# Cartan sub-C*-algebras in C*-algebras

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1. C*-algebra constructions.
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# Groupoids

## Definition

A groupoid is a small category $(G, G^{(0)})$ such that every arrow is invertible.

$$r, s : G \to G^{(0)}$$

$$\begin{array}{ccc} G^{(2)} & \to & G \\ (\gamma, \gamma') & \mapsto & \gamma\gamma' \end{array} \qquad \begin{array}{ccc} G & \to & G \\ \gamma & \mapsto & \gamma^{-1} \end{array}$$

**Example:** action of a group $\Gamma$ on a space $X$: $\begin{array}{ccc} X \times \Gamma & \to & X \\ (x, g) & \mapsto & xg \end{array}$

$$G = \{(x, g, y) \in X \times \Gamma \times X : y = xg\}$$

$$r(x, g, y) = x \qquad s(x, g, y) = y$$

$$(x, g, y)(y, h, z) = (x, gh, z) \qquad (x, g, y)^{-1} = (y, g^{-1}, x)$$

# Haar systems

## Definition

Let $G$ be locally compact Hausdorff topological groupoid. A Haar system $\lambda = (\lambda^x)$ is a family of measures $\lambda^x$ with support $G^x = r^{-1}(x)$ satisfying

- (continuity) $\forall f \in C_c(G), x \mapsto \int f d\lambda^x$ is continuous;
- (left invariance) $\forall \gamma \in G$, $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$.

In the previous example of a group action $(\Gamma, X)$, a left Haar measure $\lambda$ on $\Gamma$ defines a Haar system $(\lambda^x)$ on $G$ such that $\int f d\lambda^x = \int f(x, g, xg) d\lambda(g)$.

## Definition

We say that the topological groupoid $G$ is étale if the range map $r : G \to G^{(0)}$ is a local homeomorphism.

An étale locally compact Hausdorff groupoid has a natural Haar system, given by $\int f d\lambda^x = \sum_{r(\gamma)=x} f(\gamma)$.

# The C*-algebra $C^*_r(G)$

Let $(G, \lambda)$ be a locally compact Hausdorff groupoid endowed with a Haar system. The following operations turn the space $C_c(G)$ of compactly supported complex-valued continuous functions on $G$ into an involutive algebra:

$$f * g(\gamma) = \int f(\gamma\gamma')g(\gamma'^{-1})d\lambda^{s(\gamma)}(\gamma');$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

For each $x \in G^{(0)}$, one defines the representation $\pi_x$ of $C_c(G)$ on the Hilbert space $L^2(G_x, \lambda_x)$, where $G_x = s^{-1}(x)$ and $\lambda_x = (\lambda^x)^{-1}$, by $\pi_x(f)\xi = f * \xi$. One defines the reduced norm $\|f\|_r = \sup \|\pi_x(f)\|$. The reduced C*-algebra $C^*_r(G)$ is the completion of $C_c(G)$ for the reduced norm.

# The C*-algebra $C_r^*(G, E)$

We shall need a slight generalization of the above construction.

**Definition**

Let $G$ be a groupoid. A twist over $G$ is a groupoid extension

$$\mathbb{T} \times X \rightarrowtail E \twoheadrightarrow G$$

where $X = G^{(0)} = E^{(0)}$ and $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

For example, a 2-Čech cocycle $\sigma = (\sigma_{ijk})$ relative to an open cover $(U_i)$ of a topological space $X$ defines a twist $E_\sigma$ over the groupoid $G = \{(i, x, j) : x \in U_i \cap U_j\}$ of the open cover.

We replace the complex-valued functions by the sections of the associated complex line bundle. Essentially the same formulas as above provide the C*-algebra $C_r^*(G, E)$.

# Reconstruction

When passing from the groupoid $G$ to the C*-algebra $A = C^*_r(G)$, in general much information is lost. However, some extra piece of structure allows to recover $G$ from $A$. In the case of a group, the coproduct does the job. When $G$ is étale, the commutative C*-algebra $B = C_0(X)$, where $X = G^{(0)}$ is a subalgebra of $A = C^*_r(G, E)$. Thus our construction provides a pair $(A, B)$ where $A$ is a C*-algebra and $B$ is a commutative sub-C*-algebra rather than just a C*-algebra. We shall study the case when the pair $(A, B)$ completely determines the twisted groupoid $(G, E)$.

