^1\left(\mathbb{R}; H^{-1}(\Omega)\right)$ weak solution of

$$\left\{\begin{array}{l} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) \quad \text{in } \Omega \ , \end{array}\right.$$

in the distribution sense (see also weak solution in the transposition sense when $u = f$ on $\partial\Omega \times (0,T)$ for some $f \in L^1\left(0,T; L^2(\partial\Omega)\right)$)

$$\left\{\begin{array}{l} 0 = \left\langle \partial_t^2 u - \Delta u, \Psi(t)\Phi(x)\right\rangle \quad \forall \Psi \in C_0^1(\mathbb{R}),\ \forall \Phi \in C_0^\infty(\Omega) \ , \\ \displaystyle\int_{-\infty}^{+\infty} \Psi(t)\, u(\cdot,t)\, dt \in H_0^1(\Omega) \quad \forall \Psi \in C_0^1(\mathbb{R}) \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) \quad \text{in } \Omega \ . \end{array}\right.$$

- $\forall f \in L^1\left(0,T; L^2(\Omega)\right), \forall (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \quad \exists! u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$ solution of

$$\left\{\begin{array}{l} \partial_t^2 u - \Delta u = f \quad \text{in } \Omega \times (0,T) \ , \\ u = 0 \quad \text{on } \partial\Omega \times (0,T) \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) \quad \text{in } \Omega \ , \end{array}\right.$$

Moreover, $\exists c > 0$

$$\|u\|_{L^\infty\left(0,T; H_0^1(\Omega)\right)} + \|\partial_t u\|_{L^\infty(0,T; L^2(\Omega))} \leq c\left(\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}\right) \ .$$

Also,

$$u(x,t) = \sum_{j\geq 1}\left\{a_j^0 \cos\left(t\sqrt{\lambda_j}\right) + a_j^1 \frac{1}{\sqrt{\lambda_j}}\sin\left(t\sqrt{\lambda_j}\right) + \frac{1}{\sqrt{\lambda_j}}\int_0^t \sin\left((t-s)\sqrt{\lambda_j}\right) f_j(s)\, ds\right\} e_j(x)$$

where

$$\left\{\begin{array}{l} u_0(x) = \sum_{j\geq 1} a_j^0 e_j(x) \ , \quad \sum_{j\geq 1} \lambda_j \left|a_j^0\right|^2 < +\infty \ , \\ u_1(x) = \sum_{j\geq 1} a_j^1 e_j(x) \ , \quad \sum_{j\geq 1} \left|a_j^1\right|^2 < +\infty \ , \\ f(x,t) = \sum_{j\geq 1} f_j(t)\, e_j(x) \ , \end{array}\right.$$

the $\{e_j\}_{j\geq 1}$ is a Hilbert basis in $L^2(\Omega)$ formed by the eigenfunctions of the operator $-\Delta$, i.e.,

$$\left\{\begin{array}{l} -\Delta e_j = \lambda_j e_j \quad \text{in } \Omega \ , \\ e_j = 0 \quad \text{on } \partial\Omega \ . \end{array}\right.$$

- $\forall (u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \quad \exists! u \in C\left(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}; H_0^1(\Omega)\right) \cap C^2\left(\mathbb{R}; L^2(\Omega)\right)$ strong solution of

$$\left\{\begin{array}{l} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) \quad \text{in } \Omega \ . \end{array}\right. \tag{2.2}$$

Properties of the eigenvalues of the operator $-\Delta$ on the open, connected bounded domain Ω with dirichlet boundary conditions, are stated in the theorem below.

Theorem (Eigenvalues of the Laplace operator) .- The boundary value problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \ , \\ u = 0 & \text{on } \partial\Omega \ , \end{cases}$$

has a nontrivial solution $u \neq 0$ if and only if $\lambda \in \Sigma$, in which case λ is called an eigenvalue of $-\Delta$, u a corresponding eigenfunction.

(i) Each eigenvalues of $-\Delta$ is real.

(ii) Furthermore, if we repeat each eigenvalue according to its (finite) multiplicity, we have $\Sigma = \{\lambda_j\}_{j=1,2,...}$ where

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \ ; \\ \quad \lambda_j \to \infty \text{ as } j \to \infty \ . \end{cases}$$

(iii) Finally, there exists an orthonormal basis $\{e_j\}_{j=1,2,...}$ of $L^2(\Omega)$, where $e_j \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_j:

$$\begin{cases} -\Delta e_j = \lambda_j e_j & \text{in } \Omega \ , \\ e_j = 0 & \text{on } \partial\Omega \ , \end{cases}$$

for $j = 1, 2, ...$

2.2.2 Energy estimates

Denote by $E(u,t)$ the energy of the solution u of (2.2) with initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$:

$$E(u,t) = \frac{1}{2} \int_\Omega \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx \ .$$

Proposition 1.- $E(u,t) = E(u,0) \ \forall t \in \mathbb{R}$.

Proof.- First, we consider a smooth solution with initial data in $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Therefore, for any $T > 0$, $u \in C^1\left(0, T; H_0^1(\Omega)\right)$ and $\Delta u \in C\left(0, T; L^2(\Omega)\right)$. Clearly, by integrations by parts,

$$\frac{d}{dt} E(u,t) = 0 \text{ in } [0,T] \ .$$

Then, in the general case $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and by density, there exists $(u_{0,m}, u_{1,m}) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ such that $(u_{0,m}, u_{1,m}) \underset{m \to +\infty}{\longrightarrow} (u_0, u_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$. Thus, $u_m \underset{m \to +\infty}{\longrightarrow} u$ in $C\left(0, T; H_0^1(\Omega)\right) \cap C^1\left(0, T; L^2(\Omega)\right)$ where u_m is solution of (2.2) with initial data $(u_{0,m}, u_{1,m})$. Finally, $E(u_m, t) \underset{m \to +\infty}{\longrightarrow} E(u,t)$ in $C([0,T])$.

Another proof consists to compute it from the formula

$$u(x,t) = \sum_{j \geq 1} \left\{ a_j^0 \cos\left(t\sqrt{\lambda_j}\right) + a_j^1 \frac{1}{\sqrt{\lambda_j}} \sin\left(t\sqrt{\lambda_j}\right) \right\} e_j(x) \ .$$

Proposition 2.-

$$\|\partial_t u(\cdot,t)\|_{H^{-1}(\Omega)}^2 + \|u(\cdot,t)\|_{L^2(\Omega)}^2 = \|(u_0, u_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \ .$$

Proof.- Recall that $\|u\|_{H^{-1}(\Omega)}^2 = \left((-\Delta)^{-1} u, u \right)_{H_0^1(\Omega), H^{-1}(\Omega)}$.

2.2.3 Normal derivative

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain of class C^k, $k \geq 1$. Let $n = n(x)$ denote the outward unit normal vector to the point $x \in \partial\Omega$.

Lemma .- There exists a vector field $H : \overline{\Omega} \to \mathbb{R}^N$ of class C^{k-1} such that

$$H = n \text{ on } \partial\Omega \ .$$

Proof .- Since Ω is of class C^k, for every fixed $x^o \in \partial\Omega$ there is an open neighbourhood $\mathcal{V}$ of x^o in $\mathbb{R}^N$ and a function $\varphi : \mathcal{V} \to \mathbb{R}$ of class C^k such that

$$\begin{cases} \nabla\varphi \neq 0 \text{ on } \mathcal{V} \ ; \\ \varphi(x) = 0 \Leftrightarrow x \in \mathcal{V} \cap \partial\Omega \ . \end{cases}$$

Replacing φ by $-\varphi$ if needed, we may assume that

$$n(x^o) \cdot \nabla\varphi(x^o) > 0 \ .$$

Choosing $\mathcal{V}$ sufficiently small we may assume also that $\mathcal{V} \cap \partial\Omega$ is connected. Then the function $\psi : \mathcal{V} \to \mathbb{R}^N$ defined by $\psi = \frac{\nabla\varphi}{\|\nabla\varphi\|}$ is of class C^{k-1} and $\psi = n$ on $\mathcal{V} \cap \partial\Omega$.

Since Ω is bounded, Γ_0 is compact. Therefore it can be covered with a finite number of neighbourhoods $\mathcal{V}_1$, ..., $\mathcal{V}_m$ of this type. Denoting by ψ_1, ..., ψ_m the corresponding functions we have

$$\begin{cases} \Gamma_0 \subset \mathcal{V}_1 \cup ... \cup \mathcal{V}_m \ ; \\ \psi_j = n \text{ on } \mathcal{V}_j \cap \partial\Omega \ , \text{ for } j = 1, ..., m \ . \end{cases}$$

Next, we fix an open set $\mathcal{V}_0$ in $\mathbb{R}^N$ and define $\psi_0 : \mathcal{V}_0 \to \mathbb{R}^N$ such that

$$\begin{cases} \overline{\Omega} \subset \mathcal{V}_1 \cap ... \cap \mathcal{V}_m \ ; \\ \mathcal{V}_0 \cap \partial\Omega = \emptyset \ ; \\ \psi_0 = 0 \text{ on } \mathcal{V}_0 \ . \end{cases}$$

Let ℓ_0, ..., ℓ_m be a partition of unity of class C^k, corresponding to the covering $\mathcal{V}_0$, ..., $\mathcal{V}_m$ of $\overline{\Omega}$ that is

$$\begin{cases} \ell_j \in C_0^k(\mathcal{V}_j) \ , \text{ for } j = 1, ..., m \ ; \\ 0 \leq \ell_j \leq 1 \ , \text{ for } j = 1, ..., m \ ; \\ \ell_0 + ... + \ell_m = 1 \text{ on } \overline{\Omega} \ . \end{cases}$$

Therefore,

$$H(x) = \sum_{j=0,\dots,m} \ell_j(x)\,\psi_j(x) \ , \forall x \in \overline{\Omega}$$

has the desired properties.

Exercice .- trace theorem.

For the solutions of the wave equation, we have a better result than the classical trace theorem.

Proposition 3.- *For any* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *the unique solution* $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$ *of (2.2) satisfies* $\partial_n u \in L^2(\partial\Omega \times (0,T))$ *and*

$$\exists c > 0 \quad \int_0^T \int_{\partial\Omega} |\partial_n u(x,t)|^2 \, d\sigma dt \leq c \, \|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \quad \forall (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \ .$$

Proof.- Let $H \in C^1\left(\overline{\Omega}\right)$ be a vector field. For a strong solution u, we multiply the equation $\partial_t^2 u - \Delta u = 0$ by $H \cdot \nabla u$ and integrate over $\Omega \times (0,T)$. It comes by integrations by parts

$$\begin{array}{rl} 0 & = \int_0^T \int_\Omega \left(\partial_t^2 u - \Delta u\right) H \cdot \nabla u \\ & = \left[\int_\Omega \partial_t u H \cdot \nabla u\right]_0^T + \frac{1}{2}\int_0^T \int_\Omega \mathrm{div} H \left(\left|\partial_t u\right|^2 - \left|\nabla u\right|^2\right) + \int_0^T \int_\Omega \partial_{x_i} u \partial_{x_i} H_j \partial_{x_j} u \\ & - \int_0^T \int_{\partial\Omega} \partial_n u H \cdot \nabla u + \frac{1}{2}\int_0^T \int_{\partial\Omega} H \cdot n \left|\nabla u\right|^2 \ . \end{array} \tag{2.3}$$

We choose H such that $H = n$ on $\partial\Omega$. And recall that $\nabla u = \partial_n u n$ on $\partial\Omega$ because $u = 0$ on $\partial\Omega$. Consequently,

$$\frac{1}{2}\int_0^T \int_{\partial\Omega} \left|\partial_n u\right|^2 = \left[\int_\Omega \partial_t u H \cdot \nabla u\right]_0^T + \frac{1}{2}\int_0^T \int_\Omega \mathrm{div} H \left(\left|\partial_t u\right|^2 - \left|\nabla u\right|^2\right) + \int_0^T \int_\Omega \partial_{x_i} u \partial_{x_i} H_j \partial_{x_j} u \ .$$

Now, by a density argument, the above equality is still true for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ which gives the desired inequality where $c > 0$ depends only on Ω and $T > 0$.

2.3 Exact controllability for the wave equation

Two methods allow to get both internal and boundary exact controllability for wave equations from an observability inequality: the HUM method (Hilbert Uniqueness Method); the variational approach.

2.3.1 Preliminary properties

Here, we establish three properties concerning the observation for the wave equation. Proposition 4 below shows that observability holds for different norms. Proposition 5 below says that observability can deal with a term in a lower norm. Proposition 6 says that it still has a interest to get an observation with a small term in a higher norm.

Proposition 4.- . *Let $\omega \subset \Omega$ be a nonempty open subset of Ω. Let u be a solution of (2.2) with initial data (u_0, u_1). The following two statements are equivalent:*

i) there exists $c > 0$ such that

$$\left\|(u_0, u_1)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)}^2 \leq c \int_0^T \int_\omega \left|u(x,t)\right|^2 dxdt \ ,$$

for any $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$;

ii) there exists $c > 0$ such that

$$\left\|(u_0, u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}^2 \leq c \int_0^T \int_\omega \left|\partial_t u(x,t)\right|^2 dxdt \ ,$$

for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Proof of $i) \Rightarrow ii)$.- We fix $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, we apply $i)$ to $\varphi(x,t) = \partial_t u(x,t)$.

Proof of $ii) \Rightarrow i)$.- We fix $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. Then, we apply $ii)$ to $\varphi(x,t) = \int_0^t u(x,s)\, ds - (-\Delta)^{-1} u_1(x)$.

Lemma .- Let X, Y, and Z be Banach spaces, let $A : X \to Y$ be a closed linear operator with domain $D(A)$, and let $K : X \to Z$ be a compact linear operator. Suppose that

$$\|f\|_X \leq C(\|Af\|_Y + \|Kf\|_Z) \qquad \forall f \in D(A) . \tag{2.4}$$

If A is injective, then

$$\|f\|_X \leq C \|Af\|_Y \qquad \forall f \in D(A) .$$

Proof. We show first that one can assume that A is bounded. Indeed, let $\|\cdot\|_{D(A)}$ denote the graph norm given by

$$\|f\|_{D(A)} = \left(\|f\|_X^2 + \|Af\|_Y^2\right)^{1/2} .$$

Note that $A : X \to Y$ with the norm $\|\cdot\|_{D(A)}$ on X is bounded since

$$\|Af\|_Y \leq \left(\|f\|_X^2 + \|Af\|_Y^2\right)^{1/2} = \|f\|_{D(A)} .$$

Furthermore, (2.4) implies that

$$\|f\|_{D(A)} \leq C(\|Af\|_Y + \|Kf\|_Z) \qquad \forall f \in D(A) .$$

Assuming the lemma is true for bounded operators, we then have $\|f\|_{D(A)} \leq C \|Af\|_Y$. Since $\|f\|_X \leq \|f\|_{D(A)}$, the result follows.

For bounded A, assume on the contrary that such an estimate does not hold. Then there exists a sequence f_m in X with $\|f_m\|_X = 1$ and $Af_m \to 0$ in Y. Since $K : X \to Z$ is compact, there exists a subsequence, which we still denote by f_m, such that Kf_m converges in Z, and is therefore a Cauchy sequence in Z. Applying (2.4) to $f_m - f_p$, we have that $\|f_m - f_p\|_X \to 0$ as $m, p \to \infty$. That is, f_m is a Cauchy sequence in X. Therefore, there exists $f \in X$ such that $f_m \to f$, which implies that $\|f\|_X = 1$. Since A is closed, we have $Af_m \to Af = 0$, which contradicts the injectivity of A.

Proposition 5.- "a standard uniqueness-compactness argument". *Let $T > 0$. Let $\omega \subset \Omega$ be a nonempty open subset of Ω. Introduce*

$$\mathcal{N} = \left\{ \begin{array}{c} u \in H^1(\Omega \times (0,T)) \text{ being a solution of the wave equation (1.1)} \\ \text{such that } \partial_t u = 0 \text{ on } \omega \times (0,T) \end{array} \right\} .$$

Let u be a solution of (2.2) with initial data (u_0, u_1). Suppose that $\mathcal{N} = \{0\}$. The following two statements are equivalent:

i) there exist $c > 0$ and $d > 0$ such that

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq c \int_0^T \int_\omega |\partial_t u(x,t)|^2\, dxdt + d \|(u_0, u_1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} ,$$

for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$;

ii) there exists $c > 0$ such that

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq c \int_0^T \int_\omega |\partial_t u(x,t)|^2\, dxdt ,$$

for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Proof.- Suppose that $ii)$ is false. Then there exists a sequence $(u_{0,m},u_{1,m})_{m\in\mathbb{N}}\in H_0^1(\Omega)\times L^2(\Omega)$ such that

$$\left\|(u_{0,m},u_{1,m})\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}=1 \quad \text{and} \quad \int_0^T\int_\omega \left|\partial_t u_m\right|^2 dxdt \underset{m\rightarrow+\infty}{\longrightarrow} 0 \ ,$$

where u_m is the solution of (2.2) with initial data $(u_{0,m},u_{1,m})$. By Rellich compactness theorem, there exists a subsequence still denoted by $(u_{0,m},u_{1,m})_{m\in\mathbb{N}}\in H_0^1(\Omega)\times L^2(\Omega)$ such that

$$\left\|(u_{0,m},u_{1,m})\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \underset{m\rightarrow+\infty}{\longrightarrow} \left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ ,$$

where $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$. $(u_{0,m},u_{1,m})$ is a Cauchy sequence in $L^2(\Omega)\times H^{-1}(\Omega)$. Applying $i)$ to $(u_{0,p}-u_{0,m},u_{1,p}-u_{1,m})_{p,m\in\mathbb{N}}\in H_0^1(\Omega)\times L^2(\Omega)$, $(u_{0,m},u_{1,m})$ is a Cauchy sequence in $H_0^1(\Omega)\times L^2(\Omega)$. Therefore, $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ and $\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}=1$. Further, $\partial_t u=0$ on $\omega\times(0,T)$.
This contradicts $\mathcal{N}=\{0\}$.

A method to get $\mathcal{N}=\{0\}$ is as follows. We check that $\mathcal{N}$ is finite dimensional and prove that $\mathcal{N}$ is stable by ∂_t.

Recall that by Riesz Theorem, if X is a normed space and the closed unit ball centered at zero is compact, then X is finite dimensional. Let B be the closed unit ball centered at zero that is $B=\left\{u\in\mathcal{N};\left\|u\right\|_{L^2(\Omega\times(0,T))}\leq 1\right\}$. By $i)$ and Rellich compactness theorem, B is compact. Therefore $\mathcal{N}$ is finite dimensional.

Then, if $\mathcal{N}\neq\{0\}$, there will exist an eigenfunction u for ∂_t on $\mathcal{N}$ associated to the eigenvalue λ. In other words, there is ℓ such that $\ell\neq 0$, $\ell\in\mathcal{N}$,

Finally, we get a contradiction using uniqueness theorem for second order elliptic operator in a connected domain Ω with $\omega\subset\Omega$.

Proposition 6.- *Let $\omega\subset\Omega$ be a nonempty open subset of Ω. Let u be a solution of (2.2) with initial data (u_0,u_1). The following two statements are equivalent:*

i) there exists $c>0$ such that

$$\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\leq e^{c/\varepsilon}\int_0^T\int_\omega \left|u(x,t)\right|^2 dxdt+\varepsilon\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \ ,$$

for any $\varepsilon>0$ and $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$;

ii) there exists $c>0$ such that

$$\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\leq ce^{c\frac{\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}}\int_0^T\int_\omega \left|u(x,t)\right|^2 dxdt \ ,$$

for any $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$.

Proof of $i)\Rightarrow ii)$.- We chose $\varepsilon=\frac{1}{2}\frac{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}$.

Proof of $ii)\Rightarrow i)$.- We distinguish two cases: when $\frac{\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}\leq\frac{1}{\varepsilon}$, then $\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\leq ce^{c/\varepsilon}\int_0^T\int_\omega \left|u(x,t)\right|^2 dxdt$; when $\frac{\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}{\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}>\frac{1}{\varepsilon}$, then $\left\|(u_0,u_1)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\leq\varepsilon\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}$.

