Reconstruction of initial data for parabolic equations via impulse controls

Kim Dang PHUNG

Université d'Orléans.

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Our goal

Ω is a C^2 bounded domain in $\mathbb{R}^n$,

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 & \in H_0^1(\Omega) \ , \\ \omega \text{ is} & \text{a non empty open subset of } \Omega \ . \end{cases}$$

Can we recover u_0 from a noisy knowledge of $u(\cdot,T)$ restricted to ω?

Our method also applies to $\partial_t u + Pu = 0$ where $Pu = -\text{div}(A\nabla u)$ with $D(P) = H^2 \cap H_0^1(\Omega)$
and $A \in C^1\left(\overline{\Omega}\right)$ $n \times n$ symmetric positive-definite matrix,
or $Pu = \Delta^2 u$ with $D(P) = H^4 \cap H_0^2(\Omega)$,

Our tools

Eigenfunction expansion

Impulse control

Backward inverse problem

Eigenfunction expansion

$$\begin{cases} -\Delta e_i = \lambda_i e_i \quad \text{in } \Omega \ , \\ e_i \in H^2 \cap H_0^1(\Omega) \ , \quad 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \ \lambda_j \to +\infty \end{cases} .$$

Multiply $\partial_t u - \Delta u = 0$ by e_i, we solve an ODE and get

$$u(\cdot, t) = \sum_{j \geq 1} \langle u_0, e_j \rangle e^{-\lambda_j t} e_j \quad \forall t \geq 0$$

$$u(\cdot, t_2) = \sum_{j \geq 1} \langle u(\cdot, t_1), e_j \rangle e^{-\lambda_j (t_2 - t_1)} e_j \quad \forall t_2 \geq t_1$$

False for $t_2 < t_1$!

Impulse control for each eigenfunction

For any e_j there is an impulse control $f_j \in L^2(\Omega)$,

$$\begin{cases} \partial_t \varphi_j - \Delta \varphi_j = 1_\omega f_j \otimes \delta_{t=T} & \text{in } \Omega \times (0, 2T) \ , \\ \delta_{t=T} \text{ is} & \text{the Dirac measure at time } t = T \ , \\ \varphi_j(\cdot, 0) = e_j & \text{in } \Omega \ , \end{cases}$$

$\left\|\varphi_j(\cdot, 2T)\right\|_{L^2(\Omega)} \leq \varepsilon$ and $\|f_j\|_{L^2(\omega)} \leq \frac{c}{\varepsilon^\theta} e^{\frac{C}{T}}$

We will make an expansion with φ_j

Impulse control for each eigenfunction

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$$\begin{cases} \partial_t \varphi_j - \Delta \varphi_j = 0 & \text{in } \Omega \times (0, 2T) \setminus \{T\} \quad , \\ \varphi_j(\cdot, T_+) & = \varphi_j(\cdot, T_-) + 1_\omega f_j \text{ in } \Omega \; , \\ \varphi_j(\cdot, 0) = e_j & \text{in } \Omega \; , \end{cases}$$

$$\left\| \varphi_j(\cdot, 2T) \right\|_{L^2(\Omega)} \leq \varepsilon \text{ and } \left\| f_j \right\|_{L^2(\omega)} \leq \frac{c}{\varepsilon^\theta} e^{\frac{C}{T}}$$

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Impulse control for each eigenfunction

$$\left\|\varphi_j\left(\cdot,2T\right)\right\|_{L^2(\Omega)} \leq \varepsilon \text{ and } \left\|f_j\right\|_{L^2(\omega)} \leq \frac{c}{\varepsilon^\theta} e^{\frac{C}{T}}$$

Multiply $\partial_t u - \Delta u = 0$ by $\varphi_j\left(2T - t\right)$, and integrate over $(T, 2T)$ and over $(0, T)$ to get

$$\left\langle u\left(\cdot,2T\right), e_j\right\rangle = \left\langle u\left(\cdot,0\right), \varphi_j\left(\cdot,2T\right)\right\rangle + \int_\omega u\left(\cdot,T\right) f_j$$

Backward inverse problem (filtering)