**Proposition (R 80)**

*Let $G$ be an étale second countable locally compact Hausdorff groupoid. Then $B = C_0(X)$ is maximal abelian self-adjoint in $A = C^*_r(G, E)$ iff $G$ is topologically principal.*

Let $X$ be a topological space. A partial homeomorphism of $X$ is a homeomorphism $S : \mathrm{D}(S) \to \mathrm{R}(S)$, where $\mathrm{D}(S)$ and $\mathrm{R}(S)$ are open subsets of $X$. One defines the composition $ST$ and the inverse $S^{-1}$. A **pseudogroup** on $X$ is a family $\mathcal{G}$ of partial homeomorphisms of $X$ closed under composition and inverse.

Given a partial homeomorphism $S$ and $y \in \mathrm{D}(S)$, we denote by $[Sy, S, y]$ the germ of $S$ at $y$.

One can associate to a pseudogroup $\mathcal{G}$ on $X$ its **groupoid of germs** $G$. Its elements are the germs of $\mathcal{G}$. Its groupoid structure is:

$$r([x, S, y]) = x \quad s([x, S, y]) = y$$

$$[x, S, y][y, T, z] = [x, ST, z]$$

$$[x, S, y]^{-1} = [y, S^{-1}, x]$$

Its topology is the topology of germs. It turns $G$ into a topological groupoid, which is locally compact if $X$ is so, but not necessarily Hausdorff.

Conversely, let $G$ be an étale groupoid with $G^{(0)} = X$. Its open bisections define a pseudogroup $\mathcal{G}$ on $X$, hence a groupoid of germs $H$. We have the groupoid extension:

$$\mathrm{Int}(G') \rightarrowtail G \twoheadrightarrow H$$

where $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$ and $\mathrm{Int}(G')$ is its interior.

## Definition

An étale groupoid $G$ is said to be

- effective if it is isomorphic to its groupoid of germs;
- topologically principal if the set of units without isotropy is dense in $G^{(0)}$.

**Proposition**

*Let $G$ be an étale groupoid.*

- *if $G$ is Hausdorff and topologically principal, then it is effective;*
- *if $G$ is second countable, if its unit space is a Baire space and if it is effective, then it is topologically principal.*

## Examples

- Tranverse holonomy groupoids of foliated manifolds.
- The groupoid of a topologically free semi-group action

$$T : X \times N \to X :$$

$$G(X,T) = \{(x, m-n, y) : T(m)x = T(n)y\}.$$

- Minimal Cantor systems.
- Markov chains satisfying Cuntz-Krieger condition $(I)$.
- One sided-shifts on infinite path spaces on graphs satisfying exit condition $(L)$.

# Cartan subalgebras

**Definition (Kumjian 86)**

Let $B$ be a sub-C*-algebra of a C*-algebra $A$. One says that $B$ is regular if its normalizer $N(B) = \{a \in A : aBa^* \subset B \quad a^*Ba \subset B\}$ generates $A$ as a C*-algebra.

**Definition (cf. Vershik, Feldman-Moore 77)**

Let $B$ be an abelian sub-C*-algebra of a C*-algebra $A$ containing an approximate unit of $A$. One says that $B$ is a Cartan subalgebra if

- $B$ is maximal abelian self-adjoint (i.e. $B' = B$);
- $B$ is regular;
- there exists a faithful conditional expectation of $A$ onto $B$.

# Main theorem

## Theorem (R 08)

- *Let* $(G, E)$ *be a twist with* $G$ *étale, second countable locally compact Hausdorff and topologically principal. Then* $C_0(G^{(0)})$ *is a Cartan subalgebra of* $C^*_r(G, E)$*.*
- *Let* $B$ *be a Cartan sub-algebra of a separable C*-algebra* $A$*. Then, there exists a twist* $(G, E)$ *with* $G$ *étale, second countable locally compact Hausdorff and topologically principal and an isomorphism of* $C^*_r(G, E)$ *onto* $A$ *carrying* $C_0(G^{(0)})$ *onto* $B$*.*

This theorem is a C*-algebraic version of a well-known theorem of Feldman-Moore (77) about von Neumann algebras. The main difference is that the measured equivalence relation of the von Neumann case has to be replaced by a topologically principal groupoid.

It is an improvement of a theorem of Kumjian (86) who deals with the principal case (i.e. étale equivalence relation) and introduces the stronger notion of a diagonal.