2.3.2 Gaussian beam

We present a numerical approximation of a solution of the wave equation in a square domain in $\mathbb{R}^2$ with the Dirichlet boundary condition. From the computations done by Ralston, the solution $u(x_1,x_2,t) = a_0(x_1,x_2,t)\, e^{ik\Phi(x_1,x_2,t)}$ given below solves $\partial_t^2 u - \Delta u = O\left(1/\sqrt{k}\right)$ and is concentrated on the curve $\gamma = \left\{\left(x_1^0, x_2^0 + t\right), t \geq 0\right\}$ where $\left(x_1^0, x_2^0\right) \in \mathbb{R}^2$: for $\alpha > 0$, $\beta > 0$,

$$a_0(x_1,x_2,t) = \frac{1}{\sqrt{1+2i\alpha t}} \ ,$$

$$\Phi(x_1,x_2,t) = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ t \end{pmatrix} \cdot \begin{pmatrix} \frac{i\alpha}{1+2i\alpha t} & 0 & 0 \\ 0 & i\beta & -i\beta \\ 0 & -i\beta & i\beta \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ t \end{pmatrix} \ ,$$

therefore

$$\begin{aligned} u(x_1,x_2,t) &= \tfrac{1}{\sqrt{1+2i\alpha t}} \exp\left(i\tfrac{k}{2}\left[\left(x_2 - x_2^0\right) - t\right] + i\tfrac{k\alpha^2 t}{1+(2\alpha t)^2}\left(x_1 - x_1^0\right)^2\right) \\ &\quad \exp\left(-\tfrac{k\alpha}{2\left(1+(2\alpha t)^2\right)}\left(x_1 - x_1^0\right)^2 - \tfrac{k\beta}{2}\left[\left(x_2 - x_2^0\right) - t\right]^2\right) \ . \end{aligned}$$

By taking its real part, we get

$$\begin{aligned} u(x_1,x_2,t) &= \left(\tfrac{1}{1+(2\alpha t)^2}\right)^{1/4} \exp\left(-\tfrac{k\alpha}{2\left(1+(2\alpha t)^2\right)}\left(x_1 - x_1^0\right)^2 - \tfrac{k\beta}{2}\left[\left(x_2 - x_2^0\right) - t\right]^2\right) \\ &\quad \cos\left(\tfrac{k\alpha^2 t}{1+(2\alpha t)^2}\left(x_1 - x_1^0\right)^2 + \tfrac{k}{2}\left[\left(x_2 - x_2^0\right) - t\right] - \tfrac{1}{2}\arctan(2\alpha t)\right) \ . \end{aligned}$$

Now, the initial data are

$$\begin{aligned} u_0(x_1,x_2) &= \exp\left(-\tfrac{k\alpha}{2}\left(x_1 - x_1^0\right)^2 - \tfrac{k\beta}{2}\left(x_2 - x_2^0\right)^2\right) \cos\left(\tfrac{k}{2}\left(x_2 - x_2^0\right)\right) \\ u_1(x_1,x_2) &= \exp\left(-\tfrac{k\alpha}{2}\left(x_1 - x_1^0\right)^2 - \tfrac{k\beta}{2}\left(x_2 - x_2^0\right)^2\right) \\ &\quad \left[k\beta\left(x_2 - x_2^0\right)\cos\left(\tfrac{k}{2}\left(x_2 - x_2^0\right)\right) - \left(k\alpha^2\left(x_1 - x_1^0\right)^2 - \tfrac{k}{2} - \alpha\right)\sin\left(\tfrac{k}{2}\left(x_2 - x_2^0\right)\right)\right] \ , \end{aligned}$$

and using a Galerkin approximation, we may get a visual idea of a localized solution of the wave equation for $(x,t) \in (0,1)^2 \times [0, 0.8]$ ...(to be completed)...

2.3.3 HUM method

We apply the HUM method to get boundary controllability for the wave equation.

Step 1.- Let us introduce the operator

$$\mathcal{C} : f \in L^2(\Gamma \times (0,T)) \longrightarrow (\partial_t v(\cdot,0), -v(\cdot,0)) \in H^{-1}(\Omega) \times L^2(\Omega) \ ,$$

where $v \in C\left([0,T], L^2(\Omega)\right) \cap C^1\left([0,T], H^{-1}(\Omega)\right)$ is the solution of

$$\left\{ \begin{array}{ll} \partial_t^2 v - \Delta v = 0 & \text{in } \Omega \times (0,T) \ , \\ v = f_{|\Gamma \times (0,T)} & \text{on } \partial\Omega \times (0,T) \ , \\ (v(\cdot,T), \partial_t v(\cdot,T)) = (0,0) & \text{in } \Omega \ . \end{array} \right.$$

Then $\mathcal{C}$ is a linear continuous operator. We define $H^{-1}(\Omega) \times L^2(\Omega) \supset \mathcal{F} = \mathrm{Im}\mathcal{C}$, be the range of $\mathcal{C}$, the space of exact controllable data at time T by acting on Γ. Now, we will need to construct the dual operator of $\mathcal{C}$. Let us introduce the operator

$$\mathcal{K} : (u_1, u_0) \in H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \partial_n u_{|\Gamma \times (0,T)} \in L^2(\Gamma \times (0,T)) \ ,$$

where $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$ is the solution of

$$\left\{\begin{array}{l} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times (0,T) \ , \\ u = 0 \quad \text{on } \partial\Omega \times (0,T) \ , \\ \left(u(\cdot,0), \partial_t u(\cdot,0)\right) = (u_0, u_1) \quad \text{in } \Omega \ . \end{array}\right.$$

Then $\mathcal{K}$ is a linear continuous operator.

Step 2.- We have the following duality result between $\mathcal{C}$ and $\mathcal{K}$.

Theorem 1.- *For all* $f \in L^2(\Gamma \times (0,T))$ *and* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\int_0^T \int_\Gamma f \mathcal{K}(u_0,u_1)\, d\sigma dt = \left\langle (u_0,u_1), \mathcal{C}(f) \right\rangle_{H_0^1(\Omega)\times L^2(\Omega), H^{-1}(\Omega)\times L^2(\Omega)} \ ,$$

where $\left\langle (u_0,u_1), \mathcal{C}(f) \right\rangle_{L^2(\Omega)\times H_0^1(\Omega), L^2(\Omega)\times H^{-1}(\Omega)} = \left\langle u(\cdot,0), \partial_t v(\cdot,0) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega \partial_t u(\cdot,0)\, v(\cdot,0)\, dx.$

Step 3.- Now we have the following approximate controllability result.

Theorem 2.- $\mathcal{F} = \mathrm{Im}\mathcal{C}$ *is dense in* $H^{-1}(\Omega) \times L^2(\Omega)$ *if and only if* $\mathrm{Ker}\mathcal{K} = \{(0,0)\}$.

Proof.- We use the formula $\overline{\mathrm{Im}\mathcal{C}} = H^{-1}(\Omega) \times L^2(\Omega) \Leftrightarrow \mathcal{F}^\perp = \{(0,0)\}$ and $\mathcal{F}^\perp = \mathrm{Ker}\mathcal{K}$ where $\mathcal{F}^\perp$ denotes the orthogonal to $\mathcal{F}$ in $H^{-1}(\Omega) \times L^2(\Omega)$.

Step 4.- Now we have the following exact controllability result.

Theorem 3.- *The following two statements are equivalent.*

i) $\mathcal{F} = \mathrm{Im}\mathcal{C} = H^{-1}(\Omega) \times L^2(\Omega)$;

ii) there exists $c > 0$ *such that* $\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq c \int_0^T \int_\Gamma \left|\partial_n u(x,t)\right|^2 d\sigma dt$ *for any* $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *where* u *is the solution of (2.2).*

Proof of *i)* $\Rightarrow$ *ii)*.- First, remark that by a classical functional analysis theorem, if $\mathrm{Im}\mathcal{C} = H^{-1}(\Omega) \times L^2(\Omega)$, then $\exists \eta > 0$ such that

$$B(0,\eta)_{H^{-1}(\Omega)\times L^2(\Omega)} \subset \mathcal{C}\left(B(0,1)_{L^2(\Gamma\times(0,T))}\right) \ .$$

Next, let $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. We construct $(v_0,v_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such that

$$\left\{\begin{array}{l} \left\|(v_0,v_1)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} = 1 \\ \left\langle u_0, v_1 \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega u_1 v_0 dx = \left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} \ . \end{array}\right.$$

Then, we take $f \in L^2(\Gamma \times (0,T))$ be such that $\mathcal{C}(f) = (v_1, -v_0)$ and $\left\|f\right\|_{L^2(\Gamma\times(0,T))} \leq \frac{1}{\eta}$ in order to get

$$\begin{array}{ll} \left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} & = \left\langle u_0, \partial_t v(\cdot,0) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega u_1 v(\cdot,0)\, dx \\ & = \int_0^T \int_\Gamma f \mathcal{K}(u_0,u_1)\, d\sigma dt \\ & \leq \left\|f\right\|_{L^2(\Gamma\times(0,T))} \left\|\mathcal{K}(u_0,u_1)\right\|_{L^2(\Gamma\times(0,T))} \\ & \leq \frac{1}{\eta} \left\|\mathcal{K}(u_0,u_1)\right\|_{L^2(\Gamma\times(0,T))} \ . \end{array}$$

Proof of $ii) \Rightarrow i)$.- We look for a control function f in the following particular form. Denote $\mathcal{B} = \mathcal{C} \circ \mathcal{K}$ and suppose that $f = \mathcal{K}(\varphi_0, \varphi_1)$ for some $(\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then for all $(\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\int_0^T \int_\Gamma \mathcal{K}(\varphi_0, \varphi_1) \mathcal{K}(u_0, u_1)\, d\sigma dt = \langle (u_0, u_1), \mathcal{B}(\varphi_0, \varphi_1) \rangle_{H_0^1(\Omega)\times L^2(\Omega), H^{-1}(\Omega)\times L^2(\Omega)} \ .$$

In particular,

$$\int_0^T \int_\Gamma |\mathcal{K}(u_0, u_1)|^2\, d\sigma dt = \langle (u_0, u_1), \mathcal{B}(u_0, u_1) \rangle_{H_0^1(\Omega)\times L^2(\Omega), H^{-1}(\Omega)\times L^2(\Omega)} \ .$$

If we can prove that the bilinear form given by

$$((u_0, u_1), (\varphi_0, \varphi_1)) \longmapsto \langle (u_0, u_1), \mathcal{B}(\varphi_0, \varphi_1) \rangle_{H_0^1(\Omega)\times L^2(\Omega), H^{-1}(\Omega)\times L^2(\Omega)}$$

is coercive then by Lax-Milgram theorem, $\forall (v_1, v_0) \in H^{-1}(\Omega) \times L^2(\Omega) \quad \exists (\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that $\mathcal{B}(\varphi_0, \varphi_1) = (v_1, -v_0)$ that is $(v(\cdot, 0), \partial_t v(\cdot, 0)) = (v_0, v_1)$. Now, it is sufficient to see that the coercivity of the bilinear form is deduced from the observability estimate $ii)$.

2.3.4 Variational approach

We apply the variational approach to get the internal controllability for the wave equation.

Step 1.- Let us define the duality product between $L^2(\Omega) \times H^{-1}(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$ by

$$\langle (u_0, u_1), (v_0, v_1) \rangle = \langle u_1, v_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega u_0(x) v_1(x)\, dx \ ,$$

for all $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then we have the following result.

Theorem 4.- *The initial data $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ may be driven to zero at time $T > 0$ if and only if there exists $f \in L^2(\omega \times (0, T))$ such that*

$$0 = \int_0^T \int_\omega u(x, t) f(x, t)\, dx dt - \langle (u_0, u_1), (v_0, v_1) \rangle \ ,$$

for any $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, where u is the corresponding solution of (2.2).

Step 2.- Introduce the functional $\mathcal{J} : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$, defined by

$$\mathcal{J}(u_0, u_1) = \frac{1}{2} \int_0^T \int_\omega |u(x, t)|^2\, dx dt - \langle (u_0, u_1), (v_0, v_1) \rangle \ ,$$

where $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and u is the solution of (2.2) with initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. Then we have the following result.

Theorem 5.- *Let $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. If (φ_0, φ_1) is a minimizer of $\mathcal{J}$, then $f = \varphi_{|\omega \times (0,T)}$ is a control which leads (v_0, v_1) to zero at time T, where φ is the solution of*

$$\begin{cases} \partial_t^2 \varphi - \Delta \varphi = 0 \quad in\ \Omega \times \mathbb{R} \ , \\ \varphi = 0 \quad on\ \partial\Omega \times \mathbb{R} \ , \\ (\varphi(\cdot, 0), \partial_t \varphi(\cdot, 0)) = (\varphi_0, \varphi_1) \quad in\ \Omega \ . \end{cases}$$

Step 3.- It remains to prove that

Theorem 6.- . *If we have an observability estimate, that is*

$$\exists c>0 \quad \|(u_0,u_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq c\int_0^T\int_\omega |u(x,t)|^2\,dxdt\ ,$$

for any u the solution of (2.2) with initial data $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, then the functional $\mathcal{J}$ has an unique minimizer $(\varphi_0,\varphi_1)\in L^2(\Omega)\times H^{-1}(\Omega)$.

Step 4.- Moreover, we have

Theorem 7.- *Let $f=\varphi_{|\omega\times(0,T)}$ be the control given by minimizing the functional $\mathcal{J}$. If $g\in L^2(\omega\times(0,T))$ is any other control driving to zero the initial data $(v_0,v_1)\in H^1_0(\Omega)\times L^2(\Omega)$, then*

$$\|f\|_{L^2(\omega\times(0,T))}\leq\|g\|_{L^2(\omega\times(0,T))}\ .$$

2.4 Stabilization

...(to be completed)...

2.5 Inverse source problem

Two methods allow to reconstruct the initial data of a wave equation from the point of view of controllability: the forth-back algorithm; the iterated time reversal method.

Let Ω be a open bounded connected set of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2. Let ω be a nonempty open subset of Ω. Let given $u=u(x,t)$ the solution of the wave equation with Dirichlet boundary condition

$$\begin{cases}\partial_t^2u-\Delta u=0 & \text{in }\Omega\times\mathbb{R}\\ u=0 & \text{on }\partial\Omega\times\mathbb{R}\ ,\end{cases}$$

with initial data $(u(\cdot,0),\partial_t u(\cdot,0))=(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega)$. Denote $w=w(x,t)$ the solution of the damped wave equation

$$\begin{cases}\partial_t^2w-\Delta w+1_\omega\partial_t w=0 & \text{in }\Omega\times(0,+\infty)\\ w=0 & \text{on }\partial\Omega\times(0,+\infty)\ ,\end{cases}$$

with initial data $(w(\cdot,0),\partial_t w(\cdot,0))=(w_0,w_1)\in H^1_0(\Omega)\times L^2(\Omega)$. We define the energy by

$$E(g,t)=\|(g(\cdot,t),\partial_t g(\cdot,t))\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\qquad\forall g\in C\left([0,+\infty),H^1_0(\Omega)\right)\cap C^1\left([0,+\infty),L^2(\Omega)\right)\ .$$

Consequently, $E(u,t)=E(u,0)$ (i.e., energy conservation) and $E(w,t)-E(w,0)+2\int_\omega\int_0^t|\partial_t w|^2=0$.

Let us make the following assumption (O):

$$\exists C,T>0\qquad\int_\Omega\int_0^T|u(x,t)|^2\,dxdt\leq C\int_\omega\int_0^T|u(x,t)|^2\,dxdt$$

for any solution u of the wave equation with Dirichlet boundary condition.

The above observability estimate holds under suitable geometric condition on ω and Ω. Under such assumption (O), we have:

$i)$ If $u=0$ on $\omega\times(0,T)$, then $u\equiv 0$ sur $\overline{\Omega}\times\mathbb{R}$.

$ii)$ $\exists C,T>0$, $\forall(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, $E(u,0)\leq C\left\|\partial_t u\right\|^2_{L^2(\omega\times(0,T))}$.

$iii)$ $\exists C,T>0$, $\forall(w_0,w_1)\in H_0^1(\Omega)\times L^2(\Omega)$, $E(w,0)\leq C\left\|\partial_t w\right\|^2_{L^2(\omega\times(0,T))}$.

$iv)$ There is $0<\varepsilon<1$, precisely $\varepsilon=1/(1+2/C)$, such that

$$E(w,T)\leq\varepsilon E(w,0)\ .$$

2.5.1 Algorithm of forth-back

Let u be the solution of the wave equation with Dirichlet boundary condition and initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$. The algorithm of forth-back to reconstruct (u_0,u_1) from the knowledge of $1_{\omega\times(0,T)}\partial_t u$ is as follows.

Let

$$\left(v_B^0,\partial_t v_B^0\right)(\cdot,0)=(0,0)$$

and for any $m\geq 1$,

$$\left\{\begin{array}{rl}\partial_t^2 v_F^m-\Delta v_F^m+1_\omega\partial_t v_F^m=1_\omega\partial_t u & \text{in } \Omega\times(0,T)\ ,\\ v_F^m=0 & \text{on } \partial\Omega\times(0,T)\ ,\\ \left(v_F^m,\partial_t v_F^m\right)(\cdot,0)=\left(v_B^{m-1},\partial_t v_B^{m-1}\right)(\cdot,0) & \text{in } \Omega\ ,\end{array}\right.$$

$$\left\{\begin{array}{rl}\partial_t^2 v_B^m-\Delta v_B^m-1_\omega\partial_t v_B^m=-1_\omega\partial_t u & \text{in } \Omega\times(0,T)\ ,\\ v_B^m=0 & \text{on } \partial\Omega\times(0,T)\ ,\\ \left(v_B^m,\partial_t v_B^m\right)(\cdot,T)=\left(v_F^m,\partial_t v_F^m\right)(\cdot,T) & \text{in } \Omega\ .\end{array}\right.$$

Theorem 8.- *Under the assumption (O), there is $\varepsilon\in(0,1)$ such that for any $K\geq 1$,*

$$\left\|\left(v_B^K,\partial_t v_B^K\right)(\cdot,0)-(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\leq\varepsilon^{2K}\left\|(u_0,u_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\ .$$

Proof .- Introduce

$$w_F^m(\cdot,t)=v_F^m(\cdot,t)-u(\cdot,t)\ \text{ and }\ w_B^m(\cdot,t)=v_B^m(\cdot,T-t)-u(\cdot,T-t)\ .$$

We obtain that for any $m\geq 1$,

$$\left\{\begin{array}{rl}\partial_t^2 w_F^m-\Delta w_F^m+1_\omega\partial_t w_F^m=0 & \text{in } \Omega\times(0,T)\ ,\\ w_F^m=0 & \text{on } \partial\Omega\times(0,T)\ ,\\ \left(w_F^m,\partial_t w_F^m\right)(\cdot,0)=\left(v_B^{m-1},\partial_t v_B^{m-1}\right)(\cdot,0)-(u_0,u_1) & \text{in } \Omega\ ,\\ \left(w_F^m,\partial_t w_F^m\right)(\cdot,T)=\left(v_F^m,\partial_t v_F^m\right)(\cdot,T)-(u,\partial_t u)(\cdot,T) & \text{in } \Omega\ ,\end{array}\right.$$

$$\left\{\begin{array}{rl}\partial_t^2 w_B^m-\Delta w_B^m+1_\omega\partial_t w_B^m=0 & \text{in } \Omega\times(0,T)\ ,\\ w_B^m=0 & \text{on } \partial\Omega\times(0,T)\ ,\\ \left(w_B^m,\partial_t w_B^m\right)(\cdot,0)=\left(v_F^m,-\partial_t v_F^m\right)(\cdot,T)-(u,-\partial_t u)(\cdot,T) & \text{in } \Omega\ ,\\ \left(w_B^m,\partial_t w_B^m\right)(\cdot,T)=\left(v_B^m,-\partial_t v_B^m\right)(\cdot,0)-(u_0,-u_1) & \text{in } \Omega\ .\end{array}\right.$$

The sequence $\left(w_B^m,\partial_t w_B^m\right)(\cdot,T)$ is bounded $H_0^1(\Omega)\times L^2(\Omega)$. Further, under the assumption (O),

$$\begin{array}{ll}\left\|\left(w_B^m,\partial_t w_B^m\right)(\cdot,T)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} & \leq\varepsilon\left\|\left(w_B^m,\partial_t w_B^m\right)(\cdot,0)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\ & =\varepsilon\left\|\left(w_F^m,\partial_t w_F^m\right)(\cdot,T)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\ & \leq\varepsilon^2\left\|\left(w_F^m,\partial_t w_F^m\right)(\cdot,0)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\\ & =\varepsilon^2\left\|\left(w_B^{m-1},\partial_t w_B^{m-1}\right)(\cdot,T)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\ .\end{array}$$

Therefore,

$$\left\| \left(w_B^K, \partial_t w_B^K \right)(\cdot, T) \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon^{2K} \left\| \left(w_B^0, \partial_t w_B^0 \right)(\cdot, T) \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} .$$

Finally,

$$\left\| \left(v_B^K, \partial_t v_B^K \right)(\cdot, 0) - (u_0, u_1) \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon^{2K} \left\| (u_0, u_1) \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} .$$

2.5.2 Time reversal focusing

Let u be the solution of the wave equation with Dirichlet boundary condition and initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. The iterated time reversal method to reconstruct (u_0, u_1) from the knowledge of $1_{\omega\times(0,T)} u$ is as follows.

