Bounded groupoid cocycles

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2 Statement of the theorem
3 Existence of weakly continuous equivariant sections
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Let us first recall the Gottschalk-Hedlund theorem.

**Theorem**

Let $T$ be a minimal continuous map on a compact space $X$. For a continuous function $f : X \to \mathbb{R}$, the following conditions are equivalent:

1. there exists a continuous function $g : X \to \mathbb{R}$ such that for all $x \in X$, $f(x) = g(x) - g(Tx)$;
2. there exists $x_0 \in X$ such that the sums $\sum_{k=0}^{n-1} f(T^k x_0)$ are bounded;
3. for all $x \in X$, the sums $\sum_{k=0}^{n-1} f(T^k x)$ are bounded.

Give a sketch of the proof!
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There exist easy generalizations of this theorem. One appeared in my thesis and uses the language of groupoids and cocycles.

In the previous theorem, the function \( f \) defines a cocycle on the groupoid

\[
G(X, T) = \{(x, m - n, y) : x, y \in X, m, n \in \mathbb{N} \quad T^n x = T^m y\}
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according to

\[
c(x, m - n, y) = \sum_{k=0}^{n-1} f(T^k x) - \sum_{k=0}^{m-1} f(T^k x)
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and it is of the form \( f = g - g \circ T \) if and only if \( c \) is a coboundary, i.e. of the form \( c = g \circ r - g \circ s \), where \( r(x, k, y) = x \) and \( s(x, k, y) = y \).
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A groupoid version of the G-H theorem

Theorem (R, 1980)

Let $G$ be a topological groupoid on a compact space $X$ and let $A$ be a topological abelian group endowed with trivial $G$-action. Assume that $G$ is minimal and that $A$ has no compact subgroups. For a continuous cocycle $c : G \to A$, the following conditions are equivalent:

1. $c$ is a continuous coboundary;
2. there exists $x \in X$ such that $c(G_x)$ is relatively compact;
3. $c(G)$ is relatively compact.
The space of coefficients

The above setting is unsatisfactory: the natural data for continuous groupoid cohomology consist of:

- a topological groupoid $G$ over a topological space $X$,
- a space of coefficients (or $G$-module) $A$, which is a continuous bundle of topological abelian groups $A_x$ over $X$ endowed with a continuous $G$-action, i.e. $G$ acts by isomorphisms $L(\gamma) : A_{s(\gamma)} \to A_{r(\gamma)}$ and the action map $G \times A \to A$ is continuous.

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Continuous groupoid cohomology

Definition

Let $G$ be a topological groupoid and let $A$ be a continuous $G$-module. We define $H^1(G, A)$ as the group of isomorphism classes of $G$-equivariant $A$-principal bundles over $G^{(0)}$.

- A continuous cocycle is a continuous map $c : G \to A$ such that $c(\gamma) \in A_{r(\gamma)}$ and $c(\gamma \gamma') = c(\gamma) + L(\gamma)c(\gamma')$. It defines the $A$-principal bundle $A(c) = A$, where $G$ acts on the left by $\gamma z = L(\gamma)z + c(\gamma)$.
- $c$ is a continuous coboundary if and only if $A(c)$ is trivial (equivalently, there exists an equivariant continuous section).
- Replacing $G$ by an equivalent groupoid, each $G$-equivariant $A$-principal bundle can be realized by a continuous cocycle.
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Continuous $G$-Hilbert modules

We are interested in the case when the space of coefficients is a $G$-Hilbert module $A = H$.

We denote by

- $H$ the total bundle space;
- $\pi : H \to X$ the projection;
- $\mathcal{H} = C(X, H)$ the space of its continuous sections. Recall that $\mathcal{H}$ is a $C^*$-module over $C(X)$.

**Definition**

A $G$-Hilbert module is a continuous field of Hilbert spaces $(H_x)_{x \in X}$ on which $G$ acts by isometries $L(\gamma) : H_s(\gamma)) \to H_r(\gamma)$ and such that the action map $G \ast H \to H$ is continuous.
Let $G$ be a minimal topological groupoid on a compact space $X$. Let $c : G \to H$ be a continuous cocycle, where $H$ is a $G$-Hilbert module and let $H(c)$ be the associated affine bundle. Then the following conditions are equivalent:

1. $c$ is a continuous coboundary;
2. $H(c)$ admits an equivariant continuous section;
3. $H(c)$ admits a bounded orbit;
4. $\|c\|$ is bounded.
Statement of the theorem

Special cases

- When $G$ is a group, this is a well-known result which goes back to B. Johnson 1967. In fact the result is true for a much larger class of Banach spaces than Hilbert spaces (we still assume that the action is isometric). U. Bader, T. Gelander and N. Monod have recently shown in [A fixed point theorem for $L^1$ spaces, Math arXiv:1012.1488v1, (2010)] that it is true for Banach spaces which are $L$-embedded.

