Introduction

This set of notes was prepared for a mini-course given at the 27th Colóquio Brasileiro de Matemática. The purpose of the course is to introduce some key notions of the theory of C*-algebras and to illustrate them by examples originating from dynamical systems. It is very close in spirit to [32], written more than thirty years ago. Of course the subject has undergone many exciting developments since then, but at the elementary level of these notes, the basic ideas remain unchanged and I have liberally borrowed material from [32].

The theory of operator algebras was initiated in a series of papers by Murray and von Neumann ([26]) in the 1930's and 1940's. One motivation was undoubtedly to provide a mathematical foundation for the young and budding quantum mechanics of these days. As it is well known, observables of a quantum mechanical system are represented in this theory by operators on a Hilbert space. They generate operator algebras which encode the symmetries of the system. The interplay with quantum theory, including quantum field theory and quantum statistical mechanics has been present ever since. The notion of KMS states, briefly studied in Chapter 3, is an example of this interaction. These notes will deal almost exclusively with C*-algebras. They are norm closed sub-$*$-algebras of the algebra of all bounded operators on a Hilbert space. The original work of Murray and von Neumann focused on weak operator closed sub-$*$-algebras, now called von Neumann algebras. The theory of C*-algebras was introduced in the 1940's by Gelfand and Naimark. Another important motivation was the theory of the unitary representations of groups. Now and then, we will encounter the C*-algebra of a group, which captures this theory completely. It is often said that the von Neumann algebra theory is non-commutative measure theory and that the C*-algebra theory is non-commutative topology. Indeed the most general commutative von Neumann algebra is the algebra of (classes of) essentially bounded measurable functions on a measure space while the most general commutative C*-algebra is the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space.

A number of excellent textbooks on the theory of operator algebras are available. When it comes to C*-algebras, I just quote [14] as one of the earliest and [5] as one of the most recent. They may be intimidating for some. An effective approach to the subject is offered in [13], where a large number of examples are presented along with the theory. These notes present many similarities with [13]; in particular, most examples are the same. The main difference is the more systematic approach which is adopted here: all our examples arise from dynamical systems and our leitmotiv is to relate properties of the C*-algebra to those of the dynamical system. There is nothing new about this. Murray and von Neumann introduced the group measure space construction to produce factors (builiding blocks in von Neumann algebras theory). The work of A. Connes in the early 1970's on the classification of factors is very much related to the work of W. Krieger on ergodic theory. Many notions, amenability or property T to name a few, have had parallel developments for operator algebras and in ergodic theory. The recent and spectacular developments in factor theory by S. Popa et alii still demonstrate the richness of the interplay with ergodic theory. There have also been fruitful interactions between C*-algebra theory and topological dynamics, for example [15, 17, 38] to name a few. Nevertheless, the reader should be told that the theory of C*-algebras has a life of its own; the point of view presented here is partial and somewhat biased; it should be complemented by a more intrinsic presentation to take full advantage of the power of C*-algebraic techniques. I also highly recommend [9] to get a grasp of the richness of the subject.

Due to time limitations, some important topics of my initial plan are not covered. These are mainly foliation algebras and index theorems. The examples concern almost exclusively "étale locally compact Hausdorff groupoids"; they generalize discrete group actions. Most of the theory extends to " not necessarily Hausdorff locally compact groupoids with Haar systems", in particular to holonomy groupoids of foliations, although this involves some technical complications. Index theorems underly the introduction of the K-theory of C*-algebras and give rise to some of the most challenging problems like the Baum-Connes conjecture, but this would have taken us too far away.

The reader may find the style uneven, with many proofs left as exercises, especially towards the end. I am afraid that the text contains many errors, typographical and mathematical. I hope the latter are not too serious. Notices of errors and misprints may be sent to the author (Jean.Renault@univ-orleans.fr). The reader should also be aware of the following implicit assumptions: Hilbert spaces are separable, locally compact spaces are Hausdorff and second countable, C*-algebras are usually separable (the standard separability assumption for von Neumann algebras is that they act on a separable Hilbert space).

Acknowledgments. I am grateful to Valentin Deaconu for his help in improving the manuscript.

Contents

Chapter 1

C*-Algebras: Basics

Unless stated otherwise, all linear spaces and algebras are over the field of complex numbers. The definition of a C*-algebra is amazingly short:

Definition 1.0.1. A *C*-algebra* is an algebra A endowed with a norm $a \to \|a\|$ and an involution $a \to a^*$ such that
(i) $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$;
(ii) $\|a^*a\| = \|a\|^2$ for all $a \in A$;
(iii) the norm is complete.

Note that (i) and (ii) imply that $\|a^*\| = \|a\|$ for all $a \in A$. An algebra endowed with a norm satisfying the conditions (i) and (iii) of the above definition is called a *Banach algebra*. We shall say that a norm on an involutive algebra is a *C*-norm* if it satisfies (i) and (ii). The class of C*-algebras is rather special among the class of Banach algebras. To get an idea of this, compare the spectral theory of a self-adjoint matrix and that of an arbitrary matrix. Here are two fundamental examples.

Example 1.0.2. Let n be an integer. Then the algebra $\mathcal{A} = M_n(\mathbf{C})$ of $n \times n$ matrices with complex entries, endowed with the operator norm $\|A\| = \sup_{\|x\|\leq 1} \|Ax\|$, where $\|x\| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$, and where A^* is the adjoint of A, is a C*-algebra. More generally, given a Hilbert space $\mathcal{H}$, the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, endowed with the operator norm and where A^* is the adjoint of A

is a C*-algebra. As a consequence, every norm-closed sub-$*$-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a C*-algebra. The converse is also true: the Gelfand-Naimark Theorem says that every C*-algebra can be realized as a norm-closed sub-$*$-algebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. When $\mathcal{H}$ is infinite dimensional, we also have another natural generalization of a matrix algebra, namely the C*-algebra $\mathcal{K}(\mathcal{H})$ of all compact linear operators on $\mathcal{H}$. It is defined as the norm-closure of the $*$-algebra of all finite rank linear operators on $\mathcal{H}$.

Example 1.0.3. Let X be a locally compact Haudorff space. Then the space $C_0(X)$ of complex-valued continuous functions which vanish at infinity, endowed with the pointwise multiplication $fg(x) = f(x)g(x)$, the norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$ and the involution $f^*(x) = \overline{f(x)}$ is a commutative C*-algebra. We shall see below that the converse is true: this is the content of the Gelfand Theorem.

Let us introduce some special elements of a C*-algebra A. One says that $a \in A$ is *self-adjoint* if $a = a^*$. One says that $e \in A$ is a (self-adjoint) *projection* if $e = e^* = e^2$. If A has a unit 1, one says that $u \in A$ is *unitary* if $u^*u = uu^* = 1$. One says that $v \in A$ is a *partial isometry* if v^*v (or vv^*) is a projection.

1.1 Commutative C*-algebras

Let us first recall some definitions from the general theory of Banach algebras. Suppose first that A is an algebra with unit 1. Given $a \in A$, one defines its *resolvent set* $\Omega(a)$ as the set of $\lambda \in \mathbf{C}$ such that $\lambda 1 - a$ is invertible, its *resolvent function* $R_a(\lambda) = (\lambda 1 - a)^{-1}$ for $\lambda \in \Omega(a)$, its *spectrum* $\mathrm{Sp}(a) = \mathbf{C} \setminus \Omega(a)$, and its *spectral radius* $\rho(a) = \sup_{\lambda\in\mathrm{Sp}(a)} |\lambda|$. These notions depend on the ambient algebra. In case of ambiguity, we write for example $\mathrm{Sp}_A(a)$.

Lemma 1.1.1. *Let a be an element of a unital algebra A and let $p \in \mathbf{C}[X]$. Then* $\mathrm{Sp}(p(a)) = p(\mathrm{Sp}(a))$.

Proof. Let $\lambda \in C$. We write the factorization

$$p(X) - \lambda = c\prod_1^n (X - \alpha_i).$$

Substituting a for X, we have $p(a) - \lambda 1 = c\prod_1^n(a - \alpha_i 1)$. Therefore, $p(a) - \lambda 1$ is invertible if and only if $a - \alpha_i 1$ is invertible for all $i = 1, \dots, n$. Equivalently, $\lambda \in \mathrm{Sp}(p(a))$ if and only if there exists i such that $\alpha_i \in \mathrm{Sp}(a)$, i.e. if and only if $\lambda \in p(\mathrm{Sp}(a))$. □

Proposition 1.1.2. *Let a be an element of a unital Banach algebra A. Then its spectrum* $\mathrm{Sp}(a)$ *is a non-empty compact subset of* $\mathbf{C}$ *and* $\rho(a) = \inf \|a^n\|^{1/n}$.

Proof. Let us show first that $\mathrm{Sp}(a)$ is closed and bounded. If u is an element of the Banach algebra A of norm strictly less than one, then $1-u$ is invertible: the Neumann series $\sum u^n$ is absolutely convergent and its sum $\sum_0^\infty u^n$ is the inverse of $1-u$. For later purpose, let us observe that $\|(1-u)^{-1}\| \le (1 - \|u\|)^{-1}$. One deduces that the set of invertible elements of A is open. Since the resolvent set $\Omega(a)$ is its inverse image by the continuous function $\lambda \mapsto \lambda 1 - a$, it is also open and its complement $\mathrm{Sp}(a)$ is closed. Suppose that $|\lambda| > \|a\|$. Since $\|\lambda^{-1}a\|$ is strictly less than 1, $1 - \lambda^{-1}a$ is invertible and so is $\lambda 1 - a = \lambda(1 - \lambda^{-1}a)$. Therefore, if $\lambda \in \mathrm{Sp}\, a$, then $|\lambda| \le \|a\|$. This shows that $\mathrm{Sp}(a)$ is contained in the ball of radius $\|a\|$. The above also shows that the resolvent function $R_a(\lambda) = (\lambda 1 - a)^{-1}$ is analytic on $\Omega(a)$ and satisfies $\|R_a(\lambda)\| \le (|\lambda| - \|a\|)^{-1}$ for $|\lambda| > \|a\|$. In particular, R_a is bounded and tends to 0 at infinity. Suppose that $\mathrm{Sp}(a)$ is empty. For every continuous linear form f on A, $f \circ R_a$ is a bounded analytic function on $\mathbf{C}$. By Liouville's theorem, it is a constant function. Since it tends to 0 at infinity, it is the zero function. By the Hahn-Banach theorem, we obtain $R_a(\lambda) = 0$ for all $\lambda \in \mathbf{C}$, which is not true. Therefore, $\mathrm{Sp}(a)$ is non-empty.

Let us prove next the formula for the spectral radius. We deduce from the lemma that, for every $n \in \mathbf{N}$, we have $\rho(a^n) = \rho(a)^n$. Combined with the inequality $\rho(a^n) \le \|a^n\|$ which we have just shown, this gives the inequality $\rho(a) \le \inf \|a^n\|^{1/n}$. On the other hand, since the function $f(\mu) = R_a(1/\mu) = \mu(1 - \mu a)^{-1}$ is defined and analytic on the open ball $|\mu| < 1/\rho(a)$, the series $\mu \sum_{n=0}^\infty \mu^n a^n$ converges for $|\mu| < 1/\rho(a)$. Its radius of convergence is at least $1/\rho(a)$; hence the inequality $\limsup \|a^n\|^{1/n} \le \rho(a)$. In conclusion, we have the convergence of the sequence $(\|a^n\|^{1/n})$ and the equalities $\rho(a) = \inf \|a^n\|^{1/n} = \lim \|a^n\|^{1/n}$.

□

Remark 1.1.3. Suppose that A is a unital C*-algebra and that $a \in A$ is self-adjoint (i.e. $a^* = a$). We have the equality $\|a^2\| = \|a^*a\| = \|a\|^2$, hence $\|a^n\| = \|a\|^n$ for $n = 2^k$. Therefore $\rho(a) = \|a\|$. For an arbitrary $a \in A$, we have $\|a\| = \|a^*a\|^{1/2} = \rho(a^*a)^{1/2}$. This is a remarkable property of a C*-algebra: its norm is determined by its algebraic structure.

The above proposition is the key to Gelfand's theory of commutative Banach algebras. One says that a Banach algebra A is *simple* if it has no other closed ideals than $\{0\}$ and A.

Theorem 1.1.4. *(Gelfand-Mazur theorem) Every simple unital commutative Banach algebra is isomorphic to* $\mathbf{C}$.

Proof. Note that the unital algebra A is isomorphic to $\mathbf{C}$ if and only if $A = \mathbf{C}1$. Suppose that this is not so. Let $a \in A \setminus \mathbf{C}1$. Pick $\lambda \in \mathrm{Sp}(a)$ and consider the closed ideal $I = \overline{(\lambda 1 - a)A}$ generated by $\lambda 1 - a$. On one hand $I \neq \{0\}$ because $\lambda 1 - a \neq 0$. On the other hand $I \neq A$ because I does not contain 1. Indeed the elements $(\lambda 1 - a)b$ of $(\lambda 1 - a)A$ are not invertible, hence at a distance ≥ 1 from the unit 1; this is still true for the elements of I.

□

Let A be a unital commutative Banach algebra. A *character* of A is a non-zero linear form $\chi : A \to \mathbf{C}$ such that $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in A$. By choosing $a = 1$ and b such that $\chi(b) = 1$, one sees that $\chi(1) = 1$. Moreover, a character is necessarily continuous of norm 1. We want to show that $|\chi(a)| \leq 1$ for all a in the unit ball of A. If not, there would exist $u \in A$ such that $\|u\| < 1$ and $\chi(u) = 1$. On one hand, $\chi(1 - u) = \chi(1) - \chi(u) = 0$. On the other hand $1 - u$ is invertible: there is v such that $(1 - u)v = 1$. This is not possible since we would have $0 = 1$. The set $X(A)$ of all characters of A is called the *spectrum* of A. It is a closed subset of the unit ball of the dual A' of A, endowed with the weak* topology. Therefore, it is compact. Every $a \in A$ defines a function $\widehat{a}$ on $X(A)$ such that $\widehat{a}(\chi) = \chi(a)$. This function is clearly continuous. The map $\mathcal{G} : A \to C(X(A))$ which sends a to $\widehat{a}$ is called the *Gelfand transform* of A. Here are some properties of the Gelfand transform.

Proposition 1.1.5. *Let* $\mathcal{G} : A \to C(X(A))$ *be the Gelfand transform of a unital commutative Banach algebra* A*. Then*

(i) $\mathcal{G}$ *is an algebra homomorphism.*
(ii) The range of $\mathcal{G}$ *separates the points of* $X(A)$*.*
(iii) For all $a \in A$, $\widehat{a}(X(A)) = \mathrm{Sp}(a)$ *and* $\|\widehat{a}\|_\infty = \rho(a)$*.*

Proof. The assertions (i) and (ii) are easy. For example, for (ii) suppose that χ and χ' are distinct characters. Then, there exists $a \in A$ such that $\chi(a) \neq \chi'(a)$. In other words, $\widehat{a}(\chi) \neq \widehat{a}(\chi')$. To prove (iii), let us first show that $b \in A$ is invertible if and only if $\chi(b) \neq 0$ for all $\chi \in X(A)$. If b has an inverse c, then $bc = 1$ and we have $\chi(b)\chi(c) = \chi(1) = 1$ for all χ. If b is not invertible, it belongs to a maximal I (this uses Zorn's lemma). Endowed with the quotient norm A/I is a unital commutative Banach algebra. Moreover, it is simple. According to the above theorem, it is isomorphic to $\mathbf{C}$. Composing this isomorphism with the quotient map gives a character χ such that $\chi(b) = 0$. Therefore, given $\lambda \in \mathbf{C}$, $\lambda 1 - a$ is invertible if and only if $\chi(\lambda 1 - a) \neq 0$ for all $\chi \in X(A)$. This says that $\widehat{a}(X(A)) = \mathrm{Sp}(a)$ and implies that $\|\widehat{a}\|_\infty = \rho(a)$. □

Remark 1.1.6. There is a similar result for a commutative Banach algebra A which does not have a unit. One defines its *spectrum* $X(A)$ as the set of its characters, where a character is again defined as a non-zero multiplicative linear form. One can use the above theory by adjoining a unit to A: one sets $\tilde{A} = A \oplus \mathbf{C}$. Its elements can be written $a + \lambda 1$, where $a \in A$ and $\lambda \in \mathbf{C}$. The product is given by $(a + \lambda 1)(a' + \lambda' 1) = aa' + \lambda a' + \lambda' a + \lambda\lambda' 1$. The characters of $\tilde{A}$ are either $\chi(a + \lambda 1) = \chi(a) + \lambda$, where $\chi \in X(A)$ or $\chi_0(a + \lambda 1) = \lambda$. There exist equivalent Banach algebra norms on $\tilde{A}$ such that the injection of A into $\tilde{A}$ given by $a \to a + 01$ is isometric, for example $\|a + \lambda 1\| = \|a\| + |\lambda|$. We may identify $X(A)$ with the open subset $X(\tilde{A}) \setminus \{\chi_0\}$ and $C_0(X(A))$ with the ideal of $C(X(\tilde{A}))$ consisting of functions vanishing at χ_0. The *Gelfand transform*

$$\mathcal{G}_A : A \to C_0(X(A))$$

is the restriction of the Gelfand transform $\mathcal{G}_{\tilde{A}} : \tilde{A} \to C(X(\tilde{A}))$. One defines the *spectrum* of an element a of a Banach algebra A without a unit as $\mathrm{Sp}_{\tilde{A}}(a)$.

Lemma 1.1.7. *A character* χ *of a unital commutative C*-algebra A necessarily respects the involution, i.e. satisfies* $\chi(a^*) = \overline{\chi(a)}$ *for all* $a \in A$*.*

Proof. Suppose first that $a = a^*$. Fix $t \in \mathbf{R}$. The series $\sum_{n=0}^{\infty} \frac{(ita)^n}{n!}$ is absolutely convergent. Let u be its sum. A direct computation shows that u is unitary: $u^* = u^{-1}$. In particular $\|u\|^2 = \|u^*u\| = \|1\| = 1$. Therefore $|\chi(u)| \leq 1$. Write $\chi(a) = \lambda + i\mu$ with $\lambda, \mu \in R$. Then

$$e^{-t\mu} = |e^{it\chi(a)}| = \Big|\sum_{n=0}^{\infty} \frac{(it\chi(a))^n}{n!}\Big| = \Big|\chi\Big(\sum_{n=0}^{\infty} \frac{(ita)^n}{n!}\Big)\Big| \leq 1.$$

This being true for all $t \in \mathbf{R}$, we have $\mu = 0$. We have shown that $\chi(a)$ is real. Consider next an arbitrary $a \in A$. It can be written $a = b + ic$ with $b = b^*$ and $c = c^*$(explicitly, $b = (a + a^*)/2, c = (a - a^*)/2i$). Then $a^* = b - ic$. Taking into account that $\chi(b)$ and $\chi(c)$ are real, we obtain the desired equality $\chi(a^*) = \overline{\chi(a)}$. □

Here is the main result about commutative C*-algebras.

Theorem 1.1.8. *(Gelfand theorem) Let A be a unital commutative C*-algebra. Then, its Gelfand transform $\mathcal{G} : A \to C(X(A))$ is an isometric $*$-isomorphism.*

Proof. We already know that $\mathcal{G}$ is an algebra homomorphism. The above lemma says that it also respects the involution. Let us show that it is isometric. For all $a \in A$, we have

$$\|a\|^2 = \|a^*a\| = \rho(a^*a) = \|\mathcal{G}(a^*a)\|_\infty = \|\mathcal{G}(a)\|_\infty^2.$$

It remains to show that $\mathcal{G}$ is surjective. According to the Stone-Weierstrass theorem, $\mathcal{G}(A)$, which is a subalgebra separating the points and stable under complex conjugation, is dense in $C(X(A))$. Moreover, being isometric to A, it is also complete. Therefore, it is closed. □

Remark 1.1.9. The Gelfand theorem admits a non-unital version. Suppose that the C*-algebra A does not have a unit, we equip the algebra $\tilde{A}$ defined earlier with the involution $(a + \lambda 1)^* = a^* + \overline{\lambda} 1$ and the norm $\|a + \lambda 1\| = \sup \|a\xi + \lambda\xi\|$, where the supremum is taken over the unit ball of A. The fact that it is a C*-algebra norm will be established later in the general framework of C*-modules. One deduces that the Gelfand transform

$$\mathcal{G} : A \to C_0(X(A))$$

is an isometric $*$-isomorphism.

Example 1.1.10. **The Banach algebras $L^1(\mathbf{R})$ and $C^*(\mathbf{R})$.**

Endowed with the convolution product and its usual norm, $L^1(\mathbf{R})$ is a commutative Banach algebra. It spectrum $X(L^1(\mathbf{R}))$ can be identified with $\mathbf{R}$. Indeed for all $t \in \mathbf{R}$, we have the group character $\chi_t : \mathbf{R} \to U(1)$ (the group of complex numbers of module 1), given by $\xi_t(x) = \exp(-itx)$, and the algebra character $\chi_t : L^1(\mathbf{R}) \to \mathbf{C}$ given by $\xi_t(f) = \int f(x)\chi_t(x)dx$. By using an approximate unit in $L^1(\mathbf{R})$, one can show that every character of $L^1(\mathbf{R})$ is given by a group character. Moreover, it is known that every (measurable) group character of $\mathbf{R}$ is of the above form. The outcome is the identification (as topological spaces) $\mathbf{R} = \hat{\mathbf{R}} = X(L^1(\mathbf{R}))$, where $\hat{\mathbf{R}}$ is the group of characters of $\mathbf{R}$. With this identification, the Gelfand transform $\mathcal{G}_1 : L^1(\mathbf{R}) \to C_0(\hat{\mathbf{R}})$ is the L^1-Fourier transform. We let $L^1(\mathbf{R})$ act by left convolution on the Hilbert space $L^2(\mathbf{R})$: for $f \in L^1(\mathbf{R})$, $L(f)$ is the operator on $L^2(\mathbf{R})$ defined by $L(f)\xi = f*\xi$ for $\xi \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$. Let $\mathcal{F} : L^2(\mathbf{R}) \to L^2(\hat{\mathbf{R}})$ be the L^2-Fourier transform, where $\hat{\mathbf{R}}$ is endowed with the measure $dt/2\pi$ so that $\mathcal{F}$ is isometric. We have the intertwining relation $L(f) = \mathcal{F}^{-1}M(\mathcal{G}_1(f))\mathcal{F}$, where $M(\varphi)$ is the multiplication operator $M(\varphi)\eta = \varphi\eta$ for $\eta \in L^2(\hat{\mathbf{R}})$. If $\varphi \in L^\infty(\hat{\mathbf{R}})$, $M(\varphi)$ is a bounded operator with $\|M(\varphi)\| = \|\varphi\|_\infty$. In particular, the convolution operator $L(f)$ is bounded with $\|L(f)\| = \|\mathcal{G}(f)\|_\infty$. One defines the reduced C*-algebra $C^*_r(\mathbf{R})$ of the group $\mathbf{R}$ as the norm closure of $\{L(f), f \in L^1(\mathbf{R})\}$. There is a unique linear isometry $\mathcal{G} : C^*_r(\mathbf{R}) \to C_0(\hat{\mathbf{R}})$ such that $\mathcal{G}(L(f)) = \mathcal{G}_1(f)$ for $f \in L^1(\mathbf{R})$. By continuity of the operations, it is a $*$-algebra homomorphism. One deduces that $\mathcal{G}$ is a C*-algebra isomorphism, which is a concrete realization of the Gelfand transform of $C^*_r(\mathbf{R})$. The same results hold for an arbitrary locally compact abelian groups G and give the Gelfand isomorphism $\mathcal{G} : C^*_r(G) \to C_0(\hat{G})$, where $\hat{G}$ is the character group of G.

1.2 Continuous functional calculus

Let a be an element of a unital C*-algebra A which is self-adjoint ($a = a^*$) or more generally *normal* ($a^*a = aa^*$). The sub-C*-algebra $B = C^*(a, 1)$ generated by a and 1 is commutative. It is the norm closure of the $*$-algebra consisting of elements of the form $P(a, a^*)$, where $P \in \mathbf{C}[z, \overline{z}]$. If χ is a character of B, then $\chi(P(a, a^*) = P(\chi(a), \overline{\chi(a)})$. By density, χ is uniquely determined by $\chi(a)$. Moreover, $\chi(a)$ belongs to $\mathrm{Sp}_B(a)$ and every element of $\mathrm{Sp}_B(a)$ is of that form. Thus, we get a bijective continuous map from the spectrum $X(B)$ of B onto $\mathrm{Sp}_B(a)$. Since $X(B)$ is compact, this map is a homeomorphism and we can view the Gelfand transform $\mathcal{G}_B$ as a C*-algebra isomorphism from B onto $C(\mathrm{Sp}_B(a))$. In fact, the spectrum $\mathrm{Sp}_B(a)$ of a relative to B coincide with the spectrum $\mathrm{Sp}(a)$ of a relative to A. Indeed, since an element which is invertible in B is invertible in A, we have the inclusion $\mathrm{Sp}(a) \subset \mathrm{Sp}_B(a)$. Conversely, let $\lambda \in \mathrm{Sp}_B(a)$. Suppose that $a - \lambda 1$ has an inverse in A. Choose $0 < \epsilon$ such that $\epsilon^{-1} > \|(a - \lambda 1)^{-1}\|$. There exists $f \in C(\mathrm{Sp}_B(a))$ such that $\|f\|_\infty = 1$ and $f(\mu) = 0$ whenever $|\mu - \lambda| > \epsilon$. There exists $b \in B$ such that $f = \mathcal{G}_B(b)$. Then,

$$1 = \|b\| = \|(a - \lambda 1)^{-1}(a - \lambda 1)b\| \le \|(a - \lambda 1)^{-1}\| \|(a - \lambda 1)b\|.$$

Since $\|(a - \lambda 1)b\| = \sup_{\mu \in \mathrm{Sp}_B(a)} |(\mu - \lambda) f(\mu)| \le \epsilon$, we would have $\|(a - \lambda 1)^{-1}\| \ge \epsilon^{-1}$, which is not true. Therefore, λ belongs to $\mathrm{Sp}(a)$. The inverse map $\mathcal{G}_B^{-1} : C(\mathrm{Sp}(a)) \to A$ is called the *continuous functional calculus* map. The image of $f \in C(\mathrm{Sp}(a))$ in A is denoted by $f(a)$, which agrees with the definition when $f(z) = P(z, \overline{z})$ is a polynomial in z and $\overline{z}$. If A does not have a unit, we replace it by $\tilde{A}$. If a is not invertible, the Gelfand transform sends the C*-algebra $C^*(a)$ generated by a onto $C_0(\mathrm{Sp}(a) \setminus \{0\})$.

According to the above, a normal element of a C*-algebra A is self-adjoint if and only it its spectrum is contained in $\mathbf{R}$. A self-adjoint element a is called *positive* if its spectrum is contained in $\mathbf{R}_+$. This is one of the fundamental notions in the theory. One writes $a \ge 0$. One denotes by $A+$ the set of positive elements of A and by A_{sa} the real linear space of self-adjoint elements of A.

Proposition 1.2.1. *The set A_+ is a convex cone in A_{sa} such that $A_+ - A_+ = A_{sa}$ and $A_+ \cap (-A_+) = \{0\}$.*

Proof. In order to show that the sum of two positive elements is positive, one uses the following characterization of positive elements, which is easily established by continuous functional calculus: a self-adjoint element a is positive if and only if $\|\lambda 1 - a\| \le \lambda$ for some or for all $\lambda \geq \|a\|$. Suppose that $a, b \in A_+$. Then,

$$\begin{aligned} \|(\|a\| + \|b\|)1 - (a+b)\| &= \|(\|a\|1 - a)(\|b\|1 - b)\| \\ &\leq \|\|a\|1 - a\|\|\|b\|1 - b\| \leq \|a\| + \|b\|. \end{aligned}$$

Since $\|a\| + \|b\| \geq \|a+b\|$, $a+b \in A_+$. By continuous functional calculus, every self-adjoint element a can be written $a = a_+ - a_-$ with $a_+, a_- \in A_+$ and $a_+ a_- = 0$. If $a \in A_+ \cap (-A_+)$, then $\mathrm{Sp}(a) = \{0\}$ and $\|a\| = \rho(a) = 0$, hence $a = 0$. □

This defines an order on the real linear space A_{sa}: $a \leq b$ if and only if $b - a \geq 0$.

Proposition 1.2.2. *Let a be a self-adjoint element of a C*-algebra A. The following conditions are equivalent:*
(i) $a \geq 0$;
(ii) there exists $b \in A$ such that $a = b^* b$;
(iii) there exists h self-adjoint such that $a = h^2$.