If

$$\|u(\cdot,L)-y\|_{L^2(\Omega)} \leq \mu \text{ and } \mu \leq \tfrac{1}{4}\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)}$$

Then

$$\|u(\cdot,0)-V_\Omega\|_{L^2(\Omega)} \leq \sqrt{\frac{3}{2}}\frac{\sqrt{L}\|u_0\|_{H_0^1(\Omega)}}{\sqrt{\ln\left(\frac{1}{\mu}\sqrt{L}\|u_0\|_{H_0^1(\Omega)}\right)}}$$

$V_\Omega = \sum_{j\geq 1} \min\left(e^{\lambda_j L},\beta\right)\langle y,e_j\rangle e_j$ and β is a function of $\frac{1}{\mu}\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)}$

State of art

1. Usual controllability is:
$\forall u_0 \in L^2(\Omega)$ there is a control localized on $\omega \times (0,T)$ such that $u(\cdot,T)=0$.
See Lebeau-Robbiano, Fursikov-Imanuvilov (1995)

2. Impulse output rapid stabilization for heat equations.
See Phung-Wang-Xu, JDE (2017)

3. A spectral inequality for degenerated operators and applications.
See Buffe-Phung, (CRM 2018)

4. Optimal filtering for the backward heat equation.
See Seidman, (SIAM 1996)

5. The local backward heat problem.
See Vo, (arXiv 2017)

New result

If

$$\|u(\cdot,T)-z\|_{L^2(\omega)} \leq \delta \text{ and } \delta \ll C_T \|u_0\|_{H_0^1(\Omega)}$$

Then

$$\|u(\cdot,0)-V\|_{L^2(\Omega)} \leq c\frac{\sqrt{1+T}\|u_0\|_{H_0^1(\Omega)}}{\sqrt{\ln\left(\frac{1}{\delta}C_T\|u_0\|_{H_0^1(\Omega)}\right)}}$$

$$V=\sum_{j\geq 1}\min\left(e^{\lambda_j(2T+1)},\beta\right)e^{-\lambda_j}\int_\omega z(x)\,f_j(x)\,dx e_j$$

and β is a function of $\frac{1}{\delta}C_T\|u_0\|_{H_0^1(\Omega)}$

Proof

One start with

$$\langle u(\cdot,2T),e_j\rangle = \left\langle u(\cdot,0),\varphi_j(\cdot,2T)\right\rangle + \int_\omega u(\cdot,T)\,f_j$$

that is

$$\langle u(\cdot,2T),e_j\rangle - \int_\omega z f_j = \left\langle u(\cdot,0),\varphi_j(\cdot,2T)\right\rangle + \int_\omega (u(\cdot,T)-z)\,f_j$$

Then one uses

$$u(\cdot,2T+1) = \sum_{j\geq 1} e^{-\lambda_j}\langle u(\cdot,2T),e_j\rangle e_j$$

Multiply by $e^{-\lambda_j}$ and sum over j,

$$u(\cdot, 2T+1) - \sum_{j\geq 1} e^{-\lambda_j} \int_\omega z f_j e_j =$$
$$\sum_{j\geq 1} e^{-\lambda_j} \left\langle u(\cdot, 0), \varphi_j(\cdot, 2T) \right\rangle e_j + \sum_{j\geq 1} e^{-\lambda_j} \int_\omega (u(\cdot, T) - z) f_j e_j$$

Next, compute the norm

$$\left\| u(\cdot, 2T+1) - \sum_{j\geq 1} e^{-\lambda_j} \int_\omega z f_j e_j \right\| \leq C \left(\|u_0\| \varepsilon + \frac{c}{\varepsilon^\theta} e^{\frac{C}{T}} \delta \right)$$

Choose $\|u_0\| \varepsilon = \frac{c}{\varepsilon^\theta} e^{\frac{C}{T}} \delta$ that is $\varepsilon = \left(c e^{\frac{C}{T}} \delta \frac{1}{\|u_0\|} \right)^{1/1+\theta}$ and apply Backward inverse problem.