Introduce the time reversal operator

$$\mathcal{T} g(x,t) = g(x, T-t) \quad \forall g \in C\left([0,T]; L^2(\Omega)\right).$$

Let

$$\begin{cases} \partial_t^2 v^{(0)} - \Delta v^{(0)} + 1_\omega \partial_t v^{(0)} = -1_\omega \mathcal{T}(\partial_t u) & \text{in } \Omega \times (0,T) \\ v^{(0)} = 0 & \text{on } \partial\Omega \times (0,T) \\ v^{(0)}(\cdot,0) = \partial_t v^{(0)}(\cdot,0) = 0 & \text{in } \Omega . \end{cases}$$

and

$$w^{(0)} = v^{(0)} - \mathcal{T} u \quad \text{in } \omega \times (0,T) .$$

For any $j \geq 1$, $v^{(j)} = v^{(j)}(x,t)$ solves

$$\begin{cases} \partial_t^2 v^{(j)} - \Delta v^{(j)} + 1_\omega \partial_t v^{(j)} = -1_\omega \left[2\mathcal{T}\left(\partial_t w^{(j-1)} \right) \right] & \text{in } \Omega \times (0,T) \\ v^{(j)} = 0 & \text{on } \partial\Omega \times (0,T) \\ v^{(j)}(\cdot,0) = \partial_t v^{(j)}(\cdot,0) = 0 & \text{in } \Omega \end{cases}$$

and we consider $w^{(j)} = w^{(j)}(x,t)$ given by

$$w^{(j)} = v^{(j)} - \mathcal{T}\left(w^{(j-1)} \right) \quad \text{in } \omega \times (0,T) .$$

Theorem 9.- *Under the assumption (O), the initial data satisfies the formula*

$$\left(\nabla u(\cdot,0), \partial_t u(\cdot,0) \right) = \left(\sum_{k=0}^{+\infty} \nabla v^{(2k)}(\cdot,T), -\sum_{k=0}^{+\infty} \partial_t v^{(2k)}(\cdot,T) \right)$$

Further, there is $\beta > 0$ such that for any $K \geq 0$,

$$\begin{aligned} &\left\| \left(\sum_{k=0}^{K} \nabla v^{(2k)}(\cdot,T) - \nabla u(\cdot,0), \sum_{k=0}^{K} \partial_t v^{(2k)}(\cdot,T) + \partial_t u(\cdot,0) \right) \right\|_{(L^2(\Omega))^2} \\ &\leq e^{-\beta K} \left\| \left(\nabla u(\cdot,0), \partial_t u(\cdot,0) \right) \right\|_{(L^2(\Omega))^2} . \end{aligned}$$

Proof.

step 0.- Let $w_{(0)}(x,t) = v^{(0)}(x,t) - u(x,T-t)$. It solves

$$\left\{ \begin{array}{ll} \partial_t^2 w_{(0)} - \Delta w_{(0)} + 1_\omega \partial_t w_{(0)} = 0 & \text{in } \Omega \times (0,T) \\ w_{(0)} = 0 & \text{on } \partial\Omega \times (0,T) \end{array} \right. .$$

Under the assumption (O), we have, $w_{(0)} \equiv w^{(0)}$ (because $w_{(0)} = w^{(0)}$ in $\omega \times (0,T)$) and for some $\varepsilon \in (0,1)$,

$$E\left(w_{(0)}, T\right) \leq \varepsilon E\left(w_{(0)}, 0\right) .$$

Since $\left(w_{(0)}(\cdot,0), \partial_t w_{(0)}(\cdot,0)\right) = \left(-u(\cdot,T), \partial_t u(\cdot,T)\right)$ in Ω, we get

$$E\left(w_{(0)}, T\right) \leq \varepsilon E(u,T) = \varepsilon E(u,0) .$$

step 1.- Let $w_{(1)}(x,t) = v^{(1)}(x,t) - w_{(0)}(x,T-t)$. It solves

$$\left\{ \begin{array}{ll} \partial_t^2 w_{(1)} - \Delta w_{(1)} + 1_\omega \partial_t w_{(1)} = 0 & \text{in } \Omega \times (0,T) \\ w_{(1)} = 0 & \text{on } \partial\Omega \times (0,T) \end{array} \right. .$$

Under the assumption (O), we have, $w_{(1)} \equiv w^{(1)}$ (because $w_{(1)} = w^{(1)}$ in $\omega \times (0,T)$) and for some $\varepsilon \in (0,1)$,

$$E\left(w_{(1)}, T\right) \leq \varepsilon E\left(w_{(1)}, 0\right) .$$

Since $\left(w_{(1)}(\cdot,0), \partial_t w_{(1)}(\cdot,0)\right) = \left(-w_{(0)}(\cdot,T), \partial_t w_{(0)}(\cdot,T)\right)$ in Ω, we get

$$E\left(w_{(1)}, T\right) \leq \varepsilon E\left(w_{(0)}, T\right) .$$

Next, by induction for any $j = 2, 3, \cdots$,

step j.- Let $w_{(j)}(x,t) = v^{(j)}(x,t) - w_{(j-1)}(x,T-t)$. It solves

$$\left\{ \begin{array}{ll} \partial_t^2 w_{(j)} - \Delta w_{(j)} + 1_\omega \partial_t w_{(j)} = 0 & \text{in } \Omega \times (0,T) \\ w_{(j)} = 0 & \text{on } \partial\Omega \times (0,T) \end{array} \right. .$$

Under the assumption (O), we have, $w_{(j)} \equiv w^{(j)}$ (because $w_{(j)} = w^{(j)}$ in $\omega \times (0,T)$) and for some $\varepsilon \in (0,1)$,

$$E\left(w_{(j)}, T\right) \leq \varepsilon E\left(w_{(j)}, 0\right) .$$

Since $\left(w_{(j)}(\cdot,0), \partial_t w_{(j)}(\cdot,0)\right) = \left(-w_{(j-1)}(\cdot,T), \partial_t w_{(j-1)}(\cdot,T)\right)$ in Ω, we get

$$E\left(w_{(j)}, T\right) \leq \varepsilon E\left(w_{(j-1)}, T\right) .$$

We conclude by induction that

$$E\left(w_{(j)}, T\right) \leq \varepsilon^{j+2} E(u,0) .$$

It remains to compute $w_{(2N)}(x,t)$. Recall that

$$w_{(1)}(x,t) = v^{(1)}(x,t) - v(x,T-t) + u(x,t) ,$$

$$\begin{array}{lll} \forall j \geq 2, & w_{(j)}(x,t) & = v^{(j)}(x,t) - w_{(j-1)}(x,T-t) \\ & & = v^{(j)}(x,t) - v^{(j-1)}(x,T-t) + w_{(j-2)}(x,t) . \end{array}$$

Therefore, by induction, for any $K \geq 1$,

$$\begin{array}{ll} w_{(2K)}(x,t) & = \sum\limits_{k=2,\cdots,K} \left[v^{(2k)}(x,t) - v^{(2k-1)}(x,T-t)\right] + w_{(2)}(x,t) \\ & = \sum\limits_{k=2,\cdots,K} \left[v^{(2k)}(x,t) - v^{(2k-1)}(x,T-t)\right] + v^{(2)}(x,t) - w_{(1)}(x,T-t) \\ & = \sum\limits_{k=2,\cdots,K} \left[v^{(2k)}(x,t) - v^{(2k-1)}(x,T-t)\right] + v^{(2)}(x,t) - v^{(1)}(x,T-t) + w_{(0)}(x,t) \\ & = \sum\limits_{k=1,\cdots,K} \left[v^{(2k)}(x,t) - v^{(2k-1)}(x,T-t)\right] + v(x,t) - u(x,T-t) , \end{array}$$

$$\begin{aligned} w_{(2K+1)}(x,t) &= v^{(2K+1)}(x,t) - w_{(2K)}(x,T-t) \\ &= v^{(2K+1)}(x,t) - \sum_{k=1,\cdot\cdot,K}\left[v^{(2k)}(x,T-t) - v^{(2k-1)}(x,t)\right] - v(x,T-t) + u(x,t) \\ &= \sum_{k=1,\cdot\cdot,K}\left[v^{(2k+1)}(x,t) - v^{(2k)}(x,T-t)\right] + v^{(1)}(x,t) - v(x,T-t) + u(x,t) \ . \end{aligned}$$

In particular, at time $t=T$,

$$w_{(2K)}(x,T) = \sum_{k=1,\cdot\cdot,K} v^{(2k)}(x,T) + v(x,T) - u(x,0) \ ,$$

$$\partial_t w_{(2K)}(x,T) = \sum_{k=1,\cdot\cdot,K} \partial_t v^{(2k)}(x,T) + \partial_t v(x,T) + \partial_t u(x,0) \ .$$

This completes the proof because $w_{(2K)} \equiv w^{(2K)}$.

2.6 Observability for the wave equation

2.6.1 Multipliers method

Let $x^o \in \mathbb{R}^N$, $\Gamma(x^o) = \{x \in \partial\Omega; (x-x^o)\cdot n(x) > 0\}$ and $R(x^o) = \max_{x\in\overline{\Omega}} |x-x^o|$.

Theorem 10.- *Assume that $T > 2R(x^o)$. Then there exists $c>0$, such that any solution u of (2.2) with initial data $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, satisfies*

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq c\int_0^T \int_{\Gamma(x^o)} |\partial_n u(x,t)|^2 \, d\sigma dt \ .$$

Proof.-

Step 1.- We apply the identity (2.3) in the proof of Proposition 3 with $H(x) = x - x^o$ in order to get

$$\begin{aligned} &\tfrac{N}{2}\int_0^T \int_\Omega \left(|\partial_t u|^2 - |\nabla u|^2\right) + \int_0^T \int_\Omega |\nabla u|^2 + \left[\int_\Omega \partial_t u\,(x-x^o)\cdot\nabla u\right]_0^T \\ &= \tfrac{1}{2}\int_0^T \int_{\partial\Omega} (x-x^o)\cdot n\,|\partial_n u|^2 \\ &\leq \tfrac{R(x^o)}{2}\int_0^T \int_{\Gamma(x^o)} |\partial_n u|^2 \ . \end{aligned}$$

Step 2.- We multiply the equation $\partial_t^2 u - \Delta u = 0$ by u and integrate over $\Omega\times(0,T)$. It comes by integrations by parts

$$\int_0^T \int_\Omega \left(|\partial_t u|^2 - |\nabla u|^2\right) = \left[\int_\Omega \partial_t u u\right]_0^T \ .$$

Consequently,

$$\begin{aligned} &\tfrac{N}{2}\left[\int_\Omega \partial_t u u\right]_0^T + \tfrac{1}{2}\int_0^T \int_\Omega |\nabla u|^2 + \frac{1}{2}\int_0^T \int_\Omega |\partial_t u|^2 - \frac{1}{2}\left[\int_\Omega \partial_t u u\right]_0^T + \left[\int_\Omega \partial_t u\,(x-x^o)\cdot\nabla u\right]_0^T \\ &\leq \tfrac{R(x^o)}{2}\int_0^T \int_{\Gamma(x^o)} |\partial_n u|^2 \ . \end{aligned}$$

From the conservation of energy, we finally get

$$
\begin{aligned}
&T \left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \\
&\leq R\left(x^o\right)\int_0^T\int_{\Gamma(x^o)}\left|\partial_n u\right|^2-(N-1)\left[\int_\Omega \partial_t u u\right]_0^T-2\left[\int_\Omega \partial_t u\left(x-x^o\right)\cdot\nabla u\right]_0^T \\
&\leq R\left(x^o\right)\int_0^T\int_{\Gamma(x^o)}\left|\partial_n u\right|^2+\left|\left[2\int_\Omega \partial_t u\left(\frac{N-1}{2}u+\left(x-x^o\right)\cdot\nabla u\right)\right]_0^T\right| \\
&\leq R\left(x^o\right)\int_0^T\int_{\Gamma(x^o)}\left|\partial_n u\right|^2 \\
&\quad+\left|2\int_\Omega \partial_t u\left(\tfrac{N-1}{2}u+\left(x-x^o\right)\cdot\nabla u\right)|_{t=T}\right|+\left|2\int_\Omega \partial_t u\left(\frac{N-1}{2}u+\left(x-x^o\right)\cdot\nabla u\right)|_{t=0}\right| .
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left|2\int_\Omega \partial_t u\left(\frac{N-1}{2}u+\left(x-x^o\right)\cdot\nabla u\right)\right| \\
&\leq R\left(x^o\right)\int_\Omega\left|\partial_t u\right|^2+\frac{1}{R\left(x^o\right)}\int_\Omega\left|\frac{N-1}{2}u+\left(x-x^o\right)\cdot\nabla u\right|^2 \\
&\leq R\left(x^o\right)\int_\Omega\left|\partial_t u\right|^2+\frac{1}{R\left(x^o\right)}\left(\left(\frac{N-1}{2}\right)^2\int_\Omega|u|^2+\int_\Omega\left|\left(x-x^o\right)\cdot\nabla u\right|^2+(N-1)\int_\Omega u\left(x-x^o\right)\cdot\nabla u\right) \\
&\leq R\left(x^o\right)\int_\Omega\left|\partial_t u\right|^2+\frac{1}{R\left(x^o\right)}\left(\left(\frac{N-1}{2}\right)^2\int_\Omega|u|^2+\int_\Omega\left|\left(x-x^o\right)\cdot\nabla u\right|^2+\frac{N-1}{2}\int_\Omega\nabla\left(u^2\right)\left(x-x^o\right)\right) \\
&\leq R\left(x^o\right)\int_\Omega\left|\partial_t u\right|^2+\frac{1}{R\left(x^o\right)}\left(\left(\frac{N-1}{2}\right)^2\int_\Omega|u|^2+\int_\Omega\left|\left(x-x^o\right)\cdot\nabla u\right|^2-\frac{N\left(N-1\right)}{2}\int_\Omega|u|^2\right) \\
&\leq R\left(x^o\right)\left(\int_\Omega\left|\partial_t u\right|^2+\int_\Omega\left|\nabla u\right|^2\right)\quad\text{because }\left(\tfrac{N-1}{2}\right)^2-\tfrac{N(N-1)}{2}\leq 0 . \\
&\leq R\left(x^o\right)\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}
\end{aligned}
$$

Therefore,

$$T\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\leq R\left(x^o\right)\int_0^T\int_{\Gamma(x^o)}\left|\partial_n u\right|^2+2R\left(x^o\right)\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}$$

which implies, for $T>2R\left(x^o\right)$, the existence of $c>0$ such that

$$\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\leq c\int_0^T\int_{\Gamma(x^o)}\left|\partial_n u\right|^2 \ .$$

Theorem 11.- *Assume that $T>2R\left(x^o\right)$. Then there exists $c>0$ such that any solution u of (2.2) with initial data $(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega)$, satisfies*

$$\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\leq c\int_0^T\int_\omega\left|\partial_t u\left(x,t\right)\right|^2dxdt \ ,$$

where $\omega=\vartheta\left(x^o\right)\cap\Omega$ and $\vartheta\left(x^o\right)$ is a neighborhood of $\Gamma\left(x^o\right)$ in $\mathbb{R}^N$.

The choice of T may be improved, depending on $\vartheta\left(x^o\right)$.

Step 1.- If $\mu>0$ is such that $T-2\mu>2R\left(x^o\right)$, then we also have

$$\left\|(u_0,u_1)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\leq c\int_\mu^{T-\mu}\int_{\Gamma(x^o)}\left|\partial_n u\right|^2 \ .$$

Step 2.- It remains to prove that

$$\int_{\mu}^{T-\mu}\int_{\Gamma(x^o)}\left|\partial_n u\right|^2 \leq c\int_0^T\int_{\vartheta(x^o)\cap\Omega}\left|\partial_t u\left(x,t\right)\right|^2 + d\left\|\left(u_0,u_1\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ ,$$

by reproducing a identity like (2.2) with a suitable $H\left(x,t\right)\in C^1\left(\overline{\Omega}\times\left(0,T\right)\right)$ such that $H\left(x,\mu\right)=H\left(x,T-\mu\right)=0$. We will also need to check that

$$\int_0^T\int_{\vartheta(x^o)\cap\Omega}\left|\Phi\left(t\right)\nabla u\left(x,t\right)\right|^2 dxdt \leq c\int_0^T\int_{\vartheta(x^o)\cap\Omega}\left|\partial_t u\left(x,t\right)\right|^2 dxdt + d\left\|\left(u_0,u_1\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \ ,$$

for some suitable $\Phi\in C_0^\infty\left(0,T\right)$.

Step 3.- We conclude by using the uniqueness-compactness argument.

2.6.2 Geometric control condition [BLR]

Preliminary definitions We begin to recall some definitions.

Definitions.- Let $P\left(x,D\right)$ be a differential operator in $\mathbb{R}^N$ with a real principal symbol $p\left(x,\xi\right)$. The Hamiltonian vector field of p is given by

$$H_p\left(x,\xi\right)=\left(\frac{\partial p}{\partial\xi_1}\left(x,\xi\right),\cdots,\frac{\partial p}{\partial\xi_N}\left(x,\xi\right);-\frac{\partial p}{\partial x_1}\left(x,\xi\right),\cdots,-\frac{\partial p}{\partial x_N}\left(x,\xi\right)\right) \ .$$

A Hamiltonian curve of p is an integral curve of H_p, that is a curve

$$\gamma\left(s\right)=\left(\begin{array}{c} x\left(s\right)\\ \xi\left(s\right)\end{array}\right) \qquad \text{with } \frac{d}{ds}\gamma\left(s\right)=H_p\gamma\left(s\right) \ .$$

$x\left(s\right)$ and $\xi\left(s\right)$ are solutions of the system of ordinary differential equations

$$\left\{\begin{array}{l} \frac{d}{ds}x\left(s\right)=\nabla_\xi p\left(x\left(s\right),\xi\left(s\right)\right) \ ,\\ \frac{d}{ds}\xi\left(s\right)=-\nabla_x p\left(x\left(s\right),\xi\left(s\right)\right) \ .\end{array}\right.$$

The integral curve on which $p\left(x\left(s\right),\xi\left(s\right)\right)\equiv 0$, are called bicharacteristic of p. $\gamma\left(s\right)$ may be written as $e^{sH_p}\gamma\left(0\right)$.

Application to the wave equation.- $P=\partial_t^2-\Delta$, its principal symbol is $p\left(x,t,\xi,\tau\right)=\left|\xi\right|^2-\tau^2$. The bicharacteristics associated to the wave equations are

$$\left\{\begin{array}{ll} \frac{d}{ds}x_j\left(s\right)=2\xi_j\left(s\right) \ , & \text{for } j=1,\cdots,N \ ,\\ \frac{d}{ds}t\left(s\right)=-2\tau\left(s\right) \ , & \\ \frac{d}{ds}\xi_j\left(s\right)=0 \ , & \text{for } j=1,\cdots,N \ ,\\ \frac{d}{ds}\tau\left(s\right)=0 \ , & \\ \left|\xi\left(s\right)\right|^2-\tau\left(s\right)^2=0 \ , & \end{array}\right.$$

which gives for any initial data $\left(x^o,t^o,\xi^o,\tau^o\right)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\setminus\{0\}$ such that $\left|\xi^o\right|^2=\left(\tau^o\right)^2$,

$$\left\{\begin{array}{l} x\left(s\right)=2\xi^o s+x^o \ ,\\ t\left(s\right)=-2\tau^o s+t^o \ ,\\ \xi\left(s\right)=\xi^o \ ,\\ \tau\left(s\right)=\tau^o \ .\end{array}\right.$$

We end by giving an intuitive definition of a ray of geometric optics.

