Observation at one point for parabolic equations by a simplest way

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Our goal

Ω is a C^2 bounded convex domain in $\mathbb{R}^n$,

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 & \in L^2(\Omega) \ , \\ \omega \text{ is} & \text{a non empty open subset of } \Omega \ . \end{cases}$$

Observation at one point in time

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq Ce^{\frac{c}{T^\beta}} \|u(\cdot,0)\|^{\theta}_{L^2(\Omega)} \|u(\cdot,T)\|^{1-\theta}_{L^2(\omega)}$$

What does it imply?

$u(\cdot, T) = 0$ in $\omega \Longrightarrow u(\cdot, 0) = 0$

Observability for the heat with a positive measurable set in time

Lebeau-Robbiano spectral inequality (and conversely)

$$\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq c e^{c\lambda^{\beta'}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx$$

This inequality is also true from Lebeau-Robbiano for

Bilaplacian $\partial_t + \Delta^2$ (Le Rousseau-Robbiano (JEMS 08))

Stokes operator (ChavesSilva-Lebeau (2016))

1d degenerate operator $\partial_t - \partial_x \left(x^\gamma \partial_x \right)$ with $\gamma \in (0, 2)$
(Buffe (2018))

Time varying fractional Laplacian $\partial_t - \Delta^{\gamma(t)}$ with $\gamma\left(t\right) \geq \gamma_0 > \frac{1}{2}$
(Liang Zhang-Xin Yu (2018))

with many applications in control theory and inverse problems

What are the known extensions from the proof here

Drop the convexity of Ω by localization and propagation of smallness

Add space-time coefficient
$\partial_t u - \text{div}(A(x,t)\nabla u) + a(x,t)u + b(x,t)\cdot\nabla u = 0$

State of art for unique continuation

Log convexity for backward: Agmon-Nirenberg (67), Bardos-Tartar (73), Ghidaglia (86), FanHua Lin (90),...

Carleman estimates for observability: Lebeau-Robbiano (95), Fursikov-Imanuvilov (96), ...

Log convexity

Claim :

$$t \longmapsto \ln \|u(\cdot,t)\|^2_{L^2(\Omega)} \text{ is convex}$$

Indeed,

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 + \langle -\Delta u, u\rangle = 0$$

$$N(t) = \frac{\langle -\Delta u, u\rangle}{\|u(\cdot,t)\|^2} \text{ satisfies } \frac{d}{dt}N(t) \leq 0$$

because

$$\begin{aligned}\frac{d}{dt}N(t)\left(\|u\|^2\right)^2 &= \frac{d}{dt}\|\nabla u\|^2\|u\|^2 - \|\nabla u\|^2\frac{d}{dt}\|u\|^2 \\ &= -2\|\Delta u\|^2\|u\|^2 + 2\|\nabla u\|^4 \leq 0\end{aligned}$$

Since

$$t \longmapsto \ln \|u(\cdot,t)\|^2_{L^2(\Omega)} \text{ is convex}$$

we have

$$\begin{aligned}\ln\|u(\cdot,\theta t_1+(1-\theta)t_2)\|^2 &\leq \theta\ln\|u(\cdot,t_1)\|^2+(1-\theta)\ln\|u(\cdot,t_2)\|^2\\ &\leq \ln\left(\left(\|u(\cdot,t_1)\|^2\right)^{\theta}\left(\|u(\cdot,t_2)\|^2\right)^{1-\theta}\right)\end{aligned}$$

in order to conclude that for $0<t<T$

$$\|u(\cdot,t)\|^2 \leq \left(\|u(\cdot,0)\|^2\right)^{\theta}\left(\|u(\cdot,T)\|^2\right)^{1-\theta}$$

A win for time at the left will allow a win for space at the right

Carleman commutator

Do the same with a weight function

$$f(x,t) = u(x,t)\, e^{\Phi(x,t)/2}$$

Then it solves (Escauriaza-Kenig-Ponce-Vega (JEMS 08))

$$\partial_t f \underbrace{-\Delta f - \tfrac{1}{2} f \left(\partial_t \Phi + \tfrac{1}{2} |\nabla \Phi|^2\right)}_{Sf \text{ with } \langle Sf,g\rangle = \langle Sg,f\rangle + \int_{\partial\Omega} \partial_n f g d\sigma} + \underbrace{\nabla\Phi \cdot \nabla f + \tfrac{1}{2}\Delta\Phi f}_{-Af \text{ with } \langle Af,g\rangle = -\langle Ag,f\rangle} = 0$$

$$\begin{cases} \partial_t f + Sf = Af & \text{in } \Omega \times (0,T) \ , \\ f = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ \langle Af, f\rangle = 0 & \end{cases}$$

$$\frac{1}{2}\frac{d}{dt}\|f(\cdot,t)\|^2 + \langle Sf, f\rangle = 0$$

$$N(t) = \frac{\langle Sf, f\rangle}{\|f(\cdot,t)\|^2}$$

Next, compute $\frac{d}{dt}N(t)$

$$\begin{aligned}\frac{d}{dt}\langle Sf,f\rangle &= \langle S'f,f\rangle + \langle Sf',f\rangle + \langle Sf,f'\rangle \\ &= \langle S'f,f\rangle + 2\langle Sf,f'\rangle \\ &= \langle S'f,f\rangle + 2\langle Sf,-Sf+Af\rangle \\ &= -2\|Sf\|^2 + \langle S'f,f\rangle + 2\langle Sf,Af\rangle\end{aligned}$$