- When $G$ is the groupoid associated with a group action and when $H$ is a constant bundle over $X$, this is the recent result by D. Coronel, A. Navas, M. Ponce [Bounded orbits versus invariant sections for cocycles of affine isometries over a minimal dynamics, Math arXiv:1101.3523v2, (2011)] quoted earlier.
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We follow the proof that Coronel, Navas and Ponce give for the infinite-dimensional case. It decomposes into two parts. A compactness argument gives the existence of a weakly continuous equivariant section. Then, using minimality and a finite dimensional approximation property of C*-modules, one shows that a weakly continuous equivariant section is necessarily continuous.
In what follows, \( \pi : H \to X \) is a continuous Hilbert bundle over a compact space \( X \). We denote by \( \mathcal{H} = C(X, H) \) the \( C(X) \)-module of continuous sections.

We define the weak topology on \( H \) as follows: we embed \( H \) into \( X \times \mathcal{H}^* \) via the natural evaluation map. The weak topology is the subspace topology when \( \mathcal{H}^* \) is endowed with the \( * \)-weak topology. We write \( H_w \) to specify the weak topology.

One can observe that the space of weakly continuous sections \( C(X, H_w) \) agrees with the bounded \( C(X) \)-linear maps from \( C(X, H) \) to \( C(X) \), where the section \( x \mapsto \xi(x) \) defines the \( C(X) \)-linear map \( \eta \mapsto < \xi, \eta > \), where \( < \xi, \eta > (x) = < \xi(x), \eta(x) >_x \).
We assume now that $H$ is a continuous $G$-Hilbert bundle and that $c : G \to H$ is a continuous cocycle. We endow $H$ with the corresponding affine action. Our assumption is that there is a bounded orbit under this action. One deduces the existence of a non-empty weakly compact invariant convex subset. By Zorn, we have the existence of a minimal weakly compact invariant convex subset $M$. 
The set $M$ is the graph of a section

By minimality of $G$, $\pi(M) = X$. It remains to show that $M_x = M \cap \pi^{-1}(x)$ has exactly one element for all $x \in X$. The proof is a classical trick which uses the uniform convexity of the Hilbert spaces $H_x$ (with a constant uniform convexity constant). Thus, $M$ is the graph of a section. Since $M$ is weakly compact, this section is weakly continuous. Since $M$ is invariant under $G$, this section is equivariant.
Existence of weakly continuous equivariant sections

Details of the proof

Let $R = \sup_{\xi \in M} \|\xi\|_{\pi(\xi)}$ and $\epsilon > 0$. Choose $\zeta \in M$ such that 
$\|\zeta\| > (1 - \delta^2)R$ where $\delta = \delta(\epsilon)$ is the uniform convexity module. Choose $\eta \in H_z$ (where $z = \pi(\zeta)$) such that $\|\eta\| = 1$ and $|\langle \zeta, \eta \rangle| > (1 - \delta^2)R$. Choose $f \in C(X, H)$ such that $f(z) = \eta$. Let 
$V = \{y \in X : \|f(y)\| < 1 + \delta\}$.

Let $x \in X$ and $\xi_1, \xi_2 \in M_x$. We are going to show that $\xi_1 = \xi_2$. Let $m$ their midpoint. Then $m$ belongs to $M$. Let us show that its orbit meets the open set

$$U = \{\xi \in H : \pi(\xi) \in V, \quad |\langle \xi, f \circ \pi(\xi) \rangle| > (1 - \delta^2)R\}.$$ 

If not it would be contained in the closed convex set $M \setminus U$ and this would contradict the minimality of $M$. Let $\gamma \in G$ be such that $\gamma m \in U$. Then $\|\gamma m\| > (1 - \delta)R$. The uniform convexity inequality implies $\|\gamma \xi_1 - \gamma \xi_2\| < \epsilon R$, hence $\|\xi_1 - \xi_2\| < \epsilon R$. Hence $\xi_1 = \xi_2$. 