Proof of backward (filtering)

Write

$$V_\Omega = \sum_{j\geq 1} \min\left(e^{\lambda_j L}, \beta\right) \langle y, e_j\rangle e_j$$

and

$$W = \sum_{j\geq 1} \min\left(e^{\lambda_j L}, \beta\right) \langle u(\cdot, L), e_j\rangle e_j$$

Then

$$\|u(\cdot,0) - V_\Omega\| \leq \|u(\cdot,0) - W\| + \|V_\Omega - W\|$$

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$$\|u(\cdot, 0) - V_\Omega\| \leq \|u(\cdot, 0) - W\| + \beta\mu$$

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Then

$$\|u(\cdot,0) - V_\Omega\| \leq$$

$$\left\| \sum_{j\geq 1} \langle u(\cdot,0), e_j\rangle e_j - \sum_{j\geq 1} \min\left(e^{\lambda_j L}, \beta\right) \langle u(\cdot, L), e_j\rangle e_j \right\| + \beta\mu$$

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and

$$W = \sum_{j\geq 1} \min\left(e^{\lambda_j L}, \beta\right) \langle u(\cdot, L), e_j\rangle e_j$$

Then

$$\|u(\cdot,0) - V_\Omega\| \leq \left\| \sum_{j\geq 1} \left(1 - \min\left(e^{\lambda_j L}, \beta\right) e^{-\lambda_j L}\right) \langle u(\cdot,0), e_j\rangle e_j \right\| + \beta\mu$$

Proof of backward (filtering)

Next

$$
\begin{aligned}
&\left\| u(\cdot,0) - V_\Omega \right\| \le \\
&\le \left\| \sum_{j\ge 1} \left(1 - \min\left(e^{\lambda_j L}, \beta\right) e^{-\lambda_j L}\right) \langle u(\cdot,0), e_j\rangle e_j \right\| + \beta\mu \\
&\le \left\| \sum_{\beta < e^{\lambda_j L}} \left(1 - \beta e^{-\lambda_j L}\right) \langle u(\cdot,0), e_j\rangle e_j \right\| + \beta\mu \\
&\le \left\| \sum_{\beta < e^{\lambda_j L}} \left(1 - \beta e^{-\lambda_j L}\right) \frac{1}{\sqrt{\lambda_j L}} \sqrt{\lambda_j L} \langle u(\cdot,0), e_j\rangle e_j \right\| + \beta\mu \\
&\le \left(\sum_{\beta < e^{\lambda_j L}} \left[\left(1 - \beta e^{-\lambda_j L}\right) \frac{1}{\sqrt{\lambda_j L}} \right]^2 \lambda_j L \left| \langle u(\cdot,0), e_j\rangle \right|^2 \right)^{1/2} + \beta\mu \\
&\le \sup_{\lambda > 0} \left[\left(1 - \beta e^{-\lambda L}\right) \frac{1}{\sqrt{\lambda L}} \right] \sqrt{L} \left\| u_0 \right\|_{H_0^1(\Omega)} + \beta\mu
\end{aligned}
$$

Proof of backward (filtering)

Finally

$$\begin{aligned}
&\|u(\cdot,0) - V_\Omega\| \leq \\
&\leq \left[\left(1-\beta e^{-\overline{\lambda}L}\right)\frac{1}{\sqrt{\overline{\lambda}L}}\right]\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)} + \beta\mu \\
&\leq \left(1-\beta e^{-\overline{\lambda}L}\right)\frac{1}{\sqrt{\overline{\lambda}L}}\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)} + \left(\beta e^{-\overline{\lambda}L}\right)e^{\overline{\lambda}L}\mu \\
&\leq \frac{1}{\sqrt{\overline{\lambda}L}}\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)} \text{ with } \frac{1}{\sqrt{\overline{\lambda}L}}\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)} = e^{\overline{\lambda}L}\mu \\
&\leq \frac{1}{\sqrt{\overline{\lambda}L}}\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)} \text{ with } \overline{\lambda} \text{ is a function of } \beta \\
&\leq \sqrt{\frac{3}{2}}\frac{1}{\sqrt{\ln\left(\frac{1}{\mu}\sqrt{L}\|u_0\|_{H_0^1(\Omega)}\right)}}\sqrt{L}\,\|u_0\|_{H_0^1(\Omega)}
\end{aligned}$$