**Definition**

One says that a sub-C*-algebra $B$ of a C*-algebra $A$ has the unique extension property if pure states of $B$ extend uniquely to pure states of $A$. A Cartan subalgebra which has the unique extension property is called a diagonal.

Then one has

**Proposition (Kumjian 86, R 08)**

*Let $B$ be a Cartan sub-algebra of a separable C*-algebra $A$. Let $(G, E)$ be the associated twist. Then,*

$$G \text{ is principal} \quad \Leftrightarrow \quad B \text{ has the unique extension property.}$$

**Corollary**

*Let $B$ be a Cartan sub-algebra of a separable C*-algebra $A$. Then, the conditional expectation of $A$ onto $B$ is unique.*

This is well-known when $B$ has the unique extension property. There should be a direct proof of this result in the general case.

# Existence and uniqueness of Cartan subalgebras

There are deep theorems about the existence and the uniqueness of Cartan subalgebras in the von Neumann algebras case. For example

- the hyperfinite factors have a Cartan subalgebra which is unique up to conjugacy (Krieger+Connes-Feldman-Weiss 81);
- the free group factors $L(\mathbf{F}_n)$ do not have Cartan subalgebras for $n \geq 2$ (Voiculescu 96);
- there are $II_1$ factors which have uncountably many non-conjugate Cartan subalgebras (Popa 90).
- Ozawa and Popa have recently (07/08) a class of $II_1$ factors which have a Cartan subalgebra unique up to inner conjugacay.

Much less is known about Cartan subalgebras in C*-algebras. I will give a few examples and counter-examples.

# Two non-conjugate Cartan subalgebras

Here is an easy example which produces two non-conjugate diagonal subalgebras.

Let $G, H$ be locally compact abelian groups and a continuous homomorphism $\varphi : G \to H$. Then $G$ acts continuously on $H$ and we can form the crossed product C*-algebra $G \ltimes C_0(H)$. By dualizing, we get $\hat{\varphi} : \hat{H} \to \hat{G}$ and the crossed product C*-algebra $\hat{H} \ltimes C_0(\hat{G})$. The Fourier transform gives an isomorphism of these C*-algebras.

If $G$ is discrete and $\varphi$ is one-to-one, $C_0(H)$ is a diagonal subalgebra. Similarly, if $\hat{H}$ is discrete and $\hat{\varphi}$ is one-to-one, $C_0(\hat{G})$ is another diagonal subalgebra. Both conditions happen simultaneously if $G$ is discrete, $H$ is compact, $\varphi$ is one-to-one and has dense range. There are such examples where $C_0(H)$ and $C_0(\hat{G})$ are not isomorphic.

**Example**

$G = \mathbb{Z}^2, H = \mathbb{R}/\mathbb{Z}, \varphi(m, n) = \alpha m + \beta n + \mathbb{Z}$ where $(1, \alpha, \beta)$ are linearly independent over $\mathbb{Q}$.

# AF C*-algebras

- AF C*-algebras have a privileged AF diagonal which is unique up to conjugacy (Krieger 80).
- An example of a Cartan subalgebra which is not a diagonal in an AF C*-algebras is given in my thesis.
- Blackadar (90) writes the CAR algebra as a crossed product $C(X) \rtimes \Gamma$, where $\Gamma$ is a locally finite group acting freely on $X = \mathbf{T} \times$ Cantor space. This exhibits a diagonal of the CAR algebra which is not AF.

# Cantor minimal systems

**Theorem (Giordano-Putnam-Skau 95)**

*Two Cantor minimal systems $(X,T)$ and $(Y,S)$ are strongly orbit equivalent if and only if the C*-algebras $C^*(X,T)$ and $C^*(Y,S)$ are isomorphic.*

On the other hand, the groupoids $G(X,S)$ and $G(X,T)$ are is isomorphic (which amounts to flip conjugacy) if and only if the Cartan pairs $(C^*(X,T), C(X))$ and $(C^*(Y,S), C(Y))$ are isomorphic.

**Example (Boyle and Handelman 94)**

The strong orbit equivalence class of the dyadic adding machine contains homeomorphisms of arbitrary entropy.

These will give the same $C^*$-algebra but the corresponding Cartan subalgebras will not be conjugate.