Proof. For $(i) \Rightarrow (iii)$, we use continuous functional calculus to define $h = a^{1/2}$. The implication $(iii) \Rightarrow (ii)$ is obvious. The proof of $(ii) \Rightarrow (i)$ is more involved. One first writes $a = b^* b = a_+ - a_-$ with $a_+, a_- \in A_+$ and $a_+ a_- = 0$. Let $c = b a_-^{1/2}$. A simple computation gives that $c^* c = -a_-^{1/2}$ belongs to $-A_+$. Then, writing $c = d + ie$ with $c, d \in A_{sa}$, another computation gives that $cc^* = 2(d^2 + e^2) - c^* c$ belongs to A_+. Then one uses the standard fact that in a complex algebra with unit, $\mathrm{Sp}(xy) \setminus \{0\} = \mathrm{Sp}(yx) \setminus \{0\}$ to conclude that $\mathrm{Sp}(c^* c) = \{0\}$, hence $a_- = 0$. Therefore $a = a_+$ is positive. □

Another important application of the continuous functional calculus is the construction of an approximate unit in a C*-algebra.

Definition 1.2.3. An *approximate unit* in a Banach algebra A is a net $(u_\lambda)_{\lambda \in \Lambda}$ of elements of A such that for all $a \in A$, $\|a u_\lambda - a\|$ and $\|u_\lambda a - a\|$ tend to 0.

Proposition 1.2.4. *Each C*-algebra A contains an approximate unit. More precisely, there is an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ such that $u_\lambda \geq 0$, $\|u_\lambda\| \leq 1$ and $u_\lambda \leq u_\mu$ if $\lambda \leq \mu$. If A is separable, one can assume furthermore that the approximate unit is indexed by $\mathbf{N}$.*

Proof. Here is a sketch of a proof. Let Λ be the set of finite subsets of A ordered by inclusion. Given $\lambda = \{a_1, \ldots, a_n\}$ in Λ, define $v_\lambda = a_1^* a_1 + \ldots + a_n^* a_n$ and $u_\lambda = v_\lambda(\frac{1}{n} + v_\lambda)^{-1}$ (it is computed in $\tilde{A}$ but it belongs to A). One checks that it is an increasing approximate unit. If A is separable, one chooses a dense sequence $(a_1, a_2, \ldots)$ of A and defines $u_n = u_{\lambda_n}$, where $\lambda_n = \{a_1, \ldots, a_n\}$. $\square$

1.3 States and representations

1.3.1 Representations

Representation theory of C*-algebras is a vast domain which involves in particular integration theory and harmonic analysis. Let us begin with a basic observation.

Proposition 1.3.1. *Let A, B be C*-algebras and let $\pi : A \to B$ be a $*$-homomorphism. Then*
(i) $\|\pi(a)\| \leq \|a\|$ for all $a \in A$.
(ii) If π is one-to-one, then π is isometric.

Proof. Let us prove (i). Since π diminishes the spectral radius, the inequality holds for a self-adjoint element a. Then, it suffices to apply the inequality to a^*a. Let us prove (ii). We assume that π is one-to-one and endeavor to show that $\|\pi(a)\| = \|a\|$ for all $a \in A$. Again, it suffices to show this equality for a self-adjoint element a. Replacing A by $C^*(a)$ and B by $C^*(\pi(a))$, we may assume that A and B are commutative. If y is a character of B, then $\rho(y) = y \circ \pi$ is a character of A. The map $\rho : X(B) \to X(A)$ so defined is continuous and we have $\pi(f) = f \circ \rho$ for $f \in A = C_0(X(A))$. Since π is one-to-one, ρ has dense range. Hence $\sup_{X(B)} |f \circ \rho| = \sup_{X(A)} |f|$. $\square$

The kernel $I = \operatorname{Ker} \pi$ of a $*$-homomorphism $\pi : A \to B$ is a (norm-closed two-sided) *ideal* of A. Conversely, given an ideal I of A, the quotient A/I is a C*-algebra and the quotient map admits I as its kernel.

Definition 1.3.2. A *representation* of a $*$-algebra A in a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{H})$. A representation is *non-degenerate* if for $\xi \in \mathcal{H}$, $\pi(A)\xi = 0$ implies $\xi = 0$. A representation which is one-to-one is often called *faithful*.

In this course, *representation* will always mean *non-degenerate representation*. In particular, if A has a unit, our representations are unit-preserving.

Let π be a representation of a C*-algebra A in a Hilbert space $\mathcal{H}$. Suppose that $\mathcal{H}_1$ is a closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ which is invariant under A. Then, its orthogonal $\mathcal{H}_2 = \mathcal{H}_1^\perp$ is also invariant. This provides a decomposition $\pi = \pi_1 \oplus \pi_2$ of π. A representation is called *irreducible* if it admits no non-trivial invariant closed subspaces. One would expect that every representation can be decomposed into a direct sum of irreducible representations. The example of $C_0(\mathbf{R})$ acting by multiplication operators on $L^2(\mathbf{R}, dx)$ shows that this is not always the case. This becomes true (under the assumption that A and $\mathcal{H}$ are separable) if one replaces the direct sum by a direct integral of Hilbert spaces (see Section 3.3). There may be more useful decompositions of representations. One defines the *commutant* $\pi(A)'$ of a representation π as the set of bounded operators on $\mathcal{H}$ which commute with every element of $\pi(A)$. It is a *von Neumann algebra*, i.e. an involutive subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed in the weak operator topology. A representation is irreducible if and only if its commutant is reduced to $\mathbf{C}$ (the scalar multiple of the identity). A representation π is called *factorial* if its commutant is a *factor*, i.e. its center $\pi(A)' \cap \pi(A)''$ is reduced to $\mathbf{C}$. Under the assumption that A and $\mathcal{H}$ are separable, every representation can be decomposed into a direct integral of factorial representations. This decomposition is obtained by representing the commutative von Neumann algebra $\pi(A)' \cap \pi(A)''$, which by a measure-theoretic version of Gelfand's theorem can be viewed as $L^\infty(X, \mu)$, as multiplication operators on a space of sections of a Hilbert bundle over (X, μ).

Remark 1.3.3. The theory of von Neumann algebras provides a classification theory of factors. Here is a brief summary (we refer to [5] for unexplained terms). First, factors are classified into types I, II and III. This classification is based on the dimension range (i.e. the ordered set of classes of projections under unitary equivalence, or equivalently the range of the trace on the projections). Factors of type I are (isomorphic to) $\mathrm{I}_n = M_n(\mathbf{C})$, where $n \in \mathbf{N}$ and $\mathrm{I}_\infty = \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional. Factors of type II have dimension range $[0, 1]$ (type II_1) or $[0, \infty]$ (type II_∞). Factors of type III have no (normal semi-finite) traces. There is a further classification of type III factors into type III_0, III_λ, where $0 < \lambda < 1$ and III_1 based on the modular automorphism group which we shall meet in a later chapter. For the class of *hyperfinite* factors, i.e. factors which are the weak closure of an increasing sequence of finite dimensional C*-algebras, this classification is complete in the sense that each type contains a unique hyperfinite factor up to isomorphism, except for type III_0 which admits a further classification. The type of a factorial representation π of A is the type of the factor $\pi(A)''$.

When A is a commutative C*-algebra, irreducible representations of A correspond exactly to characters of A. It is natural to attempt to define the spectrum of a non-commutative C*-algebra in terms of the set $\mathrm{Irr}A$ of irreducible representations of A. We first have to identify representations which are unitarily equivalent in the following sense.

Definition 1.3.4. Representations π_1, π_2 of a C*-algebra A in respective Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are said to be *unitarily equivalent* if there exists an isometric isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for all $a \in A$. One writes $\pi_1 \sim \pi_2$.

One defines the *spectrum* of A as $\hat{A} = \mathrm{Irr}A/\sim$. There is also a useful generalisation of the spectrum of a commutative C*-algebra in terms of primitive ideals. By definition, a *primitive* ideal is the kernel of an irreducible representation. In the commutative case, an ideal is primitive if and only if it is maximal. We denote by $\mathrm{Prim}A$ the set of primitive ideals of A. Since unitarily equivalent representations have the same kernel, there is a map $\hat{A} \to \mathrm{Prim}A$ which associates its kernel with the class of a representation. This map is not necessarily injective (it is injective if and only if A is of type I, i.e. all its factorial representations are of type I). The primitive ideal space $\mathrm{Prim}A$ has a natural topology, the so-called hull-kernel topology, defined by the following closure operation. The closure of $Y \subset \mathrm{Prim}A$ consists of all primitive ideals which contain the ideal $I(Y) = \cap_{y \in Y} y$. This topology is locally compact and T_0 (given distinct points, there exists an open set which contains one of the points but not the other) but not necessarily T_1 (which means that singletons are closed).

1.3.2 States

Let us give the definition of a state of a C*-algebra and the relationship between states and representations. States generalize probability measures.

Definition 1.3.5. A *state* of a unital C*-algebra A is a linear functional $\varphi : A \to \mathbf{C}$ such that
(i) φ is positive: for all $a \in A$, $\varphi(a^*a) \geq 0$;
(ii) φ preserves the unit: $\varphi(1) = 1$.

Proposition 1.3.6. *Let φ be a state of a unital C*-algebra A. Then*
(i) for all $a \in A$, $\varphi(a^*a) \leq \|a\|^2$;
(ii) for all $a, b \in A$, $|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b)$;
(iii) φ is bounded of norm 1.

Proof. For the first assertion, let us first establish the inequality $a^*a \leq \|a\|^2 1$. There exists a self-adjoint element $h \in A$ such that $a^*a = h^2$. Then the inequality becomes $h^2 \leq \|h\|^2 1$, which is obviously satisfied in the commutative C*-algebra $C^*(h, 1)$, hence in A. We then apply the state φ. The second assertion is the classical Cauchy-Schwarz inequality applied to the positive sesquilinear form $\langle a, b\rangle = \varphi(a^*b)$. For $a = 1$, this gives $|\varphi(b)|^2 \leq \varphi(b^*b) \leq \|b\|^2$, hence $\|\varphi\| \leq 1$. Since $\varphi(1) = 1$, we have the equality. □

Remark 1.3.7. When the C*-algebra A does not have a unit, one defines a state of A as the restriction to A of a state of the C*-algebra $\tilde{A}$ obtained by adjoining a unit. One can show that a state of A is a positive linear functional of A which is bounded of norm 1.

Example 1.3.8. Let π be a representation of a C*-algebra A in a Hilbert space $\mathcal{H}$ and let ξ be a unit vector in $\mathcal{H}$. Then the linear functional ω_ξ defined by $\omega_\xi(a) = \langle \xi, \pi(a)\xi\rangle$ is a state of A. Such a state is called a *vector state* associated with the representation π. The celebrated GNS (after Gelfand, Naimark and Segal) construction gives a converse. Let φ be a state of a unital C*-algebra A. As earlier, we define $\langle a, b\rangle_\varphi = \varphi(a^*b)$. To get an inner product, we quotient by the subspace $N_\varphi = \{a \in A : \varphi(a^*a) = 0\}$. Let $\mathcal{H}_\varphi$ be the Hilbert space obtained by completing A/N_φ for the norm $\|a\|_2 = \langle a, a\rangle^{1/2}$. We denote the composition of the quotient map and the injection of A/N_φ into $\mathcal{H}_\varphi$ by $\eta_\varphi : A \to \mathcal{H}_\varphi$. For $a \in A$, we define the operator $\pi_\varphi(a)$ on $\eta_\varphi(A)$ by $\pi_\varphi(a)\eta_\varphi(b) = \eta_\varphi(ab)$.

Proposition 1.3.9. *Let φ be a state of a unital C*-algebra A. Let $\eta_\varphi, \mathcal{H}_\varphi$ and π_φ be as above; let $\xi_\varphi = \eta_\varphi(1)$. Then*
(i) for all $a \in A$, $\pi_\varphi(a)$ is a bounded operator on $\mathcal{H}_\varphi$;
(ii) π_φ is a representation of A in $\mathcal{H}_\varphi$;
(iii) for all $a \in A$, $\varphi(a) = \langle \xi_\varphi, \pi_\varphi(a)\xi_\varphi\rangle$;
(iv) ξ_φ is cyclic for π_φ, i.e. $\pi_\varphi(A)\xi_\varphi$ is dense in $\mathcal{H}_\varphi$.

Proof. All these verifications are straightforward. Let us check for example that $\pi_\varphi(a)$ is bounded on the dense subspace $\eta_\varphi(A)$. We have already seen the inequality $a^*a \le \|a\|^2 1$, which means that $\|a\|^2 1 - a^*a$ is of the form c^*c. One deduces that $b^*(a^*a)b \le \|a\|^2 b^*b$. Applying φ to this inequality gives

$$\|\pi_\varphi(a)\eta_\varphi(b)\|_2^2 \le \|a\|^2 \|\eta_\varphi(b)\|_2^2.$$

□

Example 1.3.10. Let μ be a probability measure on a compact space X. The GNS construction applied to the state $\varphi(f) = \int f(x)d\mu(x)$ of $C(X)$ provides the Hilbert space $L^2(X, \mu)$ and the representation of $C(X)$ by multiplication operators.

The realization of a state as a vector state is unique in the sense that two triples $(\mathcal{H}_i, \pi_i, \xi_i), i = 1, 2$ where π_i is a representation of A in $\mathcal{H}_i$ and ξ_i is a unit cyclic vector in $\mathcal{H}_i$, which give the same vector state are isomorphic: there exists an isometric isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for all $a \in A$ and $U\xi_1 = \xi_2$. One says that a state is a *factor* state if its GNS representation is a factorial representation. Then, its type is the type of this representation.

Let A be a unital C*-algebra. The set of states $S(A)$ of A is a convex subset of the unit ball of the dual of A; it is $*$-weakly closed, hence compact. Its extremal elements are called the *pure* states. The set of pure states is denoted by $P(A)$. It is not difficult to check that a state is pure if and only if the associated GNS representation is irreducible. By the Hahn-Banach theorem, a self-adjoint element a such that $\varphi(a) \geq 0$ for all $\varphi \in S(A)$ necessarily belongs to A_+. By the Krein-Milman theorem, it suffices to check this condition for all $\varphi \in P(A)$. One deduces the important theorem mentioned earlier.

Theorem 1.3.11 (Gelfand-Naimark Theorem)**.** *Every C*-algebra A admits an isometric representation in a Hilbert space $\mathcal{H}$. If A is separable, $\mathcal{H}$ can be chosen separable.*

Proof. Replacing A by $\tilde{A}$ if necessary, we assume that A has a unit. The representations $\oplus_{\varphi\in S(A)}\pi_\varphi$ and $\oplus_{\varphi\in P(A)}\pi_\varphi$ are one-to-one, hence isometric. Indeed, assume that $\pi_\varphi(a) = 0$ for all $\varphi \in P(A)$; then, we have $\varphi(a^*a) = \langle\xi_\varphi, \pi_\varphi(a^*a)\xi_\varphi\rangle = \|\pi_\varphi(a)\xi_\varphi\|^2 = 0$, hence $a^*a \in A_+ \cap (-A_+) = \{0\}$, hence $a = 0$. $\square$

1.3.3 The case of commutative C*-algebras

We have seen that in the GNS construction from a state of a commutative C*-algebra $C(X)$, the algebra acts by multiplication operators in a Hilbert space $L^2(X, \mu)$ of square-integrable functions. With suitable countability assumptions, the most general representation of a commutative C*-algebra is still by multiplication operators, but in a Hilbert space of square-integrable vector fields rather than scalar functions. The field of Hilbert spaces which appears reflects the multiplicity of the representation. The theory of measurable fields of Hilbert spaces is a bit technical and we shall be very sketchy.

Definition 1.3.12. A *measurable field of Hilbert spaces* over a measure space (X, μ), where X is a locally compact space and μ is a Radon measure, consists of a Hilbert space $\mathcal{H}_x$ for each $x \in X$ and a linear subspace $\mathcal{E} \subset \Pi_x\mathcal{H}_x$, closed under multiplication by $C_c(X)$ and such that
(i) $x \mapsto \langle\xi(x), \eta(x)\rangle_x$ is measurable for all $\xi, \eta \in \mathcal{E}$;
(ii) $\int\langle\xi(x), \xi(x)\rangle_x d\mu(x) < \infty$ for all $\xi \in \mathcal{E}$;
(iii) $\mathcal{E}$ contains a countable subset $\{\xi_n\}$ which generates $\mathcal{E}$ as a $C_c(X)$-module and such that for all $x \in X$, $\{\xi_n(x)\}$ spans a dense linear subspace of $\mathcal{H}_x$. We say that $\mathcal{E}$ is a *fundamental space of sections.*

A basic example is the constant field $\mathcal{H}_x = \mathcal{H}$, where $\mathcal{H}$ is a fixed (separable) Hilbert space and where $\mathcal{E}$ is the algebraic tensor product

$C_c(X) \otimes \mathcal{H}$.

Given a measurable field of Hilbert spaces $H = \{(\mathcal{H}_x), \mathcal{E}\}$, we denote by $L^2(X, \mu, H) = \int^{\oplus} \mathcal{H}_x d\mu(x)$ the completion of $\mathcal{E}$ with respect to the inner product $\langle \xi, \eta \rangle = \int \langle \xi(x), \eta(x) \rangle_x d\mu(x)$. It is called the *direct integral* of the Hilbert spaces. Its elements can be identified with (equivalence classes of) measurable sections of the field H. We have the representation of $C_0(X)$ by multiplication operators in $L^2(X, \mu, H)$ defined by $(M(f)\xi)(x) = f(x)\xi(x)$ for $f \in C_c(X)$ and $\xi \in \Gamma$. In other words, the operators $M(f)$ appear as *diagonal operators.*

We give without a proof the main result about representations of commutative C*-algebras, which says that all representations are of that form.

Theorem 1.3.13. *Let X be a second countable locally compact space and let L be a representation of $C_0(X)$ in a separable Hilbert space. Then, there is a Radon measure μ on X and a measurable field of Hilbert spaces $H = \{(\mathcal{H}_x), \mathcal{E}\}$ on (X, μ) such that L is unitarily equivalent to the representation by multiplication operators on $L^2(X, \mu, H)$.*

1.4 C*-modules

Definition 1.4.1. Let B be a C*-algebra. A right *C*-module* over B is a linear space E endowed with

1. a structure of right-B-module

$$(x, b) \in E \times B \ \mapsto \ xb \in E$$

2. a B-valued "inner product"

$$(x, y) \in E \times E \ \mapsto \ \langle x, y \rangle \in B$$

satisfying

(a) it is B-linear in the second variable y,

(b) $\langle y, x \rangle^* = \langle x, y \rangle$

(c) $\langle x, x\rangle$ is a positive element of B

(d) $\|x\| = \|\langle x, x\rangle\|^{1/2}$ is a complete norm on E

We say that a B-C*-module E is *full* if the linear span of the range of the inner product is dense in B.

Examples 1.4.2. When $B = \mathbf{C}$, this is the definition of a Hilbert space. Note also that $E = B$, equipped with right multiplication and the inner product $\langle x, y\rangle = x^* y$ is a full C*-module over B.

There is no space here to give a complete exposition of the theory of C*-modules. Let us just mention the following version of the Cauchy-Schwarz inequality, which can be deduced from the scalar case by checking it against states.

Lemma 1.4.3 (Cauchy-Schwarz inequality)**.** *Let E be a C*-module over the C*-algebra B. For $x, y \in E$, the following inequality holds in the C*-algebra B:*

$$\langle x, y\rangle^* \langle x, y\rangle \leq \|\langle x, x\rangle\| \langle y, y\rangle.$$

The notions of bounded and compact linear operators can be defined in this new setting.

Definition 1.4.4. Let E and F be C*-modules over the same C*-algebra B. A map $T : E \to F$ is called *adjointable* if there exists a map $S : F \to E$ such that for all $(x, y) \in E \times F$, $\langle y, T(x)\rangle = \langle S(y), x\rangle$.

Lemma 1.4.5. *Let $T : E \to F$ be an adjointable map, where E and F are C*-modules over B. Then*
(i) T is **C**- *and B-linear;*
(ii) it is bounded;
(iii) the adjoint map S of the above definition is unique and denoted $S = T^*$.

Proof. Assertion (i) is a consequence of the linearity of the inner product; assertion (ii) results from the closed graph theorem; assertion (iii) holds because the inner product is non-degenerate. □

One defines $\mathcal{L}_B(E, F)$ as the space of adjointable operators $T : E \to F$. The composition of adjointable operators is an adjointable operator. For $F = E$, one gets the *-algebra $\mathcal{L}_B(E) = \mathcal{L}_B(E, E)$. It can be checked that the operator norm turns it into a C*-algebra. The inequality $\|T^*T\| \geq \|T\|^2$ can be deduced from the above Cauchy-Schwarz inequality.

One defines $\mathcal{K}_B(E, F)$ as the norm-closure of the linear span of the rank-one operators $\theta_{x,y}$, where $x \in E, y \in E$ and for $z \in E$,

$$\theta_{x,y}(z) = x\langle y, z\rangle.$$

The elements of $\mathcal{K}_B(E, F)$ are called *compact operators*. One defines $\mathcal{K}_B(E) = \mathcal{K}_B(E, E)$; it is a closed ideal of $\mathcal{L}_B(E)$.

Remark 1.4.6. We have already given $E = B$, where B is a C*-algebra, as a basic example of a right C*-module over B. The rank-one operators are the left multiplication operators $L(xy^*) = \theta_{x,y}$, where $L(b)z = bz$. Since in a C*-algebra, each element can be written xy^*, $\mathcal{K}_B(B)$ is isomorphic to B. If B has a unit, each $T \in \mathcal{L}_B(B)$ is of the form $L(b)$ with $b = T1$. Thus $\mathcal{L}_B(B) = \mathcal{K}_B(B)$. If B does not have a unit, $\mathcal{L}_B(B) = M(B)$ is by definition the *multiplier algebra* of B. The isomorphism $B \sim \mathcal{K}_B(B)$ embeds B as a closed ideal of $M(B)$. This ideal is *essential* in the sense that it has a non-zero intersection with every non-zero closed ideal of $M(B)$. When $B = C_0(X)$, $M(B)$ can be identified with $C_b(X)$, the algebra of continuous and bounded functions on X. In the general case, $\mathcal{L}_B(E)$ can be identified with the multiplier algebra of $\mathcal{K}_B(E)$.

The next result, the proof of which is left to the reader, gives a necessary and sufficient condition for $\mathcal{K}_B(E) = \mathcal{L}_B(E)$. Of course, this is equivalent to the identity map being a compact operator. Before stating it, let us introduce the C*-module eB^n, where e is a projection in the $*$-algebra $M_n(B)$ of n by n matrices with coefficients in B. The right action of B and the B-valued inner product are given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} b = \begin{pmatrix} x_1 b \\ \vdots \\ x_n b \end{pmatrix} \qquad \langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \rangle = \sum_{i=1}^n x_i^* y_i$$

Proposition 1.4.7. *Let E be a right C*-module over a C*-algebra B. The following conditions are equivalent:*

(i) There exist n and a projection $e \in M_n(B)$ such that E is isomorphic to eB^n.
(ii) There exist n and a projection $e \in M_n(B)$ such that the C-algebra of compact operators $\mathcal{K}_B(E)$ is isomorphic to $eM_n(B)e$.*
(iii) There exist n and $x_1, \ldots, x_n \in E$ such that $Id_E = \theta_{x_1,x_1} + \ldots + \theta_{x_n,x_n}$.
(iv) The C-algebra $\mathcal{K}_B(E)$ has a unit.*
(v) $\mathcal{K}_B(E) = \mathcal{L}_B(E)$.

Remark 1.4.8. The integer n appearing in (i), (ii) and (iii) can be chosen to be the same; given $x_1, \ldots, x_n \in E$ satisfying (iii), one can obtain the projection of (i) and (ii) as $e = (\langle x_i, x_j \rangle)$.
When B is an algebra with unit, a module of the form eB^n, where e is an idempotent in $M_n(B)$ is called *finitely generated projective*.

Definition 1.4.9. Let A and B be C*-algebras. An (A, B)-*C*-correspondence* (or a correspondence from A to B) is a right B-C*-module E together with a $*$-homomorphism $\pi : A \to \mathcal{L}_B(E)$.

We shall usually view an (A, B)-C*-correspondence as an (A, B)-bimodule. A $*$-homomorphism $\pi : A \to B$ defines an (A, B)-C*-correspondence, by considering $E = B$ as a right B-C*-module (then $\mathcal{L}_B(E) = B$). It is useful to view C*-correspondences as generalized $*$-homomorphisms. There is a composition of correspondences extending composition of $*$-homomorphisms: given a C*-correspondence E from A to B and a C*-correspondence F from B to C, one can construct the C*-correspondence $E \otimes_B F$ from A to C. It is the C-C*-module obtained by separation and completion of the ordinary tensor product $E \otimes F$ with respect to the inner product

$$< x \otimes y, x' \otimes y' >=< y, < x, x' >_B y' >_C, \qquad x, x' \in E, y, y' \in F;$$

the left A action is given by $a(x \otimes y) = ax \otimes y$ for $a \in A$. Associativity holds up to isomorphism. More precisely, given C*-algebras A, B, C, D and C*-correspondences E from A to B, F from B to C and G from C to D, there is a canonical isomorphism $(E \otimes_B F) \otimes_C G \to E \otimes_B (F \otimes_C G)$. We say that an (A, A)-correspondence E is an identity if it is isomorphic to the correspondence A (defined by the identity map) and that an (A, B)-correspondence E is *invertible* if there exists a (B, A)-correspondence F such that $E \otimes_B F$ and

$F \otimes_A F$ are identity correspondences. One has the following easy characterization of invertible correspondences.

Proposition 1.4.10. *Let A and B be C^*-algebras and let E be a linear space. Then the following conditions are equivalent:*

1. *E is an invertible (A, B)-C^*-correspondence,*
2. *E is a full right B-C^*-module and there is a $*$-isomorphism $\pi : A \to \mathcal{K}_B(E)$ map π,*
3. *E is a full right B-C^*-module and a full left A-C^*-module such that for a in A, x, y, z in E and b in B,*
   - *(a)* $(ax)b = a(xb)$
   - *(b)* ${}_A\!< x, y > z = x < y, z >_B$
   - *(c)* $< ax, ax >_B \leq \|a\|^2 < x, x >_B$
   - *(d)* ${}_A\!< xb, xb > \leq \|b\|^2 \; {}_A\!< x, x >$

Definition 1.4.11. An (A, B)-bimodule as in the above proposition is called an (A, B)-*Morita equivalence*. Two C*-algebras A and B are said to be *Morita equivalent* if there exists an (A, B)-Morita equivalence.

Example 1.4.12. We leave to the reader to check that A^n is a $(M_n(A), A)$-Morita equivalence (where $M_n(A)$ is the C*-algebra of $n \times n$ matrices over A). Another related basic example is the following: let C be a C*-algebra and let e, f be two projections in C such that $CeC = CfC$. Then eCf is a (eCe, fCf)-Morita equivalence.

Let E be an (A, B)-Morita equivalence. We define an inverse (B, A)-Morita equivalence E^* by a bijection $x \in E \mapsto x^* \in E^*$ and the following operations:

1. $x^* + y^* = (x + y)^* \quad (\lambda x)^* = \overline{\lambda} x^* \quad$ for $\quad x, y \in X \quad$ and $\quad \lambda \in \mathbf{C}$,
2. $b^* x^* a^* = (axb)^* \quad$ for $\quad (a, x, b) \in A \times E \times B$,
3. $< x^*, y^* >_A = {}_A\!< x, y >$ and ${}_B\!< x^*, y^* > = < x, y >_B$.

Then $E \otimes_B E^*$ [resp. $E^* \otimes_A E$] is isomorphic to the identity correspondence A [resp. B] via the inner product.

Chapter 2

Some Examples

In this chapter, we shall illustrate the theory of C*-algebras by some important examples. Our point of view is somewhat biased: these examples are intimately related to dynamical systems and can be obtained by the groupoid C*-algebra construction described below. The connection between dynamical systems and operator algebras goes back to the early days of the theory of operator algebras, when Murray and von Neumann introduced their group measure construction to produce interesting examples of factors. It has been extremely fruitful ever since.