Link with log convexity

Suppose

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u(\cdot,0) = u_0 & \in H_0^1(\Omega) \end{cases} ,\ .$$

Then

$$\frac{d}{dt}\frac{\|\nabla u(\cdot,t)\|^2}{\|u(\cdot,t)\|^2} \leq 0$$

and

$$\|u(\cdot,0)\| \leq e^{\frac{\|\nabla u(\cdot,t)\|^2}{\|u(\cdot,t)\|^2}T} \|u(\cdot,T)\|$$

or equivalently

$$\|u(\cdot,0)\| \leq \frac{1}{\sqrt{\ln\left(\frac{\|u(\cdot,0)\|}{\|u(\cdot,T)\|}\right)}}\sqrt{T}\,\|u_0\|_{H_0^1(\Omega)}$$

Same with a weight function

To prove observation at one point in time, introduce

$$f(x,t) = u(x,t)\, e^{\Phi(x,t)/2}$$

Then

$$\begin{cases} \partial_t f + Sf = Af & \text{in } \Omega \times (0,T) \;, \\ f = 0 & \text{on } \partial\Omega \times (0,T) \;, \\ \langle Af, f \rangle = 0 & \\ \langle Sf, f \rangle = \langle f, Sf \rangle & \end{cases}$$

and

$$\frac{1}{2}\frac{d}{dt}\left\| f(\cdot,t) \right\|^2 + \langle Sf, f \rangle = 0$$

$$N(t) = \frac{\langle Sf, f \rangle}{\left\| f(\cdot,t) \right\|^2}$$

$$\frac{d}{dt} N(t) \left\| f(\cdot,t) \right\|^2 \leq \left(\left\langle \tfrac{d}{dt} Sf, f \right\rangle + 2 \langle Sf, Af \rangle \right)$$

Instead of $\frac{d}{dt}N(t) \leq 0$, we choose the weight function $\Phi(x,t) = \dfrac{\varphi(x)}{T-t+h}$ such that

$$\frac{d}{dt}N(t)\|f(\cdot,t)\|^2 \leq \left\langle \frac{d}{dt}Sf, f\right\rangle + 2\langle Sf, Af\rangle \leq \frac{\langle Sf, f\rangle}{T-t+h}$$

that is $\frac{d}{dt}N(t) \leq \dfrac{1}{T-t+h}N(t)$. Next, solve an ODE

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|f(\cdot,t)\|^2 + N(t)\|f(\cdot,t)\|^2 = 0, \\ \frac{d}{dt}N(t) \leq \dfrac{1}{T-t+h}N(t) \ . \end{cases}$$

to have for $t_1 < t_2 < T$

$$\|f(\cdot,t_2)\|^{1+M} \leq C\|f(\cdot,t_1)\|^M \|f(\cdot,T)\|$$

The localization is given from $\|f(\cdot,T)\|$ as follows:

$$\begin{aligned}
\|f(\cdot,T)\|^2 &= \int_\Omega |u(x,t)|^2 e^{\dfrac{\varphi(x)}{T-t+h}} dx \text{ with } t=T,\\
&\leq \int_{\Omega\backslash\omega} |u(x,T)|^2 e^{\dfrac{\varphi(x)}{h}} dx + \int_\omega |u(x,T)|^2 e^{\dfrac{\varphi(x)}{h}} dx \ ,\\
&\leq e^{-\dfrac{\kappa}{h}} \int_{\Omega\backslash\omega} |u(x,T)|^2 dx + \int_\omega |u(x,T)|^2 dx \text{ with } \varphi(x)\leq 0,\\
&\leq e^{-\dfrac{\kappa}{h}} \int_\Omega |u(x,0)|^2 dx + \int_\omega |u(x,T)|^2 dx \ ,
\end{aligned}$$

in order that

$$\|u(\cdot,T)\|^{1+M} \leq c e^{C/T} \|u(\cdot,0)\|^M \left\| u(\cdot,T)_{|\omega} \right\|$$

Thank You