Control and stabilization of PDEs

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Our PDEs today

Ω is a C^2 bounded open connected set in $\mathbb{R}^N$, ω is a non empty open subset of Ω

$$\begin{cases} \partial_t^2 u - \Delta u + 1_{|\omega} \partial_t u = 0 & \text{in } \Omega \times]0,+\infty[\ , \\ u = 0 & \text{on } \partial\Omega \times]0,+\infty[\ , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) & \in H_0^1(\Omega) \times L^2(\Omega) \ . \end{cases}$$

Damped wave equation

$$\begin{cases} \partial_t a - d_1 \Delta a = 1_{|\omega}(b-a) & \text{in } \Omega \times]0,+\infty[\ , \\ \partial_t b - d_2 \Delta b = -1_{|\omega}(b-a) & \text{in } \Omega \times]0,+\infty[\ , \\ \partial_n a = \partial_n b = 0 & \text{on } \partial\Omega \times]0,+\infty[\ , \\ (a,b)(\cdot,0) = (a_0, b_0) & \in L^2(\Omega) \times L^2(\Omega) \ . \end{cases}$$

Reaction-diffusion coming out of reversible chemistry with $0 < d_1 < d_2$

Recall $u = u(x,t) = u(x_1, \cdot\cdot, x_N, t)$ and $\Delta u = \partial_{x_1}^2 u + \cdot\cdot + \partial_{x_N}^2 u = \frac{\partial^2}{\partial x_1^2} u + \cdot\cdot + \frac{\partial^2}{\partial x_N^2} u$ and $\partial_n a = \nabla a \cdot n$

Main results

Theorem 1. Under Geometrical Conditions

$$\left\|(u,\partial_t u)(\cdot,t)\right\|_{H_0^1\times L^2(\Omega)} \leq Ce^{-\beta t}\left\|(u,\partial_t u)(\cdot,0)\right\|_{H_0^1\times L^2(\Omega)}$$

Theorem 2. Suppose $|\Omega| = 1$ and $\int_\Omega a_0 + b_0 = 2$

$$\left\|(a-1,b-1)(\cdot,t)\right\|_{L^2\times L^2(\Omega)} \leq Ce^{-\beta t}\left\|(a_0-1,b_0-1)\right\|_{L^2\times L^2(\Omega)}$$

Integration by parts

$$\int_\Omega \Delta u \partial_t u = -\int_\Omega \nabla u \partial_t \nabla u + \int_{\partial\Omega} \partial_n u \partial_t u = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \ ,$$

$$\int_\Omega \Delta a a = -\int_\Omega |\nabla a|^2 + \int_{\partial\Omega} \partial_n a a \ ,$$

$$\int_\Omega \Delta a = \int_{\partial\Omega} \partial_n a a \ ,$$

Poincare inequality

$$\int_\Omega \left| f - \frac{1}{|\Omega|}\int_\Omega f \right|^2 \leq C_P \int_\Omega |\nabla f|^2 \ .$$

$$E(t) = \int_\Omega |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \, dx := \|(u, \partial_t u)(\cdot, t)\|^2_{H_0^1 \times L^2(\Omega)} \ .$$

Multiply $\partial_t^2 u - \Delta u + 1_{|\omega} \partial_t u = 0$ by $\partial_t u$ and integrate by parts

$$\frac{1}{2}\frac{d}{dt} E(t) + \int_\omega |\partial_t u(x,t)|^2 \, dx = 0 \ .$$

Integrate over Ω, $\partial_t a - d_1 \Delta a = 1_{|\omega}(b - a)$ and $\partial_t b - d_2 \Delta b = -1_{|\omega}(b - a)$

$$\frac{d}{dt} \int_\Omega a(x,t) + b(x,t) = 0$$

and multiply by (a, b) and integrate by parts

$$\frac{1}{2}\frac{d}{dt} \int_\Omega |a(x,t)|^2 + |b(x,t)|^2 \, dx$$

$$+ \int_\Omega d_1 |\nabla a(x,t)|^2 + d_2 |\nabla b(x,t)|^2 \, dx + \int_\omega |(b-a)(x,t)|^2 \, dx = 0 \ .$$

Control estimate and observation

Observability theorem . Under Geometrical Conditions

$$E(T) \leq K \int_0^T \int_\omega |\partial_t u(x,t)|^2 \, dx \ .$$

Observation at one point in time theorem . $\forall \ t \geq 1$

$$\int_\Omega |(a,b)(x,t)|^2 \, dx \leq C \left(\int_\omega |(a,b)(x,t)|^2 \, dx \right)^\theta \|(a_0,b_0)\|^{2(1-\theta)}_{L^2 \times L^2(\Omega)} \ .$$