2.1 The C*-algebra of a discrete group

Let Γ be a discrete group. A *unitary representation* of Γ is a homomorphism π of Γ into the group $\mathcal{U}(\mathcal{H})$ of unitary operators of some Hilbert space $\mathcal{H}$. The range $\pi(\Gamma)$ generates a C*-algebra $C^*_\pi(\Gamma)$. For example, if π_{reg} is the so-called *left regular* representation on the Hilbert space $l^2(\Gamma)$ of square-integrable families indexed by Γ, defined by $\pi_{reg}(s)\xi_t = \xi_{s^{-1}t}$, the corresponding C*-algebra is the *reduced* C*-algebra of Γ, denoted by $C^*_{\mathrm{red}}(\Gamma)$. It is useful to introduce the *full* C*-algebra $C^*(\Gamma)$, admitting each $C^*_\pi(\Gamma)$ as quotient. It corresponds to a universal representation of Γ. Concretely, one forms the linear space $\mathbf{C}[\Gamma]$ of all finite sums $\sum a_\gamma u_\gamma$ where $a_\gamma \in \mathbf{C}$

and u_γ is a symbol. It has a unique $*$-algebra structure such that $u_\gamma u_{\gamma'} = u_{\gamma\gamma'}$ and $u_\gamma^* = u_{\gamma^{-1}}$. Unitary representations of Γ are in a one-to-one correspondence with unit preserving representations of $\mathbf{C}[\Gamma]$: given $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, one defines $\tilde{\pi} : \mathbf{C}[\Gamma] \to \mathcal{L}(\mathcal{H})$ by $\tilde{\pi}(\sum a_\gamma u_\gamma) = \sum a_\gamma \pi(\gamma)$. Conversely, given $\tilde{\pi}$, one recovers π by $\pi(\gamma) = \tilde{\pi}(u_\gamma)$. We shall often write π instead of $\tilde{\pi}$. The full C*-norm is defined by

$$\|\sum a_\gamma u_\gamma\| = \sup\{\|\sum a_\gamma \pi(\gamma)\| \mid \pi \text{ unitary representation of } \Gamma\}.$$

The inequalities

$$\|\sum a_\gamma L(\gamma)\| \leq \|\sum a_\gamma u_\gamma\| \leq \sum |a_\gamma|$$

show that it is indeed a norm. The completion of $\mathbf{C}[\Gamma]$ for the full C*-norm is the full C*-algebra $C^*(\Gamma)$. Its unit preserving representations are still in a one-to-one correspondence with the unitary representations of Γ.

Let us look at the example of $\Gamma = \mathbf{Z}$. The unitary representations of $\mathbf{Z}$ are given by a single unitary operator U. Since the group is commutative, so is the C*-algebra $A = C^*(\mathbf{Z})$. Its irreducible representations are one-dimensional: they are the characters of the C*-algebra (and of the group $\mathbf{Z}$). They are given by a complex number z of module one. This identifies $X(A)$ with the space $\mathbf{T}$ of complex numbers of module 1, as a set but also as a topological space. The Gelfand transform is the isomorphism $\mathcal{G} : C^*(\mathbf{Z}) \to C(\mathbf{T})$ sending $\sum a_n u^n \in \mathbf{C}[\mathbf{Z}]$ into the trigonometric polynomial $f(z) = \sum a_n u^n$.

2.2 The irrational rotation algebra

2.2.1 The non-commutative torus

Instead of introducing this algebra directly, let us consider the following problem. What can be said about a pair (U, V) of unitary operators in a Hilbert space $\mathcal{H}$ which satisfy the commutation relation $UV = e^{i\theta} VU$, where $\theta \in \mathbf{R}$? Our strategy is to introduce the C*-algebra $C^*(U, V)$ generated by U and V, or rather an abstract C*-algebra $C^*(u, v)$ generated by two unitary elements u and v subject to the relation $uv = e^{i\theta}vu$ which is universal in the sense that its representations are in a one-to-one correspondence with pairs of unitary operators (U, V) as above; this correspondence associates to the representation π the pair $(\pi(u), \pi(v))$. If $e^{i\theta} = 1$, then U and V commute and we are in the above situation. They define a unitary representation of the commutative group $\mathbf{Z}^2$. Unitary representations of the group $\mathbf{Z}^2$ are in one-to-one correspondence with representations of the group C*-algebra $C^*(u, v) = C^*(\mathbf{Z}^2)$. The Gelfand isomorphism identifies $C^*(\mathbf{Z}^2)$ with $C(\mathbf{T}^2)$, where $\mathbf{T}^2$ is the 2-dimensional torus. When $e^{i\theta} \neq 1$, the C*-algebra $A_\theta = C^*(u, v)$, which can be constructed just as in the group case as the completion of the $*$-algebra of Laurent polynomials $\sum a_{mn}u^m v^n$, is no longer commutative and is called a *non-commutative torus.*

2.2.2 Covariant representations

Let us reinterpret the commutation relation $V^{-1}UV = e^{i\theta}U$ as a covariant representation.

Definition 2.2.1. A (discrete) *transformation group* consists of a triple (Γ, X, T), where Γ is a discrete group, X is a compact space and T is a homomorphism of Γ into the group of homeomorphisms of X. A *covariant representation* of (Γ, X, T) consists of a pair (π, M), where π is a unitary representation of Γ in a Hilbert space $\mathcal{H}$, M is a representation of $C(X)$ in $\mathcal{H}$ satisfying the relation

$$\pi(\gamma)^{-1}M(f)\pi(\gamma) = M(f \circ T_\gamma) \quad \text{for all} \quad \gamma \in \Gamma, f \in C(X).$$

Lemma 2.2.2. *There is a one-to-one correspondence between pairs (U, V) of unitary operators in a Hilbert space $\mathcal{H}$ satisfying the commutation relation $UV = e^{i\theta}VU$ and covariant representations of $(\mathbf{Z}, \mathbf{T}, T)$ where T_1 is the rotation on the circle $\mathbf{T}$ of angle θ.*

Proof. Let U and V be unitary operators in a Hilbert space $\mathcal{H}$ satisfying the commutation relation $UV = e^{i\theta}VU$. The unitary operator U defines a unitary representation of $\mathbf{Z}$, hence a representation of $C^*(\mathbf{Z})$ which we identify with $C(\mathbf{T})$. Explicitly, we obtain a representation M of $C(\mathbf{T})$ in $\mathcal{H}$ such that $M(f) = \sum a_n U^n$ for all trigonometric polynomial $f(z) = \sum a_n z^n$. The above commutation relation takes the form $V^{-1}M(f)V = f \circ T_1$, where $T_1(z) = e^{i\theta}z$. By continuity, this holds for all $f \in C(\mathbf{T})$. Therefore (π, M), where $\pi(1) = V$ is a covariant representations of $(\mathbf{Z}, \mathbf{T}, T)$. Conversely, let (π, M) be a covariant representations of $(\mathbf{Z}, \mathbf{T}, T)$. Let $V = \pi(1)$ and let $U = M(f)$, where $f(z) = z$. Then (U, V) is a pair of unitary operators satisfying the commutation relation $UV = e^{i\theta}VU$. □

This lemma suggests another possible definition of the C*-algebra A_θ: it is a C*-algebra which is universal with respect to covariant representations of the dynamical system $(\mathbf{Z}, \mathbf{T}, T)$, where T_1 is the rotation $T_1(z) = e^{i\theta}z$. More precisely, there exists a covariant representation $(\tilde{\pi}, \tilde{M})$ of $(\mathbf{Z}, \mathbf{T}, T)$ into A_θ such that every covariant representation is of the form $(L \circ \tilde{\pi}, L \circ \tilde{M})$ for a unique representation L of A_θ. The existence and uniqueness (up to isomorphism) of such a C*-algebra can be shown for arbitrary transformation groups (Γ, X, T). It is called the *crossed product C*-algebra* and often denoted by $C(X) \rtimes \Gamma$. Let us sketch the construction of the crossed product C*-algebra, which is quite analogous to the construction of the group C*-algebra. One considers the linear space $\mathcal{A} = \mathcal{A}(X, T)$ of finite sums $\sum f_\gamma u_\gamma$ where $f_\gamma \in C(X)$ and u_γ is a symbol. It has a unique $*$-algebra structure such that $u_\gamma u_{\gamma'} = u_{\gamma\gamma'}, u_\gamma^* = u_{\gamma^{-1}}$ for all $\gamma, \gamma' \in \Gamma$ and $u_\gamma^{-1} f u_\gamma = f \circ T_\gamma$ for all $\gamma \in \Gamma, f \in C(X)$. Then $\tilde{\pi}(\gamma) = u_\gamma$ and $\tilde{M}(f) = f u_1$ define a covariant representation $(\tilde{\pi}, \tilde{M})$ in $\mathcal{A}$. Every covariant representation (π, M) in a Hilbert space $\mathcal{H}$ extends to a representation L of $\mathcal{A}$ in $\mathcal{H}$ according to $L(\sum f_\gamma u_\gamma) = \sum M(f_\gamma)\pi(u_\gamma)$ and satisfies $M = L \circ \tilde{M}$ and $\pi = L \circ \tilde{\pi}$. One defines a C*-norm on $\mathcal{A}$ by

$$\| \sum f_\gamma u_\gamma \| = \sup \| \sum M(f_\gamma)\pi(u_\gamma) \|,$$

where the supremum is taken over all covariant representations (π, M). The crossed product C*-algebra $C(X) \rtimes_T \Gamma$ is defined as the completion of $\mathcal{A}$ with respect to this norm. We will let the reader prove the following proposition.

Proposition 2.2.3. *Let $\theta \in R$ and let T be the rotation of angle θ on the circle $\mathbf{T}$. There is an isomorphism of the C*-algebra A_θ defined above and the crossed product C*-algebra $C(\mathbf{T}) \rtimes_T \mathbf{Z}$ which sends u to the function $z \mapsto z$ in $C(\mathbf{T})$ and v to the generator u_1.*

2.3 Groupoids and groupoid C*-algebras

2.3.1 Groupoids

For later purposes, it is convenient to introduce the following object in the construction of the crossed product. Let (Γ, X, T) be a transformation group as above. We define

$$G = G(X,T) = \{(x,\gamma,y) \in X \times \Gamma \times X : x = T_\gamma(y)\}.$$

It is equipped with some algebraic structure:

- two maps, called respectively *range* and *source* maps, $r, s : G \to X$ given by $r(x,\gamma,y) = x$ and $s(x,\gamma,y) = y$;
- a map, called *inclusion* map $i : X \to G$, given by $i(x) = (x, 1, x)$;
- an involutive map, called *inverse* map $G \to G$ given by $((x,\gamma,y)^{-1} = (y, \gamma^{-1}, x)$;
- a map, called *multiplication* map, $m : G^{(2)} \to G$, where

  $$G^{(2)} = \{((x,\gamma,y),(y,\gamma',z)) \in G \times G\} \subset G \times G,$$

  such that $m((x,\gamma,y),(y,\gamma',z)) = (x, \gamma\gamma', z)$.

An element of G of the form $(x, 1, x)$ is called a *unit*. The set of units, denoted by $G^{(0)}$, will be often identified with X through the inclusion map i; in particular, we shall often view r and s as maps from G to $G^{(0)}$. The properties of the units, the inverse and the multiplication are the same as those of a group. The main difference is that there is a set of units instead of a single unit and that the multiplication gg' of two elements $g, g' \in G$ is possible only if $s(g) = r(g')$. The set G equipped with this algebraic structure is an example of a *groupoid*. The following definition of a groupoid is concise but it is recommended to think of the above example rather than of this abstract definition.

Definition 2.3.1. A *groupoid* is a small category such that every morphism is invertible.

A small category means that we have a set $G^{(0)}$ of *objects* and a set of morphisms G. In our example, an object is a unit $x \in X$ and a morphism is an element (x, γ, y) which should be thought of as a morphism (or an arrow) from y to x. Note that a groupoid G is a group if and only if its unit space $G^{(0)}$ has a single element. In the general case, we define the *isotropy group* at $x \in G^{(0)}$ as $G(x) = \{\gamma : r(\gamma) = s(\gamma) = x\}$. We define the *isotropy bundle* $G' \subset G$ as the union of the isotropy groups. We say that G is *principal* if for all $x \in G^{(0)}$, $G(x) = \{x\}$. A groupoid G on $X = G^{(0)}$ induces an equivalence relation on X: two units are equivalent if there is an arrow joining them. A groupoid is principal if there is at most one arrow joining two units. Equivalently, G can be identified with the graph of the equivalence relation. When $G = G(X, T)$ as above, G is principal if and only if the action is free in the sense that $T_\gamma(x) = x$ for some x implies that $\gamma = 1$. As in our example $G^{(2)}$ denotes the set of composable pairs, i.e. elements $(\gamma, \gamma') \in G \times G$ such that $s(\gamma) = r(\gamma')$. The multiplication map is defined on $G^{(2)}$. A useful operation is the *restriction* of a groupoid: given a subset $Y \subset G^{(0)}$, $G_{|Y} = r^{-1}(Y) \cap s^{-1}(Y)$ is a subgroupoid of G with unit space Y.

In our example, $G \subset X \times \Gamma \times X$ inherits the topology of $X \times \Gamma \times X$. Since it is a closed subset, this topology is locally compact. Moreover, all the structure maps are continuous. This turns G into a locally compact topological groupoid. In the general case of a topological groupoid, we shall always assume that $G^{(0)}$ is given the relative topology of G and that $G^{(2)}$ is given the relative topology of $G \times G$. The fact that G arises from the action of a discrete group is reflected by the fact that the range and source maps are local homeomorphisms, since they send the open subsets $\{(T_\gamma(x), \gamma, x), x \in X\}$ homeomorphically onto X. This property is important for our purpose.

Definition 2.3.2. A topological groupoid G is *étale* if the range and source maps $r, s : G \to G^{(0)}$ are local homeomorphisms.

The unit space $G^{(0)}$ of an étale groupoid is an open subset of G. We have a more precise statement, the proof of which is left as an exercise.

Exercise 2.3.3. Let G be a topological groupoid. Show that $G^{(0)}$ is an open subset of G if and only if the range map $r : G \to G^{(0)}$ is locally injective, in the sense that each $\gamma \in G$ has a neighborhood on which r is one-to-one.

We have already given the definition of a principal groupoid (recall that it means that all the isotropy groups are trivial). In the topological setting, the following weaker definition is useful.

Definition 2.3.4. We say that a topological groupoid G is *topologically principal* if the set of units $x \in G^{(0)}$ such that $G(x) = \{x\}$ is dense in $G^{(0)}$.

2.3.2 Groupoid algebras

Let G be a locally compact étale groupoid. Then the space $C_c(G)$ of continuous and compactly supported complex-valued functions has a structure of $*$-algebra given by

$$f * g(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2);$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Let us spell out these formulas for our example:

$$f * g(x, \gamma, z) = \sum_{\gamma_1\gamma_2=\gamma, y=T_{\gamma_2}(z)} f(x, \gamma_1, y)g(y, \gamma_2, z);$$

$$f^*(x, \gamma, y) = \overline{f(y, \gamma^{-1}, x)}.$$

These formulas combine the usual group convolution and involution and the usual matrix multiplication and adjoint.

Recall that $G^{(0)}$ is an open subset. Therefore we may (and shall) view $C_c(G^{(0)})$ as a subspace of $C_c(G)$. In fact, it is a sub-$*$-algebra; moreover the multiplication of an element of $C_c(G^{(0)})$ and an arbitrary element of $C_c(G)$ has a pleasant expression: for $h, k \in C_c(G^{(0)})$ and $f \in C_c(G)$, we have

$$h * f * k(\gamma) = h \circ r(\gamma)f(\gamma)k \circ s(\gamma).$$

We shall usually omit the $*$ and simply write hfk.

Proposition 2.3.5. *Let (Γ, X, T) be a transformation group as above. Then the $*$-algebras $\mathcal{A}(X,T)$ and $C_c(G(X,T))$ are isomorphic.*

Proof. Recall that an element of $\mathcal{A}(X,T)$ is a finite sum $\sum f_\gamma u_\gamma$ where $f_\gamma \in C(X)$. We associate to it the function f on $G(X,T)$ such that $f(x,\gamma,y) = f_\gamma(x)$. It is clear that f belongs to $C_c(G(X,T))$. We leave to the reader to check that this is a $*$-algebra isomorphism. $\square$

2.3.3 Representations of groupoid algebras

In this section, G denotes a locally compact étale groupoid. We recommend the reader unfamiliar with the groupoid language to consider first the case $G = G(X,T)$, where (Γ, X, T) is a transformation group (and for example a rotation of the circle).

Recall that a representation of $C_c(G)$ in a Hilbert space $\mathcal{H}$ is a $*$-algebra homomorphism $L : C_c(G) \to \mathcal{L}(\mathcal{H})$. First we give the ingredients to construct a large class of representations. We are guided both by the commutative case, where a representation is given by a measure and a measurable field of Hilbert spaces, and by the group case, where a representation is given by a unitary representation of the group. We also have the important case of the crossed product by a group.

Definition 2.3.6. A *bisection* of G is a subset $S \subset G$ such that $r_{|S}$ and $s_{|S}$ are one-to-one. A bisection S defines a map $\alpha_S : s(S) \to r(S)$ such that $\alpha_S(x) = r(Sx)$ for $x \in s(S)$ and where Sx is the unique element of S such that $s(Sx) = x$. If S is open, this map is a homeomorphism.

Example 2.3.7. In the case $G = G(X,T)$, where (Γ, X, T) is a transformation group, each $\gamma \in \Gamma$ defines the open bisection $S = \{(T_\gamma(y), \gamma, y), y \in X\}$. The corresponding map α_S is T_γ.

The following definition is familiar in the case of a transformation group.

Definition 2.3.8. A Radon measure μ on $G^{(0)}$ is *invariant* [resp. *quasi-invariant*] with respect to G if for all open bisection S, the measures $(\alpha_S)_*\mu_{|s(S)}$ and $\mu_{|r(S)}$ are equal [resp. equivalent, i.e. have the same nullsets]. Here $\alpha_*\mu$ is the measure defined by $\alpha_*\mu(B) = \mu(\alpha^{-1}(B))$.

Exercise 2.3.9. Show that μ is quasi-invariant if and only if the measures $r^*\mu$ and $s^*\mu$ are equivalent, where $r^*\mu$ is the measure on G defined for $f \in C_c(G)$ by

$$\int f d(r^*\mu) = \int \sum_{\gamma \in r^{-1}(x)} f(\gamma) d\mu(x)$$

and similarly for $s^*\mu$.

Proposition 2.3.10. *Let μ be a quasi-invariant measure. There exists a measurable function $D : G \to \mathbf{R}_+$ such that for all open bisections S, and all $f \in C_c(X)$ with supp$(f) \subset r(S)$,*

$$\int_{r(S)} f(x) d\mu(x) = \int_{s(S)} f(r(Sy)) D(Sy) d\mu(y).$$

Moreover D satisfies the cocycle relation $D(\gamma\gamma') = D(\gamma)D(\gamma')$ for a.e. $(\gamma, \gamma') \in G^{(2)}$ (where $G^{(2)}$ carries the measure class of $(s \circ \pi_1)_*\mu$) and is essentially unique.

Proof. This result is well-known in the case of a transformation group (Γ, X, T). Then, one can define D by $D(T_\gamma y, \gamma, y) = d(T_{\gamma^{-1}})_*\mu / d\mu(y)$. In the general case, it is convenient to use the above exercise and to define D as the Radon-Nikodým derivative $D = d(r^*\mu)/d(s^*\mu)$. Then one checks that D has the desired properties. $\square$

Definition 2.3.11. This function D is called the *Radon-Nikodým derivative*, or simply the *derivative* of the quasi-invariant measure μ. We shall write it D_μ or $D_{\mu,G}$ if there is some ambiguity about G.

Definition 2.3.12. A *unitary representation* (or *representation* for short) of G consists of:
• a Radon measure μ on $G^{(0)}$, which is quasi-invariant with respect to G with Radon-Nikodym derivative D;
• a measurable field of Hilbert spaces $H = \{(\mathcal{H}_x), \mathcal{E}\}$ over $(G^{(0)}, \mu)$;
• a measurable action of G on H by isometries. This means that for all $\gamma \in G$, we have an isometric isomorphism

$$L(\gamma) : \mathcal{H}_{s(\gamma)} \to \mathcal{H}_{r(\gamma)}$$

such that the following properties hold:
(*i*) for all $x \in G^{(0)}$, $L(x)$ is the identity map of $\mathcal{H}_x$,
(*ii*) for all $(\gamma, \gamma') \in G^{(2)}$, $L(\gamma\gamma') = L(\gamma)L(\gamma')$,
(*iii*) $\gamma \mapsto \langle \xi \circ r(\gamma), L(\gamma)\eta \circ s(\gamma)\rangle_{r(\gamma)}$ is measurable for all $\xi, \eta \in \mathcal{E}$.

When G is a group, we have a single Hilbert space and we retrieve the definition of a unitary representation. On the other hand, when $G = G^{(0)}$, we have the ingredients to construct a $*$-representation of the commutative $*$-algebra $C_c(G^{(0)})$.

Exercise 2.3.13. Let (μ, H, L) be a representation of $G(X,T)$ where (Γ, X, T) is a transformation group. Let M be the representation of $C_c(G^{(0)})$ by multiplication operators on $\mathcal{H} = L^2(G^{(0)}, H, \mu)$. For $\gamma \in \Gamma$ and $\xi \in \mathcal{H}$, define $\pi(\gamma)\xi$ by

$$\pi(\gamma)\xi(x) = D(x,\gamma,y)^{-1/2}L(x,\gamma,y)\xi(y) \quad \text{where} \quad x = T_\gamma(y).$$

(*i*) Show that $\pi(\gamma)$ is a unitary operator on $\mathcal{H}$.
(*ii*) Show that π is a unitary representation of Γ.
(*iii*) Show that (π, M) is a covariant representation of (X,T).

Just as in the above case of a transformation group, a representation of G can be integrated to produce a representation of $C_c(G)$. One forms the Hilbert space $\mathcal{H} = L^2(G^{(0)}, \mu, H)$ of square integrable sections and one defines for $f \in C_c(G)$ the operator $L(f)$ on $\mathcal{H}$ by

$$\langle \xi, L(f)\eta\rangle = \int \sum_{r(\gamma)=x} \langle \xi(x), f(\gamma)D^{-1/2}(\gamma)L(\gamma)\eta \circ s(\gamma)\rangle_x d\mu(x)$$

for $\xi, \eta \in \mathcal{H}$. The following estimate, which is a straightforward application of the Cauchy-Schwarz inequality shows that $L(f)$ is a bounded operator.

Proposition 2.3.14. *Given $f \in C_c(G)$, the operator $L(f)$ is bounded and satisfies $\|L(f)\| \le \max(\|f\|_{I,r}, \|f\|_{I,s})$, where*
$\|f\|_{I,r} = \sup_{x\in G^{(0)}} \sum_{r(\gamma)=x} |f(\gamma)|$, $\|f\|_{I,s} = \sup_{x\in G^{(0)}} \sum_{s(\gamma)=x} |f(\gamma)|$.

Proof. We introduce the measures $\nu = r^*\mu$ and $\nu^{-1} = s^*\mu$ and recall that $\nu = D\nu^{-1}$. We have for $\xi, \eta \in \mathcal{H}$:

$$\begin{aligned} |\langle \xi, L(f)\eta\rangle| &\leq \int |f(\gamma)| \|\xi \circ r(\gamma)\|_{r(\gamma)} \|\eta \circ s(\gamma)\|_{s(\gamma)} D^{-1/2}(\gamma) d\nu(\gamma) \\ &\leq \Big(\int |f(\gamma)| \|\xi \circ r(\gamma)\|^2_{r(\gamma)} d\nu(\gamma)\Big)^{1/2} \times \\ &\quad \Big(\int |f(\gamma)| \|\eta \circ s(\gamma)\|^2_{s(\gamma)} D^{-1}(\gamma) d\nu(\gamma)\Big)^{1/2} \end{aligned}$$

Moreover,

$$\begin{aligned} \int |f(\gamma)| \|\xi \circ r(\gamma)\|^2_{r(\gamma)} d\nu(\gamma) &= \int \sum_{r(\gamma)=x} |f(\gamma)| \|\xi(x)\|^2_x d\mu(x) \\ &\leq \|f\|_{I,r} \int \|\xi(x)\|^2_x d\mu(x) \\ &\leq \|f\|_{I,r} \|\xi\|^2 \end{aligned}$$

Similarly,

$$\int |f(\gamma)| \|\eta \circ s(\gamma)\|^2_{s(\gamma)} d\nu^{-1}(\gamma) \leq \|f\|_{I,s} \|\eta\|^2$$

Thus, we obtain $|\langle \xi, L(f)\eta\rangle| \leq \|f\|^{1/2}_{I,r} \|f\|^{1/2}_{I,s} \|\xi\| \|\eta\|$ and therefore $\|L(f)\| \leq \|f\|^{1/2}_{I,r} \|f\|^{1/2}_{I,s} \leq \max(\|f\|_{I,r}, \|f\|_{I,s})$. □

One also checks that L is a $*$-homomorphism. One then says that L is an *integrated representation*. We define the *support of* L as the support of the measure μ. It is a closed invariant subset of $G^{(0)}$.

The main technical result is the disintegration theorem, which says that every representation of $C_c(G)$ is obtained in that fashion.

Theorem 2.3.15. *Assume that G is a second countable locally compact étale groupoid. Then every representation of $C_c(G)$ on a separable Hilbert space is an integrated representation.*

Proof. The proof is a bit technical but not difficult. We just sketch the main steps. We first consider the restriction M of L to the sub $*$-algebra $C_c(G^{(0)})$. We apply Theorem 1.3.13, which is also valid for the $*$-algebra $C_c(G^{(0)})$, to obtain a measure μ on $G^{(0)}$ and a measurable field of Hilbert spaces $H = \{(\mathcal{H}_x), \mathcal{E}\}$ on (X, μ) such that M is unitarily equivalent to the representation by multiplication operators on $L^2(X, \mu, H)$. Measure theory tells us how to extend M to a representation of the von Neumann algebra $L^\infty(G^{(0)}, \mu)$. Consider an element $a \in C_c(G)$ supported in an open bisection S. For all $b \in C_c(G^{(0)})$, a^*ba belongs to $C_c(G^{(0)})$and we have $a^*ba(x) = a^*a(x)b \circ \alpha_S(x)$ for all $x \in s(S)$. One deduces the covariance relation $L(a)^*M(b_{|r(S)})L(a) = M(a^*a)M(b \circ \alpha_S)$ for all $b \in L^\infty(G^{(0)}, \mu)$ and the quasi-invariance of μ. Let D be its Radon-Nikodým derivative. It remains to define the isometries $L(\gamma)$. Let a be as above. Let $Y = \{x \in G^{(0)} : a^*a(x) > 0\}$ and $Z = \{x \in G^{(0)} : aa^*(x) > 0\}$. The operator $L(a)$ admits a *polar decomposition* of the form $L(a) = VM(|a|)$ where $|a| = (a^*a)^{1/2}$ and V is a partial isometry. We view V as a unitary operator from $L^\infty(Y, H, \mu)$ onto $L^\infty(Z, H, \mu)$. It satisfies $V^*M(h)V = M(h \circ \alpha_S)$ for all $h \in L^\infty(Z, \mu)$. One can show that V is of the form

$$V\xi(x) = D(xS)^{-1/2}L(xS)\xi(\alpha_S^{-1}(x)) \quad \text{for} \quad x \in Z$$

where $L(xS)$, defined for μ a.e. $x \in Z$, is an isometry from $\mathcal{H}_{\alpha_S^{-1}}(x)$ onto $\mathcal{H}_x$. This provides the isometry $L(xS)$ for μ a.e. $x \in Z$. We omit further details. □

Corollary 2.3.16. *Every representation L of $C_c(G)$ in a separable Hilbert space satisfies the estimate $\|L(f)\| \leq \max(\|f\|_{I,r}, \|f\|_{I,s})$ for all f in $C_c(G)$.*

Proposition 2.3.17. *Let (μ, H, L) be a representation of an an étale locally compact groupoid G. We still denote by L the integrated representation of $C_c(G)$ on $\mathcal{H} = L^2(G^{(0)}, \mu, H)$. Let $\underline{x}$ be a point in the support of L with trivial isotropy. Then,*
(i) there exists a sequence of unit vectors (ξ_n) in $\mathcal{H}$ such that, for all $f \in C_c(G)$, $(\xi_n, L(f)\xi_n)$ tends to $f(\underline{x})$;
(ii) for all $f \in C_c(G)$, we have $|f(\underline{x})| \leq \|L(f)\|$.