Definition.- A generalyzed bicharacteristic ray associated to $\partial_t^2 - \Delta$ is a continuous trajectory

$$s \longmapsto \left(\begin{array}{c} x(s) \\ t(s) \end{array} \right) \text{ satisfying } x(s) \in \overline{\Omega},\ t(s) = s \text{ and}$$

when $x(t) \in \Omega$ then it propagates in straight line with speed one until it hits $\partial\Omega$ at time t_0. If it hits $\partial\Omega$ at time t_0 transversally, then the reflection off the boundary is subject to the optic geometric rules (the angle of incidence equals the angle of reflection) as light rays or billiard balls. If it hits $\partial\Omega$ at time t_0 tangentially, then either there exists another trajectory $\widetilde{x}(t)$ such that $x(t_0) = \widetilde{x}(t_0)$ and $\frac{d}{dt}x(t_0) = \frac{d}{dt}\widetilde{x}(t_0)$ living in Ω and then $x(t) = \widetilde{x}(t)$ for $t > t_0$ until it hits the boundary; or there is no such kind of $\widetilde{x}$ and then it glides along the boundary until it may branch onto a trajectory $\widetilde{x}(t)$ in Ω.

The existence of a unique generalized bicharacteristic ray holds under one of the following assumptions: $\partial\Omega$ is analytic; $\partial\Omega$ is C^∞ and $\partial\Omega$ has no contacts of infinite order with its tangents; $\partial\Omega$ is C^k for some integer $k \geq 3$ and $\partial\Omega$ has no contacts of order $k-1$ with its tangents.

Necessary and sufficient condition Here, we begin to describe the result of [BG].

Let $\Theta \in C_0(\partial\Omega \times (0,T))$ a compactly supported continuous function. We say that Θ exactly controls Ω if for any $(v_0, v_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists $g \in L^2(\partial\Omega \times \mathbb{R})$ such that the solution of

$$\left\{ \begin{array}{l} \partial_t^2 v - \Delta v = 0 \quad \text{in } \Omega \times \mathbb{R} , \\ v = \Theta g \quad \text{on } \partial\Omega \times \mathbb{R} , \\ (v(\cdot,0), \partial_t v(\cdot,0)) = (v_0, v_1) \quad \text{in } \Omega , \end{array} \right.$$

satisfies $v_{|t \geq T} \equiv 0$. We say that Θ geometrically controls Ω if any generalized bicharacteristic ray meets the set $\{(x,t) \in \partial\Omega \times \mathbb{R}; \Theta(x,t) \neq 0\}$ on a non-diffractive point.

Theorem ([BG]).- *Assume that Ω is of class C^∞ and $\partial\Omega$ has no contacts of infinite order with its tangents. Then the following statements are equivalent.*

i) the function Θ exactly controls Ω ;

ii) the function Θ geometrically controls Ω ;

iii) there exists $c > 0$ such that

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq c \int_{\partial\Omega \times \mathbb{R}} |\Theta \partial_n u(x,t)|^2 \, d\sigma dt ,$$

for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ where u is the solution of (2.2).

Previously, Bardos, Lebeau and Rauch proved that

Theorem ([BLR]).- *Assume that Ω is of class C^∞ and $\partial\Omega$ has no contacts of infinite order with its tangents. Let $T > 0$ and $\Gamma \subset \partial\Omega$ be a nonempty subset of $\partial\Omega$ such that any generalized bicharacteristic ray meets $\Gamma \times (0,T)$ in a non-diffractive point, then there exists $c > 0$ such that*

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq c \int_0^T \int_\Gamma |\partial_n u(x,t)|^2 \, d\sigma dt ,$$

for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ where u is the solution of (2.2).

A similar result holds for internal observability.

Theorem ([BLR]).- *Assume that Ω is of class C^{∞} and $\partial\Omega$ has no contacts of infinite order with its tangents. Let $T > 0$ and $\omega \subset \Omega$ be a nonempty open subset of Ω such that any generalized bicharacteristic ray meets $\omega \times (0,T)$, then there exists $c > 0$ such that*

$$\left\| (u_0, u_1) \right\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq c \int_0^T \int_\omega |u(x,t)|^2 \, dxdt \ ,$$

for any $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ where u is the solution of (2.2).

2.7 Comments on BLR's geometric control condition

There are in the literature two ways to get an observability estimate for the wave equation in a bounded open set Ω in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$:

- multipliers method [Li] (see also [Li2])-[K] ($\partial\Omega$ of class C^2);
- geometric optics [BLR] (see also [Le])-[BG] ($\partial\Omega$ of class C^{∞}, later reduced to C^3 domains by Burq).

The above two techniques come from scattering problems (study of hyperbolic systems in exterior domains) (see e.g. PhD of Pauen). The multiplier techniques can be seen as a generalization of the Morawetz energy method. The geometric optic techniques are based on microlocal analysis and the theorem of propagation of singularities of Melrose and Sjöstrand which allows to answer the conjecture of Lax and Phillips. More recently, using defect measure, Burq and Gérard established that the geometric control condition of Bardos, Lebeau and Rauch is a necessary and sufficient condition for the exact controllability of the wave equations with Dirichlet boundary conditions.

When no geometric condition is required, Robbiano [R] proves a quantitative unique continuation estimate for hyperbolic equations from a local Carleman inequality for elliptic operators and a Fourier-Bros-Iagolnitzer transform. Then, the cost of the approximate controllability for hyperbolic equations is deduced. Application to boundary stabilization without geometric control condition is established in [LR]. The optimal result without geometrical hypothesis is given in [B].

2.8 Comments on approximate controllability

The approximate controllability can be established by a variational approach as follows. Let $\varepsilon > 0$ and $(z_0, z_1) \in H^1_0(\Omega) \times L^2(\Omega)$. By time-reversibility, we are reduced to look for a control function $f_\varepsilon \in L^2(\omega \times (0,T))$ such that

$$\left\| (v(\cdot,T), \partial_t v(\cdot,T)) - (z_0, z_1) \right\|_{H^1_0(\Omega) \times L^2(\Omega)} \leq \varepsilon \ ,$$

where

$$\begin{cases} \partial_t^2 v - \Delta v = f_{\varepsilon | \omega \times (0,T)} & \text{in } \Omega \times \mathbb{R} \ , \\ v = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (v(\cdot,0), \partial_t v(\cdot,0)) = (0,0) & \text{in } \Omega \ . \end{cases}$$

Introduce the functional $\mathcal{J}_\varepsilon : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$, defined by

$$\mathcal{J}_\varepsilon(u_0, u_1) = \frac{1}{2} \int_0^T \int_\omega |u|^2 \, dxdt + \varepsilon \left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} - \left\langle (u_0, u_1), (z_0, z_1) \right\rangle \ ,$$

where u is the the solution of (2.2) with initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. Then under *UCP*, the functional $\mathcal{J}_\varepsilon$ has a minimizer $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $f_\varepsilon = \varphi_{|\omega \times (0,T)}$ is an approximate control which leads the solution of

$$\begin{cases} \partial_t^2 v - \Delta v = f_{\varepsilon|\omega\times(0,T)} \quad \text{in } \Omega \times \mathbb{R} \ , \\ v = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (v(\cdot, 0), \partial_t v(\cdot, 0)) = (0, 0) \quad \text{in } \Omega \ . \end{cases}$$

to $(v(\cdot, T), \partial_t v(\cdot, T))$ such that

$$\|(v(\cdot, T), \partial_t v(\cdot, T)) - (z_0, z_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon \ ,$$

where φ is the solution of

$$\begin{cases} \partial_t^2 \varphi - \Delta \varphi = 0 \quad \text{in } \Omega \times \mathbb{R} \ , \\ \varphi = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \ , \\ (\varphi(\cdot, 0), \partial_t \varphi(\cdot, 0)) = (\varphi_0, \varphi_1) \quad \text{in } \Omega \ . \end{cases}$$

2.9 Comments on weak stabilization

...(to be completed)...

3 The heat equation in bounded domain of $\mathbb{R}^d$

We consider the heat equation in the solution $u = u(x,t)$

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \ , \\ u(\cdot,0) \in L^2(\Omega) & , \end{cases}$$

living in a bounded open set Ω in $\mathbb{R}^d$, $d \geq 1$, either convex or C^2 and connected, with boundary $\partial\Omega$. It is well-known that the above problem is well-posed and have a unique solution $u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$ for all $T > 0$.

3.1 Wellposedness

3.1.1 Spectral theory

Properties of the eigenvalues of the operator $-\Delta$ on the open, connected bounded domain Ω with dirichlet boundary conditions, are stated in the theorem below.

Theorem (Eigenvalues of the Laplace operator) .- The boundary value problem

$$\begin{cases} -\Delta u = \lambda u \quad \text{in } \Omega \ , \\ u = 0 \quad \text{on } \partial\Omega \ , \end{cases}$$

has a nontrivial solution $u \neq 0$ if and only if $\lambda \in \Sigma$, in which case λ is called an eigenvalue of $-\Delta$, u a corresponding eigenfunction.

(i) Each eigenvalues of $-\Delta$ is real.

(ii) Furthermore, if we repeat each eigenvalue according to its (finite) multiplicity, we have $\Sigma = \{\lambda_j\}_{j=1,2,...}$ where

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \ ; \\ \quad \lambda_j \to \infty \text{ as } j \to \infty \ . \end{cases}$$

(iii) Finally, there exists an orthonormal basis $\{e_j\}_{j=1,2,...}$ of $L^2(\Omega)$, where $e_j \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_j:

$$\begin{cases} -\Delta e_j = \lambda_j e_j \quad \text{in } \Omega \ , \\ e_j = 0 \quad \text{on } \partial\Omega \ , \end{cases}$$

for $j = 1, 2, ...$

When $u_0 \in L^2(\Omega)$ and $u_0(x) = \sum_{j\geq 1} a_j e_j(x)$ with $\sum_{j\geq 1} |a_j|^2 < +\infty$, then

$$u(x,t) = \sum_{j\geq 1} a_j e^{-\lambda_j t} e_j(x)$$

solves

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \ , \\ u(\cdot,0) = u_0 & \text{in } \Omega \ . \end{cases}$$

3.1.2 Semigroup theory

Let A be the operator on $L^2(\Omega)$ defined by

$$\begin{cases} D(A) = \left\{u_0 \in H_0^1(\Omega)\,;\Delta u_0 \in L^2(\Omega)\right\} \\ Au_0 = -\Delta u_0 \quad \forall u_0 \in D(A)\ . \end{cases}$$

When the bounded open set Ω in $\mathbb{R}^d$, $d \geq 1$, is either convex or C^2 and connected, with boundary $\partial\Omega$, then $D(A) = H^2 \cap H_0^1(\Omega)$ and A is maximal monotone on $L^2(\Omega)$. By Hille-Yosida theorem, $-A$ is the generator of a semigroup of contractions on $L^2(\Omega)$, which we denote by $\left(e^{-tA}\right)_{t\geq 0}$. If $u_0 \in L^2(\Omega)$, then the solution $u(\cdot,t) = e^{-tA}u_0 = e^{t\Delta}u_0$ solves the heat equation and $u \in C\left([0,T]\,;L^2(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right)$ for all $T > 0$. Moreover, if $u_0 \in H^2 \cap H_0^1(\Omega)$, then the equations

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,+\infty)\ , \\ u(\cdot,0) = u_0 & \text{in } \Omega\ , \end{cases}$$

are satisfied in $L^2(\Omega)$ for any $t > 0$ and $u \in C^1\left([0,T]\,;L^2(\Omega)\right) \cap C\left([0,T]\,;H^2 \cap H_0^1(\Omega)\right)$ for all $T > 0$.

3.2 Energy estimates

The following inequalities hold

$$\int_\Omega |u(\cdot,t)|^2\,dx \leq \int_\Omega |u_0|^2\,dx \text{ and } \int_\Omega |\nabla u(\cdot,t)|^2\,dx \leq \frac{1}{2t}\int_\Omega |u_0|^2\,dx\ .$$

The $L^p - L^q$ regularization effect for the heat equation gives: for all $t > 0$ and $1 \leq p \leq q \leq +\infty$,

$$\left\|e^{t\Delta}u_0\right\|_{L^q(\Omega)} \leq (4\pi t)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|u_0\right\|_{L^p(\Omega)} \qquad \forall u_0 \in L^p(\Omega)\ .$$

3.3 Backward estimates

3.3.1 Initial data in $H_0^1(\Omega)$

We look for an estimate of the form

$$\|u(\cdot,0)\|^2_{L^2(\Omega)} \leq constant\ \ \|u(\cdot,T)\|^2_{L^2(\Omega)}\ ,$$

where u solves the heat equation and $T > 0$. Here the "constant" will depend on initial data. This is a backward estimate for the heat equation. To do this, we apply the ideas in [BT] : Let us consider for almost all $t \in [0,T]$ such that $u(x,t) \neq 0$ the following quantity :

$$\Phi(t) = \frac{\|u(\cdot,t)\|^2_{H_0^1(\Omega)}}{\|u(\cdot,t)\|^2_{L^2(\Omega)}}\ .$$

We first prove that

$$\frac{d}{dt}\Phi(t) \leq 0\ .$$

Indeed we have the two following energy equalities:

$$\frac{1}{2}\frac{d}{dt}\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}+\left\|u\left(\cdot,t\right)\right\|_{H_{0}^{1}(\Omega)}^{2}=0\ ,$$

$$\frac{1}{2}\frac{d}{dt}\left\|u\left(\cdot,t\right)\right\|_{H_{0}^{1}(\Omega)}^{2}+\left\|\Delta u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}=0\ .$$

Then we get

$$\frac{d}{dt}\Phi\left(t\right)=\frac{2}{\left\|u\right\|_{L^{2}(\Omega)}^{4}}\left(-\left\|\Delta u\right\|_{L^{2}(\Omega)}^{2}\left\|u\right\|_{L^{2}(\Omega)}^{2}+\left\|\nabla u\right\|_{L^{2}(\Omega)}^{4}\right)\ .$$

By Cauchy-Schwarz, we get the desired inequality. Consequently, we obtain for all $t\in(0,T)$,

$$\Phi\left(t\right)\leq\Phi\left(0\right)\ .$$

Secondly, remark that

$$\begin{array}{ll}0 & =\frac{1}{2}\frac{d}{dt}\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}+\Phi\left(t\right)\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}\\ & \leq\frac{1}{2}\frac{d}{dt}\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}+\Phi\left(0\right)\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}\ .\end{array}$$

Consequently, integrating on $(0,T)$, we get the desired estimate :

$$\left\|u\left(\cdot,0\right)\right\|_{L^{2}(\Omega)}\leq\exp\left(\Phi\left(0\right)T\right)\left\|u\left(\cdot,T\right)\right\|_{L^{2}(\Omega)}\ .$$

3.3.2 Initial data in $L^{2}\left(\Omega\right)$

We look for an estimate of the form

$$\left\|u\left(\cdot,0\right)\right\|_{H^{-1}(\Omega)}^{2}\leq constant\ \left\|u\left(\cdot,T\right)\right\|_{H^{-1}(\Omega)}^{2}\ ,$$

where u solves the heat equation and $T>0$. Here the "constant" will depend on initial data. This is a backward estimate for the heat equation. To do this, we apply the ideas in [BT] : Let us consider for almost all $t\in[0,T]$ such that $u\left(x,t\right)\neq0$ the following quantity :

$$\Psi\left(t\right)=\frac{\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}}{\left\|u\left(\cdot,t\right)\right\|_{H^{-1}(\Omega)}^{2}}\ .$$

We first prove that

$$\frac{d}{dt}\Psi\left(t\right)\leq0\ .$$

Indeed we have the two following energy equalities:

$$\frac{1}{2}\frac{d}{dt}\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}+\left\|u\left(\cdot,t\right)\right\|_{H_{0}^{1}(\Omega)}^{2}=0\ ,$$

$$\frac{1}{2}\frac{d}{dt}\left\|u\left(\cdot,t\right)\right\|_{H^{-1}(\Omega)}^{2}+\left\|u\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}=0\ .$$

Then we get

$$\frac{d}{dt}\Psi\left(t\right)=\frac{2}{\left\|u\right\|_{H^{-1}(\Omega)}^{4}}\left(-\left\|u\right\|_{H_{0}^{1}(\Omega)}^{2}\left\|u\right\|_{H^{-1}(\Omega)}^{2}+\left\|u\right\|_{L^{2}(\Omega)}^{4}\right)\ .$$

By Cauchy-Schwarz, we get the desired inequality. Consequently, we obtain for all $t\in(0,T)$,

$$\Psi\left(t\right)\leq\Psi\left(0\right)\ .$$

Secondly, remark that

$$\begin{aligned} 0 &= \tfrac{1}{2}\tfrac{d}{dt}\left\|u(\cdot,t)\right\|_{H^{-1}(\Omega)}^2 + \Psi(t)\left\|u(\cdot,t)\right\|_{H^{-1}(\Omega)}^2 \\ &\leq \tfrac{1}{2}\tfrac{d}{dt}\left\|u(\cdot,t)\right\|_{H^{-1}(\Omega)}^2 + \Psi(0)\left\|u(\cdot,t)\right\|_{H^{-1}(\Omega)}^2 \ . \end{aligned}$$

Consequently, integrating on $(0,T)$, we get the desired estimate :

$$\left\|u(\cdot,0)\right\|_{H^{-1}(\Omega)} \leq \exp\left(\Psi(0)\,T\right)\left\|u(\cdot,T)\right\|_{H^{-1}(\Omega)} \ .$$

3.3.3 Application: Nash inequality

Let $f \in H_0^1(\Omega)$. Consider the heat equation with Dirichlet boundary condition and initial data f. The backward estimate says that for any $t > 0$,

$$\left\|f\right\|_{L^2(\Omega)} \leq \exp\left(\frac{\left\|f\right\|_{H_0^1(\Omega)}^2}{\left\|f\right\|_{L^2(\Omega)}^2}t\right)\left\|e^{t\Delta}f\right\|_{L^2(\Omega)} \ .$$

The $L^p - L^q$ regularization for the heat equation gives when $p = 1$ and $q = 2$,

$$\left\|e^{t\Delta}f\right\|_{L^2(\Omega)} \leq (4\pi t)^{-\frac{d}{4}}\left\|f\right\|_{L^1(\Omega)} \ .$$

Therefore, for any $t > 0$,

$$\left\|f\right\|_{L^2(\Omega)} \leq \exp\left(\frac{\left\|f\right\|_{H_0^1(\Omega)}^2}{\left\|f\right\|_{L^2(\Omega)}^2}t\right)(4\pi t)^{-\frac{d}{4}}\left\|f\right\|_{L^1(\Omega)} \ .$$

Now, choose $t = \frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{H_0^1(\Omega)}^2}$. Then,

$$\left\|f\right\|_{L^2(\Omega)} \leq \left(\frac{1}{4\pi}\frac{\left\|f\right\|_{H_0^1(\Omega)}^2}{\left\|f\right\|_{L^2(\Omega)}^2}\right)^{\frac{d}{4}} e\left\|f\right\|_{L^1(\Omega)}$$

that is the Nash inequality

$$\left\|f\right\|_{L^2(\Omega)} \leq \left(e\left\|f\right\|_{L^1(\Omega)}\right)^{\frac{2}{2+d}}\left(\frac{1}{2\sqrt{\pi}}\left\|f\right\|_{H_0^1(\Omega)}\right)^{\frac{d}{2+d}} \ .$$

3.4 Logarithmic convexity method

Recall that if $t \mapsto \log f$ is a convex function, then for any $t_1 < t_2$ and any $\theta \in (0,1)$,

$$\log f\left(\theta t_2 + (1-\theta)t_1\right) \leq \theta \log f(t_2) + (1-\theta)\log f(t_1)$$

which implies

$$f\left(\theta t_2 + (1-\theta)t_1\right) \leq \left[f(t_2)\right]^{\theta}\left[f(t_1)\right]^{1-\theta} \ .$$

Further, when $f \in C^2$, $\log f$ is a convex function if and only if $(\log f)'' = \frac{f''f - (f')^2}{f^2} \geq 0$.