Existence of a continuous equivariant section

The relative norm function of a section

We are going to show that a weakly continuous equivariant section is necessarily continuous. This will be done by showing that its oscillation is zero. However, how can we define the oscillation of a section $f$, since its values $f(x)$ and $f(y)$ live in different spaces?

**Definition**

Let $f : X \rightarrow E$ be a section of a Banach bundle $E \rightarrow X$. Its relative norm function $N_f : E \rightarrow \mathbb{R}_+$ is the scalar function defined by $N_f(e) = \|e - f \circ \pi(e)\|$.

**Proposition**

Let $f$ be a section of $E$. Then the following conditions are equivalent

1. $f : X \rightarrow E$ is continuous [resp. continuous at $x$] and
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Continuity of equivariant sections

Suppose now that $E$ is a $G$-Banach bundle.

**Proposition**

Let $f$ be a section of $E$. Then the following conditions are equivalent

- $f : X \to E$ is equivariant;
- $N_f : E \to \mathbb{R}_+$ is invariant.

Thus it suffices to study the continuity of the invariant scalar function $N_f$. Its oscillation is also invariant; moreover it is upper semi-continuous. One deduces that the set of points of continuity of an equivariant section of a $G$-Banach bundle is an intersection of open invariant subsets.
Since we assume that $G$ is minimal, an intersection of open invariant subsets of $X$ is either the empty set or $X$ itself. Thus in order to show that an equivariant section is continuous, it suffices to show that it has at least one point of continuity. We show now that a weakly continuous section has at least one point of continuity.

**Proposition**

Let $f : X \to H$ be a weakly continuous section of a separable Hilbert bundle $H$. Then the set of its points of continuity is a dense $G_δ$.

**Proof.** This results from a well-known approximation property (e.g. [D. Blecher, A new approach to C*-modules, 1995]) of C*-module $\mathcal{H}$ over a C*-algebra $A$. 
Existence of a continuous equivariant section

Weak continuity $\Rightarrow$ norm continuity

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He proves that a Banach module $\mathcal{E}$ over a C*-algebra $A$ is a C*-module iff there exists a directed set $I$, a net of integers $(n_i)$ and nets of contractive $A$-linear maps $\varphi_i : \mathcal{E} \to A^{n_i}$ and $\psi_i : A^{n_i} \to \mathcal{E}$ such that for all $e \in \mathcal{E}$, $\psi_i \circ \varphi_i(e)$ tends to $e$.

We only use the easy part $\Rightarrow$ which is the fact that the C*-algebra of compact operators $\mathcal{K}(\mathcal{H})$ has an approximate unit of the form $e_i = \sum <\xi_k, \xi_k>$. If $\mathcal{H}$ is countably generated, one can choose $I = \mathbb{N}$.
Let $f$ be a weakly continuous section of $H$. It is the pointwise limit of the sequence of the continuous sections $f_i = \psi_i \circ \varphi_i \circ f$. Indeed $\varphi_i \circ f$ is continuous since the weak and the norm topology agree on the finite-dimensional vector bundle $X \times \mathbb{C}^{n_i}$ and so is $f_i$. This proves the proposition.

One deduces from the above discussion that the weakly continuous equivariant section we found in the first part is continuous.

**Remark.** The proof relies of a bounded approximation property which is specific to C*-modules, hence to Hilbert bundles, but which is stronger than what we really need.
Existence of a continuous equivariant section

End of the proof

Let $f$ be a weakly continuous section of $H$. It is the pointwise limit of the sequence of the continuous sections $f_i = \psi_i \circ \varphi_i \circ f$. Indeed $\varphi_i \circ f$ is continuous since the weak and the norm topology agree on the finite-dimensional vector bundle $X \times \mathbb{C}^{n_i}$ and so is $f_i$. This proves the proposition.

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Some questions

1) The result should be valid for more general Banach bundles than the Hilbert bundles considered here.

2) It seems reasonable to define a property (FH) (existence of a continuous equivariant section for isometric affine actions on Hilbert bundles) for topological groupoids extending the classical notion for groups. Such a property has been defined and studied by C. Anantharam-Delaroche for ergodic measured groupoids [Cohomology of property T groupoids and applications, Ergod. Th. & Dynam. Sys, 25 (2005), 465–471].
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