Proof. One chooses a decreasing sequence (V_n) of neighborhoods of $\underline{x}$ whose intersection is reduced to $\{\underline{x}\}$ and a sequence of unit vectors $\xi_n \in H = L^2(G^{(0)}, \mu, \mathcal{H})$ supported in V_n. Let us show that assertion

(i) holds. By linearity, it suffices to check it when f is supported on an open bisection S. We then have

$$\langle \xi_n, L(f)\xi_n\rangle = \int \langle \xi_n(x), f(xS)D^{-1/2}(xS)L(xS)\xi_n(\alpha_S^{-1}(x))\rangle_x d\mu(x).$$

If $\underline{x}$ does not belong to $\alpha_S(V_n)$ for some n, this is equal to 0, as well as $f(\underline{x})$. Let us assume that $\underline{x}$ belongs to $\alpha_S(V_n)$ for all n. Then $\underline{x} = \alpha_S(\underline{x})$. Since $\underline{x}$ has no isotropy, this implies that $\underline{x} = \underline{x}S$ and that $V_nS \subset G^{(0)}$ for n sufficiently large. Then,

$$\langle \xi_n, L(f)\xi_n\rangle = \int f(x)\langle \xi_n(x), \xi_n(x)\rangle_x d\mu(x)$$

tends to $f(\underline{x})$. Assertion (ii) is an immediate consequence of (i). □

2.3.4 The full and the reduced C*-algebras

Let G be a locally compact étale groupoid G. In order to obtain a C*-algebra, we proceed as before: we define a C*-norm on $C_c(G)$ and complete it. The most canonical choice is the *full* norm, which is defined as $\|f\| = \sup_L \|L(f)\|$, where the supremum is taken over all representations L of $C_c(G)$ in (separable) Hilbert spaces. One has to check that the supremum is finite. This results from the estimate of Corollary 2.3.16. One also has to check that if $\|f\| = 0$, then $f = 0$. This is done by exhibiting the following faithful family of representations, the so-called *regular* representations: for each $x \in G^{(0)}$, one defines the representation π_x of $C_c(G)$ on the Hilbert space $\ell^2(G_x)$, where $G_x = s^{-1}(x)$, by $\pi_x(f)\xi = f * \xi$; more accurately, we should write:

$$\pi_x(f)\xi(\gamma) = \sum_{\eta\in G_x} f(\gamma\eta^{-1})\xi(\eta).$$

The family of representations $\{\pi_x\}_{|x\in G^{(0)}}$ is faithful in the sense that, if $\pi_x(f) = 0$ for all $x \in G^{(0)}$, then $f = 0$. One defines the *reduced* norm $\|f\|_{\mathrm{red}} = \sup_{x\in G^{(0)}} \|\pi_x(f)\|$.

Definition 2.3.18. The *(full) C*-algebra* of a locally compact étale groupoid G is the completion with respect to the full C*-norm of the $*$-algebra $C_c(G)$. It is denoted by $C^*(G)$. The *reduced C*-algebra* of G is the completion with respect to the reduced C*-norm of the $*$-algebra $C_c(G)$. It is denoted by $C^*_{\mathrm{red}}(G)$.

Note that the identity map on $C_c(G)$ induces a $*$-homomorphism from $C^*(G)$ onto $C^*_{\mathrm{red}}(G)$. Both algebras have their own advantages. The reduced C*-algebra is more concrete, in the sense that its elements are still functions on G. The full C*-algebra has better functorial properties. We get the best of both worlds when these algebras are equal, which will be the case for the rotation C*-algebra and most of our examples.

After this digression about étale groupoids, we have from the above proposition and from the definition:

Corollary 2.3.19. *Let (Γ, X, T) be a transformation group as above. Then the C*-algebras $C(X) \rtimes_T \Gamma$ and $C^*(G(X,T))$ are isomorphic.*

2.3.5 The reduced C*-algebra

In order to better understand the irrational rotation algebra A_θ and some further examples, let us establish some general facts about the reduced C*-algebra $C^*_{\mathrm{red}}(G)$ of a locally compact étale groupoid G. A good intuition is given by the algebra $M_n(\mathbf{C})$ of $n \times n$ matrices. It corresponds to the groupoid $G = \{1, \dots, n\} \times \{1, \dots, n\}$ where $G^{(0)}$ is the diagonal subset $\{(i,i), i = 1, \dots, n\}$. In this section G denotes a locally compact étale groupoid G.

Proposition 2.3.20. *The inclusion $C_c(G) \subset C_0(G)$ extends to a norm-decreasing linear injection $j : C^*_{\mathrm{red}}(G) \to C_0(G)$.*

Proof. Let $f \in C_c(G)$, $\gamma \in G$ and $x = s(\gamma)$. We use the regular representation π_x in $\ell^2(G_x)$ to write

$$\pi_x(f)\delta_\gamma = f_\gamma, \qquad f(\gamma) = \langle \delta_\gamma, \pi_x(f)\delta_x \rangle$$

where $\delta_\gamma \in \ell^2(G_x)$ is the point mass at $\gamma \in G_x$ and $f_\gamma \in \ell^2(G_x)$ is defined by $f_\gamma(\gamma') = f(\gamma'\gamma^{-1})$. From the second inequality, we deduce that $|f(\gamma)| \le \|\pi_x(f)\| \le \|f\|_r$, hence $\|f\|_\infty \le \|f\|_r$. We deduce the existence of a norm-decreasing linear extension $j : C^*_r(G) \to C_0(G)$. By continuity, the equality $\pi_x(a)\delta_\gamma = j(a)_\gamma$ still holds for $a \in C^*_{\mathrm{red}}(G)$. In particular, if $j(a) = 0$, then $\pi_x(a) = 0$ for all $x \in G^{(0)}$. This implies that $a = 0$. $\square$

Since $G^{(0)}$ is an open subset, we view $C_c(G^{(0)})$ as a subspace of $C_c(G)$. In fact, it is a sub-$*$-algebra; moreover the full and the reduced norm coincide on it with the sup norm. Therefore, we view its completion $C_0(G^{(0)})$ as a sub-C*-algebra of $C^*(G)$ and of $C^*_{\mathrm{red}}(G)$. Let $P : C^*_{\mathrm{red}}(G) \to C_0(G^{(0)})$ be the restriction map: $P(a) = a_{|G^{(0)}}$ for $a \in C^*_{\mathrm{red}}(G)$. It is linear and positive in the sense that it sends positive elements to positive elements; moreover, it is a conditional expectation in the following sense, which is familiar in probability theory.

Definition 2.3.21. A *conditional expectation* P of a C*-algebra A onto a sub-C*-algebra B is a positive linear map $P : A \to B$ such that its restriction to B is the identity map. It is *faithful* if $P(a^*a) = 0$ implies $a = 0$.

One can show that such a map is bounded and satisfies $P(b_1ab_2) = b_1P(a)b_2$ for all $a \in A$, $b_1, b_2 \in B$.

Proposition 2.3.22. *Let G be an étale locally compact groupoid. Let $P : C^*_{\mathrm{red}}(G) \to C_0(G^{(0)})$ be the restriction map. Then,*
(i) P is a faithful conditional expectation;
(ii) if G is topologically principal, it is the unique conditional expectation onto $C_0(G^{(0)})$.

Proof. For (i), we are just left to show that P is faithful. By continuity, the above formulas for the product and the involution still hold for $a \in C^*_{\mathrm{red}}(G)$. They give:

$$P(a^*a)(x) = \sum_{\gamma \in G_x} |j(a)(\gamma)|^2.$$

Therefore, if $P(a^*a) = 0$, then $j(a) = 0$. We conclude from Proposition 2.3.20 that $a = 0$. We give the proof of (ii) when G is principal and leave the general case, which uses the same idea, to the reader. Let Q be a conditional expectation onto $C_0(G^{(0)})$. We will show that Q and P agree on $C_c(G)$. By continuity, they will agree on $C^*_{\mathrm{red}}(G)$. We write $a \in C_c(G)$ as $a = a_1 + a_2$, where $a_1 = P(a)$ and $a_2 = a - P(a)$. Since $G^{(0)}$ is open, the support of a_2 is a compact subset K disjoint from $G^{(0)}$. By compactness of K, there exists a finite open cover $(U_1, \ldots, U_n)$ of $r(K) \subset G^{(0)}$ such that for all $i = 1, \ldots, n$,

U_i cannot contain simultaneously the range and the source of an element of K. Let $(h_1, \ldots, h_n)$ be a partition of unity subordinate to this cover: $0 \leq h_i \leq 1$, $h_i \in C_c(U_i)$ and $\sum_1^n h_i(x) = 1$ for $x \in r(K)$. Then

$$\begin{aligned} Q(a_2) &= Q((\sum_1^n h_i)a_2) = (\sum_1^n h_i)Q(a_2) = \sum_1^n h_i^{1/2} Q(a_2) h_i^{1/2} \\ &= Q(\sum_1^n h_i^{1/2} a_2 h_i^{1/2}) = Q(0) = 0. \end{aligned}$$

Therefore, $Q(a) = Q(a_1) = a_1 = P(a)$. □

Corollary 2.3.23. *Let $\pi : C^*(G) \to C^*_{\mathrm{red}}(G)$ be the quotient map. Then π is an isomorphism if and only if $P \circ \pi$ is faithful.*

Another property enjoyed by the subalgebra $C_0(G^{(0)})$ when G is topologically principal is that it is *maximal abelian self-adjoint* or *masa* for short. It means that it is not strictly contained in a commutative sub-$*$-algebra.

Theorem 2.3.24. *Let G be an étale locally compact groupoid. Then, the following conditions are equivalent*
(i) G is topologically principal;
(ii) $C_0(G^{(0)})$ is a masa in $C^*_{\mathrm{red}}(G)$;
Moreover, these conditions imply
(iii) every non-zero ideal of $C^*_{\mathrm{red}}(G)$ has a non-zero intersection with $C_0(G^{(0)})$.

Proof. We shall only prove here that (i) implies (ii) and (iii), which is sufficient for our applications. That (ii) implies (i) is left as an exercise. To show (ii), first recall that $a \in C^*_{\mathrm{red}}(G)$ commutes with $b \in C_0(G^{(0)})$ simply means that for all $\gamma \in G$, $b \circ r(\gamma) j(a)(\gamma) = j(a)(\gamma) b \circ s(\gamma)$. Suppose that $j(a)(\gamma) = 0$ for all γ such that $r(\gamma) \neq s(\gamma)$. Then the above commutation relation is satisfied for all $b \in C_0(G^{(0)})$. Conversely, suppose that a commutes with all $b \in C_0(G^{(0)})$. Let γ be such that $r(\gamma) \neq s(\gamma)$. Choose $b \in C_0(G^{(0)})$ such that $b \circ r(\gamma) = 1$ and $b \circ s(\gamma) = 0$. Above relation shows that $a(\gamma) = 0$. This shows that the commutant of $C_0(G^{(0)})$ in $C^*_{\mathrm{red}}(G)$ consists of the elements a such that $j(a)$ vanishes off the isotropy bundle G'. Since $j(a)$ is a continuous function, this happens if and only if $j(a)$ vanishes off the interior of G'. However, our assumption (i) implies that the interior of G' is $G^{(0)}$. Indeed, let U be an open set in G contained in G'. Since $G^{(0)}$ is closed in G, $U \setminus G^{(0)}$ is open. Its image by the range map, $r(U \setminus G^{(0)})$, is open in $G^{(0)}$ because a local homeomorphism is an open map. If it were non empty, it would contain x such that $G(x) = \{x\}$. This would be a contradiction since we would also have $x = r(\gamma)$ with $\gamma \in G' \setminus G^{(0)}$. Therefore, an element which commutes with each element of $C_0(G^{(0)})$ belongs to $C_0(G^{(0)})$. This says that $C_0(G^{(0)})$ is maximal abelian. To prove (iii), consider an ideal I of $C^*_{\text{red}}(G)$ which has a zero intersection with $C_0(G^{(0)})$. Let L be a representation of $C^*_{\text{red}}(G)$ such that $\text{Ker}L = I$. Let $L^{(0)}$ be its restriction to $C_0(G^{(0)})$. Then $\text{Ker}L^{(0)} = I \cap C_0(G^{(0)}) = \{0\}$. Therefore, the support of L is $G^{(0)}$. Assume that a belongs to I. This means that $L(a) = 0$. Observe that, by continuity, Proposition 2.3.17 is still valid for $a \in C^*_{\text{red}}(G)$ and a representation L of $C^*_{\text{red}}(G)$. More precisely, it gives the estimates $a^*a(x) \le \|L(a^*a)\|$ for all points x in the support of L which have trivial isotropy. In our case, this gives $a^*a(x) = 0$ for all points of $G^{(0)}$ with trivial isotropy, hence by continuity, for all $x \in G^{(0)}$. Since the conditional expectation $P : C^*_{\text{red}}(G) \to C_0(G^{(0)})$ is faithful, this implies that $a = 0$. Therefore I is the zero ideal. $\square$

Definition 2.3.25. One says that a subset $Y \subset G^{(0)}$ of the unit space of a groupoid G is *invariant* if $r^{-1}(Y) = s^{-1}(Y)$. One says that a topological groupoid is *minimal* if its unit space does not contain non-trivial invariant open subsets.

One defines the *orbit* of $x \in G^{(0)}$ as $[x] = s(r^{-1}(x))$; it is the smallest invariant subset containing x. More generally, the saturation of $Y \subset X$ is $[Y] = s(r^{-1}(Y))$. When G is topological and the source map is open, the saturation of an open set is open. In particular, G is minimal if and only if all the orbits are dense.

Definition 2.3.26. One says that a C*-algebra is *simple* it it has no non-trivial (closed, two-sided) ideal.

Theorem 2.3.27. *Let G be an étale locally compact groupoid which is topologically principal and minimal. Then $C^*_{\text{red}}(G)$ is a simple C^*-algebra.*

We assume as always that G is Hausdorff. The theorem would be wrong without this assumption.

Proof. Let I be a non-zero ideal of $C^*_{\mathrm{red}}(G)$. According to Theorem 2.3.24, its intersection J with $C_0(G^{(0)})$ is a non-zero ideal of $C_0(G^{(0)})$. Let U be the set of x's in $G^{(0)}$ such that there exists $b \in J$ such that $b(x) \neq 0$. It is clearly open. Let us show that it is invariant. Suppose that $b \in J$ and $b(x) \neq 0$. Let $\gamma \in r^{-1}(x)$. Choose an open bisection S and $a \in C_c(G)$ supported in S such that $\gamma = xS$ and $a^*a(s(\gamma)) > 0$. Then a^*ba belongs to J. The equality $a^*ba(s(\gamma)) = a^*a(s(\gamma))b(x)$ shows that $s(\gamma) \in U$. Because of our assumption of minimality, $U = G^{(0)}$. This implies that $J = C_0(G^{(0)})$. □

Corollary 2.3.28. *Let $\theta \in R$ and let T be the rotation of angle θ on the circle $\mathbf{T}$. If $\theta/2\pi$ is irrational, then $C^*_{\mathrm{red}}(\mathbf{T}, T)$ is a simple C^*-algebra.*

Proof. By construction, $C^*_{\mathrm{red}}(\mathbf{T}, T) = C^*_{\mathrm{red}}(G(\mathbf{T}, T))$. Since the action of $\mathbf{Z}$ on the circle by T is free, $G(\mathbf{T}, T)$ is principal. It is well known that the orbits are dense, therefore $G(\mathbf{T}, T)$ is principal. □

Remark 2.3.29. We shall see later that in this case, $C^*(G(\mathbf{T}, T)) = C^*_{\mathrm{red}}(G(\mathbf{T}, T))$. This gives the uniqueness, up to isomorphism, of the C*-algebra generated by two unitary operators U and V which satisfy $UV = e^{i\theta}VU$, where $\theta/2\pi$ is irrational.

Definition 2.3.30. A state τ of a C*-algebra A is *tracial* if it satisfies $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

Proposition 2.3.31. *Let G be an étale locally compact groupoid. Then,*
(i) Let μ be a probability measure on $G^{(0)}$. Then $\tau = \mu \circ P$ is a tracial state of $C^*_{\mathrm{red}}(G)$ if and only if μ is invariant with respect to G;
(ii) if G is principal, each tracial state of $C^*_{\mathrm{red}}(G)$ is of the form $\tau = \mu \circ P$ for some invariant probability measure μ.

Proof. Let us prove (i). We first check that, for each probability measure μ on $G^{(0)}$, $\tau = \mu \circ P$ is a state. It is clearly a positive linear functional. Its norm is majorized by $\|\mu\|\|P\| = 1$. Moreover, since $\tau_{|C_0(G^{(0)})} = \mu$, it is of norm one. Recall that μ is invariant if and only if $r^*\mu = s^*\mu$. From the definition, we find that $\tau(f * g) = r^*\mu(f\tilde{g})$, where $(f\tilde{g})(\gamma) = f(\gamma)g(\gamma^{-1})$. If μ is invariant, we have $\tau(f * g) = \tau(g * f)$ for $f, g \in C_c(G)$. We conclude that τ is tracial. Conversely, if τ is tracial, $r^*\mu$ and $s^*\mu$ agree on functions $F \in C_c(G)$ of the form $F = f\tilde{g}$, where $f, g \in C_c(G)$. By considering linear combinations, they agree on $C_c(G)$. This says that the measure μ is invariant. Again, we only give the proof of (ii) when G is principal and leave the general case, which uses the same idea, to the reader. Let τ be a tracial state of $C^*_{\mathrm{red}}(G)$. We will show that $\tau = \mu \circ P$ where μ is the restriction of τ to $C_0(G^{(0)})$. It suffices to show that τ and $\mu \circ P$ agree on $C_c(G)$. We proceed as in the proof of Proposition 2.3.22. We write $a \in C_c(G)$ as $a = a_1 + a_2$, where $a_1 = P(a)$ and $a_2 = a - P(a)$. We show that $\tau(a_2) = 0$. With $U_1, \ldots, U_n$ and $h_1, \ldots, h_n$ as before, we have:

$$\begin{aligned} \tau(a_2) &= \tau((\sum_1^n h_i)a_2) = \sum_1^n \tau(h_i a_2) = \sum_1^n \tau(h_i^{1/2} a_2 h_i^{1/2}) \\ &= 0. \end{aligned}$$

Therefore, $\tau(a) = \tau(a_1) = \mu \circ P(a)$. □

Corollary 2.3.32. *Let $\theta \in R$ and let T be the rotation of angle θ on the circle $\mathbf{T}$. If $\theta/2\pi$ is irrational, then $C^*_{\mathrm{red}}(\mathbf{T}, T)$ has a unique tracial state.*

Proof. It is known that the Lebesgue measure is the only probability measure on $\mathbf{T}$ which is invariant under an irrational rotation. □

2.4 The Toeplitz algebra

Recall that an element u of a C*-algebra with 1 is unitary if it satisfies $u^*u = uu^* = 1$. We have seen in 2.1 that $C^*(\mathbf{Z}) \simeq C(\mathbf{T})$ is the universal C*-algebra generated by a unitary element: the C*-algebra generated by an arbitrary unitary operator in a Hilbert space is a quotient of $C(\mathbf{T})$. An element u which satisfies $u^*u = 1$ and $uu^* \neq 1$ is called a non-unitary isometry. The prototype of a non-unitary isometry is the one-sided shift $\underline{S}$ on the Hilbert space $\ell^2(\mathbf{N})$ which acts on the canonical orhonormal basis (e_n) by $\underline{S}e_n = e_{n+1}$. The

Toeplitz algebra which we are going to define is the universal C*-algebra generated by a non-unitary isometry. In fact, as we shall see, it is isomorphic to $C^*(\underline{S})$, the C*-algebra generated by $\underline{S}$. Let us give a dynamical system construction of $C^*(\underline{S})$. We consider the space $X = \mathbf{Z} \cup \{+\infty\}$ where the subsets $[N, +\infty], N \in \mathbf{Z}$ form a fundamental family of neighborhoods of $+\infty$. We define $+\infty + k = +\infty$ for $k \in \mathbf{Z}$. We define $T : X \to X$ by $Tn = n + 1$. Let $Y = \mathbf{N} \cup \{+\infty\}$. Consider the following groupoid, which we call the *Toeplitz groupoid*:

$$G = G(X,T)_{|Y} = \{(m,k,n) \in Y \times \mathbf{Z} \times Y : m = n + k\}.$$

Endowed with the topology of $G(X,T)$, it is a locally compact étale groupoid with compact unit space Y. Its subset

$$\Sigma = \{(m,1,n) : m \in Y, n \in Y, m = n + 1\}$$

is compact and open. It is a bisection. It satisfies

$$r(\Sigma) = \Sigma\Sigma^{-1} = [1, +\infty]; \qquad s(\Sigma) = \Sigma^{-1}\Sigma = Y.$$

Moreover, for $k \in \mathbf{Z}$, we have

$$\Sigma^k = \{(m,k,n) : m \in Y, n \in Y, m = n + k\}.$$

In particular, $G = \bigcup_{k\in\mathbf{Z}} \Sigma^k$.

Proposition 2.4.1. *The indicator function $S = \mathbf{1}_\Sigma$ is a non-unitary isometry of the $*$-algebra $C_c(G)$ which generates it as a $*$-algebra.*

Proof. Since Σ is compact and open, S belongs to $C_c(G)$. We have the relations $\mathbf{1}_\Sigma^* \mathbf{1}_\Sigma = 1$ and $\mathbf{1}_\Sigma \mathbf{1}_\Sigma^* = 1 - e_0$ where $e_n = \delta_n \in C(Y)$. More generally, for $n \in \mathbf{N}^*$, $\mathbf{1}_\Sigma^n = \mathbf{1}_{\Sigma^n}$ is an isometry such that $\mathbf{1}_\Sigma^n (\mathbf{1}_\Sigma^n)^* = 1 - e_0 - \ldots - e_{n-1}$. One deduces that all the e_n's, where $n \in \mathbf{N}$ lie in the $*$-algebra generated by $\mathbf{1}_\Sigma$. Hence the result. $\square$

We consider the following representation $(\mu, H, \underline{L})$ of G: μ is the counting measure on $\mathbf{N} \subset Y$ (it is invariant); H consists of the constant Hilbert spaces $\mathcal{H}_x = \mathbf{C}$ and $\underline{L}(\gamma)$ is the identity. The Hilbert space of the integrated representation is $\ell^2(\mathbf{N})$, $C(Y)$ acts by multiplication and $\underline{L}(\mathbf{1}_\Sigma)$ is precisely the above one-sided shift $\underline{S}$. For a general $f \in C_c(G)$, we have the formula

$$\underline{L}(f)\xi(m) = \sum_{n \in \mathbf{N}} f(m, m-n, n)\xi(n).$$

In particular, when f depends only on k, let us say $f(m,k,n) = a(k)$, where $a \in C_c(\mathbf{Z})$, $T(a) = L(f)$ is a *Toeplitz matrix.*

Proposition 2.4.2. *The above representation $\underline{L} : C_c(G) \to \mathcal{L}(\ell^2(\mathbf{N}))$ is one-to-one and the closure of its image is the C^*-algebra $C^*(\underline{S})$ generated by the one-sided shift.*

Proof. As in Theorem 2.3.24, a non-zero ideal I of $C_c(G)$ must have a non-zero intersection with $C(Y)$. In our case, this intersection is the kernel of the representation of $C(Y)$ by multiplication on $\ell^2(\mathbf{N})$, which is zero. Therefore $C_c(G)$ is isomorphic to the $*$-algebra by S and the closure of $C_c(G)$ is $C^*(\underline{S})$. □

Here is a general fact about the full C*-algebra of an étale locally compact groupoid.

Theorem 2.4.3. *Let G be an étale locally compact groupoid and let U be an invariant open subset of $G^{(0)}$. Then*
(i) the restricted groupoids $G_{|U}$ and $G_{|F}$, where F is the complement of U in $G^{(0)}$, are étale locally compact groupoids.
(ii) With natural identifications, $C^*(G_{|U})$ is an ideal of $C^*(G)$ and the quotient algebra is $C^*(G_{|F})$. In other words, we have an exact sequence of C^*-algebras:

$$0 \to C^*(G_{|U}) \xrightarrow{j} C^*(G) \xrightarrow{p} C^*(G_{|F}) \to 0$$

where $j(f)$ is the extension by 0 of $f \in C_c(G_{|U})$ and $p(g)$ is the restriction to $G_{|F}$ of $g \in C_c(G)$.

Proof. We have already seen that the restriction of a groupoid to a subset of the unit space is a groupoid. It is also clear that $G_{|U}$ and $G_{|F}$ are locally compact and that $G_{|U}$ is étale. But this is also true for $G_{|F}$: let us show that the range map $r : G_{|F} \to F$ is a local homeomorphism. Let $\gamma \in G_{|F}$. There is an open set S in G containing γ such that $r_{|S}$ is a homeomorphism onto $r(S)$ open subset of $G^{(0)}$. Then $r(S \cap G_{|F}) = r(S) \cap F$ is an open subset of F and $r_{|S \cap G_{|F}}$ is a homeomorphism onto $r(S) \cap F$. Let us prove (ii). The maps $j : C_c(G_{|U}) \to C_c(G)$ and $p : C_c(G) \to C_c(G_{|F})$ are $*$-homomorphisms. If L is a representation of $C_c(G_{|F})$, $L \circ p$ is a representation of $C^*(G)$ and we have the inequality $\|L \circ p(f)\| \leq \|f\|$ for all $f \in C_c(G)$. We deduce that $\|p(f)\| \leq \|f\|$ and that p extends to a $*$-algebra homomorphism from $C^*(G)$ into $C^*(G_{|F})$. It is onto since its range contains $C_c(G_{|F})$. Similarly, j extends to a $*$-algebra homomorphism from $C^*(G_{|U})$ into $C^*(G)$. It is one-to-one: each representation (μ, H, L) of $G_{|U}$ can be viewed as a representation of G; therefore for each representation L of $C^*(G_{|U})$, there is a representation $\tilde{L}$ of $C^*(G)$ such that $L = \tilde{L} \circ j$. The inclusion $\mathrm{Im} j \subset \mathrm{Ker} p$ is clear. Suppose that a representation L of $C^*(G)$ vanishes on $\mathrm{Im} j$. Then, the associated quasi-invariant measure is supported on F and L factors through p. This gives the reverse inclusion. $\square$

We apply this theorem to our Toeplitz groupoid G and the open invariant subset $\mathbf{N} \subset X$. The groupoid $G_{|\mathbf{N}}$ is isomorphic to the trivial groupoid $\mathbf{N} \times \mathbf{N}$ through the map $(m, k, n) \mapsto (m, n)$.

Exercise 2.4.4. Show that the C*-algebra $C^*(\mathbf{N} \times \mathbf{N})$ is isomorphic to the C*-algebra $\mathcal{K}(\ell^2(\mathbf{N}))$ of all compact operators on $\ell^2(\mathbf{N})$.

Therefore the C*-algebra $C^*(G_{|\mathbf{N}})$ is isomorphic to $\mathcal{K}(\ell^2(\mathbf{N}))$. In fact, in the representation $\underline{L}$ constructed earlier, $\underline{L}(C^*(G_{|\mathbf{N}})) = \mathcal{K}(\ell^2(\mathbf{N}))$. On the other hand $F = X \setminus \mathbf{N} = \{+\infty\}$ and $G_{|F} = G(+\infty) = \mathbf{Z}$. This shows that $C^*(G)$ satisfies the so-called *Toeplitz extension*:

$$0 \to \mathcal{K}(\ell^2(\mathbf{N})) \to C^*(G) \xrightarrow{p} C^*(\mathbf{Z}) \to 0.$$

The map which associates to $a \in C_c(\mathbf{Z})$ its Toeplitz matrix $T(a)$ extends to a positive linear map from $C^*(\mathbf{Z})$ to $C^*(G)$; it is a section for p.

Exercise 2.4.5. Show that the representation $\underline{L}$ of $C^*(G)$ on $\ell^2(\mathbf{N})$ defined earlier is faithful and that is image is $C^*(\underline{S})$.

The C*-algebra $\mathcal{T} = C^*(G)$ has the following universal property: given an isometry $S \in \mathcal{L}(\mathcal{H})$, there exists a unique representation L of $C^*(G)$ such that $L(\mathbf{1}_\Sigma) = S$. We call it the *Toeplitz algebra*.

2.5 Cuntz algebras

Definition 2.5.1. Let d be an integer strictly greater than 1. The *Cuntz algebra* O_d is the C*-algebra generated by d isometries $S_1, \dots, S_d$ in a Hilbert space subject to the relation

$$\sum_{k=1}^{d} S_k S_k^* = 1 \tag{C}$$

This relation means that these isometries are non-unitary and have mutually orthogonal ranges. This C*-algebra was introduced and studied in [11]. Note that O_d does not have any tracial state. This would lead to the contradiction

$$1 = \tau(1) = \sum_{k=1}^{d} \tau(S_k S_k^*) = \sum_{k=1}^{d} \tau(S_k^* S_k) = \sum_{k=1}^{d} \tau(1) = d.$$

One says the Cuntz algebra because, for d fixed, all these C*-algebras are isomorphic. We are going to show this by using the same approach as in our study of the non-commutative torus and of the Toeplitz algebra. First we will give a dynamical system model for the Cuntz algebra.