Now take $t=\theta T$, $t_1=0$, $t_2=T$ and $f(t)=\int_\Omega |u(x,t)|^2 dx$. Therefore,

$$\begin{aligned}(\log f(t))'' &= \frac{1}{f^2}\left[4\int_\Omega |\Delta u(x,t)|^2 dx \int_\Omega |u(x,t)|^2 dx - \left(-2\int_\Omega |\nabla u(x,t)|^2 dx\right)^2\right] \\ &\geq 0 \quad \text{by Cauchy-Schwarz inequality,}\end{aligned}$$

which implies

$$\int_\Omega |u(x,t)|^2 dx \leq \left[\int_\Omega |u(x,T)|^2 dx\right]^{t/T} \left[\int_\Omega |u(x,0)|^2 dx\right]^{1-t/T} .$$

This estimate has similar form than a quantitative Hölder continuous dependence.

3.5 Observability

Denote $u(x,t)=e^{t\Delta}u_0(x)$ the solution of the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \ , \\ u(\cdot,0) = u_0 \in L^2(\Omega) \ . & \end{cases}$$

Theorem .- *Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geq 1$, either convex or C^2 and connected. Let $\omega \subset \Omega$ be a non-empty open subset. Then, there exist $C>0$ and $\mu \in (0,1)$ such that for any $u_0 \in L^2(\Omega)$ and any $T>0$,*

i)

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \frac{1}{T} e^{C\left(1+\frac{1}{T}\right)} \left\|e^{t\Delta}u_0\right\|_{L^1(\omega\times(0,T))} .$$

ii)

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq e^{C\left(1+\frac{1}{T}\right)} \left\|e^{T\Delta}u_0\right\|^{\mu}_{L^1(\omega)} \left\|u_0\right\|^{1-\mu}_{L^2(\Omega)} .$$

iii)

$$\left\|u_0\right\|_{L^2(\Omega)} \leq \frac{1}{T^{1/(2\mu)}} e^{C\left(1+\frac{1}{T}+T\left(\frac{\|u_0\|_{L^2(\Omega)}}{\|u_0\|_{H^{-1}(\Omega)}}\right)^2\right)} \left\|e^{T\Delta}u_0\right\|_{L^1(\omega)} \text{ , when } u_0 \neq 0 .$$

Remark 1 .- The above theorem describes three type of estimate which give a quantitative uniqueness result for the heat equation: *i)* is an observability estimate; *ii)* is a Hölder estimate; *iii)* is a logarithmic estimate.

Remark 2 .- *i)* is a direct consequence of *ii)* by the telescoping series method. We will give a very simple and self-contained proof of *ii)* when Ω is convex. Another proof of *ii)* can be done using a uniqueness result for elliptic equation and a standard transformation. *iii)* is obtained by combining *ii)* and the backward estimate

$$\left\|u_0\right\|_{H^{-1}(\Omega)} \leq \exp\left(\frac{\left\|u_0\right\|^2_{L^2(\Omega)}}{\left\|u_0\right\|^2_{H^{-1}(\Omega)}} T\right) \left\|e^{T\Delta}u_0\right\|_{H^{-1}(\Omega)} .$$

More recently, in [AEWZ] the authors show that the above theorem still holds when Ω is a bounded lipschitz and locally star-shaped domain and $\omega \subset \Omega$ is a measurable set of positive measure.

3.5.1 Observability estimate (proof of $ii) \Rightarrow i)$)

Observe that

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq e^{C\left(1+\frac{1}{T}\right)} \left\|e^{T\Delta}u_0\right\|^{\mu}_{L^1(\omega)} \left\|u_0\right\|^{1-\mu}_{L^2(\Omega)}$$

implies by Young's inequality,

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon^{(1-\mu)/\mu}} e^{C\left(1+\frac{1}{T}\right)/\mu} \left\|e^{T\Delta}u_0\right\|_{L^1(\omega)} + \varepsilon \left\|u_0\right\|_{L^2(\Omega)} \quad \forall \varepsilon > 0 \ .$$

Let $z > 1$. Introduce the decreasing sequence $\{\ell_m\}_{m\geq 1}$, which converges to 0, given by $\ell_{m+1} = T/z^m$. Thus

$$\ell_m - \ell_{m+1} = \frac{1}{z^m}(z-1)T > 0 \ .$$

We start with the following interpolation estimate which is given by $ii)$ and the above observation. There exist three constants $C_1, C_2, \gamma > 0$ such that for any $0 \leq t_1 < t_2 \leq T$,

$$\left\|u(\cdot,t_2)\right\|_{L^2(\Omega)} \leq \frac{C_1}{\varepsilon^\gamma} e^{\frac{C_2}{t_2-t_1}} \left\|u(\cdot,t_2)\right\|_{L^1(\omega)} + \varepsilon \left\|u(\cdot,t_1)\right\|_{L^2(\Omega)} \quad \forall \varepsilon > 0 \ .$$

Let $0 < \ell_{m+2} < \ell_{m+1} \leq t < \ell_m < T$. By applying the above estimate with $t_2 = t$ and $t_1 = \ell_{m+2}$, we get

$$\left\|u(\cdot,t)\right\|_{L^2(\Omega)} \leq \frac{C_1}{\varepsilon^\gamma} e^{\frac{C_2}{t-\ell_{m+2}}} \left\|u(\cdot,t)\right\|_{L^1(\omega)} + \varepsilon \left\|u(\cdot,\ell_{m+2})\right\|_{L^2(\Omega)} \quad \forall \varepsilon > 0 \ .$$

Recall that

$$\left\|u(\cdot,\ell_m)\right\|_{L^2(\Omega)} \leq \left\|u(\cdot,t)\right\|_{L^2(\Omega)} \ .$$

Therefore,

$$\left\|u(\cdot,\ell_m)\right\|_{L^2(\Omega)} \leq \frac{C_1}{\varepsilon^\gamma} e^{\frac{C_2}{t-\ell_{m+2}}} \left\|u(\cdot,t)\right\|_{L^1(\omega)} + \varepsilon \left\|u(\cdot,\ell_{m+2})\right\|_{L^2(\Omega)} \quad \forall \varepsilon > 0 \ .$$

Integrating it over $t \in (\ell_{m+1}, \ell_m)$, it gives

$$\begin{aligned}(\ell_m - \ell_{m+1})\left\|u(\cdot,\ell_m)\right\|_{L^2(\Omega)} \ &\leq \tfrac{C_1}{\varepsilon^\gamma} e^{\frac{C_2}{\ell_{m+1}-\ell_{m+2}}} \int_{\ell_{m+1}}^{\ell_m} \left\|u(\cdot,t)\right\|_{L^1(\omega)} dt \\ &\quad +\varepsilon(\ell_m - \ell_{m+1})\left\|u(\cdot,\ell_{m+2})\right\|_{L^2(\Omega)} \quad \forall \varepsilon > 0 \ .\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|u(\cdot,\ell_m)\right\|_{L^2(\Omega)} \ &\leq \tfrac{1}{\varepsilon^\gamma} \tfrac{1}{z} \tfrac{C_1}{C_2} e^{2C_2\left[\frac{1}{T}\frac{z^{m+1}}{z-1}\right]} \int_{\ell_{m+1}}^{\ell_m} \left\|u(\cdot,t)\right\|_{L^1(\omega)} dt \\ &\quad +\varepsilon \left\|u(\cdot,\ell_{m+2})\right\|_{L^2(\Omega)} \quad \forall \varepsilon > 0 \ .\end{aligned}$$

Denote $\sigma = 2C_2\left[\frac{1}{T}\frac{1}{z(z-1)}\right]$. It gives

$$\begin{aligned}&\varepsilon^\gamma e^{-\sigma z^{m+2}} \left\|u(\cdot,\ell_m)\right\|_{L^2(\Omega)} - \varepsilon^{1+\gamma} e^{-\sigma z^{m+2}} \left\|u(\cdot,\ell_{m+2})\right\|_{L^2(\Omega)} \\ &\leq \tfrac{1}{z}\tfrac{C_1}{C_2} \int_{\ell_{m+1}}^{\ell_m} \left\|u(\cdot,t)\right\|_{L^1(\omega)} dt \quad \forall \varepsilon > 0 \ .\end{aligned}$$

Take $\varepsilon = e^{-\sigma z^{m+2}}$, then

$$\begin{aligned}&e^{-(\gamma+1)\sigma z^{m+2}} \left\|u(\cdot,\ell_m)\right\|_{L^2(\Omega)} - e^{-(2+\gamma)\sigma z^{m+2}} \left\|u(\cdot,\ell_{m+2})\right\|_{L^2(\Omega)} \\ &\leq \tfrac{1}{z}\tfrac{C_1}{C_2} \int_{\ell_{m+1}}^{\ell_m} \left\|u(\cdot,t)\right\|_{L^2(\omega)} dt \ .\end{aligned}$$

Take $z = \sqrt{\frac{\gamma+2}{\gamma+1}}$, it implies

$$\begin{aligned}&e^{-(2+\gamma)\sigma z^{m}} \left\|u(\cdot,\ell_m)\right\|_{L^2(\Omega)} - e^{-(2+\gamma)\sigma z^{m+2}} \left\|u(\cdot,\ell_{m+2})\right\|_{L^2(\Omega)} \\ &\leq \tfrac{1}{z}\tfrac{C_1}{C_2} \int_{\ell_{m+1}}^{\ell_m} \left\|u(\cdot,t)\right\|_{L^1(\omega)} dt \ .\end{aligned}$$

Change m to $2m'$ and sum the above from $m'=1$ to infinity give the desired result. Indeed,

$$\begin{array}{l} e^{-(2+\gamma)2C_2\left[\frac{1}{T}\frac{z}{(z-1)}\right]}\left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)} \\ \leq e^{-(2+\gamma)\sigma z^2}\left\|u\left(\cdot,\ell_2\right)\right\|_{L^2(\Omega)} \\ \leq \sum_{m'\geq 1}\left(e^{-(2+\gamma)\sigma z^{2m'}}\left\|u\left(\cdot,\ell_{2m'}\right)\right\|_{L^2(\Omega)} - e^{-(2+\gamma)\sigma z^{2m'+2}}\left\|u\left(\cdot,\ell_{2m'+2}\right)\right\|_{L^2(\Omega)}\right) \\ \leq \sum_{m'\geq 1}\frac{1}{z}\frac{C_1}{C_2}\int_{\ell_{2m'+1}}^{\ell_{2m'}}\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt \\ \leq \frac{1}{z}\frac{C_1}{C_2}\int_0^{\ell_2}\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt\ . \end{array}$$

Finally, with $D_1=\sqrt{\frac{\gamma+1}{\gamma+2}\frac{C_1}{C_2}}$ and $D_2=2C_2\left(2+\gamma\right)\sqrt{\gamma+2}\left[\sqrt{\gamma+2}+\sqrt{\gamma+1}\right]$, we have for any $T>0$,

$$\left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)}\leq D_1 e^{D_2\frac{1}{T}}\int_0^T\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt\ .$$

In particular, for any integer number $m\geq 1$,

$$\left\|u\left(\cdot,m\right)\right\|_{L^2(\Omega)}\leq D_1 e^{D_2}\int_{m-1}^{m}\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt\ .$$

Now, take $T>1$ such that $M<T\leq M+1$ for some $M\geq 1$,

$$\begin{array}{ll} \frac{T}{2}\left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)} & \leq M\left\|u\left(\cdot,M\right)\right\|_{L^2(\Omega)} \\ & \leq \sum_{m=1}^{M}\left\|u\left(\cdot,m\right)\right\|_{L^2(\Omega)} \\ & \leq D_1 e^{D_2}\sum_{m=1}^{M}\int_{m-1}^{m}\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt \\ & \leq D_1 e^{D_2}\int_0^M\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt \\ & \leq D_1 e^{D_2}\int_0^T\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt\ . \end{array}$$

We deduce that

$$\forall T>1,\quad \left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)}\leq \frac{2}{T}D_1 e^{D_2}\int_0^T\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt\ .$$

Since

$$\forall T\leq 1,\quad \left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)}\leq \frac{1}{T}D_1 e^{D_2\frac{1}{T}}\int_0^T\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt\ ,$$

by combining the case $T\leq 1$ and the case $T>1$, we get the desired observability estimate

$$\left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)}\leq \frac{2}{T}D_1 e^{D_2\left(1+\frac{1}{T}\right)}\int_0^T\left\|u\left(\cdot,t\right)\right\|_{L^1(\omega)}dt$$

for any $T>0$.

3.5.2 Hölder estimate (proof of *ii*) when Ω is convex)

Let $\lambda>0$ and $x_0\in\Omega$. Denote for $t\in[0,T]$,

$$G_\lambda\left(x,t\right)=\frac{1}{\left(T-t+\lambda\right)^{d/2}}e^{-\frac{\left|x-x_0\right|^2}{4(T-t+\lambda)}}\ .$$

Define for $t\in(0,T]$,

$$N_\lambda\left(t\right)=\frac{\int_\Omega\left|\nabla u\left(x,t\right)\right|^2 G_\lambda\left(x,t\right)dx}{\int_\Omega\left|u\left(x,t\right)\right|^2 G_\lambda\left(x,t\right)dx},\text{ whenever }\int_\Omega\left|u\left(x,t\right)\right|^2dx\neq 0\ .$$

Here, recall that $u(x,t) = e^{t\Delta} u_0(x)$ and in particular, $u_{|\partial\Omega} = 0$.

We divide the proof into two steps. In the first step, we study the monotonicity of the function $N_\lambda(t)$ in order to bound $N_\lambda(T)$. In the second step, we make appear $\|u(\cdot,T)\|_{L^2(\omega)}$.

Step 1 .- Get good sign for a derivative. We claim that if Ω is convex or star-shaped w.r.t. x_0, then

$$\frac{d}{dt} N_\lambda(t) \leq \frac{1}{T-t+\lambda} N_\lambda(t) \ .$$

That is

$$\frac{d}{dt}\left[(T-t+\lambda) N_\lambda(t)\right] \leq 0 \ .$$

Consequently, after integrating over (t,T), it gives that for all $t \in (0,T)$,

$$\lambda N_\lambda(T) \leq (T+\lambda) N_\lambda(t) \ .$$

Recall that by a simple application of Green formula using the fact that $\partial_t G_\lambda + \Delta G_\lambda = 0$ and $u_{|\partial\Omega} = 0$, the following identity holds

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)|^2 G_\lambda(x,t)\, dx + \int_\Omega |\nabla u(x,t)|^2 G_\lambda(x,t)\, dx = 0 \ .$$

In other words,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)|^2 G_\lambda(x,t)\, dx + N_\lambda(t)\int_\Omega |u(x,t)|^2 G_\lambda(x,t)\, dx = 0 \ .$$

Therefore,

$$\frac{d}{dt}\int_\Omega |u(x,t)|^2 G_\lambda(x,t)\, dx + \frac{2\lambda N_\lambda(T)}{T+\lambda}\int_\Omega |u(x,t)|^2 G_\lambda(x,t)\, dx \leq 0 \ .$$

Consequently, after multiplying by $e^{\frac{2t\lambda N_\lambda(T)}{T+\lambda}}$ and integrating over $(0,T/2)$, it gives

$$e^{\frac{T\lambda N_\lambda(T)}{T+\lambda}}\int_\Omega |u(x,T/2)|^2 G_\lambda(x,T/2)\, dx \leq \int_\Omega |u(x,0)|^2 G_\lambda(x,0)\, dx \ .$$

But

$$\begin{aligned}\int_\Omega |u(x,T)|^2\, dx &\leq \int_\Omega |u(x,T/2)|^2\, dx \\ &\leq e^{\frac{m_0}{2T}}\int_\Omega |u(x,T/2)|^2 e^{-\frac{|x-x_0|^2}{4(T/2+\lambda)}}\, dx \ .\end{aligned}$$

where $m_0 = \max_{x\in\Omega} |x-x_0|^2$. Therefore,

$$\begin{aligned} e^{\frac{T\lambda N_\lambda(T)}{T+\lambda}} &\leq \frac{\int_\Omega |u(x,0)|^2 G_\lambda(x,0)\, dx}{\int_\Omega |u(x,T/2)|^2 G_\lambda(x,T/2)\, dx} \\ &\leq \frac{\int_\Omega |u(x,0)|^2\, dx}{\int_\Omega |u(x,T/2)|^2 e^{-\frac{|x-x_0|^2}{4(T/2+\lambda)}}\, dx} \\ &\leq \frac{e^{\frac{m_0}{2T}}\int_\Omega |u(x,0)|^2\, dx}{\int_\Omega |u(x,T)|^2\, dx} \ .\end{aligned}$$

That is

$$\lambda N_\lambda(T) \leq \left(1+\frac{\lambda}{T}\right)\log\frac{e^{\frac{m_0}{2T}}\int_\Omega |u(x,0)|^2\, dx}{\int_\Omega |u(x,T)|^2\, dx} \ .$$

On the other hand,

$$\frac{d}{4}\leq\frac{d}{4}\left(1+\frac{\lambda}{T}\right)\log\frac{e^{1+\frac{m_0}{2T}}\displaystyle\int_\Omega\left|u\left(x,0\right)\right|^2dx}{\displaystyle\int_\Omega\left|u\left(x,T\right)\right|^2dx}\ .$$

Finally, the two above estimate lead to

$$\frac{d}{4}+\lambda N_\lambda\left(T\right)\leq\left(\frac{d}{4}+1\right)\left(1+\frac{\lambda}{T}\right)\log\frac{e^{1+\frac{m_0}{2T}}\displaystyle\int_\Omega\left|u\left(x,0\right)\right|^2dx}{\displaystyle\int_\Omega\left|u\left(x,T\right)\right|^2dx}\ .$$

Now, we prove the claim saying that if Ω is convex or star-shaped w.r.t. x_0, then

$$\frac{d}{dt}N_\lambda\left(t\right)\leq\frac{1}{T-t+\lambda}N_\lambda\left(t\right)\ .$$

First, by an integration by parts and by using $\nabla G_\lambda\left(x,t\right)=-\frac{x-x_0}{2(T-t+\lambda)}G_\lambda\left(x,t\right)$, we have

$$\begin{aligned}\int_\Omega\left|\nabla u\right|^2G_\lambda dx&=-\int_\Omega\left(\Delta uuG_\lambda+\nabla uu\nabla G_\lambda\right)dx+\int_{\partial\Omega}\partial_\nu uuG_\lambda d\sigma\\&=-\int_\Omega\left(\partial_tu-\frac{x-x_0}{2\left(T-t+\lambda\right)}\cdot\nabla u\right)uG_\lambda dx\ .\end{aligned}$$

We have denoted ν the outward unit normal vector to $\partial\Omega$. Next, we compute $\frac{d}{dt}\int_\Omega\left|\nabla u\right|^2G_\lambda dx$.

$$\begin{aligned}\tfrac{d}{dt}\int_\Omega\left|\nabla u\right|^2G_\lambda dx&=\int_\Omega\left(2\nabla u\partial_t\nabla uG_\lambda+\left|\nabla u\right|^2\partial_tG_\lambda\right)dx\\&=-2\int_\Omega\left(\Delta u\partial_tuG_\lambda+\nabla u\partial_tu\nabla G_\lambda\right)dx-\int_\Omega\left|\nabla u\right|^2\Delta G_\lambda dx\\&=-2\int_\Omega\left|\partial_tu\right|^2G_\lambda dx+\int_\Omega\partial_tu\frac{x-x_0}{T-t+\lambda}\cdot\nabla uG_\lambda dx\\&\quad+\int_\Omega\nabla\left(\left|\nabla u\right|^2\right)\nabla G_\lambda dx+\int_{\partial\Omega}\left|\nabla u\right|^2\frac{x-x_0}{2\left(T-t+\lambda\right)}\cdot\nu G_\lambda d\sigma\ .\end{aligned}$$