Stabilization for the damped wave equation

From

$$\frac{1}{2}\frac{d}{dt}E(t)+\int_{\omega}|\partial_t u(x,t)|^2\,dx=0$$

and

$$E(T)\leq K\int_0^T\int_{\omega}|\partial_t u(x,t)|^2\,dx$$

we have

$$E(T)\leq K\frac{1}{2}(E(0)-E(T))$$

which implies

$$E(T)\leq\frac{K/2}{1+K/2}E(0):=\delta E(0)\ \text{ with } \delta\in\,]0,1[\ .$$

Similarly for any integer $n \geq 0$

$$E\left((n+1)\,T\right) \leq \delta E\left(nT\right) \text{ with } \delta \in \left]0,1\right[$$

and by induction

$$E\left(nT\right) \leq \delta^{n} E\left(0\right) \text{ with } \delta \in \left]0,1\right[\ .$$

Let $t > 0$, there are $nT < t \leq (n+1)\,T$

$$E\left(t\right) \leq E\left(nT\right) \text{ and } \delta^{n} \leq e^{-\left|\ln \delta\right|\left(\frac{t}{T}-1\right)}$$

which implies

$$E\left(t\right) \leq c e^{-\beta t} E\left(0\right) \ .$$

Stabilization for reaction-diffusion system

Let $u_1 = a - 1$ and $u_2 = b - 1$. If $\int_\Omega a_0 + b_0 = 2$, then

$$\int_\Omega u_1(x,t) + u_2(x,t) = 0$$

and let

$$y(t) = \int_\Omega |u_1(x,t)|^2 + |u_2(x,t)|^2 \, dx := \|(a-1, b-1)(\cdot, t)\|^2_{L^2 \times L^2(\Omega)} \ .$$

$$\frac{1}{2}\frac{d}{dt} y(t) + \int_\Omega d_1 |\nabla u_1(x,t)|^2 + d_2 |\nabla u_2(x,t)|^2 \, dx + \int_\omega |(u_2 - u_1)(x,t)|^2 \, dx = 0$$

$$y(t) \le C \left(\int_\omega |(u_1, u_2)(x,t)|^2 \, dx \right)^\theta (y(0))^{1-\theta} \text{ when } t \ge 1 \ .$$

$$\int_\omega |(u_1,u_2)|^2 \le \int_\omega |u_1+u_2|^2 + \int_\omega |u_1-u_2|^2$$
$$\le \int_\Omega |u_1+u_2|^2 + \int_\omega |u_2-u_1|^2$$
$$\le C_P \int_\Omega |\nabla (u_1+u_2)|^2 + \int_\omega |u_2-u_1|^2$$
$$\le C \left(\int_\Omega d_1 |\nabla u_1|^2 + \int_\Omega d_2 |\nabla u_2|^2 + \int_\omega |u_2-u_1|^2 \right)$$
$$\le C \left(-\frac{1}{2}\frac{d}{dt} y(t) \right)$$

$$\frac{y^{1/\theta}(t)}{C y^{1/\theta - 1}(0)} \le \int_\omega |(u_1,u_2)(x,t)|^2 \, dx \text{ when } t > 1$$

implies

$$\frac{d}{dt} y(t) + \frac{2}{C^2 y^{1/\theta-1}(0)} y^{1/\theta}(t) \le 0 \; .$$

Integrating over $[1,T]$ gives

$$y\left(T\right)\leq\delta y\left(0\right)\ \text{ with } \delta\in\left]0,1\right[$$

and we conclude that

$$y\left(t\right)\leq ce^{-\beta t}y\left(0\right)\quad .$$

Wave equation

Gaussian beams=solutions of the wave equation localized around curves: `movie1.mpg` and `movie2.mpg`

Babich (1968)
Ralston (1982)
Ziolkowski (1985)

Propagation of the microlocal regularity (Fourier analysis)

Hörmander (1970)
Melrose and Sjöstrand (1982)

Measure, defect measure

Lebeau (1996)