Let $X = \{(x_0, x_1, \dots) \mid x_i = 1, 2, \dots, d\}$ endowed with the product topology. We denote by $\mathcal{X}$ the set of finite sequences $a = (a_0, a_1, \dots, a_n)$; we define the length of a as $l(a) = n + 1$; we include the empty sequence $\emptyset$ of length 0. Given a finite sequence $a = (a_0, a_1, \dots, a_n)$ and an arbitrary sequence $x = (x_0, x_1, \dots)$, we define their concatenation $xa = (a_0, a_1, \dots, a_n, x_0, x_1, \dots)$. Given $a \in \mathcal{X}$, the cylinder set $Z(a)$ consists of the sequences ax starting with a. The cylinder sets $Z(a)$, where a runs over $\mathcal{X}$ form a base for the topology of X and they are compact. We define $T : X \to X$ by $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$. The map T is the *one-sided Bernoulli shift* on d letters. It is a local homeomorphism. Note that T is one-to-one on each cylinder set $Z(a)$ where a has length 1. More generally, T^n is one-to-one on the cylinder set $Z(a)$ where a has length n. We construct the following groupoid $\mathcal{O}_d = G(X, T)$, which we shall define in a greater generality in the next chapter

$$\mathcal{O}_d = \{(x, m - n, y) \mid m, n \in \mathbf{N}, \quad x, y \in X \quad T^m x = T^n y\}$$

with the groupoid structure induced by the product structure of the trivial groupoid $X \times X$ and of the group $\mathbf{Z}$. It is convenient to write its elements as $(ax, l(a) - l(b), bx)$, where $a, b \in \mathcal{X}$ and $x \in X$. The subsets $\Sigma(a,b) = \{(ax, l(a) - l(b), bx) \mid x \in X\}$ where a, b run over $\mathcal{X}$ form a base for a topology on $\mathcal{O}_d$ which turns it into a locally compact étale groupoid. Since, as we shall see, $C^*(\mathcal{O}_d)$ is the Cuntz algebra O_d, we refer to $\mathcal{O}_d$ as the *Cuntz groupoid* $\mathcal{O}_d$. For $k = 1, 2, \ldots, d$, we define $\Sigma_k = \Sigma(k, \emptyset)$. These are compact open subsets of $\mathcal{O}_d$ which satisfy the relations: $\Sigma_k^{-1}\Sigma_k = X$ and $\Sigma_k \Sigma_k^{-1} = Z(k)$. One deduces the following proposition.

Proposition 2.5.2. *The indicator functions $\underline{S}_k = \mathbf{1}_{\Sigma_k}$ are isometries of the $*$-algebra $C_c(\mathcal{O}_d)$ which satisfy the Cuntz relation (C). They generate the $*$-algebra consisting of locally constant functions with compact support.*

Proof. These results from the corresponding properties of the Σ_k's. In particular, we have the relation

$$\Sigma(a_0 \ldots a_m, b_0 \ldots b_n) = \Sigma_{a_0} \ldots \Sigma_{a_m} \Sigma_{b_n}^{-1} \ldots \Sigma_{b_0}^{-1}.$$

□

Corollary 2.5.3. *The C^*-algebra $C^*(\mathcal{O}_d)$ is generated by d isometries $\underline{S}_1, \ldots, \underline{S}_d$ which satisfy the Cuntz relation (C).*

Proof. Let $\mathcal{A}$ be the $*$-algebra of locally constant functions with compact support and A its norm closure in $C^*(\mathcal{O}_d)$. By the Stone-Weierstrass theorem, A contains $C(X)$. Since the bisections $\Sigma(a,b)$ form a cover of $\mathcal{O}_d$ by compact open subsets, each $f \in C_c(\mathcal{O}_d)$ can be written as a sum of functions of the form $h\mathbf{1}_{\Sigma(a,b)}$, hence belongs to A. Since $C_c(\mathcal{O}_d)$ is dense in $C^*(\mathcal{O}_d)$ by construction, $A = C^*(\mathcal{O}_d)$. □

Let us show next that $C^*(\mathcal{O}_d)$ is universal for the Cuntz relation.

Theorem 2.5.4. *The correspondence which associates to a representation L of the C^*-algebra $C^*(\mathcal{O}_d)$ the isometries*
$S_1 = L(\underline{S}_1), \ldots, S_d = L(\underline{S}_d)$
is a bijection onto the d-uples of isometries $(S_1, \ldots, S_d)$ on a Hilbert space which satisfy the Cuntz relation.

Proof. It is clear that, given a representation L of $C^*(\mathcal{O}_d)$, $S_1 = L(\underline{S}_1), \dots, S_d = L(\underline{S}_d)$ are d isometries which satisfy the Cuntz relations. Moreover, because $\underline{S}_1, \dots, \underline{S}_d$ generate $C^*(\mathcal{O}_d)$, L is uniquely defined by $(S_1, \dots, S_d)$. Let us sketch the proof of the converse. Let $(S_1, \dots, S_d)$ be d isometries on a Hilbert space $\mathcal{H}$ which satisfy the Cuntz relation. There is a representation L of the $*$-algebra $\mathcal{A}$ of locally constant functions on $\mathcal{O}_d$ with compact support such that

$$L(\underline{S}_{a_0} \dots \underline{S}_{a_m} \underline{S}^*_{b_n} \dots \underline{S}^*_{b_0}) = S_{a_0} \dots S_{a_m} S^*_{b_n} \dots S^*_{b_0}$$

for all $a_0 \dots a_m, b_0 \dots b_n \in \mathcal{X}$. Consider the restriction M of L to the commutative $*$-algebra $\mathcal{B}$ of locally constant functions on X with compact support on X. By positivity, it is bounded. Hence, it extends to a representation M of $C(X)$. Then one checks that the definition of L on $C_c(\mathcal{O}_d)$ by $L(h\mathbf{1}_{\Sigma(a,b)}) = M(h)L(\mathbf{1}_{\Sigma(a,b)})$ for $h \in C(X)$ and $a, b \in \mathcal{X}$ makes sense and that L is indeed a representation of $C_c(\mathcal{O}_d)$. By definition, L extends to a representation of $C^*(\mathcal{O}_d)$. □

Theorem 2.5.5. *The C^*-algebra $C^*(\mathcal{O}_d)$ is simple.*

Proof. We admit here and shall prove later that $C^*(\mathcal{O}_d) = C^*_{\mathrm{red}}(\mathcal{O}_d)$. We can apply Theorem 2.3.27. We just have to check that the groupoid $\mathcal{O}_d$ is topologically principal and minimal. An element $x = (x_0, x_1, \dots)$ has no isotropy if and only if it is not eventually periodic, i.e. not of the form $abbb\dots$ for some $a, b \in \mathcal{X}$. The set of these elements is dense in X. The orbit of $x = (x_0, x_1, \dots)$ consists of all the elements of the form by where $b \in \mathcal{X}$ and y is a tail of x, i.e. there is $a \in \mathcal{X}$ such that $x = ay$. It is dense in X. □

Corollary 2.5.6. *The C^*-algebras generated by d isometries $S_1, \dots, S_d$ which satisfy the Cuntz relation (C) are all isomorphic.*

Proof. Let L be the representation of $C^*(\mathcal{O}_d)$ determined by $S_1, \dots, S_d$. According to the theorem, it is one-to-one, hence it is an isomorphism onto its image. Therefore the C*-algebra generated by $S_1, \dots, S_d$ is isomorphic to $C^*(\mathcal{O}_d)$. □

2.6 AF algebras

Definition 2.6.1. A C*-algebra A is *AF* (or *approximately finite dimensional*) if there exists an increasing sequence of finite dimensional sub-C*-algebras (A_n) such that A is the norm closure of $\cup_n A_n$.

Example 2.6.2. Consider the sub-C*-algebra F_d of the Cuntz algebra $O_d = C^*(S_1, \dots, S_d)$ generated by the elements $S_{a_0} \dots S_{a_n} S^*_{b_n} \dots S^*_{b_0}$ where $a = (a_0, \dots, a_n)$ and $b = (b_0, \dots, b_n)$ are finite sequences of the same length. Then B is an AF C*-algebra. To show this, fix n and consider the elements $S(a,b) = S_{a_0} \dots S_{a_n} S^*_{b_n} \dots S^*_{b_0}$. They satisfy the relations $S(a,b)S(b,c) = S(a,c)$, $S(a,b)^* = S(b,a)$ and $S(a,b)S(b',c) = 0$ if $b \neq b'$ (one says that they are *matrix units*). From this, one sees that they generate a C*-algebra B_n isomorphic to $M_{d^{n+1}}(\mathbf{C})$. The inclusion $B_n \subset B_{n+1}$ holds because

$$S(a,b) = S_{a_0} \dots S_{a_n} (\sum_{k=1}^{d} S_k S_k^*) S^*_{b_n} \dots S^*_{b_0} = \sum_{k=1}^{d} S(ak, bk).$$

The C*-algebra F_d is the norm closure of the union of the B_n's.

The most general finite dimensional C*-algebra is a finite sum of matrix algebras:

$$A = \oplus_{i=1}^{m} M_{n_i}(\mathbf{C})$$

where $n_1, \dots, n_m$ are strictly positive integers. This C*-algebra can be described as $C^*(R)$, where R is the following principal groupoid (we identify a principal groupoid with the graph of the equivalence it defines on its unit space; we write the symbol R rather than G in the case of an equivalence relation, viewed as a principal groupoid). Its unit space is the finite set

$$X = \{(i,k) \mid i = 1, \dots, m \quad k = 1, \dots, n_i\}$$

and (i,k) and (j,l) are equivalent if and only if $i = j$. Thus,

$$R = \{(i,(k,l)) \mid i = 1, \dots, m \quad k,l = 1, \dots, n_i\}.$$

The groupoid structure is given by $r(i,(k,l)) = (i,k)$, $s(i,(k,l)) = (i,l)$, $(i,(j,k))(i,(k,l) = (i,(j,l))$ and $(i,(k,l))^{-1} = (i,(l,k))$. Obviously,

$$C^*(R) = \oplus_{i=1}^{m} M_{n_i}(\mathbf{C}).$$

There is no norm problem because a C*-algebra has a unique norm.

We are going to give a similar model for AF algebras, called the *path model*. The combinatorial ingredient is a Bratteli diagram, introduced in [7].

Definition 2.6.3. A *Bratteli diagram* is an oriented graph (E, V), where E is the set of edges and V is the set of vertices, such that the vertices are stacked on levels $n = 0, 1, 2, \ldots$ and the edges run from a vertex of level n to a vertex of level $n+1$. We assume that for each n the set $V(n)$ of vertices of the level n and the set $E(n)$ of edges from level n to level $n+1$ are finite and that each vertex emits at least one edge. An *initial vertex* is a vertex which does not receive any edge.

An *infinite path* is a sequence of connected edges $x = x_l x_{l+1} \ldots$, starting at an initial vertex. The space X of infinite paths has a natural totally disconnected locally compact topology, with the cylinder sets $Z(a_l a_{l+1} \ldots a_n)$ as a base. We define on X the *tail equivalence relation*: two infinite paths $x, y \in X$ are equivalent, written $(x, y) \in R$, if there is n such that $x_i = y_i$ for $i \geq n$. The equivalence relation R (viewed as a groupoid) is the union of an increasing sequence of equivalence relations R_n, namely $(x, y) \in R_n$, if $x_i = y_i$ for $i \geq n$. We wish to define a topology on R which turns it into a locally compact étale groupoid. Let us denote by $\mathcal{R}$ the set of pairs (a, b) where a and b are finite paths starting from initial vertices and ending at the same vertex. For $(a, b) \in \mathcal{R}$ ending at vertex v, let us define

$$\Sigma(a, b) = \{(ax, bx) : x \text{ is an infinite path starting from } v\}.$$

It is a bisection of R. The family of these sets, when (a, b) runs over $\mathcal{R}$, is a base for a topology on R. These sets become open compact bisections. We deduce:

Proposition 2.6.4. *Endowed with this topology, R is a locally compact étale groupoid.*

Remark 2.6.5. This topology is strictly finer than the product topology, i.e. the topology inherited from $R \subset X \times X$. However, this topology agrees with the product topology on R_n. One can also observe that R_n is an open subset of R_{n+1} and that the topology of R is the inductive limit topology: a subset U of R is open if and only if $U \cap R_n$ is an open subset of R_n for all n.

Theorem 2.6.6. *Let (V, E) be a Bratteli diagram and let R be the tail equivalence relation on the infinite path space X of the diagram. Then $C^*(R) = C^*_{\rm red}(R)$ is an AF algebra.*

Proof. This is the same proof as in the above example. For $(a, b) \in \mathcal{R}$, we define $S(a, b) = \mathbf{1}_{\Sigma(a,b)}$. This is an element of $C_c(R)$. Given a vertex v, we let $\mathcal{R}(v)$ be the set of pairs $(a, b) \in \mathcal{R}$ ending at v. Then, $\{S(a, b), (a, b) \in \mathcal{R}(v)\}$ is a family of matrix units. Its linear span is a $*$-algebra isomorphic to $M_N(\mathbf{C})$, where N is the number of paths starting from an initial vertex and ending at v. The linear span A_n of the union of these families of matrix units over all the vertices of the level n is a finite-dimensional C*-algebra. As previously, we have the inclusion $A_n \subset A_{n+1}$ because, for $(a, b) \in \mathcal{R}(v)$, $S(a, b) = \cup S(ac, bc)$, where the union is over all the edges emitted by the vertex v. The union $A_\infty = \cup_{n\in\mathbf{N}} A_n$ is a sub-$*$-algebra of $C_c(R)$. It is also closed under the pointwise product. By the Stone-Weierstrass theorem, for all $f \in C_c(R)$, there exists a sequence (f_n) in A_∞ and a compact set $K \subset R$ containing the supports of f and the f_n's such that f_n converges uniformly to f on K. Let L be a representation L of $C_c(R)$. According to Corollary 2.3.16, $L(f_n)$ converges to $L(f)$ in norm. This shows that A_∞ is dense in $C^*(R)$. Since each A_n, being a C*-algebra, has a unique C*-norm, this is also true for A_∞. The full and reduced norms agree on A_∞ hence on $C^*(R)$. □

Remark 2.6.7. One can show that every AF algebra is isomorphic to a C*-algebra constructed from a Bratteli diagram.

Remark 2.6.8. The ideal structure of the AF algebra $C^*(R)$, where R is the tail equivalence relation on the infinite path space of a Bratteli diagram is easily visualized on the Bratteli diagram (see [7]).

Chapter 3

KMS States

The root of the notion of KMS lies in quantum statistical mechanics (KMS stands for Kubo, Martin and Schwinger, three physicists who introduced this notion). Let us look at the most elementary example. The algebra of observables is the C*-algebra $M_n(\mathbf{C})$. It is not difficult to see that every state of $M_n(\mathbf{C})$ is of the form $\varphi(A) = \mathrm{Trace}(A\Phi)$, where Trace is the usual trace of a matrix and $\Phi \in M_n(\mathbf{C})$, called the *density matrix* is self-adjoint positive of trace one. Time evolution of the system is given by a self-adjoint matrix $H \in M_n(\mathbf{C})$, called the *hamiltonian*. It implements a one-parameter automorphism group of $M_n(\mathbf{C})$ according to:

$$\sigma_t(A) = e^{itH} A e^{-itH} \quad \text{for} \quad A \in M_n(\mathbf{C}).$$

One defines the *free energy* of the state $\varphi = \mathrm{Trace}(\ .\ \Phi)$ by

$$F(\varphi) = \varphi(H) - TS(\varphi)$$

where $S(\varphi) = -\mathrm{Trace}(\Phi \log \Phi)$ is the *entropy* of the state φ and T is a real parameter called the *temperature*. It is convenient to use instead the parameter $\beta = 1/T$, called the *inverse temperature*. The *equilibrium state* of the system at inverse temperature β is the state which minimizes the free energy (where H and β are fixed). It is easily computed: it is given by the following *Gibbs condition*.

Proposition 3.0.1. *Let $H \in M_n^{s.a.}(\mathbf{C})$ and $\beta \in \mathbf{R}$. Then*
(i) $F(\varphi) \leq Tr(e^{-\beta H})$;
(ii) the equality holds if and only if $\varphi = \dfrac{Tr(\,.\, e^{-\beta H})}{Tr(e^{-\beta H})}$.

This justifies the following definition.

Definition 3.0.2. The state $\varphi = \dfrac{Tr(\,.\, e^{-\beta H})}{Tr(e^{-\beta H})}$ is called the *Gibbs state* (for the hamiltonian H at inverse temperature β). The function $\beta \mapsto Tr(e^{-\beta H})$ is called the *partition function*.

How can we extend this definition to the case of an arbitrary C*-algebra? The above formula does not necessarily make sense. For example, if we replace the C*-algebra $M_n(\mathbf{C})$ by the C*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$ of infinite dimension, this requires that $e^{-\beta H}$ is a trace-class operator, which may fail. A standard approach to the definition of Gibbs states is the *thermodynamic limit*. One computes local Gibbs states, corresponding to finite subsystems and then studies the limit when the subsystems grow. The approach that we are going to describe is global.

3.1 Definition of KMS states

Kubo, Martin and Schwinger have discovered a direct relation between the Gibbs state and the one-parameter dynamical group σ_t.

Definition 3.1.1. A *strongly continuous* one-parameter automorphism group of a C*-algebra A is a group homomorphism $\sigma : \mathbf{R} \to Aut(A)$ such that for all $a \in A$, $t \mapsto \sigma_t(a)$ is continuous from $\mathbf{R}$ to A.

Definition 3.1.2. Let A be a C*-algebra and let σ be a strongly continuous one-parameter automorphism group of A. Let $\beta \in \mathbf{R}$. A state φ of A is KMS_β for σ if for all $a, b \in A$, there exists a function F bounded and continuous on the closed strip $0 \leq Imz \leq \beta$ and holomorphic on the open strip $0 < Imz < \beta$ such that:
(*i*) $F(t) = \varphi(a\sigma_t(b))$ for all $t \in \mathbf{R}$;
(*ii*) $F(t + i\beta) = \varphi(\sigma_t(b)a)$ for all $t \in \mathbf{R}$.

When $\beta = 0$ or when σ is trivial (i.e. $\sigma_t = id_A$ for all t), this reduces to the tracial condition. Thus, the KMS condition generalizes the tracial condition.

In the elementary case of the introduction, Gibbs states and KMS states agree:

Exercise 3.1.3. Let $A = \mathcal{K}(\mathcal{H})$ be the algebra of compact operators on a Hilbert space $\mathcal{H}$, let $\beta \in \mathbf{R}$ and let H be self-adjoint operator such that $e^{-\beta H}$is trace-class. Then the Gibbs state $\varphi = \dfrac{Tr(\ .\ e^{-\beta H})}{Tr(e^{-\beta H})}$ is the unique KMS$_\beta$ state for the one-parameter automophism group $\sigma_t(A) = e^{itH}Ae^{-itH}$.

Here are some important properties of KMS states that we state without a proof (see [8]).

Proposition 3.1.4. *Let A be a separable and let $\sigma = (\sigma_t)$ be a strongly continuous one-parameter automorphism group of A. Then,*
(i) KMS states are invariant under σ_t.
(ii) For a fixed $\beta \in \mathbf{R}$, the set Σ_β of KMS$_\beta$ states is a Choquet simplex *of the dual A': it is a $*$-weakly closed convex subset of A' and each KMS$_\beta$ state is the barycenter of a unique probability measure supported on the set of extremal KMS$_\beta$ states.*
(iii) Extremal KMS$_\beta$ states are factorial *(this means that the associated GNS representation is factorial).*

The KMS problem is to determine for all $\beta \in R$ the KMS$_\beta$ states of a given system (A, σ) consisting of a C*-algebra A describing the geometry of the system and σ a strongly continuous one-parameter automorphism group of A describing its time evolution. The discontinuities of the map $\beta \mapsto \Sigma_\beta$ are interpreted as *phase transitions.* The above elementary example does not have phase transitions: for all $\beta \in \mathbf{R}$, there is a unique KMS$_\beta$ state.

3.2 The gauge group of O_d

We have seen that given two d-uple of isometries $(S_1, \ldots, S_d)$ and $(T_1, \ldots, T_d)$ satisfying the Cuntz relation (C) there is a unique isomorphism α from $C^*(S_1, \ldots, S_d)$ onto $C^*(T_1, \ldots, T_d)$ such that for all $k = 1, \ldots, d$, $\alpha(S_k) = T_k$. Let us apply this to the d-uple $(T_1 = zS_1, \ldots, T_d = zS_k)$, where $z = e^{it}$ is a complex number of modulus 1 to provide the automorphism σ_t of $O_d = C^*(S_1, \ldots, S_d)$ such that $\sigma_t(S_k) = e^{it} S_k$. It is easily checked that $\sigma = (\sigma_t)$ so defined is a strongly continuous one-parameter automorphism group of O_d.

Definition 3.2.1. This one-parameter automorphism group $\sigma = (\sigma_t)$ is called the *gauge group* of the Cuntz algebra O_d.

Theorem 3.2.2. *The gauge group of the Cuntz algebra O_d has a unique KMS state. It occurs at the inverse temperature $\beta = \log d$.*

Proof. Since the linear span of the elements of the form

$$a = S_{i_1} \ldots S_{i_m} S_{j_n}^* \ldots S_{j_1}^*$$

is dense in O_d, a state φ is uniquely determined by its values on these elements. The invariance of φ under σ_t gives $\varphi(a) = 0$ if $m \neq n$. Successive applications of the KMS condition give

$$\varphi(a) = \delta_{i_1, j_1} \ldots \delta_{i_n, j_n} e^{-n\beta}.$$

The condition $\varphi(1) = 1$ and the Cuntz relation give $1 = de^{-\beta}$, hence $\beta = \log d$. Conversely, one can check that these formulas define a KMS$_\beta$ state for $\beta = \log d$. We shall give another proof in the next section. □

3.3 Cocycles and KMS states

Let us revisit the elementary example of the introduction, where the C*-algebra is $M_n(\mathbf{C})$ and the one-parameter automorphism group is implemented by the self-adjoint matrix H. Assume that H is diagonal, with diagonal entries $(h_1, \ldots, h_n)$. Then the Gibbs state at inverse temperature β is given by

$$\varphi(A) = \sum_{i=1}^{n} A_{i,i} \rho_i$$

where the weights ρ_i are completely determined by the conditions

$$\frac{\rho_i}{\rho_j} = e^{-\beta(h_i - h_j)}, \quad \rho_i > 0 \quad \text{and} \quad \sum_{i=1}^{n} \rho_i = 1.$$

Let us reformulate this condition by writing $M_n(\mathbf{C}) = C^*(G)$ where G is the principal groupoid $G = \{1, \dots, n\} \times \{1, \dots, n\}$. We introduce the function $c : G \to \mathbf{R}$ such that $c(i,j) = h_i - h_j$ and the probability measure μ on $\{1, \dots, n\}$ given by the weights $\rho_1, \dots, \rho_n$. Then the above condition can be written

$$\frac{d(r^*\mu)}{d(s^*\mu)} = e^{-\beta c}$$

where we use the same notation as in 2.3.9: r, s are respectively the first and the second projections from $\{1, \dots, n\} \times \{1, \dots, n\}$ onto $\{1, \dots, n\}$ and for $A \in M_n(\mathbf{C})$,

$$\int A d(r^*\mu) = \sum_{i=1}^{n} (\sum_{j=1}^{n} A_{i,j}) \rho_i.$$

This says that the measure μ is quasi-invariant with respect to G and admits $e^{-\beta c}$ as its Radon-Nikodým derivative. This formulation of the KMS problem holds in a more general setting. We see that it is related to the following problem, which we call the *Radon-Nikodým problem*. Let G be a locally compact étale groupoid. We have seen that the Radon-Nikodým derivative $D_\mu = d(r^*\mu)/d(s^*\mu)$ of a quasi-invariant measure μ is a *cocycle*, i.e. a groupoid homomorphism $D_\mu : G \to \mathbf{R}_+^*$ (strictly speaking, this is only a a.e. homomorphism). Given a cocycle $D : G \to \mathbf{R}_+^*$, which we assume to be continuous, does there exist a quasi-invariant measure μ admitting D as Radon-Nikodým derivative? More generally, what can be said about the set of measures admitting D as Radon-Nikodým derivative?

Exercise 3.3.1. Let G be a locally compact étale groupoid and let $D : G \to \mathbf{R}_+^*$ be a continuous cocycle. Show that the set $M_D(G)$ of quasi-invariant probability measures admitting D as Radon-Nikodým derivative is a Choquet simplex in the dual of $C_0(G^{(0)})$.

Exercise 3.3.2. Let X, Y be compact spaces and let $\pi : X \to Y$ be a surjective local homeomorphism. Consider the equivalence relation $R = \{(x, x') \in X \times X \mid \pi(x) = \pi(x')$ endowed with the product topology.
(i) Show that for every continuous cocycle $D : R \to \mathbf{R}_+^*$, there exists a unique continuous function $\rho : X \to \mathbf{R}_+^*$ such that $\sum_{\pi(x)=y} \rho(x) = 1$

for all $y \in Y$ and $D(x, x') = \rho(x)/\rho(x')$ for all $(x, x') \in R$.
(ii) Let D and ρ as in (i). Define $E : C(X) \to C(Y)$ by $E(f)(y) = \sum_{\pi(x)=y} f(x)\rho(x)$. Show that a probability measure μ on X is quasi-invariant with respect to R with Radon-Nikodým derivative D if and only if there exists a probability measure Λ on Y such that $\mu = \Lambda \circ E$. We call ρ the *potential* of D and E its *expectation*.

Exercise 3.3.3. *The Dobrushin-Lanford-Ruelle condition.* Let (X_n) be a sequence of compact spaces indexed by $\mathbf{N}$ and for each $n \in \mathbf{N}$, let $\pi_{n+1,n} : X_n \to X_{n+1}$ be a surjective local homeomorphism. Define $\pi_n = \pi_{n,n-1} \circ \cdots \circ \pi_{2,1} \circ \pi_{1,0}$ from $X = X_0$ onto X_n. Consider the equivalence relation $R_n = \{(x, x') \in X \times X \mid \pi_n(x) = \pi_n(x')\}$ endowed with the product topology and the equivalence relation $R = \cup R_n$ endowed with the inductive limit topology (see 2.6.5). We say that R is an *approximately proper* (or *AP* for short) equivalence relation. The tail equivalence relation of a Bratteli diagram is an example of AP equivalence relation. Let $D : R \to \mathbf{R}_+^*$ be a continuous cocycle. Let D_n be its restriction to R_n. Define its potential $\rho_n \in C(X, \mathbf{R}_+^*)$ and its expectation $E_n : C(X) \to C(X_n)$ as above.
(i) Show that a probability measure μ on X is quasi-invariant with respect to R with Radon-Nikodým derivative D if and only if μ factors through every expectation E_n.
(ii) Show that for all continuous cocycle $D : R \to \mathbf{R}_+^*$, the set $M_D(R)$ of quasi-invariant probability measures admitting D as derivative is non-empty.

The following construction generalizes the gauge automorphism group of the Cuntz algebra. It starts with a locally compact étale groupoid G and a continuous cocycle $c : G \to \mathbf{R}$. For $t \in \mathbf{R}$, we define a map σ_t of $C_c(G)$ into itself by $\sigma_t(f)(\gamma) = e^{itc(\gamma)} f(\gamma)$ for $f \in C_c(G)$ and $\gamma \in G$. It is easily checked that it is an automorphism of the $*$-algebra $C_c(G)$ and that $\sigma = (\sigma_t)$ is a one-parameter automorphism group.