But, using standard summation notations,

$$\begin{aligned}&\int_\Omega\nabla\left(\left|\nabla u\right|^2\right)\nabla G_\lambda dx=\int_\Omega\partial_i\left(\left|\partial_ju\right|^2\right)\partial_iG_\lambda dx\\&=2\int_\Omega\partial_ju\partial_{ij}^2u\partial_iG_\lambda dx\\&=-2\int_\Omega\left(\partial_j^2u\partial_iu\partial_iG_\lambda+\partial_ju\partial_iu\partial_{ij}^2G_\lambda\right)dx\\&\quad+2\int_{\partial\Omega}\partial_ju\partial_iu\nu_j\partial_iG_\lambda d\sigma\\&=-2\int_\Omega\left(\Delta u\nabla u\nabla G_\lambda+\partial_ju\partial_iu\left[-\frac{\partial_j\left(x_i-x_{0i}\right)}{2\left(T-t+\lambda\right)}+\frac{\left(x_i-x_{0i}\right)\left(x_j-x_{0j}\right)}{4\left(T-t+\lambda\right)^2}\right]G_\lambda\right)dx\\&\quad-\int_{\partial\Omega}\left|\partial_\nu u\right|^2\frac{x-x_0}{\left(T-t+\lambda\right)}\cdot\nu G_\lambda d\sigma\\&=\int_\Omega\partial_tu\frac{x-x_0}{T-t+\lambda}\cdot\nabla uG_\lambda dx+\frac{1}{\left(T-t+\lambda\right)}\int_\Omega\left|\nabla u\right|^2G_\lambda dx-2\int_\Omega\left(\frac{x-x_0}{2\left(T-t+\lambda\right)}\cdot\nabla u\right)^2G_\lambda dx\\&\quad-\int_{\partial\Omega}\left|\partial_\nu u\right|^2\frac{x-x_0}{\left(T-t+\lambda\right)}\cdot\nu G_\lambda d\sigma\ .\end{aligned}$$

Therefore,

$$\begin{aligned}
&\tfrac{d}{dt}\int_{\Omega}|\nabla u|^2 G_\lambda dx\\
&= -2\int_{\Omega}|\partial_t u|^2 G_\lambda dx + 2\int_{\Omega}\partial_t u \frac{x-x_0}{T-t+\lambda}\cdot\nabla u G_\lambda dx - 2\int_{\Omega}\left(\frac{x-x_0}{2(T-t+\lambda)}\cdot\nabla u\right)^2 G_\lambda dx\\
&\quad+\tfrac{1}{(T-t+\lambda)}\int_{\Omega}|\nabla u|^2 G_\lambda dx\\
&\quad-\int_{\partial\Omega}|\partial_\nu u|^2\frac{x-x_0}{(T-t+\lambda)}\cdot\nu G_\lambda d\sigma + \int_{\partial\Omega}|\nabla u|^2\frac{x-x_0}{2(T-t+\lambda)}\cdot\nu G_\lambda d\sigma\ .
\end{aligned}$$

That is

$$\begin{aligned}
\tfrac{d}{dt}\int_{\Omega}|\nabla u|^2 G_\lambda dx &= -2\int_{\Omega}\left|\partial_t u - \frac{x-x_0}{2(T-t+\lambda)}\cdot\nabla u\right|^2 G_\lambda dx\\
&\quad+\tfrac{1}{(T-t+\lambda)}\int_{\Omega}|\nabla u|^2 G_\lambda dx\\
&\quad-\int_{\partial\Omega}|\partial_\nu u|^2\frac{x-x_0}{2(T-t+\lambda)}\cdot\nu G_\lambda d\sigma\ .
\end{aligned}$$

Finally, we compute $\frac{d}{dt}N_\lambda(t)$

$$\begin{aligned}
\tfrac{d}{dt}N_\lambda(t) &= \frac{\tfrac{d}{dt}\int_{\Omega}|\nabla u|^2 G_\lambda dx\int_{\Omega}|u|^2 G_\lambda dx - \int_{\Omega}|\nabla u|^2 G_\lambda dx\frac{d}{dt}\int_{\Omega}|u|^2 G_\lambda dx}{\left(\int_{\Omega}|u|^2 G_\lambda dx\right)^2}\\
&= \frac{1}{\left(\int_{\Omega}|u|^2 G_\lambda dx\right)^2}\left(-2\int_{\Omega}\left|\partial_t u - \frac{x-x_0}{2(T-t+\lambda)}\cdot\nabla u\right|^2 G_\lambda dx\int_{\Omega}|u|^2 G_\lambda dx\right)\\
&\quad+\frac{1}{\left(\int_{\Omega}|u|^2 G_\lambda dx\right)^2}\left(\tfrac{1}{(T-t+\lambda)}\int_{\Omega}|\nabla u|^2 G_\lambda dx\int_{\Omega}|u|^2 G_\lambda dx\right)\\
&\quad+\frac{1}{\left(\int_{\Omega}|u|^2 G_\lambda dx\right)^2}\left(-\int_{\partial\Omega}|\partial_\nu u|^2\frac{x-x_0}{2(T-t+\lambda)}\cdot\nu G_\lambda d\sigma\int_{\Omega}|u|^2 G_\lambda dx\right)\\
&\quad+\frac{1}{\left(\int_{\Omega}|u|^2 G_\lambda dx\right)^2}2\left(\int_{\Omega}|\nabla u|^2 G_\lambda dx\right)^2\\
&= \frac{1}{\left(\int_{\Omega}|u|^2 G_\lambda dx\right)^2}\left(-2\int_{\Omega}\left|\partial_t u - \frac{x-x_0}{2(T-t+\lambda)}\cdot\nabla u\right|^2 G_\lambda dx\int_{\Omega}|u|^2 G_\lambda dx\right)\\
&\quad+\tfrac{1}{(T-t+\lambda)}N_\lambda(t) - \frac{\int_{\partial\Omega}|\partial_\nu u|^2\frac{x-x_0}{2(T-t+\lambda)}\cdot\nu G_\lambda d\sigma}{\int_{\Omega}|u|^2 G_\lambda dx}\\
&\quad+\frac{1}{\left(\int_{\Omega}|u|^2 G_\lambda dx\right)^2}2\left(-\int_{\Omega}\left(\partial_t u - \frac{x-x_0}{2(T-t+\lambda)}\cdot\nabla u\right)uG_\lambda dx\right)^2\\
&\leq \tfrac{1}{(T-t+\lambda)}N_\lambda(t)
\end{aligned}$$

by Cauchy-Schwarz inequality and the fact that $(x-x_0)\cdot\nu\geq 0$ for convex domain Ω.

Step 2 .- Make appear $B_r=\left\{x\in\mathbb{R}^d;|x-x_0|<r\right\}\subset\Omega$. We start to choose $x_0\in\Omega$ and $r>0$

such that $B_{2r}=\left\{x\in\mathbb{R}^d;|x-x_0|<2r\right\}\subset\omega$. Next, notice that

$$\begin{aligned}&\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &\leq \int_{B_r}|u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx+\int_{\Omega\setminus B_r}|u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &\leq \int_{B_r}|u(x,T)|^2 dx+\frac{1}{r^2}\int_\Omega |x-x_0|^2|u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\ .\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_\Omega |x-x_0|^2|u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &=\int_\Omega (x-x_0)|u(x,T)|^2\cdot(-2\lambda)\nabla e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &=-2\lambda\int_{\partial\Omega}((x-x_0)\cdot\nu)|u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}d\sigma+2\lambda d\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &\quad+4\lambda\int_\Omega (x-x_0)u(x,T)\cdot\nabla u(x,T)e^{-\frac{|x-x_0|^2}{4\lambda}}dx\quad\text{by integration by parts}\\ &\leq 2\lambda d\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &\quad+\tfrac{1}{2}\int_\Omega 16\lambda^2|\nabla u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx+\frac{1}{2}\int_\Omega |x-x_0|^2|u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\ ,\end{aligned}$$

by Cauchy-Schwarz inequality. Therefore,

$$\begin{aligned}&\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &\leq \int_{B_r}|u(x,T)|^2 dx\\ &\quad+\tfrac{1}{r^2}\left[4\lambda d\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx+\int_\Omega 16\lambda^2|\nabla u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\right]\\ &\leq \int_{B_r}|u(x,T)|^2 dx+\frac{16\lambda}{r^2}\left[\frac{d}{4}+\lambda N_\lambda(T)\right]\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\ .\end{aligned}$$

By step 1,

$$\begin{aligned}&\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\\ &\leq \int_{B_r}|u(x,T)|^2 dx\\ &\quad+\tfrac{16\lambda}{r^2}\left[\left(\tfrac{d}{4}+1\right)\left(1+\tfrac{\lambda}{T}\right)\log\frac{e^{1+\frac{m_0}{2T}}\int_\Omega |u(x,0)|^2 dx}{\int_\Omega |u(x,T)|^2 dx}\right]\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}dx\ .\end{aligned}$$

Take

$$\lambda=-T+\left[T^2+4\frac{r^2}{16}\frac{T}{2\left(\frac{d}{4}+1\right)\log\frac{e^{1+\frac{m_0}{2T}}\int_\Omega |u(x,0)|^2 dx}{\int_\Omega |u(x,T)|^2 dx}}\right]^{1/2}$$

in order that

$$\frac{16\lambda}{r^2}\left[\left(\frac{d}{4}+1\right)\left(1+\frac{\lambda}{T}\right)\log\frac{e^{1+\frac{m_0}{2T}}\int_\Omega |u(x,0)|^2 dx}{\int_\Omega |u(x,T)|^2 dx}\right]=\frac{1}{2}\ .$$

Therefore,

$$\begin{aligned}\int_\Omega |u(x,T)|^2 dx &\leq e^{\frac{m_0}{4\lambda}} \int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\ &\leq 2e^{\frac{m_0}{4\lambda}} \int_{B_r} |u(x,T)|^2 dx \ .\end{aligned}$$

But

$$\begin{aligned}\frac{m_0}{4\lambda} &= \frac{m_0}{4}\left(-T+\left[T^2+4\frac{r^2}{16}\frac{T}{2\left(\frac{d}{4}+1\right)\log\frac{e^{1+\frac{m_0}{2T}}\int_\Omega |u(x,0)|^2 dx}{\int_\Omega |u(x,T)|^2 dx}}\right]^{1/2}\right)^{-1} \\ &\leq \frac{m_0}{4}\left(\left(1+\frac{r^2}{(d+4)m_0}\right)^{1/2}+1\right)\frac{2}{r^2}(d+4)\log\frac{e^{1+\frac{m_0}{2T}}\int_\Omega |u(x,0)|^2 dx}{\int_\Omega |u(x,T)|^2 dx} \ .\end{aligned}$$

Finally, one obtain with

$$\alpha = \frac{\left(\frac{d}{4}+1\right)\left(\left(1+\frac{r^2}{(d+4)m_0}\right)^{1/2}+1\right)\frac{2m_0}{r^2}}{1+\left(\frac{d}{4}+1\right)\left(\left(1+\frac{r^2}{(d+4)m_0}\right)^{1/2}+1\right)\frac{2m_0}{r^2}} \ ,$$

the following Hölder type estimate

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \left(e^{\frac{1}{2}\left(1+\frac{m_0}{2T}\right)}\|u(\cdot,0)\|_{L^2(\Omega)}\right)^\alpha \left(\sqrt{2}\,\|u(\cdot,T)\|_{L^2(B_r)}\right)^{1-\alpha} \ .$$

On the other hand, by Nash inequality and Poincaré inequality,

$$\|u(\cdot,T)\|_{L^2(B_r)} \leq c\left(\|u(\cdot,T)\|_{L^1(B_{2r})}\right)^{\frac{2}{d+2}}\left(\|\nabla u(\cdot,T)\|_{L^2(\Omega)}\right)^{\frac{d}{d+2}} \ .$$

Further, by an energy method,

$$\|\nabla u(\cdot,T)\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{2T}}\|u(\cdot,0)\|_{L^2(\Omega)} \ .$$

We conclude, by combining the last three above inequalities, that

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq e^{\frac{\alpha}{2}\left(1+\frac{m_0}{2T}\right)}\left(\sqrt{2}c\right)^{1-\alpha}\left(\frac{1}{\sqrt{2T}}\right)^{\frac{d(1-\alpha)}{d+2}}\left(\|u(\cdot,0)\|_{L^2(\Omega)}\right)^{\frac{2\alpha+d}{d+2}}\left(\|u(\cdot,T)\|_{L^1(B_{2r})}\right)^{\frac{2(1-\alpha)}{d+2}} \ .$$

This is the desired estimate with $\mu = \frac{2(1-\alpha)}{d+2}$ since $B_{2r} \subset \omega$.

3.5.3 Logarithmic estimate (proof of $ii) \Rightarrow iii)$)

Define for almost all $t \in (0,T]$,

$$\Psi(t) = \frac{\|u(\cdot,t)\|^2_{L^2(\Omega)}}{\|u(\cdot,t)\|^2_{H^{-1}(\Omega)}}, \text{ whenever } u(x,t) \neq 0 \ .$$

Here, recall that $u(x,t) = e^{t\Delta}u_0(x)$ with $u_0 \in L^2(\Omega)$. We first claim that

$$\|u_0\|_{H^{-1}(\Omega)} \leq \exp(\Psi(0)T)\left\|e^{T\Delta}u_0\right\|_{H^{-1}(\Omega)} \ .$$

Now, we apply $\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq e^{C\left(1+\frac{1}{T}\right)} \left\|e^{T\Delta}u_0\right\|^{\mu}_{L^1(\omega)} \|u_0\|^{1-\mu}_{L^2(\Omega)}$ to get

$$\|u_0\|_{H^{-1}(\Omega)} \leq \exp\left(\Psi\left(0\right)T\right) e^{C\left(1+\frac{1}{T}\right)} \left\|e^{T\Delta}u_0\right\|^{\mu}_{L^1(\omega)} \|u_0\|^{(1-\mu)}_{L^2(\Omega)} \ .$$

In other words,

$$\|u_0\|^{\mu}_{L^2(\Omega)} \leq \sqrt{\Psi\left(0\right)}\exp\left(\Psi\left(0\right)T\right) e^{C\left(1+\frac{1}{T}\right)} \left\|e^{T\Delta}u_0\right\|^{\mu}_{L^1(\omega)} \ .$$

Finally, we have the desired estimate

$$\|u_0\|_{L^2(\Omega)} \leq \left[\frac{1}{\sqrt{T}} e^{C\left(1+\frac{1}{T}+\Psi(0)T\right)}\right]^{1/\mu} \left\|e^{T\Delta}u_0\right\|_{L^1(\omega)} \ .$$

This completes the proof. Notice that the term involving $\Psi\left(0\right)$ can be improved as follows.

$$\|u_0\|_{L^2(\Omega)} \leq \frac{1}{T} e^{C\left(1+\sqrt{\Psi(0)}\left(T+\frac{1}{T}\right)\right)} \left\|e^{t\Delta}u_0\right\|_{L^1(\omega\times(0,T))} \ .$$

Indeed, let us denote $\lambda_1 > 0$ a constant such that $\lambda_1 \leq \Psi\left(0\right)$. Now, we choose

$$L = T\sqrt{\frac{\lambda_1}{\Psi\left(0\right)}} \leq T \ ,$$

and recall that the solution u of the heat equation satisfies if $u\left(x,t\right) \neq 0$ for almost all $t \in [0,T]$:

$$\|u_0\|_{H^{-1}(\Omega)} \leq \exp\left(\sqrt{\lambda_1\Psi\left(0\right)}T\right) \left\|e^{\left(T\sqrt{\frac{\lambda_1}{\Psi(0)}}\right)\Delta}u_0\right\|_{L^2(\Omega)} \ .$$

Finally, we apply $\left\|e^{L\Delta}u_0\right\|_{L^2(\Omega)} \leq \frac{1}{L} e^{C\left(1+\frac{1}{L}\right)} \left\|e^{t\Delta}u_0\right\|_{L^1(\omega\times(0,L))}$ to get the improved estimate.

3.6 Null controllability

3.6.1 Functional analysis via Hahn-Banach theorem

The characteristic function on a set X will be denoted by 1_X.

Let Ω be bounded open connected set in $\mathbb{R}^d$, $d \geq 1$, with a C^2 boundary $\partial\Omega$. Let ω be an open subset of Ω.

Let $T > 0$ and $E \subset (0,T)$ be a subset of positive measure.
The main result is as follows.

Theorem .- *Let $\kappa > 0$. The following two statements are equivalent.*

(C) *For any $y_0 \in L^2\left(\Omega\right)$, there is a control $v \in L^\infty\left(\Omega\times(0,T)\right)$ such that the solution $y = y\left(x,t\right)$ to*

$$\begin{cases} \partial_t y - \Delta y = 1_E 1_\omega v & \text{in } \Omega\times(0,T) \ , \\ y = 0 & \text{on } \partial\Omega\times(0,T) \ , \\ y\left(\cdot,0\right) = y_0 & \text{in } \Omega \ , \end{cases} \tag{3.6.1}$$

satisfies $y\left(\cdot,T\right) = 0$ in Ω. Moreover,

$$\|v\|_{L^\infty(\Omega\times(0,T))} \leq \kappa\|y_0\|_{L^2(\Omega)} \ . \tag{3.6.2}$$

(O) *For any* $u_0 \in L^2(\Omega)$*, the solution* $u = u(x,t)$ *to*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 & \text{in } \Omega \ , \end{cases} \tag{3.6.3}$$

satisfies

$$\|u(\cdot,T)\|_{L^2(\Omega)} \le \kappa \int_0^T \int_\Omega 1_E 1_\omega |u(x,T-t)| \, dxdt \tag{3.6.4}$$

that is

$$\|u(\cdot,T)\|_{L^2(\Omega)} \le \kappa \int_0^T \int_\Omega 1_E(T-t) 1_\omega |u(x,t)| \, dxdt \ .$$

Proof of $(C) \Rightarrow (O)$.- The goal consists to prove (3.6.4). To this ends, we multiply by $u(x,T-t)$ the equations in (3.6.1) solved by y and integrate over $\Omega \times (0,T)$. It gives

$$\int_0^T \int_\Omega (\partial_t - \Delta) y(x,t) u(x,T-t) \, dxdt = \int_0^T \int_\Omega 1_E 1_\omega v(x,t) u(x,T-t) \, dxdt \ . \tag{3.6.5}$$

On another hand, by integration by parts, we also have using the homogeneous Dirichlet boundary condition and $y(\cdot,T) = 0$ in Ω, that

$$\begin{aligned} \int_0^T \int_\Omega (\partial_t - \Delta) y(x,t) u(x,T-t) \, dxdt &= \left[\int_\Omega y(x,t) u(x,T-t) \, dx \right]_0^T \\ &= -\int_\Omega y_0(x) u(x,T) \, dx \end{aligned} \tag{3.6.6}$$

and this for any $y_0 \in L^2(\Omega)$. Now, we choose $y_0 = u(\cdot,T)$ which belongs to $L^2(\Omega)$ because (3.6.3) is well-posed with $u_0 \in L^2(\Omega)$. Consequently,

$$\begin{aligned} \int_\Omega |u(x,T)|^2 \, dx &= -\int_0^T \int_\Omega 1_E 1_\omega v(x,t) u(x,T-t) \, dxdt \quad \text{from (3.6.4) and (3.6.6)} \\ &\le \|v\|_{L^\infty(\Omega \times (0,T))} \|1_E 1_\omega u(x,T-t)\|_{L^1(\Omega \times (0,T))} \\ &\le \tfrac{1}{2\varepsilon} \|v\|^2_{L^\infty(\Omega \times (0,T))} + \tfrac{\varepsilon}{2} \|1_E 1_\omega u(x,T-t)\|^2_{L^1(\Omega \times (0,T))} \quad \text{for any } \varepsilon > 0 \\ &\le \tfrac{1}{2\varepsilon} \kappa^2 \left\|y^0\right\|^2_{L^2(\Omega)} + \tfrac{\varepsilon}{2} \|1_E 1_\omega u(x,T-t)\|^2_{L^1(\Omega \times (0,T))} \quad \text{from (3.6.2)} \ . \end{aligned} \tag{3.6.7}$$

Since $y_0 = u(\cdot,T)$, we get the desired inequality (3.6.4) by taking $\varepsilon = \kappa^2$.