# Continuous trace C*-algebras

One has:

**Proposition**

*Let $B$ be a Cartan subalgebra of a continuous-trace $C^*$-algebra $A$. Then $B$ has the unique extension property.*

We recall that:

**Theorem (Green 77, Muhly-R-Williams 94)**

*Let $(G, \lambda)$ be a locally compact principal groupoid with a Haar system . TFAE*

- $C_r^*(G, \lambda)$ *has continuous trace;*
- $G$ *is a proper groupoid.*

Combining these results we see that if $A = C^*_r(G, E)$, where $G$ is étale, second countable locally compact Hausdorff and topologically principal, has continuous trace, then $G$ must be proper and principal with $G^{(0)}/G = \hat{A}$. Moreover,

**Theorem (R 85)**

*Let $(R, E)$ and $(S, F)$ be two twists with $R, S$ étale, second countable locally compact proper and principal. Suppose that $C^*(R, E)$ and $C^*(S, F)$ are isomorphic. Then*

- *$(R, E)$ and $(S, F)$ are Morita equivalent.*
- *The Dixmier-Douady class of $C^*_r(G, E)$ is the image of $[E]$ in $H^3(G^{(0)}/G, \mathbb{Z})$.*

The proof exhibits $(R, E)$ as a reduction of the dual groupoid $R(A), E(A)$ of $A = C^*(R, E)$. This is not quite the expected uniqueness.
Question: Are two Cartan subalgebras of a continuous trace C*-algebra necessarily conjugate?

Here are two results about the existence of Cartan subalgebras in continuous trace C*-algebras.

**Proposition (Raeburn-Taylor 85, R 85)**

*Given a second countable locally compact Hausdorff space $T$ and $\delta \in H^3(T, \mathbb{Z})$, there exists a continuous trace C*-algebra $A$ possessing a Cartan subalgebra and realizing $\delta$ as its Dixmier-Douady invariant.*

**Example (Natsume in Kumjian 85)**

There exists a continuous trace C*-algebra which does not possess a Cartan subalgebra.

Natsume's example is the C*-algebra of compact operators of a continuous field of Hilbert spaces $H \to T$, where $T$ is connected and simply connected and $H$ does not decompose as a direct sum of line bundles.

# A quiz

Among the following subalgebras, which ones are Cartan subalgebras, which ones are diagonals?

$$A = \{f : [0,1] \to M_2(\mathbb{C}) \text{ continuous}\} \quad B = \{f \in A : \forall t, f(t) \in D_2(\mathbb{C})\}$$

$$A_1 = \{f \in A : f(0) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\} \quad B_1 = B \cap A_1$$

$$A_2 = \{f \in A : f(0) = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\} \quad B_2 = B \cap A_2$$

$$A_3 = \{f \in A : f(0) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\} \quad B_3 = B \cap A_3$$

# Answer to Exercise 1

$$A_1 = \{f \in A : f(0) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\}$$

$B_1$ is a Cartan subalgebra which does not have the unique extension property: the states $f \mapsto a \pm b$ both extend the pure state $f \mapsto a$ of $B$.

$A_1$ is the C*-algebra of the groupoid of germs of the map $T(x) = -x$ on $[-1, 1]$. Explicitly, $A_1 = C^*(G)$ where $G = \{(\pm x, \pm 1, x), x \in [-1, 1]\}$ is topologically principal but not principal.

# Answer to Exercise 2

$$A_2 = \{f \in A : f(0) = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\}$$

$B_2$ is a masa which is not a Cartan subalgebra.

The C*-algebra $A_2$ can be realized as $C^*(R, \lambda)$ where $R$ is the graph of the equivalence relation $y = \pm x$ on $[-1, 1]$. This is a closed subset of the product $[0, 1] \times [0, 1]$. We endow it with the product topology. It is a proper groupoid which is not étale. It has the Haar system

$$\int f d\lambda^x = f(x, x) + f(x, -x).$$

# Answer to Exercise 3

$$A_3 = \{f \in A : f(0) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}$$

$B_3$ is a diagonal in $A_3$.

The C*-algebra $A_3$ can be realized as $C^*(R_\tau)$ where $R_\tau$ is again the graph $R$ of the equivalence relation $y = \pm x$ on $[-1, 1]$. However, we endow it with a topology finer than the product topology to make it étale. Following Molberg 06, we consider the topology generated by the product topology and the diagonal $\{(x, x), x \in \mathbb{R}\}$. Then $R_\tau$ is étale but no longer proper.