Proposition 3.3.4. *The one-parameter automorphism group σ associated to the continuous cocycle $c : G \to \mathbf{R}$ extends to a strongly continuous one-parameter automorphism group of $C^*(G)$ and of $C^*_{\mathrm{red}}(G)$.*

Proof. Since for each representation L of $C_c(G)$, $L \circ \sigma_t$ is also a representation of $C_c(G)$, we have the inequality $\|L \circ \sigma_t(f)\| \le \|f\|$ for all $f \in C_c(G)$, where $\|f\|$ is the full norm. We deduce the inequality $\|\sigma_t(f)\| \leq \|f\|$, hence the equality. This shows that σ extends to a one-parameter automorphism group of the full C*-algebra $C^*(G)$. Let $x \in G^{(0)}$ and let π_x be the regular representation of $C_c(G)$ on $\ell^2(G_x)$ introduced in Section 2.3.4. One has for all $f \in C_c(G)$ the equality $\pi_x \circ \sigma_t(f) = V\pi_x(f)V^*$ where V is the unitary operator on $\ell^2(G_x)$ such that $V\xi(\gamma) = e^{itc(\gamma)}\xi(\gamma)$. One deduces the equality $\|\sigma_t(f)\|_{\text{red}} = \|f\|_{\text{red}}$. This shows that σ extends also to a one-parameter automorphism group of the reduced C*-algebra $C^*_{\text{red}}(G)$. In both cases, we have the continuity of $t \mapsto \sigma_t(f)$ for $f \in C_c(G)$. By density, we deduce the stong continuity of σ. $\square$

The *fixed point algebra* of an automorphism group σ of a C*-algebra A is defined as $A^\sigma = \{a \in A \mid \forall t, \sigma_t(a) = a\}$; it is a sub-C*-algebra of A.

Lemma 3.3.5. *Assume that σ is periodic, with $\sigma_{t+T} = \sigma_t$ for all $t \in \mathbf{R}$. Then $Q(a) = (1/T)\int_0^T \sigma_t(a)dt$ defines a faithful conditional expectation onto A^σ.*

Proof. We use here the elementary theory of integration of continuous functions with values in a Banach space. The integral can be defined as a limit of Riemann sums. It is readily checked that Q is linear positive and that it is a projection of norm one onto A^σ. If a is positive and $Q(a) = 0$, the positivity of $\sigma_t(a)$ for all t implies that $a = 0$. $\square$

We shall need the following result, which we give as an exercise.

Exercise 3.3.6. Assume that H is a closed and open subgroupoid of a locally compact étale groupoid G.
(i) Show that the inclusion map $C_c(H) \to C_c(G)$ extends to injective $*$ homomorphisms $C^*(H) \to C^*(G)$ and $C^*_{\text{red}}(H) \to C^*_{\text{red}}(G)$. Thus we may view $C^*(H)$ [resp. $C^*_{\text{red}}(H)$] as a sub-C*-algebra of $C^*(G)$ [resp. $C^*_{\text{red}}(G)$].
(ii) Show that the restriction map $C_c(G) \to C_c(H)$ extends to conditional expectations $C^*(G) \to C^*(H)$ and $C^*_{\text{red}}(G) \to C^*_{\text{red}}(H)$.

Proposition 3.3.7. *Let G be a locally compact étale groupoid and let $c : G \to \mathbf{Z}$ be a continuous cocycle. We define $H = c^{-1}(0)$. Let σ*

be the associated automorphism group. Then,
(i) we have $C^*(G)^\sigma = C^*(H)$ and $C^*_{\rm red}(G)^\sigma = C^*_{\rm red}(H)$;
(ii) the conditional expectation of the above lemma agrees with the restriction maps.

Proof. For $f \in C_c(H)$, we obviously have $\sigma_t(f) = f$. Therefore $C_c(H) \subset C^*(G)^\sigma$, which implies $C^*(H) \subset C^*(G)^\sigma$. Let us compare the conditional expectation $Q : C^*(G) \to C^*(G)^\sigma$ of the lemma and the conditional expectation $Q' : C^*(G) \to C^*(H) \subset C^*(G)^\sigma$ extending the restriction map. Since they agree on $C_c(G)$, they agree on $C^*(G)$. In particular, one has the equality $C^*(G)^\sigma = C^*(H)$. The proof is similar with the reduced C*-algebra. □

Corollary 3.3.8. *Let G be a locally compact étale groupoid and let $c : G \to \mathbf{Z}$ be a continuous cocycle. We define $H = c^{-1}(0)$ as above. If $C^*(H) = C^*_{\rm red}(H)$, then $C^*(G) = C^*_{\rm red}(G)$.*

Proof. Let $\pi : C^*(G) \to C^*_{\rm red}(G)$ be the quotient map and let $P : C^*_{\rm red}(G) \to C_0(G^{(0)})$ be the canonical conditional expectation. Then $P \circ \pi$ factors through $C^*(H)$ as a composition of two faithful maps. Therefore it is faithful and π is faithful. □

Example 3.3.9. Let $G = G(X, T)$ where T is a homeomorphism of a locally compact space X. Then $C^*(G(X, T)) = C^*_{\rm red}(G(X, T))$. One applies the above proposition to the cocycle $c(x, k, y) = k$. Then $c^{-1}(0) = \{0\}$.

Example 3.3.10. Let $\mathcal{O}_d$ be the Cuntz groupoid and let $c(x, k, y) = k$. Then $c^{-1}(0)$ is the tail equivalence relation. According to Theorem 2.6.6, $C^*(c^{-1}(0)) = C^*_{\rm red}(c^{-1}(0))$ hence $C^*(\mathcal{O}_d) = C^*_{\rm red}(\mathcal{O}_d)$.

Let $\sigma = (\sigma_t)_{t\in\mathbf{R}}$ be a strongly continuous one-parameter group of automorphisms of a C*-algebra A. One says that $a \in A$ is *entire* for σ if the function $t \mapsto \sigma_t(a)$ extends to an entire function on $\mathbf{C}$. For example, if σ is the one-parameter automorphism group of $C^*(G)$ constructed from a cocycle, the elements of $C_c(G)$ are entire for σ.

Lemma 3.3.11. *Let $\sigma = (\sigma_t)_{t\in\mathbf{R}}$ be a strongly continuous one-parameter automorphism group of a C*-algebra A. Let $\beta \in \mathbf{R}$. The following conditions are equivalent for a state φ of A:*

(i) φ *is* KMS_β *for* σ*;*
(ii) for all a, b *entire for* σ*, we have*

$$\varphi(a\sigma_{i\beta}(b)) = \varphi(ba) \tag{$*$}$$

(iii) condition $(*)$ *holds for all* a, b *in a norm dense sub-*$*$*-algebra* $\mathcal{A}$ *of* A*, invariant under* σ *and consisting of elements entire for* σ*.*

Proof. Let us first assume that the state φ is KMS_β. Let a, b entire for σ. Define $G(z) = \varphi(a\sigma_z(b))$. It is an entire function. Let F be the function given by the KMS condition. It agrees with G on the real line. It follows from the edge of the wedge theorem that it agrees with G on the closed strip $D = \{z \mid 0 \leq Imz \leq \beta\}$. In particular, we obtain that $G(i\beta) = F(i\beta) = \varphi(ba)$. This shows (ii). One can check that the set of entire elements is a norm dense sub-$*$-algebra invariant under σ. Therefore, (ii) implies (iii). Let us assume that $(*)$ is satisfied for all a, b in a norm dense sub-$*$-algebra $\mathcal{A}$ of A, invariant under σ and consisting of elements entire for σ. For $a, b \in \mathcal{A}$, the function F, defined as the restriction to D of $G(z) = \varphi(a\sigma_z(b))$ has the required property. Let $a, b \in A$. Pick sequences $(a_n), (b_n)$ in $\mathcal{A}$ converging respectively to a and b. Define as above $G_n(z) = \varphi(a_n\sigma_z(b_n))$ and let F_n be its restriction to D. By the three lines theorem, $|F_n - F_m|$ assumes its maximum value on the boundary of D. An estimate on the real line and on the line $Imz = \beta$ shows that it is a Cauchy sequence in the uniform convergence norm. Its limit F satisfies the required properties of the KMS_β condition. □

Let G be a locally compact étale groupoid. Each probability measure μ on $G^{(0)}$ defines a state $\varphi_\mu = \mu \circ P$ of $C^*_{\mathrm{red}}(G)$, where P is the canonical conditional expectation onto $C_0(G^{(0)})$. We also view φ_μ as a state on the full C*-algebra $C^*(G)$.

Theorem 3.3.12. *Let* G *be a locally compact étale groupoid, let* c *be a real-valued continuous cocycle, let* σ *be the associated automorphism group and let* $\beta \in \mathbf{R}$*. Then*
(i) A probability measure μ *on* $G^{(0)}$ *is quasi-invariant under* G *with Radon-Nikodým derivative* $dr^*\mu/ds^*\mu = e^{-\beta c}$ *if and only if the state* φ_μ *of* $C^*_{\mathrm{red}}(G)$ *(or of* $C^*(G)$*) is* KMS_β *for the automorphism group* σ*.*

(ii) If $c^{-1}(0)$ is principal, every KMS$_\beta$ state of $C^*(G)$ for σ is of the form φ_μ for some quasi-invariant probability measure μ on $G^{(0)}$ with Radon-Nikodym derivative $dr^*\mu/ds^*\mu = e^{-\beta c_\varphi}$.

Proof. Condition $(*)$ applied to the state φ_μ and $f, g \in C_c(G)$ says exactly that

$$\int Fe^{-\beta c} d(s^*\mu) = \int Fd(r^*\mu)$$

for $F(\gamma) = f(\gamma^{-1})g(\gamma)$. Since all $F \in C_c(G)$ can be written as a linear combination of functions of that form, condition $(*)$ for the state φ_μ and the $*$-algebra $\mathcal{A} = C_c(G)$ is equivalent to the quasi-invariance of μ with Radon-Nikodým derivative $e^{-\beta c}$.

Let us now assume that $c^{-1}(0)$ is principal. This precisely means that its intersection with the isotropy bundle G' is contained in $G^{(0)}$. Let φ be a state of $C^*(G)$ which is KMS$_\beta$ for σ. We view its restriction to $C_c(G)$ as complex Radon measure ν on G. We are going to show that its support is both contained in $c^{-1}(0)$ and in G'. Condition $(*)$ applied to $f \in C_c(G)$ and $h \in C_c(G^{(0)}$ gives $\nu(f(h \circ s) = \nu((h \circ s)f)$. An argument similar to the one we used in the proof of Theorem 2.3.24 gives that the support of ν is contained in G'. Next we use the fact that φ is invariant under σ_t. This implies that for all $f \in C_c(G)$ and all $t \in \mathbf{R}$, we have $\varphi(e^{itc}f) = \varphi(f)$. This implies that for all $t \in \mathbf{R}$, $e^{itc} \equiv 1$ on the support of ν. Therefore the support of ν is contained in $c^{-1}(0)$. We deduce that the support of ν is contained in $G^{(0)}$. In other words, ν is equal to its restriction μ to $G^{(0)}$. This gives the equality $\varphi(f) = \mu \circ P(f)$ for all $f \in C_c(G)$ and by density, the equality $\varphi = \varphi_\mu$. $\square$

3.4 Further examples

3.4.1 Expansive dynamical systems

They generalize the example of the Cuntz algebra. Here, I am borrowing some material from [36, 23]. Let X be a compact space, let T be a local homeomorphism of X onto X and let ψ be a continuous and strictly positive function defined on X. One defines:

- The groupoid

$$G(X,T) = \{(x, m-n, y) \mid x, y \in X;\ m, n \in \mathbf{N} \text{ and } T^m x = T^n y\}.$$

Just as in the case of the Cuntz groupoid $\mathcal{O}_d$, its topology is defined by a base of open bisections of the following form. One fixes $m, n \in \mathbf{N}$ and chooses open subsets $U, V \subset X$ on which T^m and T^n are respectively one-to-one. Then

$$S(U, m, n, V) = \{(x, m-n, y) : x \in U, y \in V \text{ and } T^m x = T^n y\}$$

is a bisection and these bisections form a base of a topology on $G(X, T)$. With this topology, $G(X, T)$ is a locally compact étale groupoid.

• The cocycle $D_\psi : G(X, T) \to \mathbf{R}_+^*$ by

$$D_\psi(x, m-n, y) = \frac{\psi(x)\psi(Tx)\ldots\psi(T^{m-1}x)}{\psi(y)\psi(Ty)\ldots\psi(T^{n-1}y)}.$$

It is continuous.

• The *transfer operator* $\mathcal{L}_\psi : C(X) \to C(X)$ by

$$\mathcal{L}_\psi f(x) = \sum_{Ty=x} \psi(y) f(y).$$

We denote by $\mathcal{L}_\psi^*$ the transposed operator acting on the space of finite measures on X.

Proposition 3.4.1. *Let μ be a probability measure on X. Then μ is quasi-invariant under $G(X, T)$ with Radon-Nikodým derivative $dr^*\mu/ds^*\mu = D_\psi$ if and only if $\mathcal{L}_\psi^*\mu = \mu$.*

Proof. Suppose that μ is a quasi-invariant probability measure admitting D as its Radon-Nikodym derivative. We have for all $f \in C(X)$:

$$\begin{aligned} < f, \mathcal{L}_\psi^*\mu > &= < \mathcal{L}_\psi f, \mu > \\ &= \int_X \sum_{Ty=x} \psi(y) f(y) d\mu(x) \\ &= \int g ds^*\mu \end{aligned}$$

where $g = ((\psi f) \circ r)\mathbf{1}_S$ and $S = \{(x, 1, Tx) : x \in X\}$. By quasi-invariance of μ, this is

$$\begin{aligned} \int g D_\psi^{-1} dr^*\mu &= \int_X \psi(x) f(x) \psi(x)^{-1} d\mu(x) \\ &= < f, \mu > \end{aligned}$$

Suppose now that the probability measure μ satisfies $\mathcal{L}_\psi^*\mu = \mu$. Because of the equality

$$\mathcal{L}_\psi(f \circ T) = (\mathcal{L}_\psi 1)f,$$

we have $T_*\mu = (\mathcal{L}_\psi 1)\mu$, where $T_*\mu(A) = \mu(T^{-1}A)$. In particular $\mu(A) = 0 \Rightarrow \mu(T^{-1}A) = 0$. If $\mu(A) = 0$, we also have

$$\int_X \sum_{Ty=x} \psi(y)\mathbf{1}_A(y)d\mu(x) = 0$$

and this implies $\mu(TA) = 0$. These two properties imply the quasi-invariance of μ under $G(X,T)$. Let D' be its Radon-Nikodym derivative and let $J(x) = D'(x,1,Tx)$ be its *Jacobian*. As above, we have for all $f \in C(X)$,

$$\begin{aligned} < f, \mu > &= < f, \mathcal{L}_\varphi^*\mu > \\ &= \int_X \psi(x)f(x)J(x)^{-1}d\mu(x) \end{aligned}$$

which implies that $J(x) = \psi(x)$ for μ-a.e. x and that $D' = D_\psi$. □

We assume that X is a compact metric space. One says that $T : X \to X$ is *positively expansive* if there is an $\epsilon > 0$ such that for all $x \neq y$ there is an $n \in \mathbf{N}$ with $d(T^n x, T^n y) \geq \epsilon$. One says that T is *exact* if for every non-empty open set $U \subset X$ there is an $n > 0$ such that $T^n(U) = X$. One says that a real-valued continuous function φ on X satisfies the *Bowen condition* with respect to T if there are $\delta, C > 0$ such that

$$\sum_{i=0}^{n-1} \varphi(T^i x) - \varphi(T^i y) \leq C$$

for all $x, y \in X$ and $n > 0$ such that $d(T^i x, T^i y) \leq \delta$ for $0 \leq i \leq n-1$. The following result is part of a version, due to Walters, of the Ruelle-Perron-Frobenius theorem (see [39]).

Theorem 3.4.2. *Let $T : X \to X$ be a local homeomorphism which is positively expansive and exact. Let $\psi \in C(X, \mathbf{R}_+^*)$ such that* $\log\psi$

satisfies Bowen condition. Then, there is a unique probability measure μ *and a unique* $\lambda > 0$ *such that* $\mathcal{L}_\psi^* \mu = \lambda\mu$*; moreover the logarithm of* λ *is the pressure of the logarithm of* ψ *(written* $P(T, \log\psi)$*).*

Proof. We just give the ideas of a proof which is developed in [36]. It uses the kernel $R(X,T) = c^{-1}(0)$ of the gauge cocycle $c : G(X,T) \to \mathbf{Z}$ given by $c(x,k,y) = k$. We note that $R(X,T)$ fits within our example 3.3.3, where $X_n = X$ and $\pi_{n+1,n} = T$ for all $n \in \mathbf{N}$. Assume that the probability measure μ satisfies $\mathcal{L}_\psi^* \mu = \lambda\mu$. Then $\mathcal{L}_{\lambda_{-1}\psi}^* \mu = \mu$. According to Proposition 3.4.1, μ is quasi-invariant under $G(X,T)$ with derivative $D_{\lambda_{-1}\psi}$. This implies that μ is quasi-invariant under $R(X,T)$ with derivative D, restriction of $D_{\lambda_{-1}\psi}$ to $R(X,T)$. Note that D is also the restriction of D_ψ to $R(X,T)$. We can use the Dobrushin-Lanford-Ruelle formulation, i.e. the sequence of expectations (E_n) constructed from D to solve this Radon-Nikodým problem. Here, E_n is related to $\mathcal{L}_\psi^n$. A simple compactness argument gives the existence of μ. A careful study of the sequence (E_n) using our assumptions gives the uniqueness of μ. We deduce from the Dobrushin-Lanford-Ruelle condition that $\mathcal{L}_\psi^* \mu$ is also a non-normalized solution. By uniqueness, $\mathcal{L}_\psi^* \mu = \lambda\mu$ with $\lambda = \mathcal{L}_\psi^* \mu(1)$. □

Corollary 3.4.3. *Let* $T : X \to X$ *be a local homeomorphism which is positively expansive and exact. Let* $\varphi \in C(X, \mathbf{R})$ *defining the cocycle* $c_\varphi : G(X,T) \to \mathbf{R}$ *and let* σ_φ *be the one-parameter automorphism group of* $C^*(G(X,T))$ *defined by* c_φ*. Assume that* $c_\varphi^{-1}(0)$ *is principal and that* φ *satisfies the Bowen condition. Let* $\beta \in R$*.*
(i) If $P(T, -\beta\varphi) \neq 0$*, there exists no* KMS_β *state for* σ_φ*.*
(ii) If $P(T, -\beta\varphi) = 0$*, there exists a unique* KMS_β *state for* σ_φ*.*

Proof. According to Theorem 3.3.12, every possible KMS state is of the form φ_μ where μ is a quasi-invariant measure admitting $e^{-\beta c}$ as Radon-Nikodým derivative. We deduce from Proposition 3.4.1 that $\mathcal{L}_\psi^* \mu = \mu$ where $\psi = e^{-\beta\varphi}$. According to Theorem 3.4.2 this is possible only if $\lambda = \exp(P(T, -\beta\varphi)) = 1$. Moreover the theorem gives the existence and the uniqueness of the quasi-invariant measure μ, hence of the KMS state φ_μ when $P(T, -\beta\varphi) = 0$. □

Example 3.4.4. The one-sided full shift. Here, we have

$$X = \{1, \dots, d\}^{\mathbf{N}} \qquad T(x_0 x_1 \dots) = x_1 x_2 \dots$$

As we have seen, $C^*(G(X,T)) = O_d$. The function $\varphi \equiv 1$ defines the gauge group σ. The hypotheses of the corollary are satisfied. Note that the equation $P(T,-\beta\varphi) = 0$ becomes here $\beta = h(T) = \log d$, where $h(T) = P(T,0)$ is the topological entropy of T.

This example admits many generalizations. For example, if $\varphi(x)$ depends only of the first letter x_0 and takes the values $\lambda_1, \ldots, \lambda_d$, the equation $P(T,-\beta\varphi) = 0$ becomes $\sum_{i=1}^n e^{-\beta\lambda_i} = 1$. It admits a solution if and only if $\lambda_1, \ldots, \lambda_d$ have same sign. Then, there is a unique KMS$_\beta$ state.

More generally one-sided subshifts of finite type fit within this framework. The associated C*-algebras are the *Cuntz-Krieger algebras* (see [12]).

3.4.2 Bost-Connes system

This is a one-parameter automorphism group σ of a C*-algebra A originating from number theory which exhibits a phase transition. It was introduced by Bost and Connes in [6]. The C*-algebra is constructed from the following so-called *Hecke pair*,

$$P^+_{\mathbf{Z}} := \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathbf{Q} \\ 0 & \mathbf{Q}^*_+ \end{pmatrix} := P^+_{\mathbf{Q}}$$

consisting of a group $P^+_{\mathbf{Q}}$ and a subgroup $P^+_{\mathbf{Z}}$ which is *almost normal.* This means that each of the double coset in $P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}/P^+_{\mathbf{Z}}$ contains a finite number of right (and left) cosets. On $C_c(P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}/P^+_{\mathbf{Z}})$, one defines the convolution product

$$f * g(\gamma) = \sum_{P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}} f(\gamma\gamma_1^{-1})g(\gamma_1)$$

and the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

One defines its regular representation on $\ell^2(P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}})$ by

$$L(f)\xi(\gamma) = \sum_{P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}} f(\gamma\gamma_1^{-1})\xi(\gamma_1).$$

One checks that $L(f)$ is bounded for $f \in C_c(P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}/P^+_{\mathbf{Z}})$. The C*-algebra A is the completion of $C_c(P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}/P^+_{\mathbf{Z}})$ for the norm $\|f\|_{\text{red}} = \|L(f)\|$.

For $t \in \mathbf{R}$, the automorphism σ_t is defined for $f \in C_c(P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}/P^+_{\mathbf{Z}})$ by

$$\sigma_t(f)(\gamma) = \Big(\frac{L(\gamma)}{R(\gamma)}\Big)^{-it} f(\gamma),$$

where $L(\gamma)$ [resp. $R(\gamma)$] is the cardinality of the image of $P^+_{\mathbf{Z}}\gamma P^+_{\mathbf{Z}}$ in $P^+_{\mathbf{Q}}/P^+_{\mathbf{Z}}$ [resp. in $P^+_{\mathbf{Z}}\backslash P^+_{\mathbf{Q}}$].

The main result of [6] is:

Theorem 3.4.5. *Let (A, σ) be as above. Then,*
(i) For all $0 < \beta \leq 1$, there exists one and only one KMS_β state. Its GNS representation generates the hyperfinite factor of type III_1. It is invariant under the action of $Aut(\mathbf{Q}/\mathbf{Z})$.
(ii) For all $1 < \beta \leq \infty$, the extremal KMS_β states are parametrized by the complex imbeddings $\chi : \mathbf{Q}^{\text{cycl}} \to \mathbf{C}$ of the subfield $\mathbf{Q}^{\text{cycl}}$ of $\mathbf{C}$ generated by the roots of unity. They generate the factor of type I_∞. The group $Aut(\mathbf{Q}/\mathbf{Z})$ acts freely and transitively on the set of extremal KMS_β.
(iii) The partition function of this system is the Riemann zeta function.

The partition function is the function $\beta \mapsto Tr(e^{-\beta H})$ occuring as a normalization factor in the definition of a Gibbs state in the case of type I factors. The hamiltonian H implementing the automorphism group is defined up to an additive constant. To fix it, we choose $H \geq 0$ with 0 in its spectrum.

Again, we only give the idea of a proof using the formalism of this chapter. The first observation is that the C*-algebra A can be written as a groupoid C*-algebra. The construction is similar to that given above in the case of a single endomorphism. The main difference is that the subsemigroup $\mathbf{N} \subset \mathbf{Z}$ is replaced by the multiplicative semigroup of positive integers $\mathbf{N}^* \subset \mathbb{Q}^*$. Indeed, for each prime number p, we view the semi-group $\mathbf{N}^*$ as a subset of the ring $\mathbf{Z}_p$ of p-adic integers, hence by the diagonal embedding as a subset of the ring of *integral adeles* $\mathcal{R} = \prod_{\mathcal{P}} \mathbf{Z}_p$, where the product is taken over the set $\mathcal{P}$ of all prime numbers. Therefore the semi-group $\mathbf{N}^*$

acts on $\mathcal{R}$ by multiplication. Endowed with the product topology $\mathcal{R}$ is a compact space and for $n \in \mathbf{N}^*$, the map $T_n : x \mapsto nx$ is a homeomorphism from $\mathcal{R}$ onto $n\mathcal{R}$. Just as above, we define

$$G = \{(x, m/n, y) \in \mathcal{R} \times \mathbf{Q}_+^* \times \mathcal{R} \mid m, n \in \mathbf{N}^* \quad mx = ny\}.$$

It has the obvious groupoid structure

$$(x, t, y)(y, t', z) = (x, tt', z) \quad (x, t, y)^{-1} = (y, t^{-1}, x)$$

and the basic open sets

$$\mathcal{U}(U; m, n; V) = \{(x, m/n, y) : (x, y) \in U \times V, \ mx = ny\}$$

where U and V are open subsets of X, define a topology which turns it into a locally compact étale groupoid. Note that $x = 0$ is the only point of $\mathcal{R}$ with non trivial isotropy. Its isotropy subgroup is ${\mathbf{Q}_+}^*$. This groupoid has a canonical cocycle $c : G \to \mathbf{R}$, analogous to the gauge cocycle of the Cuntz groupoid, namely it is given by $c(x, m/n, y) = \log(m/n)$. We associate to it the one-parameter automorphism group σ_c of $C^*_{\text{red}}(G)$.

Exercise 3.4.6. Let (A, σ) be the Bost-Connes system.
(i) Show that the character group of the discrete abelian group $\mathbf{Q}/\mathbf{Z}$ can be identified with $\mathcal{R}$. We denote by μ the normalized Haar measure of the additive group $\mathcal{R}$.
(ii) Let $L' = \int^{\oplus} \pi_x d\mu(x)$ be the regular representation of $C_{\text{red}}(G)$ on $L^2(G, s^*\mu) = \int^{\oplus} \ell^2(G_x) d\mu(x)$. Construct an isometry

$$\mathcal{F} : \ell^2(P^+_{\mathbf{Z}} \backslash P^+_{\mathbf{Q}}) \to L^2(G, s^*\mu)$$

such that $L(A) = \mathcal{F}^{-1} L'(C_{\text{red}}(G))\mathcal{F}$.
(iii) Show that the isomorphism φ of $C^*_{\text{red}}(G)$ onto A so constructed conjugates σ_c and σ.
Hint. The C*-algebra A admits the following set of generators:

- for $n \in \mathbf{N}^*$, $\mu_n = n^{-1/2} \mathbf{1}_{X_n}$, where X_n is the double coset of $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \in P^+_{\mathbf{Q}}$;
- for $\gamma \in \mathbf{Q}/\mathbf{Z}$, $e(\gamma) = \mathbf{1}_{X^\gamma}$, where X^γ is the double coset of $\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \in P^+_{\mathbf{Q}}/P^+_{\mathbf{Z}}$.

Moreover, we have

$$\sigma_t(e(\gamma)) = e(\gamma); \qquad \sigma_t(\mu_n) = n^{it}\mu_n.$$

The isomorphism φ sends $C(\mathcal{R})$ onto the C*-algebra $C^*(\mathbf{Q}/\mathbf{Z})$ generated by the $e(\gamma)$'s and sends $\mathbf{1}_{S_n}$, where S_n is the bisection

$$S_n = \{(x, n, y) \in \mathcal{R} \times \mathbf{Q}_+^* \times \mathcal{R} \mid \quad nx = y\}$$

onto μ_n.

Since $c^{-1}(0)$ is principal, the problem of finding the KMS states for σ is reduced to finding probability measures μ on $\mathcal{R}$ which are quasi-invariant under G with Radon-Nikodym derivative $e^{-\beta c}$. An essential ingredient is the group of symmetries $\mathcal{R}^*$, the multiplicative group of invertible elements of $\mathcal{R}$ acting by multiplication. Since this action commutes with the action of $\mathbf{N}^*$, $\mathcal{R}^*$ acts on G by automorphisms which preserve c. This reduces the problem to the quotient groupoid $G/\mathcal{R}^*$, which is more tractable.

Chapter 4

Amenability and Nuclearity

Given a locally compact étale groupoid G, we have defined the full C*-algebra $C^*(G)$ and its quotient $C^*_{\mathrm{red}}(G)$. We have seen a number of examples such as the Cuntz groupoids, tail equivalence relations or the irrational rotation groupoid where $C^*(G) = C^*_{\mathrm{red}}(G)$. There is a general property of G, called *amenability*, which implies the equality $C^*(G) = C^*_{\mathrm{red}}(G)$. This property has been known and studied for groups, in particular discrete or locally compact, for a long time (see for example [18, 28, 30] for a survey of the theory). A fascinating feature of this notion is that it can take many shapes. We shall limit ourselves to one among many equivalent definitions. We refer the reader to [2] for a more detailed exposition.

4.1 Amenability

Generalization of amenability from groups to group actions were given in [42, 1]. Groupoids provide a convenient setting for expressing these generalizations.