Proof of $(O) \Rightarrow (C)$.- Let $y_0 \in L^2(\Omega)$. It is enough to prove the existence of $v \in L^\infty(\Omega \times (0,T))$ with (3.6.2) such that the solution $y = y(x,t)$ to (3.6.1) satisfies

$$-\int_\Omega y_0(x) u(x,T) \, dx = \int_0^T \int_\Omega v(x,t) 1_E 1_\omega u(x,T-t) \, dxdt \quad \forall u_0 \in L^2(\Omega) \ . \tag{3.6.8}$$

Indeed, by multiplying by y the equations in (3.6.3) solved by $u(x,T-t)$ and integrate over $\Omega \times (0,T)$, it gives

$$\int_\Omega y(x,T) u_0(x) \, dx - \int_\Omega y_0(x) u(x,T) \, dx = \int_0^T \int_\Omega v(x,t) 1_E 1_\omega u(x,T-t) \, dxdt \tag{3.6.9}$$

and the fact that $\int_\Omega y(x,T) u_0(x) \, dx = 0$ for any $u_0 \in L^2(\Omega)$ means $y(\cdot,T) = 0$ in Ω. Now, let

$$Z = \left\{ 1_E 1_\omega u(x,T-t) \, ; u_0 \in L^2(\Omega) \text{ and } u \text{ solves (3.6.3)} \right\} \ . \tag{3.6.10}$$

Z is a linear subspace of $L^1(\Omega \times (0,T))$. We define a linear functional $\mathcal{F} : Z \to \mathbb{R}$ by setting

$$\mathcal{F}(1_E 1_\omega u(x,T-t)) = -\int_\Omega y_0(x) u(x,T) \, dx \ . \tag{3.6.11}$$

Then $\mathcal{F}$ is a continuous linear functional on Z. Next, it follows by (3.6.4)

$$\begin{aligned} \left|\mathcal{F}\left(1_E 1_\omega u\left(x,T-t\right)\right)\right| &\leq \int_\Omega \left|y_0\left(x\right)u\left(x,T\right)\right| dx \\ &\leq \left\|y_0\right\|_{L^2(\Omega)} \left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)} \\ &\leq \kappa \left\|y_0\right\|_{L^2(\Omega)} \left\|1_E 1_\omega u\left(x,T-t\right)\right\|_{L^1(\Omega\times(0,T))} \end{aligned} \tag{3.6.12}$$

that is

$$\left\|\mathcal{F}\right\|_{\mathcal{L}(Z;\mathbb{R})} \leq \kappa \left\|y_0\right\|_{L^2(\Omega)} \ . \tag{3.6.13}$$

By Hahn-Banach theorem, $\mathcal{F}$ can be extended to $\mathcal{G} : L^1\left(\Omega\times(0,T)\right) \to \mathbb{R}$, a continuous linear functional with the same operator norm $\left\|\mathcal{G}\right\|_{\mathcal{L}(L^1(\Omega\times(0,T));\mathbb{R})} = \left\|\mathcal{F}\right\|_{\mathcal{L}(Z;\mathbb{R})}$, such that $\mathcal{G} = \mathcal{F}$ on Z. Therefore,

$$\mathcal{G}\left(1_E 1_\omega u\left(x,T-t\right)\right) = -\int_\Omega y_0\left(x\right)u\left(x,T\right)dx \tag{3.6.14}$$

and

$$\left\|\mathcal{G}\right\|_{\mathcal{L}(L^1(\Omega\times(0,T));\mathbb{R})} \leq \kappa \left\|y_0\right\|_{L^2(\Omega)} \ . \tag{3.6.15}$$

By Riesz representation theorem, there exists $v \in L^\infty\left(\Omega\times(0,T)\right)$ such that

$$\mathcal{G}\left(f\right) = \int_0^T \int_\Omega v\left(x,t\right) f\left(x,t\right) dxdt \quad \forall f \in L^1\left(\Omega\times(0,T)\right) \tag{3.6.16}$$

and

$$\left\|v\right\|_{L^\infty(\Omega\times(0,T))} = \left\|\mathcal{G}\right\|_{\mathcal{L}(L^1(\Omega\times(0,T));\mathbb{R})} \ . \tag{3.6.17}$$

In particular, by (3.6.17) and (3.6.15), it holds

$$\left\|v\right\|_{L^\infty(\Omega\times(0,T))} \leq \kappa \left\|y_0\right\|_{L^2(\Omega)} \ . \tag{3.6.18}$$

Further, by (3.6.16) and (3.6.14), we have

$$\int_0^T \int_\Omega v\left(x,t\right) 1_E 1_\omega u\left(x,T-t\right) dxdt = -\int_\Omega y_0\left(x\right)u\left(x,T\right)dx \quad \forall u_0 \in L^2\left(\Omega\right) \ . \tag{3.6.19}$$

This completes the proof.

3.6.2 Semi-group and variational approach

We consider the abstract differential equation

$$\begin{aligned} \tfrac{d}{dt}u(\cdot,t) + Au(\cdot,t) &= 0 \quad \text{for } t>0 \ , \\ u(\cdot,0) &= u_0 \in H \end{aligned}$$

where $-A : D(A) \subset H \to H$ is the generator of a strongly continuous semigroup $\left(e^{-tA}\right)_{t\geq 0}$ on a Hilbert space H. The solution is $u\left(\cdot,t\right) = e^{-tA}u_0$.

We also consider a bounded observation operator $\mathcal{B} \in \mathcal{L}(H,F)$, i.e. $\mathcal{B}$ is a continuous operator from H to another Hilbert space F. Let A^* be the adjoint of A (and we will suppose that A is self-adjoint, i.e. $A^* = A$, and accretive. We have in mind that A is the operator on $L^2\left(\Omega\right)$ defined by

$$\left\{ \begin{array}{l} D\left(A\right) = \left\{u_0 \in H_0^1\left(\Omega\right); \Delta u_0 \in L^2\left(\Omega\right)\right\} \\ Au_0 = -\Delta u_0 \quad \forall u_0 \in D\left(A\right) \ . \end{array} \right.$$

with Ω a bounded open set in $\mathbb{R}^d$, $d \geq 1$, which is either convex or C^2 and connected). Let $T>0$. We introduce the following abstract differential equation

$$\begin{aligned} \tfrac{d}{dt}y(\cdot,t) + A^*y(\cdot,t) &= \mathcal{B}^*v(\cdot,t) \quad \text{for } t \in (0,T] \ , \\ y(\cdot,0) &= y_0 \in H \end{aligned}$$

where $v \in L^2([0,T];F)$ is the input, $\mathcal{B}^* \in \mathcal{L}(F,H)$ is the control operator (here it is the adjoint of $\mathcal{B}$). The solution is

$$y(\cdot,t) = e^{-tA^*}y_0 + \int_0^t e^{-(t-s)A^*}\mathcal{B}^* f(\cdot,s)ds \ .$$

The main result is as follows.

Theorem .- *Let $\kappa, \varepsilon > 0$. The following two statements are equivalent.*

(C) *For any $y_0 \in H$, there is an input $v \in L^2([0,T];F)$ such that the solution $y(\cdot,t)$ to*

$$\begin{cases} \frac{d}{dt}y(\cdot,t) + A^*y(\cdot,t) = \mathcal{B}^*v(\cdot,t) & \text{for } t \in (0,T] \ , \\ y(\cdot,0) = y_0 \in H & \end{cases}$$

satisfies

$$\frac{1}{\kappa}\int_0^T \|v(\cdot,t)\|_F^2 \, dt + \frac{1}{\varepsilon}\|y(\cdot,T)\|_H^2 \leq \|y_0\|_H^2 \ .$$

(O) *For any $u_0 \in H$, the solution $u(\cdot,t)$ to*

$$\begin{cases} \frac{d}{dt}u(\cdot,t) + Au(\cdot,t) = 0 & \text{for } t > 0 \ , \\ u(\cdot,0) = u_0 \in H & \end{cases}$$

satisfies

$$\|u(\cdot,T)\|_H^2 \leq \kappa \int_0^T \|\mathcal{B}u(\cdot,T-t)\|_F^2 \, dt + \varepsilon \|u_0\|_H^2 \ .$$

Proof of $(C) \Rightarrow (O)$.- We multiply the equations of (C) by $u(\cdot,T-t)$ to get

$$\begin{aligned} \left\langle \tfrac{d}{dt}y(\cdot,t), u(\cdot,T-t)\right\rangle &= \langle -A^*y(\cdot,t) + \mathcal{B}^*v(\cdot,t), u(\cdot,T-t)\rangle \\ &= \langle y(\cdot,t), -Au(\cdot,T-t)\rangle + \langle v(\cdot,t), \mathcal{B}u(\cdot,T-t)\rangle \\ &= \left\langle y(\cdot,t), \left(\tfrac{d}{dt}u\right)(\cdot,T-t)\right\rangle + \langle v(\cdot,t), \mathcal{B}u(\cdot,T-t)\rangle \\ &= -\left\langle y(\cdot,t), \tfrac{d}{dt}(u(\cdot,T-t))\right\rangle + \langle v(\cdot,t), \mathcal{B}u(\cdot,T-t)\rangle \ . \end{aligned}$$

On the other hand,

$$\left\langle \frac{d}{dt}y(\cdot,t), u(\cdot,T-t)\right\rangle + \left\langle y(\cdot,t), \frac{d}{dt}(u(\cdot,T-t))\right\rangle = \frac{d}{dt}\langle y(\cdot,t), u(\cdot,T-t)\rangle \ .$$

Therefore, integrating over $(0,T)$,

$$\langle y(\cdot,T), u(\cdot,0)\rangle - \langle y(\cdot,0), u(\cdot,T)\rangle = \int_0^T \langle v(\cdot,t), \mathcal{B}u(\cdot,T-t)\rangle \, dt \ .$$

The above computation are allowed for $(u(\cdot,0), y(\cdot,0)) \in D(A) \times D(A)$ and the last equality holds for any $(u(\cdot,0), y(\cdot,0)) \in H \times H$ by density arguments. Choosing $y(\cdot,0) = u(\cdot,T)$, we obtain by Cauchy-Schwarz inequality and using the inequality in (C)

$$\begin{aligned} \|u(\cdot,T)\|_H^2 &\leq \int_0^T |\langle v(\cdot,t), \mathcal{B}u(\cdot,T-t)\rangle| \, dt + |\langle y(\cdot,T), u(\cdot,0)\rangle| \\ &\leq \int_0^T \|v(\cdot,t)\|_F \|\mathcal{B}u(\cdot,T-t)\|_F \, dt + \|y(\cdot,T)\|_H \|u(\cdot,0)\|_H \\ &\leq \tfrac{1}{2}\tfrac{1}{\kappa}\int_0^T \|v(\cdot,t)\|_F^2 \, dt + \frac{1}{2}\frac{1}{\varepsilon}\|y(\cdot,T)\|_H^2 \\ &\quad + \tfrac{1}{2}\kappa \int_0^T \|\mathcal{B}u(\cdot,T-t)\|_F^2 \, dt + \frac{1}{2}\varepsilon \|u(\cdot,0)\|_H^2 \\ &\leq \tfrac{1}{2}\|y_0\|_H^2 + \tfrac{1}{2}\left(\kappa \int_0^T \|\mathcal{B}u(\cdot,T-t)\|_F^2 \, dt + \varepsilon \|u(\cdot,0)\|_H^2\right) \end{aligned}$$

which gives the desired estimate since $y_0 = u(\cdot,T)$.

Proof of $(O) \Rightarrow (C)$.- Let $y_0 \in H$. Consider the strictly convex C^1 functional J defined on H given by

$$J(u_0) = \frac{\kappa}{2}\int_0^T \left\| \mathcal{B}e^{-(T-t)A}u_0 \right\|_F^2 dt + \frac{\varepsilon}{2}\left\|u_0\right\|_H^2 + \left\langle y_0, e^{-TA}u_0 \right\rangle \ .$$

Notice that J is coercive and therefore J has a unique minimizer $z_0 \in H$, i.e. $J(z_0) = \min_{u_0 \in H} J(u_0)$. Let $z(\cdot,t) = e^{-tA}z_0$ and $h(\cdot,t) = e^{-tA}h_0$. Since $J'(z_0)h_0 = 0$ for any $h_0 \in H$, we have

$$\int_0^T \left\langle \kappa\mathcal{B}e^{-(T-t)A}z_0, \mathcal{B}e^{-(T-t)A}h_0 \right\rangle dt + \left\langle \varepsilon z_0, h_0 \right\rangle + \left\langle y_0, e^{-TA}h_0 \right\rangle = 0 \quad \forall h_0 \in H \ .$$

But

$$\left\langle y(\cdot,T), u(\cdot,0) \right\rangle - \left\langle y(\cdot,0), u(\cdot,T) \right\rangle = \int_0^T \left\langle v(\cdot,t), \mathcal{B}u(\cdot,T-t) \right\rangle dt \quad \forall u(\cdot,0) \in H$$

means

$$\int_0^T \left\langle v(\cdot,t), \mathcal{B}e^{-(T-t)A}h_0 \right\rangle dt - \left\langle e^{-TA}y_0, h_0 \right\rangle + \left\langle y_0, e^{-TA}h_0 \right\rangle = 0 \quad \forall h_0 \in H \ .$$

By choosing $v(\cdot,t) = \kappa\mathcal{B}e^{-(T-t)A}z_0$, we deduce that the solution $y(\cdot,t)$ to

$$\begin{cases} \frac{d}{dt}y(\cdot,t) + A^* y(\cdot,t) = \mathcal{B}^* v(\cdot,t) & \text{for } t \in (0,T] \ , \\ y(\cdot,0) = y_0 \in H & \end{cases}$$

satisfies

$$e^{-TA}y_0 = -\varepsilon z_0 \ .$$

Recall that $y(\cdot,T) = e^{-TA}y_0$. Moreover, taking $h_0 = z_0$,

$$\frac{1}{\kappa}\int_0^T \left\|v(\cdot,t)\right\|_F^2 dt + \frac{1}{\varepsilon}\left\|y(\cdot,T)\right\|_H^2 + \left\langle y_0, e^{-TA}z_0 \right\rangle = 0 \ .$$

Now, (O) implies

$$\left\| e^{-TA}z_0 \right\|_H^2 \leq \kappa \int_0^T \left\| \mathcal{B}e^{-(T-t)A}z_0 \right\|_F^2 dt + \varepsilon\left\|z_0\right\|_H^2 \leq \frac{1}{\kappa}\int_0^T \left\|v(\cdot,t)\right\|_F^2 dt + \frac{1}{\varepsilon}\left\|y(\cdot,T)\right\|_H^2 \ .$$

We conclude by Cauchy-Schwarz that

$$\frac{1}{\kappa}\int_0^T \left\|v(\cdot,t)\right\|_F^2 dt + \frac{1}{\varepsilon}\left\|y(\cdot,T)\right\|_H^2 \leq \left\|y_0\right\|_H^2 \ .$$

A direct application is given as follows.

Theorem .- *Let $\kappa > 0$. The following two statements are equivalent.*

(C) *For any $y_0 \in L^2(\Omega)$, there is a control $v \in L^2(\Omega \times (0,T))$ with*

$$\left\|v\right\|_{L^2(\Omega\times(0,T))} \leq \kappa \left\|y_0\right\|_{L^2(\Omega)} \ ,$$

such that the solution $y = y(x,t)$ to

$$\begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } \Omega \times (0,T) \ , \\ y = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ y(\cdot,0) = y_0 & \text{in } \Omega \ , \end{cases}$$

satisfies

$$y(\cdot,T) = 0 \text{ in } \Omega \ .$$

(O) *For any* $u_0 \in L^2(\Omega)$*, the solution* $u = u(x,t)$ *to*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 & \text{in } \Omega \ , \end{cases}$$

satisfies

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \kappa \|u\|_{L^2(\omega\times(0,T))} \ .$$

3.7 Inverse source problem

Recall that when $u_0 \in L^2(\Omega)$ and $u_0(x) = \sum_{j\geq 1} a_j e_j(x)$ with

$$a_j = \int_\Omega u_0(x)\, e_j(x)\, dx \text{ and } \sum_{j\geq 1} |a_j|^2 < +\infty \ ,$$

then

$$u(x,t) = \sum_{j\geq 1} a_j e^{-\lambda_j t} e_j(x)$$

solves

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \ , \\ u(\cdot,0) = u_0 & \text{in } \Omega \ . \end{cases}$$

On the other hand, by the null controllability for the heat equation, we have that for any e_j, there is a control function $f_j \in L^2(\Omega \times (0,T_j))$ such that

$$\|f_j\|_{L^2(\Omega\times(0,T_j))} \leq Ce^{C/T_j}$$

for some $C > 0$ and

$$\begin{cases} \partial_t z + \Delta z = f_{j|\omega\times(0,T_j)} & \text{in } \Omega \times (0,T_j) \ , \\ z = 0 & \text{on } \partial\Omega \times (0,T_j) \ , \\ z(\cdot,T_j) = e_j & \text{in } \Omega \ , \\ z(\cdot,0) = 0 & \text{in } \Omega \ . \end{cases}$$

We deduce the following formula.

$$u_0(x) = \sum_{j\geq 1} \left(e^{\lambda_j T_j} \int_\omega \int_0^{T_j} u(x,t)\, f_j(x,t)\, dxdt \right) e_j(x) \ .$$

Indeed, by multiplying the previous equation by u and integration over $\Omega \times (0,T_j)$, we get

$$\begin{aligned} \int_\Omega \int_0^{T_j} u f_{j|\omega\times(0,T_j)} &= \int_\Omega \int_0^{T_j} u(\partial_t z + \Delta z) \\ &= \int_\Omega u(\cdot,T_j)\, z(\cdot,T_j) - \int_\Omega u(\cdot,0)\, z(\cdot,0) - \int_\Omega \int_0^{T_j} \partial_t u z + \int_\Omega \int_0^{T_j} \Delta u z \\ &= \int_\Omega u(\cdot,T_j)\, e_j \ . \end{aligned}$$

But

$$u(x,T_j) = \sum_{k\geq 1} e^{-\lambda_k T_j} \left(\int_\Omega u_0(x)\, e_k(x)\, dx \right) e_k(x) \ .$$

Therefore,

$$\int_\omega \int_0^{T_j} u(x,t)\, f_j(x,t)\, dxdt = e^{-\lambda_j T_j} \int_\Omega u_0(x)\, e_j(x)\, dx$$

Finally,

$$u_0(x) = \sum_{j\geq 1} \left(e^{\lambda_j T_j} \int_\omega \int_0^{T_j} u(x,t) f_j(x,t)\, dxdt \right) e_j(x)$$

and

$$\begin{aligned}
\|u_0\|^2_{L^2(\Omega)} &= \textstyle\sum_{j\geq 1} e^{2\lambda_j T_j} \left| \displaystyle\int_\omega \int_0^{T_j} u(x,t) f_j(x,t)\, dxdt \right|^2 \\
&\leq \textstyle\sum_{j\geq 1} e^{2\lambda_j T_j} \left(\displaystyle\int_\omega \int_0^{T_j} \left| \sum_{k\geq 1} e^{-\lambda_k t} \left(\int_\Omega u_0 e_k dx \right) e_k \right| dxdt \right)^2 C e^{C/T_j} \\
&\leq C \textstyle\sum_{j\geq 1} e^{C\sqrt{\lambda_j}} \left(\displaystyle\int_\omega \int_0^{1/\sqrt{\lambda_j}} \left| \sum_{k\geq 1} e^{-\lambda_k t} \left(\int_\Omega u_0 e_k dx \right) e_k \right| dxdt \right)^2 \text{ if } T_j = \tfrac{1}{\sqrt{\lambda_j}} \ .
\end{aligned}$$

4 Quantitative uniqueness for the Laplacian operator

4.1 Doubling method for the Laplacian

see freephung.free.fr\kimdang\phung_n.pdf

4.1.1 The approach of Garofalo and Lin

Define for $r \in (0, R]$, $B_r = \left\{x \in \mathbb{R}^d ; |x - x_0| < r\right\}$ and

$$\mathcal{N}(r) = \frac{r \int_{B_r} |\nabla u(x)|^2 \, dx}{\int_{\partial B_r} |u(x)|^2 \, dx}, \text{ whenever } \int_{\partial B_r} |u(x)|^2 \, dx \neq 0 \ .$$

Our aim is to get good sign for the derivative of $\mathcal{N} = \mathcal{N}(r)$.