Therefore $\frac{d}{dt}N(t)\left(\|f\|^2\right)^2 = \frac{d}{dt}\langle Sf,f\rangle\|f\|^2 - \langle Sf,f\rangle\frac{d}{dt}\|f\|^2$

$$= -2\|Sf\|^2\|f\|^2 + 2\left(\langle Sf,f\rangle\right)^2 + \langle S'f,f\rangle + 2\langle Sf,Af\rangle$$

$$\frac{d}{dt}N(t)\|f(\cdot,t)\|^2 \leq \langle S'f,f\rangle + 2\langle Sf,Af\rangle$$

Choice of the weight function

Take the heat kernel $\Phi(x,t) = \dfrac{-|x-x_0|^2/4}{T-t+h} - \frac{n}{2}\ln(T-t+h)$

or take it of the form $\Phi(x,t) = \dfrac{-s\varphi(x)}{T-t+h}$ such that

$$\langle S'f, f\rangle + 2\langle Sf, Af\rangle \leq \frac{\langle Sf, f\rangle}{T-t+h}$$

that is $\frac{d}{dt}N(t) \leq \dfrac{1}{T-t+h}N(t)$. Next, solve an ODE

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|f(\cdot,t)\|^2 + N(t)\|f(\cdot,t)\|^2 = 0 \\ \frac{d}{dt}N(t) \leq \dfrac{1}{T-t+h}N(t) \end{cases}$$

to have for $t_1 < t_2 < T$

$$\|f(\cdot,t_2)\|^{1+M} \leq C\|f(\cdot,t_1)\|^M \|f(\cdot,T)\|$$

The localization for space is given from $\|f(\cdot,T)\|$ as follows:

$$\|f(\cdot,T)\|^2 = \int_{\Omega} |u(x,t)|^2 e^{\frac{-|x-x_0|^2/4}{T-t+h}} dx \text{ with } t=T,$$

$$\leq \int_{\Omega\setminus B(r,x_0)} |u(x,T)|^2 e^{\frac{-|x-x_0|^2/4}{h}} dx + \int_{B(r,x_0)} |u(x,T)|^2 e^{\frac{-|x-x_0|^2/4}{h}} dx$$

$$\leq e^{-\frac{c}{h}} \int_{\Omega\setminus B(r,x_0)} |u(x,T)|^2 dx + \int_{B(r,x_0)} |u(x,T)|^2 dx \ ,$$

$$\leq e^{-\frac{c}{h}} \int_{\Omega} |u(x,0)|^2 dx + \int_{B(r,x_0)} |u(x,T)|^2 dx \ ,$$

The left hand-side $\|f(\cdot,t_2)\|^{1+M}$ with a win for time implies:

Take t_2 closed to T,

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M} \leq \left(\int_\Omega |u(x,t_2)|^2 dx\right)^{1+M} ,$$

$$\leq \left(\int_\Omega |u(x,t_2)|^2 e^{\frac{-|x-x_0|^2/4}{T-t_2+h}} e^{\frac{|x-x_0|^2/4}{T-t_2+h}} dx\right)^{1+M} ,$$

$$\leq e^{\dfrac{C(1+M)}{T-t_2+h}} \left(\|f(\cdot,t_2)\|^2\right)^{1+M} ,$$

in order that

$$e^{\dfrac{C(1+M)}{T-t_2+h}} e^{-\dfrac{c}{h}} \leq e^{-\dfrac{\kappa}{h}}$$

to conclude

$$\|u(\cdot,T)\|^{1+M} \leq c e^{C/T} \|u(\cdot,0)\|^M \left\|u(\cdot,T)_{|\omega}\right\|$$

UCP with space-time potential

$$\left\{\begin{array}{rl} \partial_t u - \Delta u + a(x,t)\, u = 0 & \text{in } \Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 & \in L^2(\Omega) \ , \\ \omega \text{ is} & \text{a non empty open subset of } \Omega \ . \end{array}\right.$$

ODE

$$\left\{\begin{array}{l} \left| \frac{1}{2}\frac{d}{dt} \left\| f(\cdot,t) \right\|^2 + N(t) \left\| f(\cdot,t) \right\|^2 \right| = \left| \langle af, f \rangle \right| \leq \left\| a \right\|_{L^\infty} \left\| f(\cdot,t) \right\|^2 \\ \frac{d}{dt} N(t) \leq \dfrac{1}{T-t+h} N(t) + \frac{\langle af, f\rangle}{\left\| f(\cdot,t) \right\|^2} \end{array}\right.$$

UCP with space-time potential

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Observation at one point in time

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq C e^{\frac{c}{T}} \|u(\cdot,0)\|^{\theta}_{L^2(\Omega)} \|u(\cdot,T)\|^{1-\theta}_{L^2(\omega)}$$

What about $\varepsilon \partial_t^2 u - \Delta u + \partial_t u + a(x,t)\, u = 0$ for very small ε.
Can we have $u(\cdot,T) = 0$ in $\omega \Longrightarrow u(\cdot,0) = 0$

Thank You