Definition 4.1.1. ([2], 2.2.6) A locally compact étale groupoid G is *topologically amenable* if there exists a sequence (f_n) of non-negative continuous functions with compact support on G such that:

1. the functions $x \mapsto \sum_{r(\gamma)=x} f_n(\gamma)$ tend to the function 1 uniformly on the compact subsets of $G^{(0)}$.
2. The functions $\gamma \mapsto \sum_{r(\gamma')=x} |f_n(\gamma^{-1}\gamma') - f_n(\gamma')|$ tend to the zero function uniformly on the compact subsets of G.

Such a sequence (f_n) is called an *approximate invariant mean.*

Here is another convenient and equivalent definition ([2], 2.2.13) of topological amenability for a locally compact étale groupoid in terms of positive type functions which we now define.

Definition 4.1.2. A complex valued function h on a groupoid G is *of positive type* if for all integers $n \geq 1$, all $\gamma_1, \ldots, \gamma_n$ elements of G having the same range and all complex numbers $z_1, \ldots, z_n$,

$$\sum_{i,j} h(\gamma_i^{-1}\gamma_j)\overline{z}_i z_j \geq 0.$$

Suppose that (H, L) is a unitary representation of G (cf Definition 2.3.12). Here, we just mean a family of Hilbert spaces $\mathcal{H}_x$ indexed by $x \in G^{(0)}$ and a functor L, where $L(\gamma) : \mathcal{H}_{s(\gamma)} \to \mathcal{H}_{r(\gamma)}$. Then, for each section $x \mapsto \xi(x) \in \mathcal{H}_x$, the *coefficient*

$$h(\gamma) = (\xi, \xi)(\gamma) \stackrel{\text{def}}{=} \langle \xi \circ r(\gamma), L(\gamma)\xi \circ s(\gamma)\rangle$$

is of positive type. It is easily shown that all positive type functions can be written as coefficients of unitary representations. If moreover G is locally compact and étale and if h is a positive type function which is continuous with compact support, one can take in the above construction the left regular representation, where $\mathcal{H}_x = \ell^2(G^x)$ and $L(\gamma)\xi(\gamma') = \xi(\gamma^{-1}\gamma')$. This expresses h as $\eta * \eta^*$ where $\eta = \overline{\xi}$.

Proposition 4.1.3. *([2], 2.2.13) A locally compact étale groupoid G is topologically amenable if and only if there exists a sequence (h_n) of positive type continuous functions with compact support in G such that $h_{n|G^{(0)}} \leq 1$ for all n and $\lim_n h_n = 1$ uniformly on every compact subset of G.*

Proof. We refer the reader to [2] for the proof, where an auxiliary equivalent condition is introduced, namely the existence of a sequence (g_n) in $C_c(G)$ such that

1. $\sum_{r(\gamma)=x} |g_n|^2(\gamma) \leq 1$ for all $x \in G^{(0)}$ and all n,

2. $\lim_n \sum_{r(\gamma)=x} |g_n|^2(\gamma) = 1$ uniformly on every compact subset of $G^{(0)}$,

3. $\lim_n \sum_{r(\gamma')=r(\gamma)} |g_n(\gamma^{-1}\gamma') - g_n(\gamma')|^2 = 0$ uniformly on every compact subset of G.

Let's stick to our notation (f_n), (g_n) and (h_n) to distinguish these three types of sequences. Starting with g_n, $f_n = |g_n|^2$ will have the required properties, and so will $h_n = \overline{g_n} * \overline{g_n}^*$. Conversely, from the sequence (f_n), we define $g_n = |f_n|^{1/2}$. Finally, given a sequence (h_n), we can write h_n of the form $\overline{g_n} * \overline{g_n}^*$. The sequence (g_n) satisfies the required properties. □

Amenability of a discrete group Γ is often expressed as the existence of a *Følner sequence.* It is a sequence (F_n) of finite subsets of Γ such that for all $\gamma \in \Gamma$, $\dfrac{|F_n \Delta \gamma F_n|}{|F_n|}$ goes to zero as n goes to infinity, where $A\Delta B$ denotes the symmetric difference of the subsets A and B. Then the sequence $(f_n = \dfrac{1}{|F_n|}\mathbf{1}_{F_n})$ is an approximate invariant mean. The group $\mathbf{Z}$ possesses the Følner sequence $F_n = [-n, n]$. Therefore, it is amenable.

Here is a useful elementary fact.

Proposition 4.1.4. *The groupoid $G(X,T)$ of a transformation group (Γ, X, T), where Γ is amenable, is also amenable.*

Proof. Given an approximate invariant mean (f_n) for G, one obtains an approximate invariant mean $(\tilde{f}_n)$ for $G(X,T)$ by setting $\tilde{f}_n(x,s,y) = f_n(s)$ (or $h_n(x)f_n(s)h_n(y)$ where h_n is a cut-off function if X is not compact). □

When $G(X,T)$ is amenable, one also says that the action of Γ on X is amenable. The converse of the proposition does not hold. The action of an arbitrary group on itself by left translation is amenable. More interestingly, many non amenable groups admit amenable actions on a compact space.

Exercise 4.1.5. Consider the free group on two generators $F(a,b)$ and the space X of infinite reduced words $x_0x_1\dots$ where $x_i \in \{a,b,a^{-1},b^{-1}\}$ and a cannot be followed by a^{-1} and the same for b. Let $F(a,b)$ act on X by left concatenation. Show that this action is amenable by constructing an approximate invariant mean.

Let us show that most examples of groupoids introduced in the previous chapters are amenable.

Example 4.1.6. The AP equivalence relation R on the compact space X met in 3.3.3 is amenable. Recall that R is the union of an increasing sequence of sub-equivalence relations R_n, closed and open in R, which are closed subsets of $X \times X$. In particular, $\mathbf{1}_{R_n}$ belongs to $C_c(G)$; moreover, it is of positive type. These functions converge to 1 uniformly on compact subsets of R.

Exercise 4.1.7. Show that the Cuntz groupoid $\mathcal{O}_d$ is amenable by constructing an approximate invariant mean.

4.2 C*-algebraic properties

As suggested in the introduction, the equality $C^*(G) = C^*_{\mathrm{red}}(G)$ holds for amenable groupoids.

Theorem 4.2.1. *Let G be a locally compact amenable groupoid. Assume that it is amenable. Then $C^*(G) = C^*_{\mathrm{red}}(G)$.*

Proof. Let L be an arbitrary representation of $C_c(G)$. According to Theorem 2.3.15, L is the integrated representation of unitary representation (μ, H, L) of G. We realize the Hilbert space of the representation as $\mathcal{H} = L^2(G^{(0)}, \mu, H)$. For $\xi, \eta \in \mathcal{H}$ and $f \in C_c(G)$, we have

$$\langle \xi, L(f)\eta \rangle = \int \sum_{r(\gamma)=x} \langle \xi(x), f(\gamma)D^{-1/2}(\gamma)L(\gamma)\eta \circ s(\gamma)\rangle_x d\mu(x)$$

where D is the Radon-Nikodym derivative of μ. We insert $h_n = \overline{g_n} * \overline{g_n}^*$ into the integral, where $g_n \in C_c(G)$ and h_n tends to 1 uniformly on compact subsets of G. This gives, after a number of changes of variables and orders of integration:

$$\langle \xi, L(f)\eta\rangle = \lim_n \int \sum_{r(\gamma_2)=s(\gamma_1)=x} f(\gamma_1\gamma_2)\langle \xi_n(\gamma_1), \eta_n(\gamma_2^{-1})\rangle_x d\mu(x)$$

where

$$\xi_n(\gamma) = D^{1/2}(\gamma)g_n(\gamma)L(\gamma^{-1})\xi \circ r(\gamma)$$

and η_n is defined similarly from η. This equality can be written as

$$\langle \xi, L(f)\eta\rangle = \lim_n \langle \xi_n, L'(f)\eta_n\rangle$$

where L' is the representation $\int^{\oplus} \pi_x \otimes 1_{\mathcal{H}_x} d\mu(x)$ of $C_c(G)$ on the Hilbert space

$$\mathcal{H}' = L^2(G, s^*\mu, s^*H) = \int^{\oplus} \ell^2(G_x, \mathcal{H}_x) d\mu(x)$$

and where π_x is the regular representation introduced in the definition of the reduced norm in 2.3.4. In other words, L' is a regular representation with multiplicity. It factors through the reduced C*-algebra $C^*_{\mathrm{red}}(G)$. One can check that $\|\xi_n\|$ [resp. $\|\eta_n\|$] tends to $\|\eta\|$ [resp. $\|xi\|$]. This gives the inequality: $|\langle \xi, L(f)\eta\rangle| \leq \|L'(f)\|\|\xi\|\|\eta\| \leq \|f\|_{\mathrm{red}}\|\xi\|\|\eta\|$ and therefore $\|L(f)\| \leq \|f\|_{\mathrm{red}}$. We deduce that the full norm is equal to the reduced norm. $\square$

The converse holds for a locally compact group and is rather easy to obtain: since the trivial representation factors through the reduced C*-algebra, one can obtain the constant function 1 as a limit of coefficients of the regular representation. The general case of a groupoid is not known.

Now we will turn to an important approximation property for C*-algebras called nuclearity. But we need to introduce some new notions before we can define it. Given a C*-algebra A and an integer $n \geq 1$, we define the C*-algebra $A \otimes M_n(\mathbf{C}) = M_n(A)$ as the $*$-algebra of n by n matrices with coefficients in A equipped with the matrix multiplication and the involution. To define the norm, we pick a faithful representation L of A on a Hilbert space $\mathcal{H}$. Then $L \otimes Id_n$ is a representation of $A \otimes M_n(\mathbf{C})$ on the Hilbert space $\mathcal{H} \otimes \mathbf{C}^n =$

$\mathcal{H}^{\oplus n}$ and defines the unique C*-norm of $A \otimes M_n(\mathbf{C})$. Given two C*-algebras A, B and a bounded linear map $\varphi : A \to B$, we define the bounded linear map $\varphi_n = \varphi \otimes Id_n : A \otimes M_n(\mathbf{C}) \to B \otimes M_n(\mathbf{C})$ by $\varphi_n([a_{i,j}]) = [\varphi_n(a_{i,j})]$.

Definition 4.2.2. Given two C*-algebras A, B and a bounded linear map $\varphi : A \to B$, one says that
(*i*) φ is *completely bounded* if $\|\varphi\|_{cb} = \sup \|\varphi_n\| < \infty$;
(*ii*) φ is *completely contractive* if $\sup \|\varphi_n\| \leq 1$;
(*iii*) φ is *completely positive* if φ_n is a positive linear map for all n.

A completely positive map is completely bounded with $\|\varphi\|_{cb} = \|\varphi\|$.

Definition 4.2.3. A C*-algebra A is *nuclear* if the identity map $Id : A \to A$ factors approximately through $M_n(\mathbf{C})$ via completely positive contractions in the point-norm topology. More precisely, for all $\epsilon > 0$ and $a_1, \dots, a_k \in A$, there exist $n \in \mathbf{N}$ and completely positive contractions $\alpha : A \to M_n(\mathbf{C})$ and $\beta : M_n(\mathbf{C}) \to A$ such that for all $j = 1, \dots, k$, $\|\beta \circ \alpha(a_j) - a_j\| \leq \epsilon$.

An important characterization of nuclearity is the following result which we quote without a proof.

Theorem 4.2.4. *A C*-algebra A is nuclear if and only if for all C*-algebra B, the algebraic tensor product $A \otimes B$ carries a unique C*-norm.*

For groupoid C*-algebras, we have the following characterization.

Theorem 4.2.5. *Let G be a locally compact étale groupoid. Then the reduced C*-algebra $C^*_{\mathrm{red}}(G)$ is nuclear if and only if G is amenable.*

Note that if this happens, $C^*(G) = C^*_{\mathrm{red}}(G)$. Moreover, we would have obtained the same result under the assumption that $C^*(G)$ is nuclear, because a quotient of a nuclear C*-algebra is nuclear.

Proof. A proof is given in [2], Corollary 6.2.14, using the deep fact that a C*-algebra is nuclear if and only if all its factor representations are hyperfinite. We sketch here a more direct proof of the nuclearity of $C^*(G)$ under the assumption that G is amenable. Let B be an arbitrary C*-algebra. The minimal tensor product $C^*_{\rm red}(G)\otimes_r B$ is defined by a representation $L_G\otimes L_B$, where L_G and L_B are respectively faithful representations of $C^*_{\rm red}(G)$ and of B. Equivalently, it is the completion $C^*_{\rm red}(G,B)$ of the $*$-algebra $C_c(G,B)$ under the reduced norm, which is defined just as in 2.3.4 but where the representations π_x act on $\ell^2(G_x,\mathcal{K})$, where $\mathcal{K}$ is the space of the representation L_B. On the other hand, the maximal tensor product $C^*(G)\otimes B$ can be defined as the completion $C^*(G,B)$ of the $*$-algebra $C_c(G,B)$ under the full norm defined by considering all the representations of this $*$-algebra. Essentially the same proof as in Theorem 4.2.1 shows that $C^*_{\rm red}(G,B)=C^*(G,B)$. For the converse, one can extract a sequence of positive type functions with compact support converging to 1 from a sequence of completely positive contractions factorizing through a matrix algebra and converging to the identity map. □

4.3 Exactness

In this section, we shall examine the following generalization of amenability for a discrete group.

Definition 4.3.1. We say that a discrete group Γ is *amenable at infinity* if it admits an amenable action T on a compact space X, i.e. such that the groupoid $G(\Gamma,X,T)$ is topologically amenable.

Example 4.3.2. Amenable groups, but also free groups, word hyperbolic groups or discrete subgroups of connected Lie groups are amenable at infinity.

There is a characterization of the amenability at infinity of a discrete group Γ which only invokes the group Γ. Indeed, Γ admits an amenable action on a compact space X if and only if its left action on the spectrum $\beta\Gamma$ of the C*-algebra $l^\infty(\Gamma)$ is topologically amenable and this last condition can be expressed in terms of a net of functions on $\Gamma\times\Gamma$ (see [2], 5.2.3). N. Higson and J. Roe observed in [19] that, in the case of a finitely generated discrete group Γ, this is exactly the property (A), introduced by G. Yu in [40], of the metric space Γ, (where Γ is endowed with a word-length metric associated to some finite set of generators).

Definition 4.3.3. A countable discrete group Γ has *property (A)* of there exists a sequence (f_n) of non-negative functions defined on $\Gamma \times \Gamma$ such that
(i) the support of each f_n is contained in a tube, i.e. a subset of the form

$$\{(\gamma, \eta) \in \Gamma \times \Gamma \mid \gamma^{-1}\eta \in F\}$$

where F is a finite subset of Γ;
(ii) for each n and each γ, $\sum_\eta f(\gamma, \eta) = 1$;
(iii) the functions $(\gamma, \gamma') \mapsto \sum_\eta |f_n(\gamma, \eta) - f_n(\gamma', \eta)|$ tend to zero uniformly on the tubes of $\Gamma \times \Gamma$.

Proposition 4.3.4. *([19]) Let Γ be a finitely generated discrete group endowed with a word-length metric. Then, the following conditions are equivalent:*
(i) *The group Γ is amenable at infinity.*
(ii) *The metric space Γ has property (A).*

Proof. We only give a sketch of the proof. One introduces the compact space $X = \beta\Gamma$ which is the character space of the commutative C*-algebra $\ell^\infty(\Gamma)$. It admits the action of Γ by right translation. Then condition A is essentially the amenability of this action, in its formulation by an approximate invariant mean. □

An important property enjoyed by the groups which are amenable at infinity is C*-exactness.

Definition 4.3.5. A C*-algebra A is *exact* if the reduced tensor product by A transforms a short exact sequence of C*-algebras:

$$0 \to I \to B \to C \to 0$$

into a short exact sequence of C*-algebras

$$0 \to I \otimes A \to B \otimes A \to C \otimes A \to 0.$$

Since the full tensor product by A always transforms a short exact sequence of C*-algebras into a short exact sequence of C*-algebras, nuclear C*-algebras are exact. However, there exists exact C*-algebras which are not nuclear. For example, the results of this chapter show that the reduced C*-algebra $C^*_{\mathrm{red}}(\mathbf{F}_2)$ of the free group on two generators is exact but not nuclear. We shall admit and use the fact that a sub-C*-algebra of a nuclear C*-algebra is exact.

Definition 4.3.6. ([21]) A locally compact group G is called *C*-exact* if the reduced crossed product construction transforms a short exact sequence of G-C*-algebras into a short exact sequence of C*-algebras.

If G is C*-exact, its reduced C*-algebra is exact. The converse holds for a discrete group. Amenability is a convenient way to prove C*-exactness:

Proposition 4.3.7. ([2]) *Let Γ be a discrete group. If it is amenable at infinity, then it is C*-exact.*

Proof. It suffices to prove the exactness of the reduced C*-algebra $C^*_{\rm red}(\Gamma)$. Suppose that Γ admits an amenable action T on a compact space X. Let $G = G(\Gamma, X, T)$. The map $f \in C_c(\Gamma) \mapsto \tilde{f} \in C_c(G)$ such that $\tilde{f}(x, \gamma, y) = f(\gamma)$ extends to an injective $*$-homomorphism of $C^*_{\rm red}(\Gamma)$ into $C^*_{\rm red}(\Gamma)$. This exhibits $C^*_{\rm red}(\Gamma)$ as a sub-C*-algebra of a nuclear C*-algebra. Therefore, it is exact. $\square$

Chapter 5

K-theory

K-theory is one of the main invariants of C*-algebras. Because of Bott periodicity, it is composed of only two abelian groups $K_0(A)$ and $K_1(A)$, often endowed with an additional structure, as an order or a unit. When the C*-algebra A arises from a dynamical system, it is natural to try to relate these groups to properties or invariants of the dynamical system.

5.1 The abelian group $K_0(A)$

5.1.1 The unital case

It is easier to define the groups $K_i(A)$ when A has a unit. The group $K_0(A)$ is constructed from projections (elements satisfying $e = e^* = e^2$) while $K_1(A)$ is constructed from unitaries (elements satisfying $uu^* = u^*u = 1$). It is necessary to consider not only elements of the C*-algebra A but also elements of $M_n(A) = A \otimes M_n(\mathbf{C})$ for all n.

Definition 5.1.1. One says that the projections $e, f \in A$ are *equivalent*, and one writes $e \sim f$, if there exists $u \in A$ such that $u^*u = e$ and $uu^* = f$.

It is an equivalence relation.

We define

$$\mathcal{P}_n(A) = \{\text{projections in}\, M_n(A)\}.$$

We view $\mathcal{P}_n(A)$ as a subset of $\mathcal{P}_{n+1}(A)$ by sending

$$e \in \mathcal{P}_n(A) \quad \mapsto \quad \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}_{n+1}(A).$$

Let $\mathcal{P}_\infty(A) = \cup_n \mathcal{P}_n(A)$ be the union. One says that $e, f \in \mathcal{P}_\infty(A)$ are equivalent if there exists n such that $e \sim f$ in $M_n(A)$. It is an equivalence relation. The equivalence class of e is denoted by $[e]$ and the quotient space by $\mathcal{P}_\infty(A)/\sim$. One defines an addition on $\mathcal{P}_\infty(A)/\sim$: $[e] + [f] = [e' + f']$ where $e' \sim e$, $f' \sim f$ and $e'f' = 0$. Then $(\mathcal{P}_\infty(A)/\sim, +, [0])$ is an abelian semi-group with a zero element.

Definition 5.1.2. The abelian group $K_0(A)$ is the *Grothendieck group* of the semi-group $(\mathcal{P}_\infty(A)/\sim, +, [0])$. Concretely, its elements are differences $[e] - [f]$ where $e, f \in \mathcal{P}_\infty(A)$ where the following rule holds:

$$[e] - [f] = [e'] - [f'] \Leftrightarrow \exists g \in \mathcal{P}_\infty(A) \mid [e] + [f'] + [g] = [e'] + [f] + [g].$$

Given C*-algebras A, B and a $*$-homomorphism $\varphi : A \to B$, one defines $\varphi_* : K_0(A) \to K_0(B)$ by sending $e \in \mathcal{P}_n(A)$ to $\varphi_n(e) \in \mathcal{P}_n(B)$. We leave to the reader to check that:

Theorem 5.1.3. *K_0 is a covariant functor from the category of C*-algebras with unit to the category of abelian groups.*

Here are two more properties left to the reader.

Proposition 5.1.4. *(i) Let $A_1, \dots, A_n$ be C*-algebras with unit. Then $K_0(A_1 \oplus \cdots \oplus A_n) = K_0(A_1) \oplus \cdots \oplus K_0(A_n)$.*
(ii) Let $A = \overline{\cup A_n}$, where (A_n) is an increasing sequence of sub-C-algebras containing the unit. Then $K_0(A) = \varinjlim K_0(A_n)$.*

Remark 5.1.5. Suppose that τ is a tracial state on A. We extend it to a trace on $M_n(A)$ by defining $\tau_n((a_{i,j})) = \sum_{i=1}^n \tau(a_{i,i})$. By the tracial property, $\tau_n(e)$ for $e \in \mathcal{P}_n(A)$ depends only on $[e] \in \mathcal{P}_\infty(A)/\sim$. We denote it by $\tau_*([e])$. Since $\tau_*([e] + [f]) = \tau_*([e]) + \tau_*([f])$, τ_* extends to a group homomorphism $\tau_* : K_0(A) \to \mathbf{R}$.

5.1.2 The non-unital case

When A does not have a unit, we consider the C*-algebra $\tilde{A}$ obtained by adjoining a unit. It contains A as an ideal and $\tilde{A}/A = \mathbf{C}$. Let $p : \tilde{A} \to \mathbf{C}$ be the quotient map. We have $p_* : K_0(\tilde{A}) \to K_0(\mathbf{C}) = \mathbf{Z}$ and define $K_0(A)$ as its kernel.

Definition 5.1.6. When A does not have a unit, one defines $K_0(A) = \operatorname{Ker} p_*$ where $p : \tilde{A} \to \mathbf{C}$ sends $a + \lambda 1$ to λ.

One checks that K_0 is a functor and that above properties
$K_0(A_1 \oplus \cdots \oplus A_n) = K_0(A_1) \oplus \cdots \oplus K_0(A_n)$.
$K_0(\overline{\cup A_n}) = \varinjlim K_0(A_n)$
are still satisfied.

Example 5.1.7. *The K_0 group of an AF algebra.* Two projections e, f in $M_n(\mathbf{C})$ are equivalent if and only if $\dim(e) = \dim(f)$, where dim is the restriction of the usual trace Trace to projections; the dimension is an integer. Moreover, if $ef = 0$, $\dim(e + f) = \dim(e) + \dim(f)$. One deduces that for $A = M_n(\mathbf{C})$, dim establishes a semigroup isomorphism between $\mathcal{P}_\infty(A)/\sim$ and $\mathbf{N}$, hence an isomorphism between $K_0(A)$ and $\mathbf{Z}$. The image $K_0(A)^+$ of $\mathcal{P}_\infty(A)$ is then identified with $\mathbf{Z}^+ = \mathbf{N}$. The K_0 group of a finite dimensional C*-algebra $A = \oplus_{i=1}^m M_{d_i}(\mathbf{C})$ is $K_0(A) = \mathbf{Z}^m$; moreover $K_0(A)^+ = \mathbf{N}^m$ and the image of $\mathcal{P}_1(A)$, called the *dimension range* is $\oplus_{i=1}^m \{0, 1, \ldots, d_i\}$. A $*$-homomorphism

$$\varphi : \oplus_{i=1}^m M_{d_i}(\mathbf{C}) \to \oplus_{i=1}^n M_{d'_i}(\mathbf{C})$$

induces an order preserving group homomorphism $\varphi : \mathbf{Z}^m \to \mathbf{Z}^n$. Therefore, the K_0 group of an AF algebra $A = \overline{\cup A_n}$ is of the form $K_0(A) = \varinjlim \mathbf{Z}^{m_n}$, where the inductive system is given by order preserving homomorphisms. Ordered abelian groups which can be constructed in that fashion are called *dimension groups*.

Example 5.1.8. *The K_0 group of $C_0(\mathbf{R})$.* Let $\mathbf{T}$ denote the circle. A projection in $M_n(C(\mathbf{T}))$ can be viewed as a continuous map from $e : \mathbf{T}$ to $\mathcal{P}_n(\mathbf{C})$. It turns out (exercise!) that not only the dimension of $e(t)$ is constant, but the map is homotopic to a constant map. One deduces that $K_0(C(\mathbf{T})) = \mathbf{Z}$. Let us turn to $A = C_0(\mathbf{R})$. Since $\tilde{A}$ can be identified with $C(\mathbf{T})$ and since the map $p_* : K_0(C(\mathbf{T})) \to \mathbf{Z}$ is an isomorphim, $K_0(C_0(\mathbf{R})) = 0$.

5.2 The abelian group $K_1(A)$

5.2.1 The unital case

Let A be a C*-algebra A with unit. We define

$$\mathcal{U}_n(A) = \{\text{unitaries in } M_n(A)\}.$$

Definition 5.2.1. One says that the unitaries $u, v \in A$ are *homotopic*, and one writes $u \sim v$, if there exists a continuous map $\gamma : [0,1] \to \mathcal{U}_1(A)$ such that $\gamma(0) = u$ and $\gamma(1) = v$. It is an equivalence relation.

We view $\mathcal{U}_n(A)$ as a subset of $\mathcal{U}_{n+1}(A)$ by sending

$$u \in \mathcal{U}_n(A) \quad \mapsto \quad \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{U}_{n+1}(A).$$

Let $\mathcal{U}_\infty(A) = \cup_n \mathcal{U}_n(A)$ be the union. One says that $u, v \in \mathcal{U}_\infty(A)$ are equivalent if there exists n such that $u \sim v$ in $M_n(A)$. This defines an equivalence relation on $\mathcal{U}_\infty(A)$. Note that $\mathcal{U}_n(A)$ is a topological group. Endowed with the inductive limit topology $\mathcal{U}_\infty(A)$ is also a topological group. The set of elements equivalent to 1 is the connected component $\mathcal{U}_\infty^{(0)}(A)$ of this group. It is a normal subgroup. Therefore

$$\mathcal{U}_\infty(A)/\sim = \mathcal{U}_\infty(A)/\mathcal{U}_\infty^{(0)}(A)$$

is also a group.

Definition 5.2.2. The group $K_1(A)$ is $\mathcal{U}_\infty(A)/\sim$.

Lemma 5.2.3. *Let $u, v \in \mathcal{U}_1(A)$. Then $\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ are homotopic in $\mathcal{U}_2(A)$.*

Proof. One introduces the rotation matrix

$$u(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \in \mathcal{U}_2(\mathbf{C}) \subset \mathcal{U}_2(A).$$

Then $\gamma(t) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} u(t) \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} u(t)^{-1}$ is the desired homotopy. ∎

Corollary 5.2.4. *$K_1(A)$ is an abelian group.*

Proof. One uses the lemma and the equivalence

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

□

Given C*-algebras A, B with unit and a $*$-homomorphism $\varphi : A \to B$, one defines $\varphi_* : K_1(A) \to K_1(B)$ by sending $u \in \mathcal{U}_n(A)$ to $\varphi_n(u) \in \mathcal{U}_n(B)$. We leave to the reader to check that:

Theorem 5.2.5. *K_1 is a covariant functor from the category of C*-algebras with unit to the category of abelian groups.*

Example 5.2.6. We leave as an exercise the fact that every unitary matrix in $M_n(\mathbf{C})$ is homotopic to the identity matrix. One deduces that $K_1(\mathbf{C}) = K_1(M_n(\mathbf{C})) = 0$.

Example 5.2.7. Let $A = C(\mathbf{T})$. Elements of $\mathcal{U}_n(A)$ are continuous maps $u : \mathbf{T} \to \mathcal{U}_n(\mathbf{C})$. Their homotopy class depends only on the index of $\det \circ u : \mathbf{T} \to \mathbf{T}$. One deduces that $K_1(C(\mathbf{T})) = \mathbf{Z}$.

5.2.2 The non-unital case

Definition 5.2.8. When A does not have a unit, one defines $K_1(A) = K_1(\tilde{A})$ where $\tilde{A}$ is the C*-algebra obtained by adjoining the unit.

One checks that K_1 is a functor from the category of C*-algebras to the category of abelian groups.

Exercise 5.2.9. Let $\mathcal{K}$ be the C*-algebra of all compact operators on an infinite dimensional Hilbert space. Show that $K_1(\mathcal{K}) = 0$.