First, we compute the derivative of $\int_{\partial B_r} |u(x)|^2 \, dx$.

$$\begin{aligned} \frac{d}{dr} \int_{\partial B_r} |u(x)|^2 \, dx &= \frac{d}{dr} \int_{\mathbb{S}^{d-1}} |u(x_0 + rs)|^2 \, r^{d-1} ds \\ &= \frac{d-1}{r} \int_{\partial B_r} |u(x)|^2 \, dx + \frac{2}{r} \int_{\partial B_r} u(x) \nabla u(x) \cdot (x - x_0) \, dx \ . \end{aligned}$$

Notice that

$$\int_{\partial B_r} u(x) \nabla u(x) \cdot (x - x_0) \, dx = r \int_{B_r} |\nabla u(x)|^2 \, dx + r \int_{B_r} u(x) \Delta u(x) \, dx$$

using the fact that

$$\int_{\partial B_r} u(x) \nabla u(x) \cdot (x - x_0) \, dx = r \int_{\partial B_r} u(x) \partial_n u(x) \, dx \ .$$

Second, we compute the derivative of $r \int_{B_r} |\nabla u(x)|^2 \, dx$.

$$\frac{d}{dr} \left(r \int_{B_r} |\nabla u(x)|^2 \, dx \right) = \int_{B_r} |\nabla u(x)|^2 \, dx + r \int_{\partial B_r} |\nabla u(x)|^2 \, dx \ .$$

Now, $\int_{B_r} \mathrm{div}\Big(|\nabla u(x)|^2 (x - x_0)\Big) \, dx$ gives $r \int_{\partial B_r} |\nabla u(x)|^2 \, dx$ by Green formula. On the other hand,

$$\begin{aligned} &\int_{B_r} \mathrm{div}\left(|\nabla u(x)|^2 (x - x_0)\right) dx \\ &= d \int_{B_r} |\nabla u(x)|^2 \, dx + \int_{B_r} 2\partial_{x_i} u \partial^2_{x_j x_i} u (x_j - x_{0j}) \, dx \\ &= d \int_{B_r} |\nabla u(x)|^2 \, dx + \int_{\partial B_r} 2\nabla u(x) \cdot (x - x_0) \nabla u(x) \cdot n dx \\ &\quad - \int_{B_r} 2\Delta u(x) (x - x_0) \cdot \nabla u(x) \, dx - \int_{B_r} 2 |\nabla u(x)|^2 \, dx \\ &= (d-2) \int_{B_r} |\nabla u(x)|^2 \, dx + \frac{2}{r} \int_{\partial B_r} |\nabla u(x) \cdot (x - x_0)|^2 \, dx \\ &\quad - \int_{B_r} 2\Delta u(x) (x - x_0) \cdot \nabla u(x) \, dx \ . \end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dr}\left(r\int_{B_r}|\nabla u(x)|^2\,dx\right) &= (d-1)\int_{B_r}|\nabla u(x)|^2\,dx+\frac{2}{r}\int_{\partial B_r}|\nabla u(x)\cdot(x-x_0)|^2\,dx\\ &-\int_{B_r}2\Delta u(x)(x-x_0)\cdot\nabla u(x)\,dx\ .\end{aligned}$$

Combining the above equalities, the computation of the derivative of $\mathcal{N}=\mathcal{N}(r)$ gives

$$\begin{aligned}\left(\int_{\partial B_r}|u(x)|^2\,dx\right)^2\mathcal{N}'(r) &= (d-1)\int_{B_r}|\nabla u(x)|^2\,dx\int_{\partial B_r}|u(x)|^2\,dx\\ &+\frac{2}{r}\int_{\partial B_r}|\nabla u(x)\cdot(x-x_0)|^2\,dx\int_{\partial B_r}|u(x)|^2\,dx\\ &-\int_{B_r}2\Delta u(x)(x-x_0)\cdot\nabla u(x)\,dx\int_{\partial B_r}|u(x)|^2\,dx\\ &-\frac{d-1}{r}\int_{\partial B_r}|u(x)|^2\,dx\left(r\int_{B_r}|\nabla u(x)|^2\,dx\right)\\ &-\frac{2}{r}\int_{\partial B_r}u(x)\nabla u(x)\cdot(x-x_0)\,dx\left(r\int_{B_r}|\nabla u(x)|^2\,dx\right)\end{aligned}$$

that is

$$\begin{aligned}\left(\int_{\partial B_r}|u(x)|^2\,dx\right)^2\mathcal{N}'(r) &= \frac{2}{r}\int_{\partial B_r}|\nabla u(x)\cdot(x-x_0)|^2\,dx\int_{\partial B_r}|u(x)|^2\,dx\\ &-\int_{B_r}2\Delta u(x)(x-x_0)\cdot\nabla u(x)\,dx\int_{\partial B_r}|u(x)|^2\,dx\\ &-\frac{2}{r}\left|\int_{\partial B_r}u(x)\nabla u(x)\cdot(x-x_0)\,dx\right|^2\\ &+2\int_{\partial B_r}u(x)\nabla u(x)\cdot(x-x_0)\,dx\int_{B_r}u(x)\Delta u(x)\,dx\ .\end{aligned}$$

By Cauchy Schwarz,

$$0\le\frac{2}{r}\int_{\partial B_r}|\nabla u(x)\cdot(x-x_0)|^2\,dx\int_{\partial B_r}|u(x)|^2\,dx-\left|\int_{\partial B_r}u(x)\nabla u(x)\cdot(x-x_0)\,dx\right|^2$$

which implies

$$\begin{aligned}0 &\le \left(\int_{\partial B_r}|u(x)|^2\,dx\right)^2\mathcal{N}'(r)\\ &+\int_{B_r}2\Delta u(x)(x-x_0)\cdot\nabla u(x)\,dx\int_{\partial B_r}|u(x)|^2\,dx\\ &-2\int_{\partial B_r}u(x)\nabla u(x)\cdot(x-x_0)\,dx\int_{B_r}u(x)\Delta u(x)\,dx\ .\end{aligned}$$

Application: When $\Delta u=0$ in B_1, one has $\mathcal{N}'(r)\ge 0$ which gives $\mathcal{N}(r)\le\mathcal{N}(1)\ \forall r\le 1$. By

$$\frac{d}{dr}\int_{\partial B_r}|u(x)|^2\,dx=\frac{d-1}{r}\int_{\partial B_r}|u(x)|^2\,dx+2\int_{B_r}|\nabla u(x)|^2\,dx$$

we deduce that

$$\frac{d}{dr}\int_{\partial B_r}|u(x)|^2\,dx=\frac{d-1}{r}\int_{\partial B_r}|u(x)|^2\,dx+\frac{2}{r}\mathcal{N}(r)\int_{\partial B_r}|u(x)|^2\,dx$$

which gives $\forall r\le 1$

$$\frac{d}{dr}\ln\int_{\partial B_r}|u(x)|^2\,dx-(d-1)\frac{1}{r}\le 2\mathcal{N}(1)\frac{1}{r}\ .$$

By integrating between $R>0$ and $2R\geq 1$, one finds

$$\ln\left(\frac{\int_{\partial B_{2R}}|u(x)|^2\,dx}{\int_{\partial B_R}|u(x)|^2\,dx}\frac{1}{2^{d-1}}\right)\leq(2\ln 2)\mathcal{N}(1)\ ,$$

that is $\forall 0<R\leq 1/2$,

$$\int_{\mathbb{S}^{d-1}}|u(x_0+2Rs)|^2(2R)^{d-1}\,ds\leq 2^{d-1}e^{\mathcal{N}(1)\ln 4}\int_{\mathbb{S}^{d-1}}|u(x_0+Rs)|^2R^{d-1}ds\ .$$

One conclude that for any $M\leq 1/2$,

$$\begin{aligned}\int_{B_{2M}}|u(x)|^2\,dx &=\int_0^{2M}\int_{\mathbb{S}^{d-1}}|u(x_0+rs)|^2r^{d-1}drds\\ &=2\int_0^M\int_{\mathbb{S}^{d-1}}|u(x_0+2Rs)|^2(2R)^{d-1}\,dRds\\ &\leq 2^de^{\mathcal{N}(1)\ln 4}\int_0^M\int_{\mathbb{S}^{d-1}}|u(x_0+Rs)|^2R^{d-1}dRds\\ &\leq 2^de^{\mathcal{N}(1)\ln 4}\int_{B_M}|u(x)|^2\,dx\text{ with }\mathcal{N}(1)=\frac{\displaystyle\int_{B_1}|\nabla u(x)|^2\,dx}{\displaystyle\int_{\partial B_1}|u(x)|^2\,dx}\ .\end{aligned}$$

4.1.2 The approach of Kukavica

Define for $r\in(0,R]$, $B_r=\left\{x\in\mathbb{R}^d;|x-x_0|<r\right\}$ and

$$N(r)=\frac{\displaystyle\int_{B_r}|\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx}{\displaystyle\int_{B_r}|u(x)|^2\,dx},\text{ whenever }\int_{B_r}|u(x)|^2\,dx\neq 0\ .$$

Our aim is to get good sign for the derivative of $N=N(r)$.

First, we compute the derivative of $\displaystyle\int_{B_r}|u(x)|^2\,dx$.

$$\frac{d}{dr}\int_{B_r}|u(x)|^2\,dx=\int_{\partial B_r}|u(x)|^2\,dx\ .$$

Now, $\displaystyle\int_{B_r}\text{div}\Big(|u(x)|^2(x-x_0)\Big)\,dx$ gives by Green formula $r\displaystyle\int_{\partial B_r}|u(x)|^2\,dx$. On the other hand,

$$\int_{B_r}\text{div}\left(|u(x)|^2(x-x_0)\right)dx=d\int_{B_r}|u(x)|^2\,dx+\int_{B_r}2u(x)\nabla u(x)\cdot(x-x_0)\,dx\ .$$

Therefore

$$\frac{d}{dr}\int_{B_r}|u(x)|^2\,dx=\frac{d}{r}\int_{B_r}|u(x)|^2\,dx+\frac{1}{r}\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx\ .$$

Notice that

$$\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx=\int_{B_r}|\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx+\int_{B_r}u(x)\,\Delta u(x)\left(r^2-|x-x_0|^2\right)dx$$

using the fact that

$$\begin{aligned}\int_{B_r} 2u(x)(x-x_0)\cdot\nabla u(x)\,dx &= \int_{B_r} 2u(x)(-\frac{1}{2})\nabla\left(r^2-|x-x_0|^2\right)\cdot\nabla u(x)\,dx\\ &= \int_{B_r} \operatorname{div}(u(x)\nabla u(x))\left(r^2-|x-x_0|^2\right)dx\ .\end{aligned}$$

Second, we compute the derivative of $\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx$.

$$\frac{d}{dr}\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx = 2r\int_{B_r} |\nabla u(x)|^2\,dx\ .$$

Now, $\int_{B_r} \operatorname{div}\Big(|\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)(x-x_0)\Big)\,dx$ gives zero by Green formula. On the other hand,

$$\begin{aligned}&\int_{B_r} \operatorname{div}\left(|\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)(x-x_0)\right)dx\\ &= d\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx + \int_{B_r} |\nabla u(x)|^2\left(-2(x-x_0)\right)\cdot(x-x_0)\,dx\\ &\quad + \int_{B_r} 2\partial_{x_i}u\partial^2_{x_jx_i}u\left(r^2-|x-x_0|^2\right)(x_j-x_{0j})\,dx\\ &= d\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx - 2\int_{B_r} |\nabla u(x)|^2|x-x_0|^2\,dx\\ &\quad - \int_{B_r} 2\Delta u(x)(x-x_0)\cdot\nabla u(x)\left(r^2-|x-x_0|^2\right)dx - \int_{B_r} 2(-2)\left|(x-x_0)\cdot\nabla u(x)\right|^2dx\\ &\quad - \int_{B_r} 2|\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx\\ &= d\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx + \int_{B_r} |2(x-x_0)\cdot\nabla u(x)|^2\,dx - 2r^2\int_{B_r} |\nabla u(x)|^2\,dx\\ &\quad - \int_{B_r} 2\Delta u(x)(x-x_0)\cdot\nabla u(x)\left(r^2-|x-x_0|^2\right)dx\ .\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{d}{dr}\textstyle\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx\\ &= \frac{d}{r}\textstyle\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx + \frac{1}{r}\int_{B_r} |2(x-x_0)\cdot\nabla u(x)|^2\,dx\\ &\quad -\frac{1}{r}\textstyle\int_{B_r} 2\Delta u(x)(x-x_0)\cdot\nabla u(x)\left(r^2-|x-x_0|^2\right)dx\ .\end{aligned}$$

Combining the above equalities, the computation of the derivative of $N=N(r)$ gives

$$\begin{aligned}\left(\textstyle\int_{B_r}|u(x)|^2\,dx\right)^2 N'(r) &= \frac{d}{r}\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx\int_{B_r}|u(x)|^2\,dx\\ &\quad+\frac{1}{r}\int_{B_r} |2(x-x_0)\cdot\nabla u(x)|^2\,dx\int_{B_r}|u(x)|^2\,dx\\ &\quad-\frac{1}{r}\int_{B_r} 2\Delta u(x)(x-x_0)\cdot\nabla u(x)\left(r^2-|x-x_0|^2\right)dx\int_{B_r}|u(x)|^2\,dx\\ &\quad-\frac{d}{r}\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx\int_{B_r}|u(x)|^2\,dx\\ &\quad-\frac{1}{r}\int_{B_r} |\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx\\ &= \frac{1}{r}\int_{B_r} |2(x-x_0)\cdot\nabla u(x)|^2\,dx\int_{B_r}|u(x)|^2\,dx\\ &\quad-\frac{1}{r}\int_{B_r} 2\Delta u(x)(x-x_0)\cdot\nabla u(x)\left(r^2-|x-x_0|^2\right)dx\int_{B_r}|u(x)|^2\,dx\\ &\quad-\frac{1}{r}\left(\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx - \int_{B_r}u(x)\Delta u(x)\left(r^2-|x-x_0|^2\right)dx\right)\\ &\quad\times\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx\end{aligned}$$

that is

$$\begin{aligned}\left(\int_{B_r}|u(x)|^2\,dx\right)^2 N'(r) \quad &= \frac{1}{r}\int_{B_r}|2(x-x_0)\cdot\nabla u(x)|^2\,dx\int_{B_r}|u(x)|^2\,dx \\ &-\frac{1}{r}\int_{B_r}2\Delta u(x)(x-x_0)\cdot\nabla u(x)\left(r^2-|x-x_0|^2\right)dx\int_{B_r}|u(x)|^2\,dx \\ &-\frac{1}{r}\left|\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx\right|^2 \\ &+\frac{1}{r}\int_{B_r}u(x)\,\Delta u(x)\left(r^2-|x-x_0|^2\right)dx\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx\ .\end{aligned}$$

But

$$\begin{aligned}&\int_{B_r}|2(x-x_0)\cdot\nabla u(x)|^2\,dx\int_{B_r}|u(x)|^2\,dx \\ &-\int_{B_r}2\Delta u(x)(x-x_0)\cdot\nabla u(x)\left(r^2-|x-x_0|^2\right)dx\int_{B_r}|u(x)|^2\,dx \\ =&\left(\int_{B_r}\left|2(x-x_0)\cdot\nabla u(x)-\frac{1}{2}\Delta u(x)\left(r^2-|x-x_0|^2\right)\right|^2dx-\int_{B_r}\left|\frac{1}{2}\Delta u(x)\left(r^2-|x-x_0|^2\right)\right|^2dx\right) \\ &\times\int_{B_r}|u(x)|^2\,dx\ .\end{aligned}$$

and

$$\begin{aligned}&-\left|\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx\right|^2 \\ &+\int_{B_r}u(x)\,\Delta u(x)\left(r^2-|x-x_0|^2\right)dx\int_{B_r}2u(x)(x-x_0)\cdot\nabla u(x)\,dx \\ =&-\left|\int_{B_r}\left[2u(x)(x-x_0)\cdot\nabla u(x)-\frac{1}{2}u(x)\,\Delta u(x)\left(r^2-|x-x_0|^2\right)\right]dx\right|^2 \\ &+\left|\int_{B_r}\frac{1}{2}u(x)\,\Delta u(x)\left(r^2-|x-x_0|^2\right)dx\right|^2\ .\end{aligned}$$

By Cauchy Schwarz,

$$\begin{aligned}0 \quad &\le -\left|\int_{B_r}\left[2u(x)(x-x_0)\cdot\nabla u(x)-\frac{1}{2}u(x)\,\Delta u(x)\left(r^2-|x-x_0|^2\right)\right]dx\right|^2 \\ &+\int_{B_r}\left|2(x-x_0)\cdot\nabla u(x)-\frac{1}{2}\Delta u(x)\left(r^2-|x-x_0|^2\right)\right|^2dx\int_{B_r}|u(x)|^2\,dx\end{aligned}$$

which implies

$$\begin{aligned}0 \quad &\le \left(\int_{B_r}|u(x)|^2\,dx\right)^2 N'(r) \\ &-\frac{1}{r}\left|\int_{B_r}\frac{1}{2}u(x)\,\Delta u(x)\left(r^2-|x-x_0|^2\right)dx\right|^2 \\ &+\frac{1}{r}\int_{B_r}\left|\frac{1}{2}\Delta u(x)\left(r^2-|x-x_0|^2\right)\right|^2dx\int_{B_r}|u(x)|^2\,dx\ .\end{aligned}$$

Application: When $\Delta u=0$, one has $N'(r)\ge 0$. By

$$\frac{d}{dr}\int_{B_r}|u(x)|^2\,dx=\frac{d}{r}\int_{B_r}|u(x)|^2\,dx+\frac{1}{r}\int_{B_r}|\nabla u(x)|^2\left(r^2-|x-x_0|^2\right)dx$$

we deduce that

$$\frac{d}{dr}\int_{B_r}|u(x)|^2\,dx=\frac{d}{r}\int_{B_r}|u(x)|^2\,dx+\frac{1}{r}N(r)\int_{B_r}|u(x)|^2\,dx\ .$$

For $r_1 \leq r \leq r_2$,

$$\frac{d}{dr}\ln\int_{B_r}|u(x)|^2\,dx = (d+N(r))\,\frac{d}{dr}\ln r \leq (d+N(r_2))\,\frac{d}{dr}\ln r$$

which implies

$$\ln\frac{\int_{B_{r_2}}|u(x)|^2\,dx}{\int_{B_{r_1}}|u(x)|^2\,dx} \leq (d+N(r_2))\ln\frac{r_2}{r_1}\ .$$

For $r_2 \leq r \leq r_3$,

$$-\frac{d}{dr}\ln\int_{B_r}|u(x)|^2\,dx = -(d+N(r))\,\frac{d}{dr}\ln r \leq -(d+N(r_2))\,\frac{d}{dr}\ln r$$

which implies

$$(d+N(r_2))\ln\frac{r_3}{r_2} \leq \ln\frac{\int_{B_{r_3}}|u(x)|^2\,dx}{\int_{B_{r_2}}|u(x)|^2\,dx}\ .$$

Therefore

$$\frac{1}{\ln\frac{r_2}{r_1}}\ln\frac{\int_{B_{r_2}}|u(x)|^2\,dx}{\int_{B_{r_1}}|u(x)|^2\,dx} \leq \frac{1}{\ln\frac{r_3}{r_2}}\ln\frac{\int_{B_{r_3}}|u(x)|^2\,dx}{\int_{B_{r_2}}|u(x)|^2\,dx}$$

that is

$$\int_{B_{r_2}}|u(x)|^2\,dx \leq \left(\int_{B_{r_1}}|u(x)|^2\,dx\right)^{\alpha}\left(\int_{B_{r_3}}|u(x)|^2\,dx\right)^{1-\alpha} \text{ with } \alpha = \frac{\ln\frac{r_3}{r_2}}{\ln\frac{r_3}{r_2}+\ln\frac{r_2}{r_1}}\ .$$

4.2 Comments on Carleman inequalities

...(to be completed)...

4.3 Applications to wave and heat

...(to be completed)...

5 Background

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5.1 Potential

The study of hyperbolic equations with a potential in (x, t)-variable is done by Zhang [Zh] using a global Carleman inequality.

5.2 About the heat equation

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5.3 About the doubling property for elliptic operators and for parabolic operators

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6 Courses

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