Microlocalization

Under Geometrical Conditions saying every ray meets $\omega \times]0, T[$

$$\begin{aligned}
\|u\|^2_{L^2(\Omega\times]\delta,T-\delta[)} &\leq \sum_{finite} \int_{\mathbb{R}^{N+1}} |\Phi_i u|^2 \, dx dt \simeq \sum_{finite} \int_{\mathbb{R}^{N+1}} \left|\widehat{\Phi_i u}\right|^2 d\xi d\tau \\
&\leq \sum_{finite} \sum_{finite} \int_{Conical\ set\ \mathsf{V}_j} \left|\widehat{\Phi_i u}\right|^2 d\xi d\tau \\
&\leq c \sum_{\rho=\left(\widetilde{x},\widetilde{t},\widetilde{\xi},\widetilde{\tau}\right)} \|u\|^2_{L^2_\rho} + d \|u\|^2_{H^{-1}(\Omega\times]0,T[)} \\
&\leq C \|u\|^2_{L^2(\omega\times]0,T[)} + d \|u\|^2_{H^{-1}(\Omega\times]0,T[)}
\end{aligned}$$

Heat equation

Heat kernel: $\frac{1}{t^{N/2}} e^{-|x-x_0|^2/4t}$

Logarithmic convexity of the energy

Agmon-Nirenberg (1963)

Cours de P.L.-Lions (2014): Équations paraboliques et ergodicité
Cours de L. Desvillettes (2017): Collisions in plasmas

Log convexity

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times]0,+\infty[\ , \\ \partial_n u = 0 & \text{on } \partial\Omega \times]0,+\infty[\ . \end{cases}$$

$f : t \mapsto \int_\Omega |u(x,t)|^2 \, dx$ is a log convex function that is $\frac{d^2}{dt^2} \ln f(t) \geq 0$.

Indeed, $\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx = 0$ that is $\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \, dx + \mathcal{N}(t) \int_\Omega |u|^2 \, dx = 0$

where $\mathcal{N}(t) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$ is the frequency function.
$\mathcal{N}(t) = -\frac{1}{2}\frac{d}{dt} \ln f(t)$.

But $\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\Delta u|^2 \, dx = 0$ and $\mathcal{N}'(t) \|u\|^4 = -2\|\Delta u\|^2 \|u\|^2 + 2\|\nabla u\|^4 \leq 0$.

Log convexity

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times]0,+\infty[\ , \\ \partial_n u = 0 & \text{on } \partial\Omega \times]0,+\infty[\ . \end{cases}$$

$f : t \mapsto \int_\Omega |u(x,t)|^2 \, dx$ is a log convex function that is $\frac{d^2}{dt^2} \ln f(t) \geq 0$.

Therefore, $\ln f(\theta t_1 + (1-\theta) t_2) \leq \theta \ln f(t_1) + (1-\theta) \ln f(t_2) = \ln f^\theta(t_1) f^{1-\theta}(t_2)$

that is $\int_\Omega |u(x, \theta t_1 + (1-\theta) t_2)|^2 \, dx \leq \left(\int_\Omega |u(x,t_1)|^2 \, dx \right)^\theta \left(\int_\Omega |u(x,t_2)|^2 \, dx \right)^{1-\theta}$.

Choose $t_1 = T$ and $t_2 = 0$ and $t = \theta t_1$,

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|u(\cdot,T)\|_{L^2(\Omega)}^{t/T} \|u(\cdot,0)\|_{L^2(\Omega)}^{1-t/T} \ .$$

Log convexity and weight function

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times]0,+\infty[\ , \\ \partial_n u = 0 & \text{on } \partial\Omega \times]0,+\infty[\ . \end{cases}$$

$f : t \mapsto \int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx$ is it a log convex function ? we deform u to $ue^{\Phi/2}$.

Can we replace

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|u(\cdot,T)\|_{L^2(\Omega)}^{t/T} \|u(\cdot,0)\|_{L^2(\Omega)}^{1-t/T}$$

by

$$\int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx \leq C \left(\int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \right)^\theta \|u(\cdot,0)\|_{L^2(\Omega)}^{2(1-\theta)}$$

in order to make appear $L^2(\omega)$. ok if $e^{\Phi/2}$ looks like the heat kernel

Comments and open problems

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Damped wave equation

$$\begin{cases} \partial_t a - d_1 \Delta a = k(x,t)\, 1_{|\omega}\, (b-a) & \text{in } \Omega \times]0,+\infty[\ , \\ \partial_t b - d_2 \Delta b = -k(x,t)\, 1_{|\omega}\, (b-a) & \text{in } \Omega \times]0,+\infty[\ , \\ \partial_n a = \partial_n b = 0 & \text{on } \partial\Omega \times]0,+\infty[\ , \\ (a,b)(\cdot,0) = (a_0, b_0) & \in L^2(\Omega) \times L^2(\Omega) \ . \end{cases}$$

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Thank You