5.3 The six-term exact sequence

One of the basic tools to compute $K_i(A)$ is the six-term exact sequence associated with an ideal I of a C*-algebra A. Let us spell out the short exact sequence of C*-algebras, where $B = A/I$ is the quotient C*-algebra:

$$0 \longrightarrow I \xrightarrow{\mathrm{i}} A \xrightarrow{\mathrm{p}} B \longrightarrow 0.$$

Then we have:

Theorem 5.3.1. *The above short exact sequence gives the following six-term exact sequence in K-theory:*

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{p_*} & K_0(B) \\ {\scriptstyle \mathrm{ind}}\uparrow & & & & \downarrow{\scriptstyle \mathrm{exp}} \\ K_1(B) & \xleftarrow{p_*} & K_1(A) & \xleftarrow{i_*} & K_1(I) \end{array}$$

We admit this theorem. We just say a few words about the vertical arrows *exp* and *ind*. They arise from the fact that it is not always possible to lift a projection [resp. a unitary] of B to a projection [resp. a unitary] of A. Let us assume that A has a unit. Consider $e \in \mathcal{P}_1(B)$. There exists $a \in A$ self-adjoint such that $p(a) = e$. Then $u = \exp(2i\pi a)$ is a unitary such that $p(u) = 1$. Therefore it belongs to $\tilde{I}$. Its class in $K_1(I)$ depends only on $[e] \in K_0(B)$. One sets $\exp([e]) = [\exp(2i\pi a)]$. Consider now $u \in \mathcal{U}_1(B)$. Because $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ belongs to $\mathcal{U}_2^{(0)}(B)$, there exists $v \in \mathcal{U}_2^{(0)}(A)$ such that $p_2(v) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. One checks that

$$[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$$

is an element of $K_0(I)$ which depends only on $[u] \in K_1(B)$. This is $\mathrm{ind}([u])$.

Let us define the *suspension* SA and the *cone* CA of a C*-algebra A:

$$SA = \{f : [0,1] \in A \,\text{continuous} \mid f(0) = f(1) = 0\}$$

$$CA = \{f : [0,1] \in A \,\text{continuous} \mid f(0) = 0\}$$

They satisfy the short exact sequence

$$0 \longrightarrow SA \longrightarrow CA \longrightarrow A \longrightarrow 0.$$

Moreover, CA is contractile: the map $\gamma : [0,1] \to \mathrm{End}(CA)$ such that $(\gamma(t)f)(s) = f(st)$ satisfies $\gamma(0) = 0$ and $\gamma(1) = id_A$. Therefore, $K_i(CA) = 0$ and the six-term exact sequence provide isomorphisms $\mathrm{ind} : K_1(A) \to K_0(SA)$ and $\exp : K_0(A) \to K_1(SA)$. We deduce an isomorphism $K_0(A) \simeq K_0(S^2A)$. This is the statement of the *Bott periodicity theorem.* We have reversed the logical order since the Bott periodicity theorem is used in the proof of the six-term exact sequence.

5.4 K-theory and Bratteli diagrams

The dimension group of an AF algebra is one of the early occurrences of K-theory in C*-algebras. In this section, we will show how it can be expressed when the C*-algebra is $C^*(R)$, where R is the tail equivalence relation on the path space X of a Bratteli diagram. Recall that a topological space X is *totally disconnected* if it has a base of compact open subsets. Note that the path space of a Bratteli diagram is totally disconnected and that every totally disconnected space may be so obtained.

Exercise 5.4.1. Let A be a commutative C*-algebra.
(i) Show that A is AF if and only if its spectrum $X(A)$ is totally disconnected.
(ii) Assume that A is AF. Show that $K_0(A) = C_0(X(A), \mathbf{Z})$ and that $K_0(A)^+ = C_0(X(A), \mathbf{N})$.

Exercise 5.4.2. Let R be the tail equivalence relation on the path space X of a Bratteli diagram. Define $B(X, \mathbf{Z})$ as the subgroup of $C_0(X, \mathbf{Z})$ generated by the elements of the form $g\mathbf{1}_{r(S)} - (g \circ \alpha_S)\mathbf{1}_{s(S)}$ for some $g \in C(X, \mathbf{Z})$ and some compact open bisection $S \subset R$. Here $\alpha_S(x) = r(Sx)$ for $x \in s(S)$. Let $A = C^*(R)$. Show that $K_0(A) = C_0(X, \mathbf{Z})/B(X, \mathbf{Z})$ and that $K_0(A)^+$ is the image of $C_0(X, \mathbf{N})$.

The dimension group (i.e. the ordered abelian group $K_0(A)$) of the AF C*-algebra $A = C^*(R)$ arising from a Bratteli diagram can be computed from the diagram itself. Let us define the dimension group of the Bratteli diagram (V, E) as $K_0(V, E) = \varinjlim_n \mathbf{Z}^{|V(n)|}$ where $|V(n)|$ is the number of vertices at the level n and the homomorphism $\varphi_{n+1,n} : \mathbf{Z}^{|V(n)|} \to \mathbf{Z}^{|V(n+1)|}$ is given by the matrix $(a_{w,v})$

where $a_{w,v}$ is the number of edges from vertex $v \in V(n)$ to vertex $w \in V(n+1)$.

Exercise 5.4.3. Let (V, E) be a Bratteli diagram and let A be the C*-algebra of the tail equivalence relation on its path space. Then $K_0(A) = K_0(V, E)$.

One of the main results about AF algebras is that they are completely classified by their K_0 group. More precisely, recalling that the *dimension range* of a C*-algebra A is the image of $\mathcal{P}_1(A)$ in $K_0(A)$:

Theorem 5.4.4. *Let A, B be AF algebras. Then A and B are isomorphic if and only if there exists an isomorphism from $K_0(A)$ onto $K_0(B)$ sending the dimension range of A onto the dimension range of B.*

Let us use the notions introduced above to give a sketch of a proof. A direct proof would follow the same lines. For simplicity, we will only consider the case of C*-algebras with unit. The image of 1 in $K_0(A)$ is an *order unit*: for any $x \in K_0(A)$, there exists $n \in \mathbf{N}$ such that $x \leq n1$, where $x \leq y$ means $y - x \in K_0^+(A)$. The dimension range is the segment $[0, 1]$. Similarly, we will only consider Bratteli diagrams where each vertex of a level $n \geq 1$ receives and emits at least one edge. This ensures the compactness of the path space. The order unit of the group $\mathbf{Z}^{|V(n)|}$ is $1_n = (k_v)_{v \in V(n)}$, where k_v is the number of paths arriving at v. We have $\varphi_{n+1,n}(1_n) = 1_{n+1}$. We denote by 1 its image in $K_0(V, E)$.

Exercise 5.4.5. (*i*) Let $(V, E), (V', E')$ be Bratteli diagrams with respective path spaces X, X'. Show that there exists a homeomorphism from X onto X' preserving the tail equivalence relation if and only if there exists an isomorphism from $K_0(V, E)$ onto $K_0(V', E')$ sending $[0, 1]$ onto $[0, 1']$.
(*ii*) Let A, B be tail equivalence relation C*-algebras. Show that they are isomorphic if and only if there exists an isomorphism from $K_0(A)$ onto $K_0(B)$ preserving the order and the unit.

5.5 The irrational rotation algebra

We have defined in Section 2.2 the rotation algebra A_θ and we have seen that it is simple when $\theta/2\pi$ is irrational and that it has a unique

tracial state τ. Two related questions arise.
• Does A_θ contain non trivial projections? More specifically, what is the image by τ of the set of projections of A_θ ?
• Are these C*-algebras non-isomorphic? More precisely, for which pairs (θ, θ') do we have $A_\theta \simeq A_{\theta'}$?

Let us assume that $\alpha = \frac{\theta}{2\pi} \in [0,1] \setminus \mathbf{Q}$. M. Rieffel and R. Powers (see [37]) have constructed projections of trace α. Recall the link given in Proposition 1.4.7 between projections and C*-modules E such that the identity map is compact. There is a standard construction of such C*-modules over groupoid C*-algebras, which generalizes the construction of the convolution C*-algebra $C^*(G)$ of a groupoid G.

Definition 5.5.1. A *right action* of a groupoid G on a set Z consists of a surjective map $s : Z \to G^{(0)}$ and a map $(z, \gamma) \mapsto z\gamma$ from $Z * G$ to Z , where $Z * G$ is the subset of $Z \times G$ consisting of the pairs (z, γ) satisfying $s(z) = r(\gamma)$, such that
(i) $zs(z) = z$ for all $z \in Z$,
(ii) for all $(\gamma, \gamma') \in G^{(2)}$ and all $(z, \gamma') \in Z * G$, $(z\gamma)\gamma' = z(\gamma\gamma')$.
One then says that Z is a *right G-space.*

It is a topological right G-space when G is a topological groupoid, Z is a topological space and above maps are continuous.

Definition 5.5.2. Let G be a locally compact groupoid. A locally compact right G-space Z is *proper* if the map $(z, \gamma) \mapsto (z, z\gamma)$ from $Z * G$ to $Z \times Z$ is proper, in the sense that the inverse image of a compact set is a compact set. It is *free* if the above map is injective.

If the locally compact right G-space Z is proper, the quotient space Z/G is locally compact and Hausdorff. We shall denote $r : Z \to Z/G$ the quotient map. If G is étale, this map is a local homeomorphism.

Assume that G is a locally compact étale groupoid, that Z is a proper locally compact right G-space such that the map $s : Z \to G^{(0)}$ is a local homeomorphism. Then, one defines for $f \in C_c(G)$ and $h, k \in C_c(Z)$:

$$h * f(z) = \sum_{r(\gamma)=s(z)} h(z\gamma) f(\gamma^{-1}),$$

$$\langle h,k\rangle(\gamma)=\sum_{s(z)=s(\gamma)}\overline{h(z\gamma^{-1})}k(z).$$

One checks that $h*f$ belongs to $C_c(Z)$, that $\langle h,k\rangle$ belongs to $C_c(G)$ and that $C_c(Z)$ satisfies the axioms (i) and $(ii)(a,b)$ of the definition 5.5.3 of a right C*-module. We denote by $C^*(Z)$ the completion of $C_c(Z)$ with respect to the norm $\|h\|=\|\langle h,h\rangle\|^{1/2}$ where we have used the full norm on $C_c(G)$.

Theorem 5.5.3. *Let G be a locally compact étale groupoid and let Z be a proper locally compact right G-space. Assume that the map $s:Z\to G^{(0)}$ is a local homemorphism. Then $C^*(Z)$ is a right C*-module over $C^*(G)$.*

Proof. Let L be a representation of $C_c(G)$. We may assume that it is the integrated representation of a representation $(\mu,H=\{\mathcal{H}_x\},L)$ of G. Let us construct a measure μ_Z on Z/G and a measurable Hilbert bundle H_Z over Z/G according to the following construction, known as *induction*. We first consider the pull-back measure $s^*\mu$ on Z and then choose a pseudo-image μ_Z of $s^*\mu$ by the quotient map $r:Z\to Z/G$: explicitly, we choose a probability measure ν on Z equivalent to $s^*\mu$ and define $\mu_Z=r_*\nu$. The quasi-invariance of μ implies that the measures $r^*\mu_Z$ and $s^*\mu$ are equivalent. We denote by D_Z the Radon-Nikodým derivative $dr^*\mu_Z/ds^*\mu$. Similarly we consider the the pull-back s^*H of H along $s:Z\to G^{(0)}$ and take its quotient $H_Z=s^*H/G$ over Z/G. Instead of giving the details, we describe the Hilbert space of sections $\mathcal{H}_Z=L^2(Z/G,H_Z,\mu_Z)$. Its elements are measurable vector fields $\xi:z\mapsto\xi(z)\in\mathcal{H}_{s(z)}$ satisfying $\xi(z\gamma)=L(\gamma^{-1})\xi(z)$ for all $(z,\gamma)\in Z*G$ and $\int\|\xi(z)\|^2d\mu_Z([z])<\infty$ (we observe that $\|\xi(z)\|^2$ depends only on the class $[z]$ of z in Z/G). We also consider the Hilbert space $\mathcal{H}=L^2(G^{(0)},H,\mu)$. For $h\in C_c(Z)$, we define $L(h):\mathcal{H}\to\mathcal{H}_Z$ by its coefficients: given $\xi\in\mathcal{H}_Z$ and $\eta\in\mathcal{H}$ we set

$$\langle\xi,L(h)\eta\rangle=\int h(z)\langle\xi(z),\eta\circ s(z)\rangle_{s(z)}D_Z^{1/2}(z)ds^*\mu(z).$$

Just as in 2.3.3, one checks that this defines a bounded operator $L(h)$ and that for all $f\in C_c(G)$ and $h,k\in C_c(Z)$, we have the relations $L(h*f)=L(h)L(f)$ and $L(\langle h,k\rangle)=L(h)^*L(k)$. In particular

$L(\langle h, h\rangle)$ is a positive operator. One deduces that $\langle h, h\rangle$ is positive in $C^*(G)$. One also deduces that $\|h * f\| \leq \|h\|\|f\|$. Hence the operations extend to $C^*(Z)$ and $C^*(G)$ all the axioms of a C*-module are satisfied. □

In the above situation, one can form $(Z * Z)/G$, where $Z * Z = \{(z, z') \in Z \times Z \mid s(z) = s(z')\}$ and G acts diagonally on the right. If Z is a free G-space, it is a locally compact and étale groupoid with unit space Z/G. If Z is not free, $(Z * Z)/G$ is a new object which can be called a hypergroupoid. In all cases, one can define the convolution algebra $C_c((Z * Z)/G)$ and its envelopping C*-algebra $C^*((Z * Z)/G)$ as in the groupoid case (see [20]).

Corollary 5.5.4. *(cf. [25]) Let G be a locally compact étale groupoid and let Z be a free and proper locally compact right G-space. Then*
(i) the C-algebra $\mathcal{K}(C^*(Z))$ of compact operators of the C*-module $C^*(Z)$ is isomorphic to the C*-algebra $C^*((Z * Z)/G)$.*
(ii) $C^*(Z)$ is a $(C^*((Z * Z)/G), C^*(G))$-Morita equivalence.

Proof. We let $(Z * Z)/G$ act on Z on the left by $[z, z']z' = z$. This action is proper and the map $r : Z \to Z/G$ is étale. Just as before, we turn $C_c(Z)$ into a left C*-module over $C_c((Z * Z)/G)$: for $h \in C_c(Z)$ and $f \in C_c((Z * Z)/G)$, we define $f * h \in C_c(Z)$. For $h, k, l \in C_c(Z)$, we have $\theta_{h,k} l = (h * k^*) * l$ where $h * k^* \in C_c((Z * Z)/G)$ is defined by

$$h * k^*([z, z']) = \sum_{r(\gamma)=s(z)} h(z\gamma)\overline{k(z'\gamma)}.$$

One deduces that the operator $L(f)$ of left convolution by $f \in C_c((Z * Z)/G)$ is a compact operator of the C*-module $C^*(Z)$. The $*$-homomorphism L from $C_c((Z * Z)/G)$ to $\mathcal{K}(C^*(Z))$ extends to an isomorphism from $C^*((Z * Z)/G)$ onto $\mathcal{K}(C^*(Z))$. The freeness assumption ensures that $C^*(Z)$ is a full right C*-module over $C^*(G)$. It is also a full left C*-module over $C^*((Z * Z)/G)$. It is easy to check that the conditions $(iii)(a, b, c, d)$ of Proposition 1.4.10 are satisfied. □

Corollary 5.5.5. *Let G be a locally compact étale groupoid and let Z be a proper locally compact right G-space. Assume that the map $s : Z \to G^{(0)}$ is a local homeomorphism and that Z/G is compact. Then* *the C^*-module $C^*(Z)$ is isomorphic to $eC^*(G)^n$ for some integer n and some projection $e \in M_n(C^*(G))$.*

Proof. In this case $\mathcal{K}(C^*(Z)) \simeq C^*((Z * Z)/G)$ has a unit. We apply Proposition 1.4.7.

□

To construct explicitly a projection $e \in M_n(C^*(G))$ giving the module $C^*(Z)$, we find $h_1, \ldots, h_n \in C_c(Z)$ satisfying the equality $\sum_{i=1}^n h_i * h_i^* = \mathbf{1}_{\Delta_Z/G}$ in $C_c((Z * Z)/G)$, where Δ_Z is the diagonal in $Z * Z$. We proceed as follows. We choose relatively compact open subsets $V_1, \ldots, V_n$ of Z such that $s_{|V_i}$ is one-to-one and the images $r(V_i)$ in the quotient cover Z/G. We choose a partition of unity (b_i) subordinate to the open cover $(r(V_1), \ldots, r(V_n))$. For all i, there exists $h_i \in C_c(Z)$ whose support is contained in V_i and such that $\sum_{r(\gamma)=s(z)} |h_i(z\gamma)|^2 = b_i([z])$. The projection we are looking for is $e = (\langle h_i, h_j \rangle)$. In particular, if there exists a relatively compact open subset V of Z such that $s_{|V}$ is one-to-one and $r(V) = Z/G$, one obtains a projection $e = \langle h, h \rangle$ in $C^*(G)$.

Proposition 5.5.6. *Under the assumptions of the previous corollary, assume that μ is an invariant measure for G. Let τ be the trace on $C^*(G)$ defined by μ. Then $\tau_*([e]) = \nu(Z/G)$, where ν is the unique measure on Z/G such that $r^*\nu = s^*\mu$.*

Proof. The existence of the measure ν results from our assumptions and its uniqueness is clear. We have

$$\begin{aligned}
\tau_*([e]) &= \sum_{i=1}^n \tau(\langle h_i, h_i \rangle) = \sum_{i=1}^n \int |h_i|^2 d(s^*\mu) \\
&= \sum_{i=1}^n \int |h_i|^2 d(r^*\nu) = \int \sum_{i=1}^n |h_i|^2 d(r^*\nu) \\
&= \int \sum_{i=1}^n \sum_{r(\gamma)=s(z)} |h_i(z\gamma)|^2 d\nu([z]) = \nu(Z/G)
\end{aligned}$$

□

Example 5.5.7. *Modules over the irrational rotation algebra.* Let $\theta = 2\pi\alpha$, where $\alpha \in [0,1] \setminus \mathbf{Q}$ and let T be the rotation of angle θ on the circle $\mathbf{T}$. Let $G = G(T, \mathbf{T})$ be the groupoid of this action. We turn $Z = \mathbf{R}$ into a right G-space by introducing the map $s : \mathbf{R} \to \mathbf{T}$ defined by $s(t) = \exp(i2\pi t)$ and by lifting T to $\tilde{T}$ such that $\tilde{T}(t) = t+\alpha$. The quotient map $r : Z \to Z/G$ is simply the quotient map $\mathbf{R} \to \mathbf{R}/\alpha\mathbf{Z}$. This G-space Z satisfies the conditions of the Corollary 5.5.5 with $n = 1$: each open interval $V \subset \mathbf{R}$ of length strictly between α and 1 satisfies $s_{|V}$ one-to-one and $r(V) = Z/G$. Thus, we get a projection $e \in C^*(G)$. The unique tracial state τ is given by the normalized Lebesgue measure on the circle. The lifted measure on $Z = \mathbf{R}$ is the Lebesgue measure and $\tau(e)$ is the length α of $\mathbf{R}/\alpha\mathbf{Z}$. The same construction gives the construction of projections in $C^*(G)$ of dimension $p + q\alpha$ for arbitrary $p, q \in \mathbf{Z}$ such that $p + q\alpha \in [0,1]$.

This shows that the range of $\tau_* : K_0(A_\theta) \to \mathbf{R}$ contains $\mathbf{Z} + \alpha\mathbf{Z}$ but does not completely answer our initial questions. A complete answer is provided by the *Pimsner-Voiculescu exact sequence* for crossed products by $\mathbf{Z}$ [31]. The crossed product construction $B \rtimes \Gamma$ that we have described for a commutative C*-algebra $B = C(X)$ works in general. The data consist of a C*-algebra B, a discrete group Γ and a homomorphism $\sigma : \Gamma \to Aut(B)$. One builds the $*$-algebra whose elements are finite sums $\sum b_\gamma u_\gamma$ where $b_\gamma \in B$ and u_γ is a symbol and where the $*$-algebra structure is defined by $u_\gamma u_{\gamma'} = u_{\gamma\gamma'}, u_\gamma^* = u_{\gamma^{-1}}$ for all $\gamma, \gamma' \in \Gamma$ and $u_\gamma^{-1} b u_\gamma = \sigma_\gamma(b)$ for all $\gamma \in \Gamma, b \in B$. The crossed product C*-algebra $B \rtimes \Gamma$ is its completion for the full norm, defined as usual. We denote by $i : B \to B \rtimes \Gamma$ the inclusion map. When $\Gamma = \mathbf{Z}$, we write σ instead of σ_1.

Theorem 5.5.8. *Let σ be an automorphism of a C*-algebra B. Then there is a six-term exact sequence:*

$$\begin{tikzcd}
K_0(B) \arrow[r, "id-\sigma_*"] & K_0(B) \arrow[r, "i_*"] & K_0(B \rtimes_\sigma \mathbf{Z}) \arrow[d, "\partial"] \\
K_1(B \rtimes_\sigma \mathbf{Z}) \arrow[u, "\partial"] & K_1(B) \arrow[l, "i_*"'] & K_1(B) \arrow[l, "id-\sigma_*"']
\end{tikzcd}$$

Proof. We only sketch the idea of the proof. Recall that the Toeplitz algebra is defined as $\mathcal{T} = C^*(S)$ where S is the unilateral shift (or any non unitary isometry). Let us define $\mathcal{T}B$ as the sub-C*-algebra of $(B \rtimes_\sigma \mathbf{Z}) \otimes \mathcal{T}$ generated by $i(B) \otimes 1$ and $u \otimes S$, where $u = u_1 \in B \rtimes_\sigma \mathbf{Z}$. We have seen that $\mathcal{T}$ satisfies the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{p} C^*(\mathbf{Z}) \to 0.$$

Similarly, $\mathcal{T}B$ satisfies the short exact sequence

$$0 \to B \otimes \mathcal{K} \to \mathcal{T}B \xrightarrow{p} C^*(B \rtimes_\sigma \mathbf{Z}) \to 0.$$

One gets the Pimsner-Voiculescu exact sequence from the 6 term exact sequence associated with this Toplitz extension and the identifications $K_i(B \otimes \mathcal{K}) = K_i(B)$, which results from the definition and $K_i(\mathcal{T}B) = K_i(B)$. The left arrow $\partial : K_1(B \rtimes_\sigma \mathbf{Z}) \to K_0(B)$ is the index map ind. Its explicit expression is given by an index theorem. The most rudimentary index theorem corresponds to the case $B = \mathbf{C}$. It says that

$$\partial[\varphi] = \mathrm{ind}(T_\varphi)$$

where $\varphi : \mathbf{S}^1 \to \mathbf{C}^*$ is a loop which does not contain the origin, T_φ is the Toeplitz operator with symbol φ and ind is its Fredholm index. The right handside is the analytical index. On the other hand, $[\varphi] \in K^1(\mathbf{S}^1)$ is given by the winding number of the loop around the origin. $\square$

Example 5.5.9. *The irrational rotation algebra.* Let us apply the Pimsner-Voiculescu exact sequence to $A_\theta = C(\mathbf{T}) \rtimes_T \mathbf{Z}$. We write $\sigma(f) = f \circ T^{-1}$. We have seen the isomorphism $K_0(C(\mathbf{T})) \simeq \mathbf{Z}$ given by the dimension and the isomorphism $K_1(C(\mathbf{T})) \simeq \mathbf{Z}$ given by the index. Since the dimension and the index are invariant under T, $\sigma_* = id$. The Pimsner-Voiculescu exact sequence gives two short exact sequences:

$$0 \to \mathbf{Z} \to K_i(A_\theta) \to \mathbf{Z} \to 0.$$

One deduces that $K_i(A_\theta) = \mathbf{Z} \oplus \mathbf{Z}$. In particular, $K_0(A_\theta)$ has two generators: $[1]$ and $[e]$ with $\tau(e) = \alpha$ which we have constructed above. We deduce that $\tau_* : K_0(A_\theta) \to \mathbf{R}$ is an isomorphism onto $\mathbf{Z} + \alpha\mathbf{Z}$. Thus a necessary condition for $A_\theta \simeq A_{\theta'}$ is that $\mathbf{Z} + \alpha\mathbf{Z} \simeq \mathbf{Z} + \alpha'\mathbf{Z}$ as ordered groups. One can check that this condition is also sufficient.

5.6 Cuntz algebras

Now, we shall briefly give some results on the K-theory of the Cuntz algebras and some of their generalizations. Recall that O_d is the C*-algebra generated by d isometries $S_1, \ldots, S_d$ subject to the relation $\sum_{k=1}^{d} S_k S_k^* = 1$. Let $E_k = S_k S_k^*$ be the range projection of S_k. These projections are all equivalent to 1. This gives in $K_0(O_d)$ the relation

$$d[1] = \sum_{k=1}^{d} [E_k] = [1].$$

Therefore, we have a group homomorphism $\mathbf{Z}/(d-1)\mathbf{Z} \to K_0(O_d)$, sending k to $k[1]$. It is in fact an isomorphism.

Proposition 5.6.1. *$K_0(O_d) = \mathbf{Z}/(d-1)\mathbf{Z}$ and $K_1(O_d) = 0$.*

Proof. This is only an outline. It is based on various versions of the Pimsner-Voiculescu exact sequence for endomorphisms. Recall that we have defined the gauge automorphism group $\sigma \in Aut(O_d)$ such that $\sigma_t(S_k) = e^{it} S_k$. The fixed point algebra $F_d = O_d^\sigma$ is an AF algebra which can be written as $\overline{M_{d^n}(\mathbf{C})}$. One deduces that

$$\begin{aligned} &K_0(F_d) = \mathbf{Z}[1/d] = \{1/d^n \mid k \in \mathbf{Z}, n \in \mathbf{N}\} \subset \mathbf{Q} \\ &K_1(F_d) = 0. \end{aligned}$$

The C*-algebra O_d can be written as a crossed product of F_d by an endomorphism α. It is also possible to replace O_d by $O_d \otimes \mathcal{K}$, where as usual, $\mathcal{K}$ is the C*-algebra of compact operators on an infinite dimensional Hilbert space and to write $O_d \otimes \mathcal{K}$ as a crossed product of $F_d \otimes \mathcal{K}$ by an automorphism. Whatever the approach, the homomorphism $\alpha_* : \mathbf{Z}[1/d] \to \mathbf{Z}[1/d]$ induced by α is multiplication by d and the Pimsner-Voiculescu sequence gives the result: we have $K_0(O_d) = \mathbf{Z}[1/d]/(1-d)\mathbf{Z}[1/d] = \mathbf{Z}/(d-1)\mathbf{Z}$ and $K_1(O_d) = 0$. □

It is possible to give a dynamical system interpretation of these invariants. We shall limit our study to the case of one-sided subshifts of finite type. Let $d \geq 1$ be an integer and $A = (a_{i,j}) \in M_d(\mathbf{N})$. We associate to A a graph (V, E) with vertices labeled by $1, 2, \ldots d$ and having $a_{i,j}$ edges starting at vertex i and ending at vertex j. A path is a sequence $x_1x_2\dots$ of edges with matching endpoints: x_2 starts where x_1 ends. We consider the infinite path space

$$X_A = \{x_0x_1x_2\dots\}.$$

and the one-sided shift As before $T_A(x_0x_1\dots) = (x_1x_2\dots)$. It is a local homeomorphism. We define $G_A = G(X_A, T_A)$ and the gauge cocycle $c : G_A \to \mathbf{Z}$ given by $c(x,k,y) = k$ as in Section 3.4.1. We let $R_A = c^{-1}(0)$. The associated *Cuntz-Krieger* algebra is $O_A = C^*(G_A)$ and its gauge invariant subalgebra is $F_A = O_A^\sigma = C^*(R_A)$. A computation as above gives:

Proposition 5.6.2. *Let* $A = (a_{i,j}) \in M_d(\mathbf{N})$. *Then*

(i) F_A *is an AF algebra with* $K_0(A) = D_A \overset{\text{def}}{=} \varinjlim(\mathbf{Z}^d \overset{{}^tA}{\to} \mathbf{Z}^d \overset{{}^tA}{\to} \dots)$.

(ii) $K_0(O_A) = \mathbf{Z}^d/(1-{}^tA)\mathbf{Z}^d$ *and* $K_1(O_A) = \ker(1-{}^tA)$ *(viewed as an endomorphism of* $\mathbf{Z}^d$*).*

Remark 5.6.3. In topological dynamics, the group $\mathbf{Z}^d/(1-{}^tA)\mathbf{Z}^d$ is known as the *Bowen-Franks group* of A and the group D_A is called the *dimension group* of A (see [24]).

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